Md. Abdus Salam

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ISBN 978-981-4585-65-1 ISBN 978-981-4585-66-8 (eBook)
DOI 10.1007/978-981-4585-66-8
Springer Singapore Heidelberg New York Dordrecht London

Library of Congress Control Number: 2014932423

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

To all my teachers, colleagues, and students who have encouraged and helped me to develop professionally over the years. Also, to my wife Asma Ara Bagum, my son Syeed Hasan, and my daughters, Yusra binti Salam and Sundus Salam for their love, patience, and support.

Preface

Electrical engineering plays an important role in modernizing human life and encompasses wide areas such as: generation, transmission, and distribution of electrical power, digital systems, satellite communications, signal processing, robotics, mechatronics, computer, control, artificial intelligence, and networks.

A 4 year electrical and electronic engineering curriculum normally contains two modules of electromagnetic field theories during the first 2 years. However, some curricula do not have enough slots to accommodate the two modules. This book, *Electromagnetic Field Theories*, is designed for electrical and electronic engineering undergraduate students to provide fundamental knowledge of electromagnetic fields and waves in a structured manner. A comprehensive fundamental knowledge of electric and magnetic fields is required to understand the working principles of generators, motors, and transformers. This knowledge is also necessary to analyze transmission lines, substations, insulator flashover mechanism, transient phenomena, etc.

This book is written in a simple way so that the students will find it easy to understand the electromagnetic field theory and its application in electrical engineering. Several worked out examples are included to enhance the understanding of electromagnetic field theories. Each chapter also includes several practice problems with answers given at the end of the book, which would facilitate students' understanding.

The basic parameters in electromagnetic fields are discussed in Chap. 1, while vector calculus and orthogonal coordinate systems are explained in Chap. 2. In Chap. 3, the basics of electrostatics, Coulomb's law, electric field intensity, Gauss' law, Ohm's law, and energy have been discussed. Poisson's and Laplace's equations, uniqueness theorem, and their analysis on geometric shapes have been introduced in Chap. 4. The current and its density, resistance, capacitance, continuity equation, etc., have been discussed in Chap. 5. Chapter 6 explains Lorentz's force, magnetic flux density, Biot–Savart law, Ampere's circuital law, vector magnetic potential, air gap, and series and parallel magnetic circuit. Faraday's law, conduction current, displacement current, Maxwell's equation, and basics of transformer, have been discussed in Chap. 7. Chapter 8 deals with transmission line equations, velocity of wave propagation, wavelengths, lossless propagation, distortionless transmission line, power, and Smith chart. Plane waves and its analysis are included in Chap. 9, and basics of antenna have been discussed in Chap. 10.

Features

Several textbooks on electromagnetic theories already exist in the market. However, the book on Electromagnetic Field Theories for Engineering is written for electrical and electronic engineering students with the following key features.

- Easy and logical presentation of each article
- Interpretation of each theory with proper mathematical expressions
- Emphasis on engineering mathematics to understand electromagnetic field theories
- Detailed description of fundamental laws of electromagnetic field theories
- Step-by-step problem solving procedures
- Inclusion of solved examples and practice problems
- Large number of exercise problems at the end of each chapter
- Inclusion of answers to practice and exercise problems

Aids for Instructors

The solution manual will be provided to instructors who will adopt this as a textbook, and they may obtain the solution manual by directly contacting the publishers.

Acknowledgments

The author has written this textbook based on his years of teaching experience. The author would like to acknowledge with gratitude the following faculty members for their inspiration, comments, and suggestions during the preparation of the first edition of this book.

Dr. M. H. Rashid, Professor, University of West Florida, USA
Dr. Akhtar Kalam, Professor, Victoria University, Australia
Dr. M. Saifur Rahman, Professor, Virginia Tech and State University, USA
Dr. Jim Cathey, Professor, University of Kentucky, Lexington, USA
Dr. Mohammod Ali, Professor, University of South Carolina, USA
Dr. M. Bashir Uddin, Professor, Dhaka University of Engineering and Technology, Gazipur, Bangladesh
Dr. M. A. Rahman, Professor, Memorial University of Newfoundland, Canada
Dr. S. M. Islam, Professor, Curtin University of Technology, Perth, Australia
Dr. Hussein Ahmad, Professor, Universiti Teknologi Malaysia
Dr. M. M. A. Hashem, Professor, Khulna University of Engineering and Technology, Bangladesh
Dr. Khaled Ellithy, Professor, Qatar University, Doha, Qatar
Dr. Saleh Al Alawi, Professor, Sultan Qaboos University, Oman

Dr. Md. Rafiqul Islam, Associate Professor, International Islamic University, Malaysia

Dr. Mohammad A. Kashem, Associate Professor, University of Wollongong, NSW, Australia

Dr. Quazi Delwar Hossain, Assistant Professor, Chittagong University of Engineering and Technology, Bangladesh

Dr. Q. M. Rahman, Assistant Professor, Western University, London, Canada

The author would like to thank Dk. Nurul Saidatul Mirzuana and Umi Farina, B. Eng. students for typing the solution manual of this book. The author would also like to thank Dr. Loyola D'Silva, Production Manager, Springer Asia Pvt. Ltd., Singapore, and the production staff for their help in bringing the first edition of the book to completion.

Brunei Darussalam Md. Abdus Salam PhD
January 2014

Contents

Chapter 1
Basics of Electromagnetics

1.1 Introduction

The word electromagnetic field is the combination of electric and magnetic fields. An electromagnetic field is sometimes known as EM field, and it is generated when charged particles are at rest or in motion. There are two types of charges in elementary physics, namely the positive charge and the negative charge. An electric field depends mainly on these charges. The rate of change of charge generates current, which produces a magnetic field. A field is a special distribution of a parameter, which may or may not be a function of time. Time varying electric and magnetic fields are joined together to form an EM field. The time-dependent EM field produces a wave that radiates from the source.

The arcs or sparks are produced when the surface potential gradient (electric field) of a conductor exceeds the dielectric strength of the surrounding air. These arcs transmit energy to a certain distance. This kind of phenomena leads scientists or engineers to work on communication systems. Principles of EM fields are applied in designing microwaves, antennae, electric machines, communication systems, and bioelectromagnetic and remote sensing systems. Some tools are required to study EM fields. These include imagination, vector algebra, coordinate systems and transformation. In this chapter, different parameters of EM fields will be discussed.

1.2 Field Parameters and SI Units

Electric charges and currents produce electric and magnetic fields. It is very important to define all the field parameters and their standard units to get fundamental knowledge of electromagnetism. The notation and units of EM field parameters are mentioned in Table 1.1. The charge density is defined as the fixed amount of charge per unit volume. The charge density is categorized as volume, surface and line charge densities. For surface, the charge Δq is identified by an element whose area is Δs. Similarly, for line, the charge Δq is identified by an element whose length is Δl. The volume, surface and line charge densities can be expressed as

Md. A. Salam, *Electromagnetic Field Theories for Engineering*,
DOI 10.1007/978-981-4585-66-8_1,

Table 1.1 Symbols and units of field parameters

Name of field parameter	Notation/Symbol	Unit
Electric field intensity	E	V/m
Electric flux density or electric displacement	D	C/m
Magnetic field intensity	H	A/m
Magnetic flux density	B	Wb/m^2 or Tesla
Charge density	ρ	C/m^3
Current density	J	A/m^2

$$\rho_v = \lim_{\Delta v \to 0} \frac{\Delta q}{\Delta v} (\mathrm{C/m^3}) \tag{1.1}$$

$$\rho_s = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} (\mathrm{C/m^2}) \tag{1.2}$$

$$\rho_l = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} (\mathrm{C/m}) \tag{1.3}$$

The time rate of change of charge is known as current. The current is symbolised by I and its unit of measure is C/s or A.

In electromagnetic modelling, four fundamental units are required. These are length, mass, time and current. The SI (International System of Units) unit is often known as MKSA system, which is derived from the four basic units as mentioned in Table 1.2.

1.2.1 Electric Flux Density and Field Intensity

There is a direct relationship between the electric flux density and the electric field intensity. The relationship between the electric flux density and the electric field intensity is represented as

$$D = \varepsilon E, \tag{1.4}$$

where ε is the proportionality constant and it is known as permittivity of the medium. The permittivity of any medium is defined as

$$\varepsilon = \varepsilon_0 \varepsilon_\mathrm{r}, \tag{1.5}$$

where

ε_0 is the permittivity of the free space and its value is 8.854×10^{-12} F/m and
ε_r is the relative permittivity of the medium.

Substituting Eq. (1.5) into Eq. (1.4) yields

$$D = \varepsilon_0 \varepsilon_\mathrm{r} E. \tag{1.6}$$

Table 1.2 Symbols and units of basic parameters

Name of basic parameter	Notation/Symbol	Unit
Length	L	m
Mass	M	kg
Time	T	s
Current	I	A

From Eq. (1.6), it is observed that either electric field or electric flux density can be determined if the other related parameters are known.

Example 1.1 The electric field intensity of a porcelain insulator is found to be 200 V/m. Determine the value of the electric flux density. Consider that the relative permittivity of the porcelain insulator is 5.7.

Solution The relative permittivity of the porcelain insulator is given as

$\varepsilon_r = 5.7$ and

$\varepsilon_0 = 8.854 \times 10^{-12}\text{F/m}$.

The value of the electric flux density is

$D = \varepsilon_0\varepsilon_r E = 8.854 \times 10^{-12} \times 5.7 \times 200 = 1.009 \times 10^{-8}\text{C/m}$

Practice problem 1.1 The electric flux density of a glass insulator is found to be 1.05×10^{-7} C/m. Calculate the electric field intensity if $\varepsilon_r = 8$ for the glass insulator.

1.2.2 Magnetic Flux Density and Field Intensity

After a series of experiments, it is found that there is a direct relationship between the magnetic flux density and the magnetic field intensity. The relationship between the magnetic flux density and magnetic field intensity is

$$B = \mu H, \tag{1.7}$$

where μ is the proportionality constant and it is known as permeability. The permeability of any medium can be expressed as

$$\mu = \mu_0\mu_r, \tag{1.8}$$

where

μ_0 is the permeability of the free space and its value is $4\pi \times 10^7$ H/m and

μ_r is the relative permeability of medium.

Substituting Eq. (1.8) into Eq. (1.7) yields

$$B = \mu_0\mu_r H. \tag{1.9}$$

Example 1.2 The magnetic field intensity of a cobalt material is found to be 300 A/m. Determine the value of the magnetic flux density if $\mu_r = 600$ for a cobalt material.

Solution For a cobalt material, the relative permeability is given as

$\mu_r = 600$

$\mu_0 = 4\pi \times 10^{-7}\text{H/m}$

The value of the magnetic flux density can be determined as

$B = \mu_0\mu_r H = 4\pi \times 10^{-7} \times 600 \times 300 = 2.26 \times 10^{-3}\text{Wb/m}^2$

Practice problem 1.2 The magnetic flux density of a pure iron material is found to be 0.25 Wb/m^2. Determine the magnetic field intensity if $\mu_r = 4000$ for an iron material.

1.2.3 Current Density

According to Ohm's law, the current density of a material is directly related to the electric field intensity. The current density is symbolised by the letter J and its unit is A/m^2. This relation can be expressed as

$$J = \sigma E, \tag{1.10}$$

where

σ is the conductivity of material in S/m and

J is the current density in A/m^2.

The detailed analysis of current density has been discussed in Chap. 5.

Example 1.3 The electric field intensity of a copper wire is found to be 0.15 V/m. Determine the value of the current density if $\sigma = 5 \times 10^7\text{S/m}$for a copper wire.

Solution The value of the current can be determined as, $J = \sigma E = 5 \times 10^7 \times 0.15 = 7.5 \times 10^6\text{A/m}^2$

Practice problem 1.3 An aluminium wire carries a current density of 1.5×10^6 A/m^2. Calculate the electric field intensity if the conductivity of the aluminium is found to be 3.54×10^7 S/m.

1.3 Exercise Problems

1.1 The electric flux density of a porcelain insulator is found to be 1.12×10^{-8} C/m. Determine the value of the electric field intensity. The relative permittivity of the porcelain insulator is 5.7.

1.2 The electric flux density of a glass plate is found to be 1.05×10^{-8} C/m. Calculate the relative permittivity of the plate if the electric field intensity is 148.24 V/m.

1.3 The electric field intensity of a copper wire is found to be 31.43 V/m. Determine the value of the current density if $\sigma = 5 \times 10^7\text{S/m}$ for copper wire.

1.4 An aluminium wire carries a current density of 3.4×10^5 A/m^2. Calculate the electric field intensity if the conductivity of aluminium is 3.54×10^7 S/m .

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Chapter 2
Vector Analysis and Coordinate Systems

2.1 Introduction

Vector analysis is a mathematical tool that is used in expressing and simplifying the related laws and theorems of electric and magnetic fields. The electric and magnetic fields are vector quantities. The characteristics of these fields are analysed by a set of laws known as Maxwell's equations. The basic knowledge of vectors is important to formulate Maxwell's equations and to apply in the practical field. Vector addition, subtraction, multiplication and division will be discussed in this chapter. In addition, the three most orthogonal coordinate systems, namely Cartesian, cylindrical and spherical will also be discussed to deeply understand electromagnetic fields and waves.

2.2 Vectors and Scalars

Knowledge of vectors and scalars is important when analysing electromagnetic fields. A vector is a quantity that has both magnitude and direction. Vectors are represented by boldface roman-type symbols (**A**). An arrow on the top of the letter often represents vector ($\overrightarrow{A}$). The magnitude of the vector is represented by $|\mathbf{A}|$ or simply A. Displacement, velocity, force and acceleration are examples of vectors. Different vectors with directions are shown in Fig. 2.1.

A vector field is a function that specifies a vector quantity everywhere in a region. Examples are gravitational force on a body in space and the displacement of a plane in space. A scalar is a quantity with magnitude but no direction. Length, mass, time, temperature and any real number are examples of scalar quantities. A scalar field is a function that specifies a scalar quantity everywhere in a region. Examples are temperature distribution and electric potential in a room.

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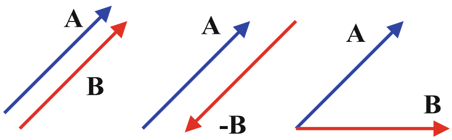


Fig. 2.1 Vectors with directions

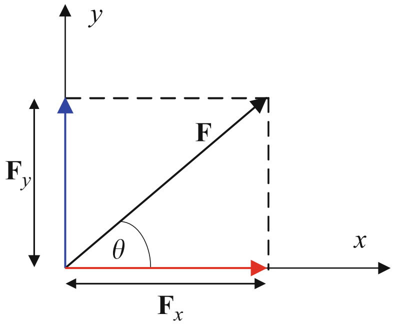


Fig. 2.2 Vectors with directions

2.3 Vector Components

A vector can be resolved into two components, namely the horizontal component and the vertical component. The addition of these two components is equal to the original vector. In Fig. 2.2, a vector **F** is working at an angle of θ with the x-axis. The x-axis component of this vector is

$$\mathbf{F}_x = \mathbf{F}\cos\theta. \tag{2.1}$$

The y-axis component is

$$\mathbf{F}_y = \mathbf{F}\sin\theta. \tag{2.2}$$

Vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ are working at angles of θ_1 and θ_2 with the x-axis, respectively, which are shown in Fig. 2.3. Here, the x-axis and y-axis components are

$$\mathbf{F}_{x1} = \mathbf{F}_1\cos\theta_1 \tag{2.3}$$

$$\mathbf{F}_{x2} = \mathbf{F}_2\cos\theta_2 \tag{2.4}$$

$$\mathbf{F}_{y1} = \mathbf{F}_1\sin\theta_1 \tag{2.5}$$

$$\mathbf{F}_{y2} = \mathbf{F}_2\sin\theta_2 \tag{2.6}$$

The sum of the horizontal components is

$$\mathbf{F}_x = \mathbf{F}_{x1} + \mathbf{F}_{x2}. \tag{2.7}$$

Substituting Eqs. (2.3) and (2.4) into Eq. (2.7) yields

$$\mathbf{F}_x = \mathbf{F}_1\cos\theta_1 + \mathbf{F}_2\cos\theta_2. \tag{2.8}$$

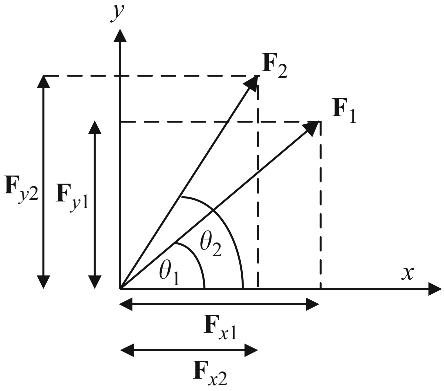


Fig. 2.3 Two vectors with directions

The sum of the vertical components is

$$\mathbf{F}_y = \mathbf{F}_{y1} + \mathbf{F}_{y2}. \tag{2.9}$$

Substituting Eqs. (2.5) and (2.6) into Eq. (2.9) yields

$$\mathbf{F}_y = \mathbf{F}_1 \sin\theta_1 + \mathbf{F}_2 \sin\theta_2. \tag{2.10}$$

Finally, the resultant vector can be determined as

$$\mathbf{F}_r = \sqrt{\mathbf{F}_x{}^2 + \mathbf{F}_y{}^2}. \tag{2.11}$$

2.4 Unit Vectors

A unit vector is a vector whose magnitude is 1. Unit vectors in three directions are $\mathbf{a}_x, \mathbf{a}_y$ and $\mathbf{a}_z$ as shown in Fig. 2.4.

The magnitudes of three unit vectors are

$$\mathbf{a}_x = (1{,}0, 0), \tag{2.12}$$

$$\mathbf{a}_y = (0{,}1, 0)\ and \tag{2.13}$$

$$\mathbf{a}_z = (0{,}0, 1). \tag{2.14}$$

A general representation of a vector $\mathbf{A}$ is shown in Fig. 2.5. The unit vector $\mathbf{a}_A$ is working in the same direction as the vector $\mathbf{A}$. The unit vector can be expressed as

$$\mathbf{a}_A = \frac{\mathbf{A}}{|\mathbf{A}|} = 1, \tag{2.15}$$

Fig. 2.4 Three unit vectors

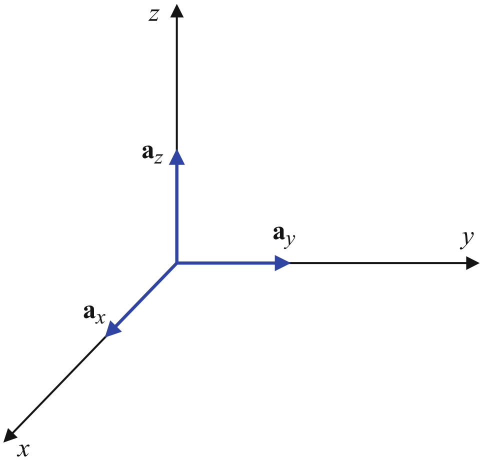
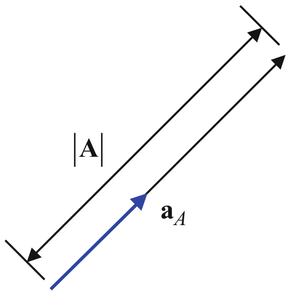


Fig. 2.5 Unit vector representation



where
$|\mathbf{A}|$ is the magnitude of the vector **A**.

2.5 Vector Addition

Consider that the vector **A** has three components A_x, A_y and A_z in the x, y and z directions, respectively. According to Fig. 2.6, the vectors $A_x\mathbf{a}_x$, $A_y\mathbf{a}_y$ and $A_z\mathbf{a}_z$ are the components of the vector **A** in the x, y and z directions, respectively. The resultant of these vectors is

$$\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z. \tag{2.16}$$

Similarly, vectors **B** and **C** can be expressed as

$$\mathbf{B} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z \; and \tag{2.17}$$

$$\mathbf{C} = C_x\mathbf{a}_x + C_y\mathbf{a}_y + C_z\mathbf{a}_z. \tag{2.18}$$

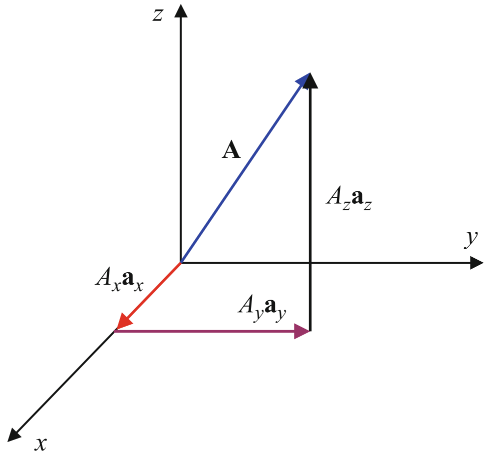


Fig. 2.6 Three component vectors

The magnitude of the vector A can be written as

$$A = |\mathbf{A}| = \sqrt{{A_x}^2 + {A_y}^2 + {A_z}^2}. \tag{2.19}$$

The unit vector in the direction of the vector A is

$$\mathbf{a}_A = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z}{\sqrt{{A_x}^2 + {A_y}^2 + {A_z}^2}}. \tag{2.20}$$

Vector addition can be obtained by parallelogram and nose-to-tail or head-to-tail rules. Two vectors **A** and **B** started from the same point as shown in Fig. 2.7. The resultant vector can be calculated as

$$\mathbf{R}_a = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}. \tag{2.21}$$

Substituting Eqs. (2.16) and (2.17) into Eq. (2.21) yields

$$\mathbf{R}_a = (A_x + B_x)\mathbf{a}_x + (A_y + B_y)\mathbf{a}_y + (A_z + B_z)\mathbf{a}_z. \tag{2.22}$$

2.6 Vector Subtraction

Vector subtraction is defined as the special addition of two vectors. Consider two vectors B and C for vector subtraction as shown in Fig. 2.8. The vector subtraction can be represented as

$$\mathbf{R}_s = \mathbf{B} - \mathbf{C} = \mathbf{B} + (-\mathbf{C}), \tag{2.23}$$

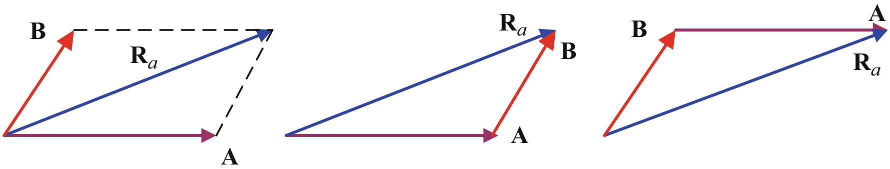
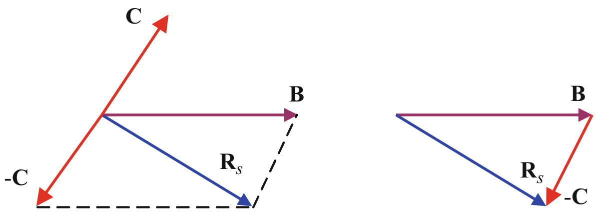


Fig. 2.7 Parallelogram and head-to-tail rules

Fig. 2.8 Parallelogram and head-to-tail rules



where
$-\mathbf{C}$ is the negative vector of $\mathbf{C}$.

Substituting Eqs. (2.17) and (2.18) into Eq. (2.23) provides

$$\mathbf{R}_s = (B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z) + (-C_x\mathbf{a}_x - C_y\mathbf{a}_y - C_z\mathbf{a}_z), \quad (2.24)$$

$$\mathrm{R}_s = (B_x - C_x)\mathrm{a}_x + (B_y - C_y)\mathrm{a}_y + (B_z - C_z)\mathrm{a}_z. \quad (2.25)$$

Example 2.1 Three vectors are given by $\mathbf{A} = 4\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z$, $\mathbf{B} = 2\mathbf{a}_x - 5\mathbf{a}_y - 4\mathbf{a}_z$ and $\mathbf{C} = -\mathbf{a}_x + 3\mathbf{a}_y + 6\mathbf{a}_z$, respectively. Determine the magnitude of (1) $\mathbf{R}_a = \mathbf{A} + \mathbf{B}$ and (2) $\mathbf{R}_s = \mathbf{B} - \mathbf{C}$.

Solution

1. The magnitude of $\mathbf{R}_a$ can be determined as

$$\mathbf{R}_a = 4\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z + 2\mathbf{a}_x - 5\mathbf{a}_y - 4\mathbf{a}_z = 6\mathbf{a}_x - 8\mathbf{a}_y - 3\mathbf{a}_z,$$

$$|\mathbf{R}_a| = \sqrt{6^2 + (-8)^2 + (-3)^2} = 10.44.$$

2. The magnitude of $\mathbf{R}_s$ can be calculated as

$$\mathbf{R}_s = 2\mathbf{a}_x - 5\mathbf{a}_y - 4\mathbf{a}_z + \mathbf{a}_x - 3\mathbf{a}_y - 6\mathbf{a}_z = 3\mathbf{a}_x - 8\mathbf{a}_y - 10\mathbf{a}_z,$$

$$|\mathbf{R}_s| = \sqrt{3^2 + (-8)^2 + (-10)^2} = 13.15.$$

Example 2.2 A unit vector is parallel to the resultant (addition) vector of $\mathbf{A} = 2\mathbf{a}_x + 3\mathbf{a}_y + 6\mathbf{a}_z$ and $\mathbf{B} = 5\mathbf{a}_x - \mathbf{a}_y - 2\mathbf{a}_z$. Determine the unit vector.

Solution The resultant vector can be determined as

$$\mathbf{R}_a = \mathbf{A} + \mathbf{B} = 7\mathbf{a}_x + 2\mathbf{a}_y + 4\mathbf{a}_z.$$

The unit vector can be calculated as

$$\mathbf{a}_u = \frac{\mathbf{R}_a}{|\mathbf{R}_a|} = \frac{7\mathbf{a}_x + 2\mathbf{a}_y + 4\mathbf{a}_z}{\sqrt{7^2 + 2^2 + 4^2}} = 0.84\mathbf{a}_x + 0.24\mathbf{a}_y + 0.48\mathbf{a}_z.$$

Practice Problem 2.1 Three vectors are given by $\mathbf{A} = 2\mathbf{a}_x + 5\mathbf{a}_y - 3\mathbf{a}_z, \mathbf{B} = 3\mathbf{a}_x - 4\mathbf{a}_y - 2\mathbf{a}_z$ and $\mathbf{C} = \mathbf{a}_x - 2\mathbf{a}_y + \mathbf{a}_z$, respectively. Determine the magnitude of (1) $\mathbf{R}_a = \mathbf{A} + \mathbf{B}$ and (2) $\mathbf{R}_s = \mathbf{B} - \mathbf{C}$.

Practice Problem 2.2 Calculate the unit vector which is parallel to the resultant (subtraction) vector of $\mathbf{A} = 3\mathbf{a}_x - 2\mathbf{a}_y + 3\mathbf{a}_z$ and $\mathbf{B} = 2\mathbf{a}_x + 5\mathbf{a}_y + \mathbf{a}_z$.

2.7 Vectors Multiplication and Division

Multiplication of a vector **A** by a positive scalar parameter k can be expressed as

$$\mathbf{R}_m = k\mathbf{A}. \tag{2.26}$$

Substituting Eq. (2.16) into Eq. (2.26) yields

$$\mathbf{R}_m = k(A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z) \tag{2.27}$$

$$\mathbf{R}_m = kA_x\mathbf{a}_x + kA_y\mathbf{a}_y + kA_z\mathbf{a}_z. \tag{2.28}$$

Division of a vector **B** by another positive scalar parameter n can be expressed as

$$\mathbf{R}_d = \frac{\mathbf{B}}{n}. \tag{2.29}$$

Substituting Eq. (2.17) into Eq. (2.29) provides

$$\mathbf{R}_d = \frac{1}{n}(B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z) \tag{2.30}$$

$$\mathbf{R}_d = \frac{B_x}{n}\mathbf{a}_x + \frac{B_y}{n}\mathbf{a}_y + \frac{B_z}{n}\mathbf{a}_z. \tag{2.31}$$

2.8 Dot Product of Two Vectors

The dot product of two vectors **A** and **B** is represented as

$$\mathbf{R}_{dot} = \mathbf{A} \bullet \mathbf{B}. \tag{2.32}$$

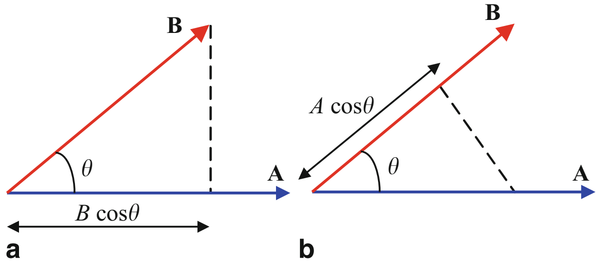


Fig. 2.9 Projections of (**a**) vectors B on A (**b**) A on B

The dot product of two vectors **A** and **B** is equal to the product of the magnitudes and the cosine of the angle between them. It can be expressed as

$$\mathbf{A} \bullet \mathbf{B} = AB \cos\theta. \tag{2.33}$$

In Fig. 2.9, the vector **A** is working in the x-axis and the vector **B** is working at an angle θ with the vector A. The projection of the vector B on the vector A is $B\cos\theta$ as shown in Fig. 2.9a. Equation (2.32) can be modified as

$$\mathbf{A} \bullet \mathbf{B} = A(B\cos\theta) = AB\cos\theta. \tag{2.34}$$

The projection of vector **A** on vector **B** is $A\cos\theta$ as shown in Fig. 2.9b. Equation (2.32) again can be modified to

$$\mathbf{A} \bullet \mathbf{B} = (A\cos\theta)B = AB\cos\theta. \tag{2.35}$$

The angle between two vectors can be determined with

$$\cos\theta = \frac{\mathbf{A} \bullet \mathbf{B}}{AB}. \tag{2.36}$$

The dot products of unit vectors are

$$\mathbf{a}_x \bullet \mathbf{a}_x = 1.1\cos 0^\circ = 1, \tag{2.37}$$

$$\mathbf{a}_x \bullet \mathbf{a}_y = 1.1\cos 90^\circ = 0, \tag{2.38}$$

$$\mathbf{a}_y \bullet \mathbf{a}_z = 1.1\cos 90^\circ = 0, \tag{2.39}$$

$$\mathbf{a}_z \bullet \mathbf{a}_x = 1.1\cos 90^\circ = 0, \tag{2.40}$$

$$\mathbf{a}_y \bullet \mathbf{a}_y = 1.1\cos 0^\circ = 1 \tag{2.41}$$

$$\mathbf{a}_z \bullet \mathbf{a}_z = 1.1\cos 0^\circ = 1. \tag{2.42}$$

Substituting Eqs. (2.16) and (2.17) into Eq. (2.32) yields

$$\mathbf{A} \bullet \mathbf{B} = (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z) \bullet (B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z), \tag{2.43}$$

$$\mathbf{A} \bullet \mathbf{B} = A_x B_x \mathbf{a}_x \bullet \mathbf{a}_x + A_y B_x \mathbf{a}_y \bullet \mathbf{a}_x + A_z B_x \mathbf{a}_z \bullet \mathbf{a}_x + A_x B_y \mathbf{a}_x \bullet \mathbf{a}_y$$
$$+ A_y B_y \mathbf{a}_y \bullet \mathbf{a}_y + A_z B_y \mathbf{a}_z \bullet \mathbf{a}_y + A_x B_z \mathbf{a}_x \bullet \mathbf{a}_z \quad (2.44)$$
$$+ A_y B_z \mathbf{a}_y \bullet \mathbf{a}_z + A_z B_z \mathbf{a}_z \bullet \mathbf{a}_z.$$

Equation (2.44) can be modified by considering unit vector properties as

$$\mathbf{A} \bullet \mathbf{B} = A_x B_x + A_y B_y + A_z B_z. \quad (2.45)$$

The dot product of two same vectors is

$$\mathbf{A} \bullet \mathbf{A} = (A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z) \bullet (A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z). \quad (2.46)$$

Equation (2.46) can be modified by applying properties of unit vectors to

$$A^2 = A_x{}^2 + A_y{}^2 + A_z{}^2. \quad (2.47)$$

Example 2.3 Two vectors are given by $\mathbf{A} = 3\mathbf{a}_x + 2\mathbf{a}_y - 4\mathbf{a}_z$ and $\mathbf{B} = 3\mathbf{a}_x - 4\mathbf{a}_y - 5\mathbf{a}_z$, respectively. Determine the dot product of two vectors.

Solution The dot product can be determined as

$$\mathbf{A} \bullet \mathbf{B} = (3\mathbf{a}_x + 2\mathbf{a}_y - 4\mathbf{a}_z) \bullet (3\mathbf{a}_x - 4\mathbf{a}_y - 5\mathbf{a}_z),$$
$$\mathbf{A} \bullet \mathbf{B} = (3)(3) + (2)(-4) + (-4)(-5) = 9 - 8 + 20 = 21.$$

Practice Problem 2.3 Determine the angle between the two vectors $\mathbf{A} = 4\mathbf{a}_x + \mathbf{a}_y - 3\mathbf{a}_z$ and $\mathbf{B} = 2\mathbf{a}_x + 4\mathbf{a}_y - 3\mathbf{a}_z$.

2.9 Cross Product of Two Vectors

The cross product is the second kind of vector multiplication. The cross product of two vectors A and B is represented as

$$\mathbf{R}_{cross} = \mathbf{A} \times \mathbf{B}. \quad (2.48)$$

The magnitude of $\mathbf{A} \times \mathbf{B}$ is defined as the product of the magnitude of **A** and **B** and the sine of the smaller angle (θ) between them. The direction of the vector $\mathbf{A} \times \mathbf{B}$ is perpendicular to both **A** and **B** as shown in Fig. 2.10. Let $\mathbf{a}_N$ be the unit vector in the direction of $\mathbf{A} \times \mathbf{B}$; then, the expression of $\mathbf{A} \times \mathbf{B}$ is

$$\mathbf{A} \times \mathbf{B} = AB \sin\theta \mathbf{a}_N. \quad (2.49)$$

In Fig. 2.10a and b, it is seen that the direction of A × B is not the same as the direction of $\mathbf{B} \times \mathbf{A}$. The direction is 180° out of phase with each other; however, the magnitude is the same. It can be expressed as

$$(\mathbf{A} \times \mathbf{B}) = -(\mathbf{B} \times \mathbf{A}). \quad (2.50)$$

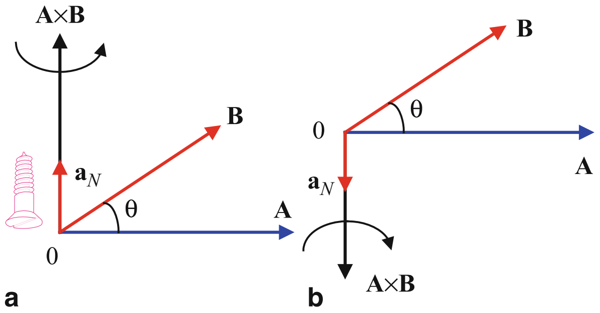


Fig. 2.10 Vectors cross product (**a**) anticlockwise direction (**b**) clockwise direction

The properties of unit vectors for the cross product can be expressed as

$$\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z, \tag{2.51}$$

$$\mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x, \tag{2.52}$$

$$\mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y \tag{2.53}$$

$$\mathbf{a}_x \times \mathbf{a}_x = \mathbf{a}_y \times \mathbf{a}_y = \mathbf{a}_z \times \mathbf{a}_z = 1.1 \sin 0° = 0. \tag{2.54}$$

The properties of unit vectors can be determined from the cyclic permutation as shown in Fig. 2.11.

Multiplying Eqs. (2.16) and (2.17) provides

$$\mathbf{A} \times \mathbf{B} = (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z) \times (B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z), \tag{2.55}$$

$$\begin{aligned} \mathbf{A} \times \mathbf{B} = {} & A_xB_x(\mathbf{a}_x \times \mathbf{a}_x) + A_yB_x(\mathbf{a}_y \times \mathbf{a}_x) + A_zB_x(\mathbf{a}_z \times \mathbf{a}_x) \\ & + A_xB_y(\mathbf{a}_x \times \mathbf{a}_y) + A_yB_y(\mathbf{a}_y \times \mathbf{a}_y) + A_zB_y(\mathbf{a}_z \times \mathbf{a}_y) \\ & + A_xB_z(\mathbf{a}_x \times \mathbf{a}_z) + A_yB_z(\mathbf{a}_y \times \mathbf{a}_z) + A_zB_z(\mathbf{a}_z \times \mathbf{a}_z), \end{aligned} \tag{2.56}$$

$$\begin{aligned} \mathbf{A} \times \mathbf{B} = {} & A_yB_x(-\mathbf{a}_z) + A_zB_x(\mathbf{a}_y) + A_xB_y(\mathbf{a}_z) + A_zB_y(-\mathbf{a}_x) \\ & + A_xB_z(-\mathbf{a}_y) + A_yB_z(\mathbf{a}_x), \end{aligned} \tag{2.57}$$

$$\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y)\mathbf{a}_x + (A_zB_x - A_xB_z)\mathbf{a}_y + (A_xB_y - A_yB_x)\mathbf{a}_z. \tag{2.58}$$

Equation (2.58) can be written in the determinant form as

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \tag{2.59}$$

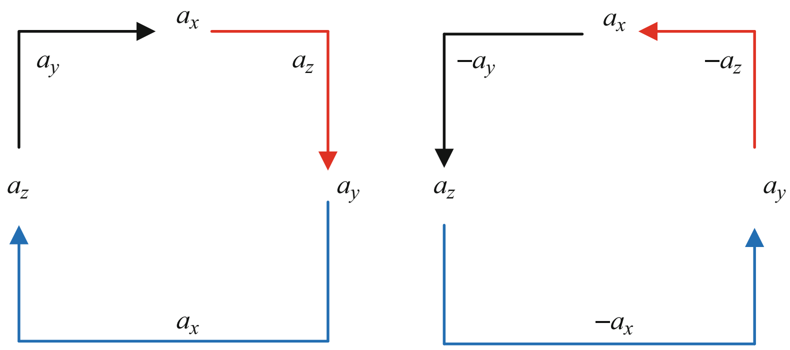


Fig. 2.11 Cyclic permutation for unit vectors

Consider three vectors **A**, **B** and **C** to analyse vector triple product. The vector triple product is defined as $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$. The triple product of vectors $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is not equal to $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$. The cross product does not have the associative property but it does have the distributive property. Therefore, this expression can be written as

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}. \tag{2.60}$$

The vector triple product can be expressed as the difference of two vectors using the back-cab rule and it can be written as

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{B}(\mathbf{A} \bullet \mathbf{C}) - \mathbf{C}(\mathbf{A} \bullet \mathbf{B}). \tag{2.61}$$

According to Eq. (2.59), the cross product of the vectors B and C is

$$\mathbf{B} \times \mathbf{C} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}. \tag{2.62}$$

The dot product of two vectors **A** and $\mathbf{B} \times \mathbf{C}$ can be expressed as

$$\mathbf{A} \bullet \mathbf{B} \times \mathbf{C} = (A_x \mathbf{a}_x + B_y \mathbf{a}_y + A_z \mathbf{a}_z) \bullet \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}. \tag{2.63}$$

Finally, Eq. (2.63) can be expressed as

$$\mathbf{A} \bullet \mathbf{B} \times \mathbf{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}. \tag{2.64}$$

Example 2.4 Two vectors are given by $\mathbf{A} = 4\mathbf{a}_x - 3\mathbf{a}_y - \mathbf{a}_z$ and $\mathbf{B} = 3\mathbf{a}_x + 2\mathbf{a}_y - 4\mathbf{a}_z$, respectively. Determine (1) $\mathbf{A} \times \mathbf{B}$ and (2) $|\mathbf{A} \times \mathbf{B}|$.

Solution

1. The cross product can be determined as

$$\mathbf{A}\times\mathbf{B}=\begin{vmatrix}\mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z\\ 4 & -3 & -1\\ 3 & 2 & -4\end{vmatrix}=(12+2)\mathbf{a}_x-(-16+3)\mathbf{a}_y+(8+9)\mathbf{a}_z,$$

$$\mathbf{A}\times\mathbf{B}=14\mathbf{a}_x+13\mathbf{a}_y+17\mathbf{a}_z.$$

2. The magnitude of the cross product can be determined as

$$|\mathbf{A}\times\mathbf{B}|=\sqrt{14^2+13^2+17^2}=25.57.$$

Example 2.5 Three vectors are given by $\mathbf{A}=2\mathbf{a}_x+3\mathbf{a}_y-4\mathbf{a}_z, \mathbf{B}=\mathbf{a}_x+3\mathbf{a}_y-5\mathbf{a}_z$ and $\mathbf{C}=3\mathbf{a}_x+4\mathbf{a}_y-6\mathbf{a}_z$, respectively. Determine the vector $\mathbf{A}\bullet\mathbf{B}\times\mathbf{C}$.

Solution (1) The cross product can be determined as

$$\mathbf{A}\bullet\mathbf{B}\times\mathbf{C}=\begin{vmatrix}2 & 3 & -5\\ 1 & 3 & -5\\ 3 & 4 & -6\end{vmatrix}=2(-18+20)-3(-6+15)-5(4-9)=2.$$

Practice Problem 2.4 Determine the magnitude of the cross product of two vectors $\mathbf{A}=2\mathbf{a}_x+3\mathbf{a}_y-4\mathbf{a}_z$ and $\mathbf{B}=\mathbf{a}_x+2\mathbf{a}_y+3\mathbf{a}_z$.

Practice Problem 2.5 Three vectors are given by $\mathbf{A}=\mathbf{a}_x+2\mathbf{a}_y+3\mathbf{a}_z, \mathbf{B}=2\mathbf{a}_x-3\mathbf{a}_y+4\mathbf{a}_z$ and $\mathbf{C}=\mathbf{a}_x+2\mathbf{a}_y-7\mathbf{a}_z$. Determine the vector $\mathbf{A}\bullet\mathbf{B}\times\mathbf{C}$.

2.10 Orthogonal Coordinate Systems

Orthogonal coordinate systems are very important to determine the electric field at a certain point in space. An orthogonal system is one in which coordinates are mutually perpendicular. Coordinate systems are normally applied to identify the specific point and its source. Many orthogonal coordinate systems are available, but the three most common and useful coordinate systems will be discussed here. These are mentioned as follows:

- Cartesian or rectangular coordinates
- Cylindrical coordinates
- Spherical coordinates

2.10.1 Cartesian Coordinate System

The rectangular or Cartesian coordinate system is the most common and simple one. Three mutually perpendicular coordinate axes are used to determine the location of

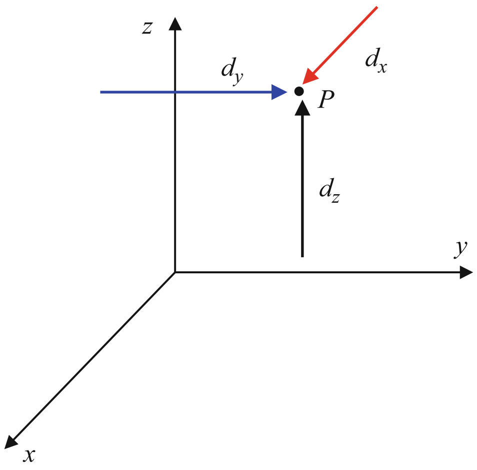


Fig. 2.12 Point in space

a point in space as shown in Fig. 2.12. Normally, two axes are used to determine the coordinate plane. From Fig. 2.12, the following points can be written:

d_x represents the x-distance from the yz-plane,
d_y represents the y-distance from the xz-plane and
d_z represents the z-distance from the yx-plane.

Two more vectors are discussed here to understand the coordinate systems. These are position and distance vectors. The position vector is defined as the direct distance between the origin and a coordinate point in space. A distance vector is defined as the direct distance between two coordinate points in space. The vector r is working from the origin to the point $P(x_1,\ y_1,\ z_1)$ as shown in Fig. 2.13. Then, the position vector can be written as

$$\mathbf{OP} = x_1\mathbf{a}_x + y_1\mathbf{a}_y + z_1\mathbf{a}_z. \tag{2.65}$$

The magnitude of the position vector is

$$|\mathbf{OP}| = \sqrt{{x_1}^2 + {y_1}^2 + {z_1}^2}. \tag{2.66}$$

The vector **A** in Cartesian coordinates can be expressed as

$$\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z = (A_x, A_y, A_z). \tag{2.67}$$

The dot product and cross product of two vectors are already mentioned in Eqs. (2.45) and (2.58), respectively. Again, consider that two position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are working from the origin to the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, respectively, as shown in Fig. 2.14. These two position vectors can be expressed as

$$\mathbf{r}_1 = x_1\mathbf{a}_x + y_1\mathbf{a}_y + z_1\mathbf{a}_z \text{ and} \tag{2.68}$$

$$\mathbf{r}_2 = x_2\mathbf{a}_x + y_2\mathbf{a}_y + z_2\mathbf{a}_z. \tag{2.69}$$

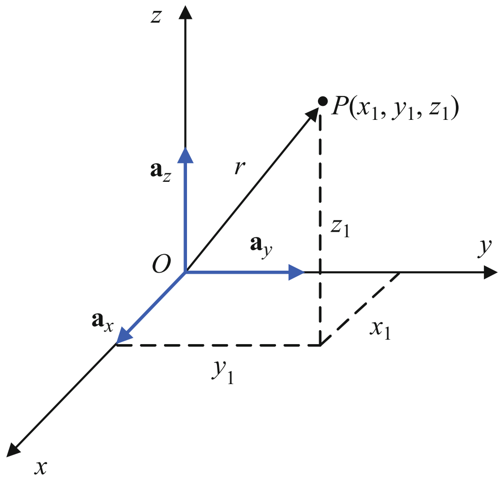


Fig. 2.13 Vector in space

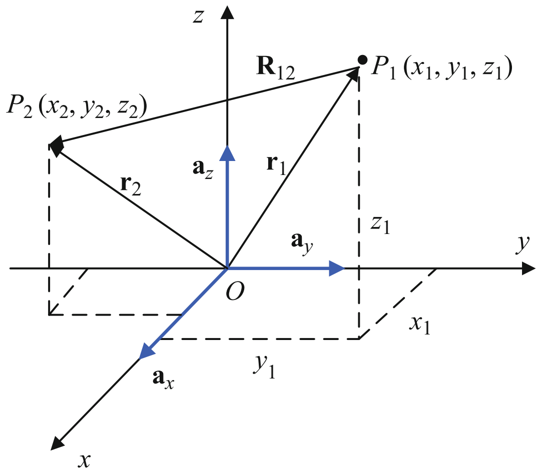


Fig. 2.14 Two position vectors

The resultant vector can be determined as

$$\mathbf{R}_{12} + \mathbf{r}_1 = \mathbf{r}_2, \tag{2.70}$$

$$\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1. \tag{2.71}$$

Substituting Eqs. (2.68) and (2.69) into Eq. (2.71) yields

$$\mathbf{R}_{12} = (x_2 - x_1)\mathbf{a}_x + (y_2 - y_1)\mathbf{a}_y + (z_2 - z_1)\mathbf{a}_z. \tag{2.72}$$

A small rectangular box is placed in the x, y and z directions as shown in Fig. 2.15. The differential surface areas of the three sides of a smaller rectangular box are

$$ds_x = dydz, \tag{2.73}$$

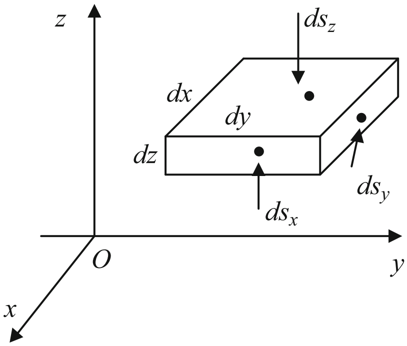


Fig. 2.15 A small rectangular box

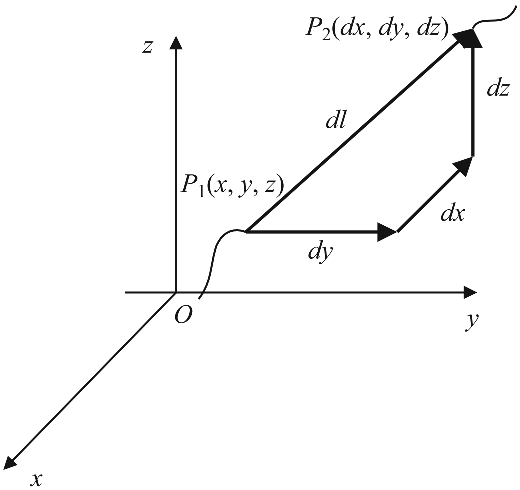


Fig. 2.16 Differential length within a path

$$ds_y = dzdx \ and \tag{2.74}$$

$$ds_z = dydx. \tag{2.75}$$

The differential volume enclosed by the small rectangular box is

$$dv = dxdydz. \tag{2.76}$$

Two points $P_1(x, y, z)$ and $P_2(dx, dy, dz)$ are considered within a path as shown in Fig. 2.16. The differential length of a path is considered as dl.

The vector for differential length is

$$d\mathbf{l} = \mathbf{a}_x dx + \mathbf{a}_y dy + \mathbf{a}_z dz. \tag{2.77}$$

From Eq. (2.77), the length of the vector can be determined as

$$dl = \sqrt{dx^2 + dy^2 + dz^2}. \tag{2.78}$$

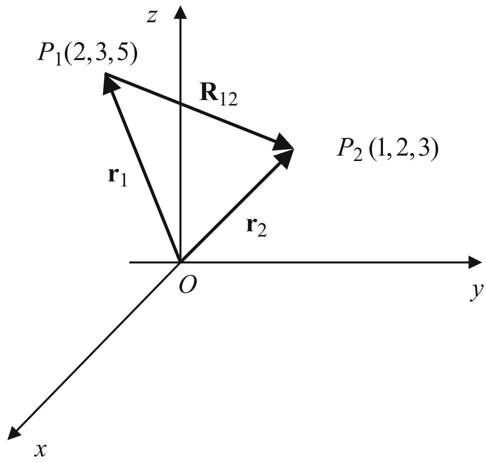


Fig. 2.17 Two points in a space

Example 2.6 Two vectors are moving from the origin to the points $P_1(2,3,5)$ and $P_2(1,2,3)$ as shown in Fig. 2.17. Write down three vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{R}_{12}$ in Cartesian coordinates and determine the distance between the two points.

Solution Three vectors in Cartesian coordinates can be written as

$$\mathbf{r}_1 = 2\mathbf{a}_x + 3\mathbf{a}_y + 5\mathbf{a}_z,$$
$$\mathbf{r}_2 = \mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z,$$
$$\mathbf{R}_{12} = (1-2)\mathbf{a}_x + (2-3)\mathbf{a}_y + (3-5)\mathbf{a}_z,$$
$$\mathbf{R}_{12} = -\mathbf{a}_x - \mathbf{a}_y - 2\mathbf{a}_z.$$

The distance between two points can be determined as

$$R_{12} = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}.$$

Practice Problem 2.6 Two vectors are moving from the origin to the points $P_1(4,2,6)$ and $P_2(2,5,3)$ as shown in Fig. 2.18. Write down the vector $\mathbf{R}_{12}$ in Cartesian coordinates and determine its distance.

2.10.2 Circular Cylindrical Coordinate System

The circular cylindrical coordinate system is normally known as the cylindrical co-ordinate system. The cylindrical coordinate usually refers to the three-dimensional polar coordinate in analytical geometry. This coordinate is represented by (ρ, ϕ, z). The ρ coordinate represents the radius of the cylinder, ϕ represents the magnitude of the circumference of the specific point on the surface of the cylinder and z represents the coordinate as represented by rectangular coordinate system. Consider a

Fig. 2.18 Points P_1 and P_2

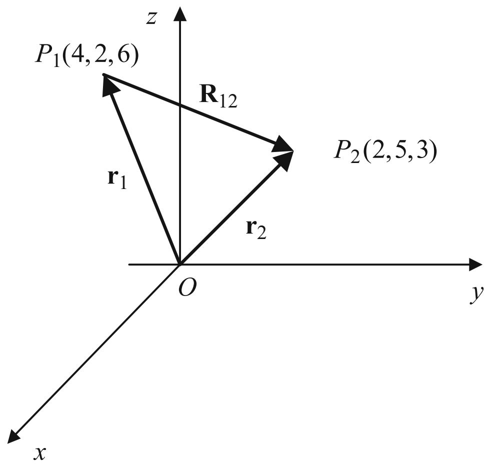
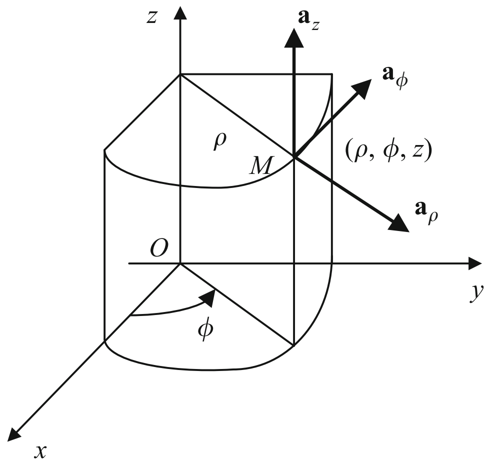


Fig. 2.19 Cylindrical coordinate with unit vectors



point $M(\rho, \phi, z)$ on the cylindrical system as shown in Fig. 2.19. The vector **A** can be written in terms of its components as

$$\mathbf{A} = A_\rho \mathbf{a}_\rho + A_\phi \mathbf{a}_\phi + A_z \mathbf{a}_z \equiv (A_\rho, A_\phi, A_z). \tag{2.79}$$

The ranges of the coordinates are

$$0 < \rho < \infty,\ 0 < \phi < 2\pi,\ -\infty < z < \infty. \tag{2.80}$$

The magnitude of the cylindrical vector is

$$|\mathbf{A}| = \sqrt{{A_\rho}^2 + {A_\phi}^2 + {A_z}^2}. \tag{2.81}$$

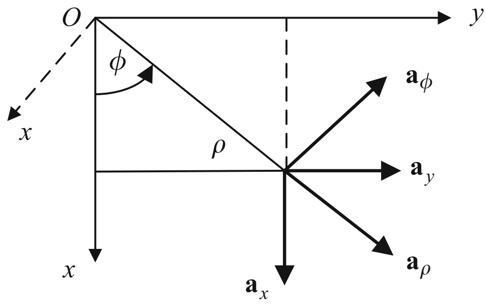


Fig. 2.20 Relation between unit vectors

In this coordinate system, the properties of unit vectors are

$$\mathbf{a}_\rho \bullet \mathbf{a}_\rho = \mathbf{a}_\phi \bullet \mathbf{a}_\phi = \mathbf{a}_z \bullet \mathbf{a}_z = 1, \tag{2.82}$$

$$\mathbf{a}_\rho \bullet \mathbf{a}_\phi = \mathbf{a}_\phi \bullet \mathbf{a}_z = \mathbf{a}_z \bullet \mathbf{a}_\rho = 0, \tag{2.83}$$

$$\mathbf{a}_\rho \times \mathbf{a}_\phi = \mathbf{a}_z, \tag{2.84}$$

$$\mathbf{a}_\phi \times \mathbf{a}_z = \mathbf{a}_\rho \tag{2.85}$$

$$\mathbf{a}_z \times \mathbf{a}_\rho = \mathbf{a}_\phi. \tag{2.86}$$

Any vector in Cartesian coordinates can be transformed into the cylindrical coordinates and vice versa. Consider Fig. 2.20 to accomplish the conversion between cylindrical and Cartesian coordinates. Replace the x-axis in a position, so that the angle between the x-axis and y-axis is 90°. From Fig. 2.20, the following relations can be written as

$$\mathbf{a}_\rho \bullet \mathbf{a}_x = \cos\phi, \tag{2.87}$$

$$\mathbf{a}_\rho \bullet \mathbf{a}_y = \cos(90^\circ - \phi) = \sin\phi, \tag{2.88}$$

$$\mathbf{a}_\phi s_y = \cos\phi, \tag{2.89}$$

$$\mathbf{a}_\phi \bullet \mathbf{a}_x = \cos(90^\circ + \phi) = -\sin\phi. \tag{2.90}$$

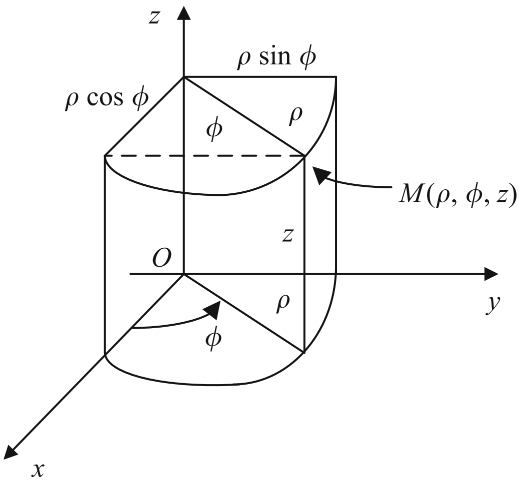
From Fig. 2.21, the relations from cylindrical to Cartesian coordinates can be written as

$$x = \rho\cos\phi, \tag{2.91}$$

$$y = \rho sin\phi \tag{2.92}$$

$$z = z. \tag{2.93}$$

Fig. 2.21 Relation between cylindrical and Cartesian coordinates



From Eqs. (2.91) and (2.92), the relations from Cartesian to cylindrical coordinates can be written as

$$\rho = \sqrt{x^2 + y^2}, \tag{2.94}$$

$$\phi = \tan^{-1}\frac{y}{x} \tag{2.95}$$

$$z = z. \tag{2.96}$$

The dot products of unit vectors in Cartesian and cylindrical coordinate systems can be determined as

$$\begin{array}{cccc} & \mathbf{a}_\rho & \mathbf{a}_\phi & \mathbf{a}_z \\ \mathbf{a}_x \bullet & \cos\phi & -\sin\phi & 0 \\ \mathbf{a}_y \bullet & \sin\phi & \cos\phi & 0 \\ \mathbf{a}_z \bullet & 0 & 0 & 1 \end{array}. \tag{2.97}$$

From Eq. (2.97), the conversion of unit vectors from cylindrical to Cartesian coordinate systems is

$$\mathbf{a}_x = \cos\phi\mathbf{a}_\rho - \sin\phi\mathbf{a}_\phi, \tag{2.98}$$

$$\mathbf{a}_y = \sin\phi\mathbf{a}_\rho + \cos\phi\mathbf{a}_\phi \tag{2.99}$$

$$\mathbf{a}_z = \mathbf{a}_z. \tag{2.100}$$

Similarly, the conversion of unit vectors from Cartesian to cylindrical coordinate systems is

$$\mathbf{a}_\rho = \cos\phi\mathbf{a}_x + \sin\phi\mathbf{a}_y, \tag{2.101}$$

$$\mathbf{a}_\phi = -\sin\phi\mathbf{a}_x + \cos\phi\mathbf{a}_y \tag{2.102}$$

$$\mathbf{a}_z = \mathbf{a}_z. \tag{2.103}$$

A vector in cylindrical coordinates can be transformed into a Cartesian coordinates as

$$A_x = \mathbf{A} \bullet \mathbf{a}_x = A_\rho \mathbf{a}_\rho \bullet \mathbf{a}_x + A_\phi \mathbf{a}_\phi \bullet \mathbf{a}_x + A_z \mathbf{a}_z \bullet \mathbf{a}_x, \tag{2.104}$$

$$A_y = \mathbf{A} \bullet \mathbf{a}_y = A_\rho \mathbf{a}_\rho \bullet \mathbf{a}_y + A_\phi \mathbf{a}_\phi \bullet \mathbf{a}_y + A_z \mathbf{a}_z \bullet \mathbf{a}_y \tag{2.105}$$

$$A_z = \mathbf{A} \bullet \mathbf{a}_z = A_\rho \mathbf{a}_\rho \bullet \mathbf{a}_z + A_\phi \mathbf{a}_\phi \bullet \mathbf{a}_z + A_z \mathbf{a}_z \bullet \mathbf{a}_z. \tag{2.106}$$

Applying Eq. (2.97) to Eqs. (2.104)–(2.106) provides

$$A_x = A_\rho \cos\phi - A_\phi \sin\phi, \tag{2.107}$$

$$A_y = A_\rho \sin\phi + A_\phi \cos\phi \tag{2.108}$$

$$A_z = A_z. \tag{2.109}$$

Equations (2.107)–(2.109) can be represented in the matrix form as

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}. \tag{2.110}$$

Similarly, a vector in Cartesian coordinates can be transformed into cylindrical coordinates as

$$A_\rho = \mathbf{A} \bullet \mathbf{a}_\rho = A_x \mathbf{a}_x \bullet \mathbf{a}_\rho + A_y \mathbf{a}_y \bullet \mathbf{a}_\rho + A_z \mathbf{a}_z \bullet \mathbf{a}_\rho, \tag{2.111}$$

$$A_\phi = \mathbf{A} \bullet \mathbf{a}_\phi = A_x \mathbf{a}_x \bullet \mathbf{a}_\phi + A_y \mathbf{a}_y \bullet \mathbf{a}_\phi + A_z \mathbf{a}_z \bullet \mathbf{a}_\phi \tag{2.112}$$

$$A_z = \mathbf{A} \bullet \mathbf{a}_z = A_x \mathbf{a}_x \bullet \mathbf{a}_z + A_y \mathbf{a}_y \bullet \mathbf{a}_z + A_z \mathbf{a}_z \bullet \mathbf{a}_z. \tag{2.113}$$

Again, applying Eq. (2.97) to Eqs. (2.111)–(2.113) provides

$$A_\rho = A_x \cos\phi + A_y \sin\phi, \tag{2.114}$$

$$A_\phi = -A_x \sin\phi + A_y \cos\phi \tag{2.115}$$

$$A_z = A_z. \tag{2.116}$$

Equations (2.114)–(2.116) can be represented in the matrix form as

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}. \tag{2.117}$$

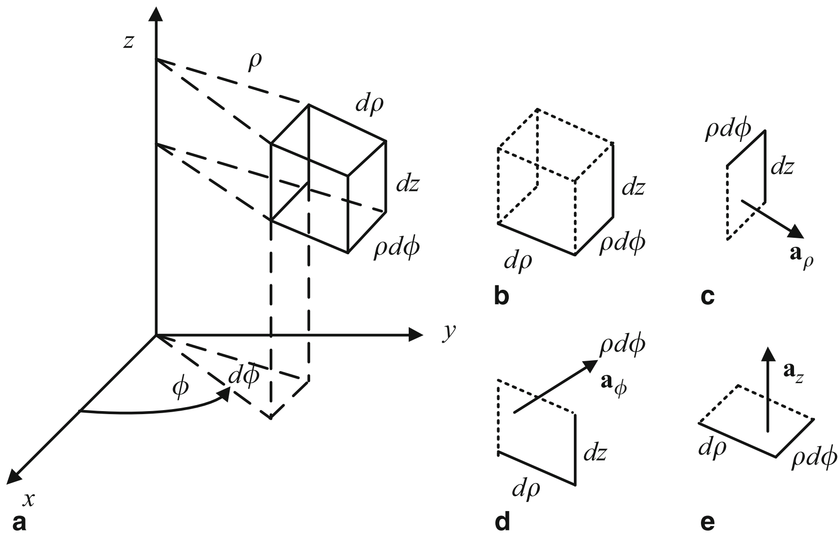


Fig. 2.22 (**a**) Cylindrical coordinates with differential element (**b**) differential element (**c**) face in $\mathbf{a}_\rho$ (d) face in $\mathbf{a}_\phi$ (**e**) face in $\mathbf{a}_z$

Consider Fig. 2.22 to derive the expressions of differential length, normal surface and volume.

From Fig. 2.22a–e, the differential length, normal surface and volume can be written as

$$d\mathbf{l} = d\rho\mathbf{a}_\rho + \rho d\phi\mathbf{a}_\phi + dz\mathbf{a}_z, \tag{2.118}$$

$$ds_\rho = \rho d\phi dz, \tag{2.119}$$

$$d\mathbf{S} = \rho d\phi dz\mathbf{a}_\rho, \tag{2.120}$$

$$ds_\phi = d\rho dz, \tag{2.121}$$

$$d\mathbf{S} = d\rho d\phi\mathbf{a}_\phi, \tag{2.122}$$

$$ds_z = \rho d\rho d\phi, \tag{2.123}$$

$$d\mathbf{S} = \rho d\rho d\phi\mathbf{a}_z \tag{2.124}$$

$$dv = \rho d\rho d\phi dz. \tag{2.125}$$

Example 2.7 A vector in Cartesian coordinates is given by $\mathbf{A} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$. Transform this vector into cylindrical coordinates.

Solution The components of cylindrical coordinates can be determined as follows

$$A_\rho = \mathbf{A} \bullet \mathbf{a}_\rho = x\mathbf{a}_x \bullet \mathbf{a}_\rho + y\mathbf{a}_y \bullet \mathbf{a}_\rho + z\mathbf{a}_z \bullet \mathbf{a}_\rho = x\cos\phi + y\sin\phi,$$

$$A_\rho = \rho\cos\phi\cos\phi + \rho\sin\phi\sin\phi = \rho,$$

$$A_\phi = \mathbf{A} \bullet \mathbf{a}_\phi = x\mathbf{a}_x \bullet \mathbf{a}_\phi + y\mathbf{a}_y \bullet \mathbf{a}_\phi + z\mathbf{a}_z \bullet \mathbf{a}_\phi = -x\sin\phi + y\cos\phi,$$

$$A_\phi = -\rho\cos\phi\sin\phi + \rho\sin\phi\cos\phi = 0,$$

$$A_z = \mathbf{A} \bullet \mathbf{a}_z = x\mathbf{a}_x \bullet \mathbf{a}_z + x\mathbf{a}_y \bullet \mathbf{a}_z + z\mathbf{a}_z \bullet \mathbf{a}_z = z.$$

A new vector in cylindrical coordinates is

$$\mathbf{A} = \rho\mathbf{a}_\rho + z\mathbf{a}_z.$$

Example 2.8 A point in cylindrical coordinates is given as $P(\rho = 2.4,\ \phi = 130°,\ z = 2)$. Convert this point into Cartesian coordinates.

Solution The point in Cartesian coordinates can be determined as

$$x = \rho\cos\phi = 2.2\cos 130° = -1.414,$$

$$y = \rho\sin\phi = 2.2\sin 130° = 1.685$$

$$z = 2.$$

The final point is

$$P(x = -1.414,\ y = 1.685,\ z = 2).$$

Practice Problem 2.7 A vector in Cartesian coordinates is given by $\mathbf{A} = 2x\mathbf{a}_x - 3y\mathbf{a}_y + z\mathbf{a}_z$. Transform this vector into cylindrical coordinates.

Practice Problem 2.8 A point in Cartesian coordinates is given as $Q(x = 1.24,\ y = 2.31,\ z = 2.4)$. Convert this point into cylindrical coordinates.

2.10.3 Spherical Coordinate System

The point in spherical coordinates is defined as $M(r,\ \theta, \phi)$. A radial line with the length of r is drawn at an angle θ with the z-axis and the unit vectors of this system are $\mathbf{a}_r$, $\mathbf{a}_\theta$ and $\mathbf{a}_\phi$ as shown in Fig. 2.23a. Here, $\mathbf{a}_r$ is parallel to the radial line and the unit vector $\mathbf{a}_\phi$ is tangent to the sphere, and it increases in the direction of increasing ϕ. The unit vector $\mathbf{a}_\theta$ is basically a tangent to the sphere, which is not shown in Fig. 2.23, and it increases in the direction of increasing θ. The vector $\mathbf{A}$ in terms of spherical components can be written as

$$\mathbf{A} = A_r\mathbf{a}_r + A_\theta\mathbf{a}_\theta + A_\phi\mathbf{a}_\phi \equiv (A_r, A_\theta, A_\phi). \tag{2.126}$$

The ranges of the coordinates are

$$0 < r < \infty,\ 0 < \theta < \pi,\ 0 < \phi < 2\pi. \tag{2.127}$$

The magnitude of the vector can be written as

$$|\mathbf{A}| = \sqrt{A_r{}^2 + A_\theta{}^2 + A_\phi{}^2}. \tag{2.128}$$

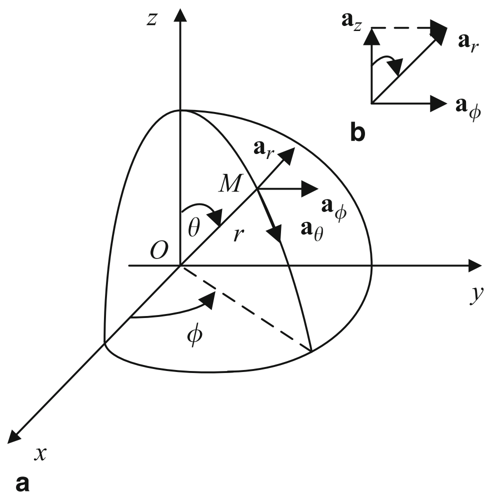


Fig. 2.23 (**a**) Spherical coordinates with unit vectors (**b**) only unit vectors

In this coordinate system, the properties of unit vectors are

$$\mathbf{a}_r \bullet \mathbf{a}_r = \mathbf{a}_\theta \bullet \mathbf{a}_\theta = \mathbf{a}_\phi \bullet \mathbf{a}_\phi = 1, \tag{2.129}$$

$$\mathbf{a}_r \bullet \mathbf{a}_\theta = \mathbf{a}_\theta \bullet \mathbf{a}_\phi = \mathbf{a}_\phi \bullet \mathbf{a}_r = 0, \tag{2.130}$$

$$\mathbf{a}_r \times \mathbf{a}_\theta = \mathbf{a}_\phi, \tag{2.131}$$

$$\mathbf{a}_\theta \times \mathbf{a}_\phi = \mathbf{a}_r \tag{2.132}$$

$$\mathbf{a}_\phi \times \mathbf{a}_r = \mathbf{a}_\theta. \tag{2.133}$$

From Fig. 2.24, the following expressions can be written

$$\rho = r \sin\theta, \tag{2.134}$$

$$x = \rho \cos\phi, \tag{2.135}$$

$$y = \rho \sin\phi \tag{2.136}$$

$$z = r \cos\theta. \tag{2.137}$$

Substituting Eq. (2.134) into Eqs. (2.135) and (2.136) yields

$$x = r \sin\theta \cos\phi \tag{2.138}$$

$$y = r \sin\theta \sin\phi. \tag{2.139}$$

Again, consider Fig. 2.24 to derive the relationship between the Cartesian and the spherical coordinates.

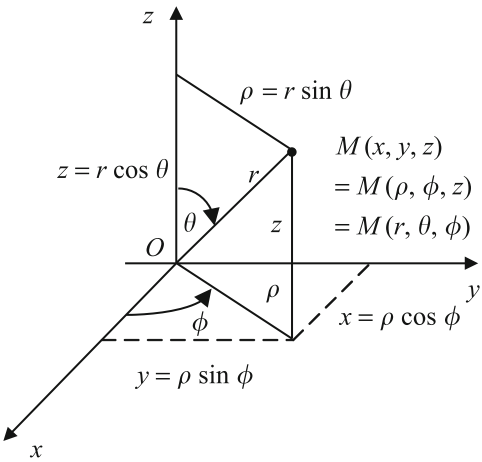


Fig. 2.24 Relation between the Cartesian and spherical coordinates

From Eqs. (2.137) to (2.139), the following relations can be written

$$r = \sqrt{x^2 + y^2 + z^2}, \tag{2.140}$$

$$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \tag{2.141}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right). \tag{2.142}$$

The unit vectors of spherical coordinates are functions of their positions. Consider the position vector in Cartesian coordinates. Then the relationship between unit vectors of spherical and Cartesian coordinates can be derived as

$$\mathbf{a}_r = \frac{\mathbf{r}}{r} = \frac{x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z}{r} = \frac{x}{r}\mathbf{a}_x + \frac{y}{r}\mathbf{a}_y + \frac{z}{r}\mathbf{a}_z. \tag{2.143}$$

From Fig. 2.23b, the cross product of two vectors is

$$\mathbf{a}_z \times \mathbf{a}_r = \mathbf{a}_\phi \sin\theta, \tag{2.144}$$

$$\mathbf{a}_\phi = \frac{\mathbf{a}_z \times \mathbf{a}_r}{\sin\theta} \tag{2.145}$$

$$\mathbf{a}_\theta = \mathbf{a}_\phi \times \mathbf{a}_r. \tag{2.146}$$

Substituting Eqs. (2.137)–(2.139) into Eq. (2.143) provides

$$\mathbf{a}_r = \sin\theta\cos\phi\mathbf{a}_x + \sin\theta\sin\phi\mathbf{a}_y + \cos\theta\mathbf{a}_z. \tag{2.147}$$

Substituting Eq. (2.147) into Eq. (2.145) yields

$$\mathbf{a}_\phi = \frac{\mathbf{a}_z \times (\sin\theta\cos\phi\mathbf{a}_x + \sin\theta\sin\phi\mathbf{a}_y + \cos\theta\mathbf{a}_z)}{\sin\theta}, \tag{2.148}$$

$$\mathbf{a}_\phi = \cos\phi(\mathbf{a}_z \times \mathbf{a}_x) + \sin\phi(\mathbf{a}_z \times \mathbf{a}_y), \tag{2.149}$$

$$\mathbf{a}_\phi = -\sin\phi\mathbf{a}_x + \cos\phi\mathbf{a}_y. \tag{2.150}$$

Substituting Eqs. (2.147) and (2.150) into Eq. (2.146) yields

$$\mathbf{a}_\theta = (-\sin\phi\mathbf{a}_x + \cos\phi\mathbf{a}_y) \times (\sin\theta\cos\phi\mathbf{a}_x + \sin\theta\sin\phi\mathbf{a}_y + \cos\theta\mathbf{a}_z), \tag{2.151}$$

$$\begin{aligned}\mathbf{a}_\theta = &-\sin\theta\sin^2\phi(\mathbf{a}_x \times \mathbf{a}_y) - \cos\theta\sin\phi(\mathbf{a}_x \times \mathbf{a}_z)\\ &+ \sin\theta\cos^2\phi(\mathbf{a}_y \times \mathbf{a}_x) + \cos\theta\cos\phi(\mathbf{a}_y \times \mathbf{a}_z).\end{aligned} \tag{2.152}$$

Applying the cross product rules to Eq. (2.152) provides

$$\begin{aligned}\mathbf{a}_\theta = &-\sin\theta\sin^2\phi\mathbf{a}_z - \cos\theta\sin\phi(-\mathbf{a}_y)\\ &+ \sin\theta\cos^2\phi(-\mathbf{a}_z) + \cos\theta\cos\phi(\mathbf{a}_x),\end{aligned} \tag{2.153}$$

$$\mathbf{a}_\theta = \cos\theta\cos\phi\mathbf{a}_x + \cos\theta\sin\phi\mathbf{a}_y - \sin\theta\mathbf{a}_z. \tag{2.154}$$

Alternative approach: The position vector in Cartesian coordinates is

$$\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z. \tag{2.155}$$

Substituting Eqs. (2.137)–(2.139) into Eq. (2.155) yields

$$\mathbf{r} = r\sin\theta\cos\phi\mathbf{a}_x + r\sin\theta\sin\phi\mathbf{a}_y + r\cos\theta\mathbf{a}_z. \tag{2.156}$$

Differentiating Eq. (2.156) with respect to the r, θ and ϕ yields

$$\frac{\partial\mathbf{r}}{\partial r} = \sin\theta\cos\phi\mathbf{a}_x + \sin\theta\sin\phi\mathbf{a}_y + \cos\theta\mathbf{a}_z, \tag{2.157}$$

$$\left|\frac{\partial\mathbf{r}}{\partial r}\right| = \sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta} = 1, \tag{2.158}$$

$$\frac{\partial\mathbf{r}}{\partial\theta} = r\cos\theta\cos\phi\mathbf{a}_x + r\cos\theta\sin\phi\mathbf{a}_y - r\sin\theta\mathbf{a}_z, \tag{2.159}$$

$$\left|\frac{\partial\mathbf{r}}{\partial\theta}\right| = r\sqrt{\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta} = r, \tag{2.160}$$

$$\frac{\partial\mathbf{r}}{\partial\phi} = -r\sin\theta\sin\phi\mathbf{a}_x + r\sin\theta\cos\phi\mathbf{a}_y, \tag{2.161}$$

$$\left|\frac{\partial\mathbf{r}}{\partial\phi}\right| = r\sqrt{\sin^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi} = r\sin\theta. \tag{2.162}$$

Table 2.1 Dot product of unit vectors of Cartesian and spherical coordinate systems

	$\mathbf{a}_r$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x \bullet$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y \bullet$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z \bullet$	$\cos\theta$	$-\sin\theta$	0

The unit vector $\mathbf{a}_r$ is defined as

$$\mathbf{a}_r = \frac{\dfrac{\partial \mathbf{r}}{\partial r}}{\left|\dfrac{\partial \mathbf{r}}{\partial r}\right|}. \tag{2.163}$$

Substituting Eqs. (2.157) and (2.158) into Eq. (2.163) yields

$$\mathbf{a}_r = \sin\theta\cos\phi\mathbf{a}_x + \sin\theta\sin\phi\mathbf{a}_y + \cos\theta\mathbf{a}_z \tag{2.164}$$

The unit vector $\mathbf{a}_\theta$ is defined as

$$\mathbf{a}_\theta = \frac{\dfrac{\partial \mathbf{r}}{\partial \theta}}{\left|\dfrac{\partial \mathbf{r}}{\partial \theta}\right|}. \tag{2.165}$$

Substituting Eqs. (2.159) and (2.160) into Eq. (2.165) yields

$$\mathbf{a}_\theta = \cos\theta\cos\phi\mathbf{a}_x + \cos\theta\sin\phi\mathbf{a}_y - \sin\theta\mathbf{a}_z. \tag{2.166}$$

The unit vector $\mathbf{a}_\phi$ is defined as

$$\mathbf{a}_\phi = \frac{\dfrac{\partial \mathbf{r}}{\partial \phi}}{\left|\dfrac{\partial \mathbf{r}}{\partial \phi}\right|}. \tag{2.167}$$

Substituting Eqs. (2.161) and (2.162) into Eq. (2.167) yields

$$\mathbf{a}_\phi = -\sin\phi\mathbf{a}_x + \cos\phi\mathbf{a}_y. \tag{2.168}$$

From Eqs. (2.164), (2.166) and (2.168), the dot product of unit vectors between Cartesian and spherical coordinate systems can be written as shown in Table 2.1.

A vector in spherical coordinates can be transformed into Cartesian coordinates as

$$A_x = \mathbf{A} \bullet \mathbf{a}_x = A_r\mathbf{a}_r \bullet \mathbf{a}_x + A_\phi\mathbf{a}_\phi \bullet \mathbf{a}_x + A_\phi\mathbf{a}_\phi \bullet \mathbf{a}_x, \tag{2.169}$$

$$A_y = \mathbf{A} \bullet \mathbf{a}_y = A_r\mathbf{a}_r \bullet \mathbf{a}_y + A_\theta\mathbf{a}_\theta \bullet \mathbf{a}_y + A_\phi\mathbf{a}_\phi \bullet \mathbf{a}_y \tag{2.170}$$

$$A_z = \mathbf{A} \bullet \mathbf{a}_z = A_r\mathbf{a}_r \bullet \mathbf{a}_z + A_\theta\mathbf{a}_\theta \bullet \mathbf{a}_z + A_\phi\mathbf{a}_\phi \bullet \mathbf{a}_z. \tag{2.171}$$

Applying the dot product rules of Eqs. (2.169)–(2.171) provides

$$A_x = A_r \sin\theta\cos\phi + A_\theta \cos\theta\cos\phi - A_\phi \sin\phi, \tag{2.172}$$

$$A_y = A_r \sin\theta\sin\phi + A_\theta \cos\theta\sin\phi + A_\phi \cos\phi \tag{2.173}$$

$$A_z = A_r \cos\theta - A_\theta \sin\theta. \tag{2.174}$$

Equations (2.172)–(2.174) can be represented in the matrix form as

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}. \tag{2.175}$$

Similarly, a vector in Cartesian coordinates can be transformed into spherical coordinates as

$$A_r = \mathbf{A} \bullet \mathbf{a}_r = A_x \mathbf{a}_x \bullet \mathbf{a}_r + A_y \mathbf{a}_y \bullet \mathbf{a}_r + A_z \mathbf{a}_z \bullet \mathbf{a}_r, \tag{2.176}$$

$$A_\theta = \mathbf{A} \bullet \mathbf{a}_\theta = A_x \mathbf{a}_x \bullet \mathbf{a}_\theta + A_y \mathbf{a}_y \bullet \mathbf{a}_\theta + A_z \mathbf{a}_z \bullet \mathbf{a}_\theta \tag{2.177}$$

$$A_\phi = \mathbf{A} \bullet \mathbf{a}_\phi = A_x \mathbf{a}_x \bullet \mathbf{a}_\phi + A_y \mathbf{a}_y \bullet \mathbf{a}_\phi + A_z \mathbf{a}_z \bullet \mathbf{a}_\phi. \tag{2.178}$$

Applying the dot product rules of Eqs. (2.176)–(2.178) provides

$$A_r = A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z \cos\theta, \tag{2.179}$$

$$A_\theta = A_x \cos\theta\cos\phi + A_y \cos\theta\sin\phi - A_z \sin\theta \tag{2.180}$$

$$A_\phi = -A_r \sin\phi + A_\theta \cos\phi. \tag{2.181}$$

Equations (2.179)–(2.181) can be represented in the matrix form as

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}. \tag{2.182}$$

Consider Fig. 2.25 to derive the expressions of differential length, area and volume. The expression of differential length is

$$d\mathbf{l} = dr\mathbf{a}_r + rd\theta\mathbf{a}_\theta + r\sin\theta d\phi\mathbf{a}_\phi. \tag{2.183}$$

From Fig. 2.25a–d, the expressions of differential area and volume can be written as

$$ds_r = (rd\theta)(r\sin\theta d\phi) = r^2 \sin\theta d\theta d\phi, \tag{2.184}$$

$$d\mathbf{S} = r^2 \sin\theta d\theta d\phi \mathbf{a}_r, \tag{2.185}$$

$$ds_\theta = (dr)(r\sin\theta d\phi) = r\sin\theta dr d\phi, \tag{2.186}$$

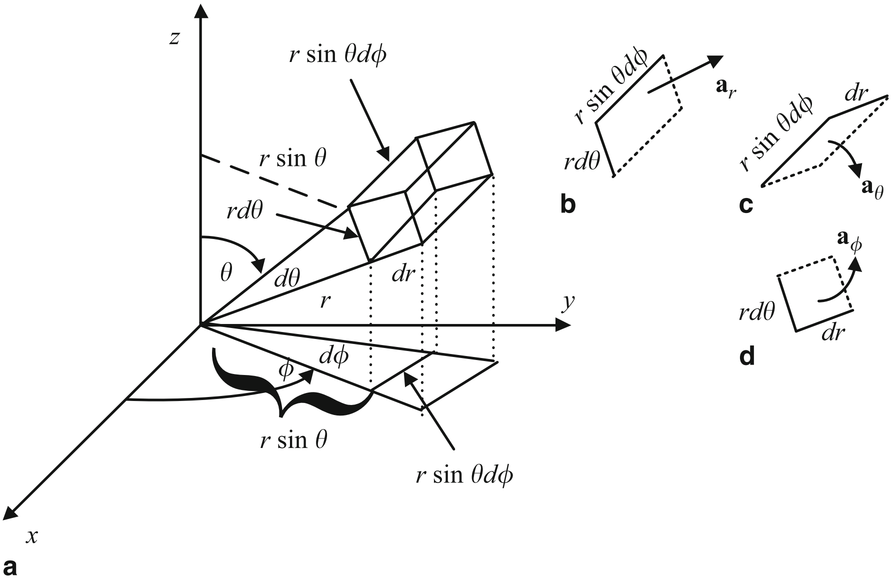


Fig. 2.25 (**a**) Spherical coordinates with differential elements (**b**) face in $\mathbf{a}_r$ (**d**) face in $\mathbf{a}_\theta$ (**e**) face in $\mathbf{a}_\phi$

$$d\mathbf{S} = r\sin\theta drd\phi\mathbf{a}_\theta, \tag{2.187}$$

$$ds_\phi = (dr)(rd\theta) = rdrd\theta, \tag{2.188}$$

$$d\mathbf{S} = rdrd\theta\mathbf{a}_\phi, \tag{2.189}$$

$$dv = (dr)(r\sin\theta d\phi)(rd\theta) = r^2\sin\theta drd\theta d\phi. \tag{2.190}$$

Example 2.9 A vector in Cartesian coordinates is given by $\mathbf{A} = \frac{2x}{y}\mathbf{a}_x$. Transform this vector into spherical coordinates.

Solution The components of cylindrical coordinates can be determined as follows:

$$\begin{aligned}
A_r &= \mathbf{A} \bullet \mathbf{a}_r = \frac{2x}{y}\mathbf{a}_x \bullet \mathbf{a}_r = \frac{2x}{y}\sin\theta\cos\phi \\
&= \frac{2r\sin\theta\cos\phi}{r\sin\theta\sin\phi}\sin\theta\cos\phi = 2\cot\phi\sin\theta\cos\phi, \\
A_\theta &= \mathbf{A} \bullet \mathbf{a}_\theta = \frac{2x}{y}\mathbf{a}_x \bullet \mathbf{a}_\theta = \frac{2x}{y}\cos\theta\cos\phi \\
&= \frac{2r\sin\theta\cos\phi}{r\sin\theta\sin\phi}\cos\theta\cos\phi = 2\cot\phi\cos\theta\cos\phi, \\
A_\phi &= \mathbf{A} \bullet \mathbf{a}_\phi = \frac{2x}{y}\mathbf{a}_x \bullet \mathbf{a}_\phi = \frac{2x}{y}(-\sin\phi) = \frac{2r\sin\theta\cos\phi}{r\sin\theta\sin\phi}(-\sin\phi) = -2\cos\phi.
\end{aligned}$$

A new vector in spherical coordinates is

$$\mathbf{A} = 2\cot\phi\sin\theta\cos\phi\mathbf{a}_r + 2\cot\phi\cos\theta\cos\phi\mathbf{a}_\theta - 2\cos\phi\mathbf{a}_\phi.$$

Example 2.10 A point in Cartesian coordinates is given by $P(1, 3, 2)$. Convert this point into spherical coordinates.

Solution The components of spherical coordinates can be determined as

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{1^2 + 3^2 + 2^2} = 3.74,$$

$$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right) = 57.69^\circ \text{ and}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{3}{1}\right) = 71.57^\circ.$$

Practice Problem 2.9 A vector in Cartesian coordinates is given by $\mathbf{A} = x\mathbf{a}_x$. Transform this vector into spherical coordinates.

Practice Problem 2.10 A point in spherical coordinates is given as $Q(r = 4,\ \theta = 130^\circ,\ \phi = 300^\circ)$. Convert this point into Cartesian and cylindrical coordinates.

2.11 Potential Gradient and Gradient of a Scalar Field

The potential difference is defined as the work done in moving a unit charge from one point to another point in an electric field. In general, it can be written as

$$V = -\int_{initial}^{final} \mathbf{E}.d\mathbf{L}. \tag{2.191}$$

For a short element of length ($\Delta\mathbf{L}$), the incremental potential is ΔV, but the electric field is constant. Then, Eq. (2.191) can be modified as

$$\Delta V = -\mathbf{E}.\Delta\mathbf{L}. \tag{2.192}$$

The total differential of potential in terms of partial differentiation with a function of x, y and z can be written as

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz. \tag{2.193}$$

Differentiating the Eq. (2.192) yields

$$dV = -\mathbf{E}.d\mathbf{L}. \tag{2.194}$$

Expressing the electric field in Cartesian coordinates and substituting Eq. (2.77) into Eq. (2.194) provides

$$dV = -(E_x\mathbf{a}_x + E_y\mathbf{a}_y + E_z\mathbf{a}_z).(dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z), \tag{2.195}$$

$$dV = -(E_x dx + E_y dy + E_z dz). \tag{2.196}$$

From Eqs. (2.193) and (2.196), following equations can be written

$$E_x = -\frac{\partial V}{\partial x}, \tag{2.197}$$

$$E_y = -\frac{\partial V}{\partial y} \tag{2.198}$$

$$E_z = -\frac{\partial V}{\partial z}. \tag{2.199}$$

The gradient of a scalar field is a vector field whose magnitude depends on the maximum magnitude of the directional derivative and direction depends on the direction of the directional derivative. The gradient is represented by the symbol ∇ (del). This vector operator del is represented as

$$\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z. \tag{2.200}$$

The gradient in Cartesian, cylindrical and spherical coordinates can be written as

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z, \tag{2.201}$$

$$\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z, \tag{2.202}$$

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi. \tag{2.203}$$

Using the total differential of Eqs. (2.91) and (2.92) to derive Eq. (2.202) as

$$dx = \cos\phi d\rho - \rho\sin\phi d\phi \tag{2.204}$$

$$dy = \sin\phi d\rho + \rho\cos\phi d\phi. \tag{2.205}$$

Substituting Eqs. (2.204) and (2.205) into Eq. (2.193) yields

$$dV = \frac{\partial V}{\partial x}[\cos\phi d\rho - \rho\sin\phi d\phi] + \frac{\partial V}{\partial y}[\sin\phi d\rho + \rho\cos\phi d\phi] + \frac{\partial V}{\partial z}dz, \tag{2.206}$$

$$dV = \left[\cos\phi\frac{\partial V}{\partial x} + \sin\phi\frac{\partial V}{\partial y}\right]d\rho + \left[-\rho\sin\phi\frac{\partial V}{\partial x} + \rho\cos\phi\frac{\partial V}{\partial y}\right]d\phi + \frac{\partial V}{\partial z}dz, \tag{2.207}$$

$$dV = \frac{\partial V}{\partial \rho} d\rho + \frac{\partial V}{\partial \phi} d\phi + \frac{\partial V}{\partial z} dz, \tag{2.208}$$

where

$$\frac{\partial V}{\partial \rho} = \cos\phi \frac{\partial V}{\partial x} + \sin\phi \frac{\partial V}{\partial y} \tag{2.209}$$

$$\frac{\partial V}{\partial \phi} = -\rho \sin\phi \frac{\partial V}{\partial x} + \rho \cos\phi \frac{\partial V}{\partial y}. \tag{2.210}$$

Applying the rules (2.219) $\times \cos\phi -$ (2.220) $\times \frac{\sin\phi}{\rho}$ provides

$$\frac{\partial V}{\partial x} = \cos\phi \frac{\partial V}{\partial \rho} - \frac{\sin\phi}{\rho} \frac{\partial V}{\partial \phi}. \tag{2.211}$$

Applying the rules (2.219) $\times \sin\phi +$ (2.220) $\times \frac{\cos\phi}{\rho}$ provides

$$\frac{\partial V}{\partial y} = \sin\phi \frac{\partial V}{\partial \rho} + \frac{\cos\phi}{\rho} \frac{\partial V}{\partial \phi}. \tag{2.212}$$

Substituting Eqs. (2.98), (2.99), (2.211) and (2.212) into Eq. (2.201) yields

$$\begin{aligned} \nabla V = &\left[\cos\phi \frac{\partial V}{\partial \rho} - \frac{\sin\phi}{\rho} \frac{\partial V}{\partial \phi}\right] [\cos\phi \mathbf{a}_\rho - \sin\phi \mathbf{a}_\phi] + \\ &\left[\sin\phi \frac{\partial V}{\partial \rho} + \frac{\cos\phi}{\rho} \frac{\partial V}{\partial \phi}\right] [\sin\phi \mathbf{a}_\rho + \cos\phi \mathbf{a}_\phi] + \frac{\partial V}{\partial z} \mathbf{a}_z. \end{aligned} \tag{2.213}$$

Equation (2.213) can be simplified as

$$\begin{aligned} \nabla V = &\cos^2\phi \frac{\partial V}{\partial \rho} \mathbf{a}_\rho - \frac{\sin\phi\cos\phi}{\rho} \frac{\partial V}{\partial \phi} \mathbf{a}_\phi - \sin\phi\cos\phi \frac{\partial V}{\partial \rho} \mathbf{a}_\phi + \frac{\sin^2\phi}{\rho} \frac{\partial V}{\partial \phi} \mathbf{a}_\phi \\ &+ \sin^2\phi \frac{\partial V}{\partial \rho} \mathbf{a}_\rho + \frac{\sin\phi\cos\phi}{\rho} \frac{\partial V}{\partial \phi} \mathbf{a}_\rho + \sin\phi\cos\phi \frac{\partial V}{\partial \rho} \mathbf{a}_\phi + \frac{\cos^2\phi}{\rho} \frac{\partial V}{\partial \phi} \mathbf{a}_\phi, \end{aligned} \tag{2.214}$$

$$\nabla V = (\sin^2\phi + \cos^2\phi) \frac{\partial V}{\partial \rho} \mathbf{a}_\rho + \frac{(\sin^2\phi + \cos^2\phi)}{\rho} \frac{\partial V}{\partial \phi} \mathbf{a}_\phi + \frac{\partial V}{\partial z} \mathbf{a}_z, \tag{2.215}$$

$$\nabla V = \frac{\partial V}{\partial \rho} \mathbf{a}_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \mathbf{a}_\phi + \frac{\partial V}{\partial z} \mathbf{a}_z.$$

Example 2.11 A scalar electric potential is given as $V = x^2 y^2 + z^2$. Determine the gradient of the scalar potential.

Solution The gradient of scalar potential can be determined as

$$\nabla V = \frac{\partial}{\partial x}(x^2y^2 + z^2)\mathbf{a}_x + \frac{\partial}{\partial y}(x^2y^2 + z^2)\mathbf{a}_y + \frac{\partial}{\partial z}(x^2y^2 + z^2)\mathbf{a}_z,$$

$$\nabla V = 2xy^2\mathbf{a}_x + 2x^2y\mathbf{a}_y + 2z\mathbf{a}_z.$$

Example 2.12 A scalar electric potential and a point are given as $V = 3x^3y + 7z$ and $P(1, 3, 2)$, respectively. Find the (1) ∇V and (2) value of ∇V.

Solution (1) The gradient of potential is

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z,$$

$$\nabla V = 9x^2y\mathbf{a}_x + 3x^3\mathbf{a}_y + 7\mathbf{a}_z.$$

(2) The value of the gradient of potential at point $P(1, 3, 2)$ is

$$\nabla V = 27\mathbf{a}_x + 3\mathbf{a}_y + \mathbf{7}a_z.$$

Example 2.13 Scalar electric potentials are given by $V(\rho, \phi, z) = \rho\cos\phi + 2z$ and $V(r, \theta, \phi) = r^2 + 2\theta + \sin\phi$.

Determine the gradient of the potential at the points $P(\rho = 1,\ \phi = 140°,\ z = 3)$ and $Q(r = 1.4,\ \theta = 80°,\ \phi = 130°)$, respectively.

Solution The gradient of potential in cylindrical coordinates is

$$\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z,$$

$$\nabla V = \cos\phi\mathbf{a}_\rho - \sin\phi\mathbf{a}_\phi + 2\mathbf{a}_z.$$

The electric field at point $P(\rho = 1,\ \phi = 140°,\ z = 3)$ is

$$\nabla V = 0.77\mathbf{a}_\rho + 0.64\mathbf{a}_\phi - 2\mathbf{a}_z.$$

The gradient of electric potential in spherical coordinates is

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi,$$

$$\nabla V = 2r\mathbf{a}_r + \frac{2}{r}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\cos\phi\mathbf{a}_\phi.$$

The value of the gradient at point $Q(r = 1.4,\ \theta = 80°,\ \phi = 130°)$ is

$$\nabla V = 2.8\mathbf{a}_r + 1.43\mathbf{a}_\theta - 0.47\mathbf{a}_\phi = 3.76.$$

Practice Problem 2.11 A scalar electric potential is given by $V = yx^2 + 2y^2z$. Determine the gradient of scalar potential.

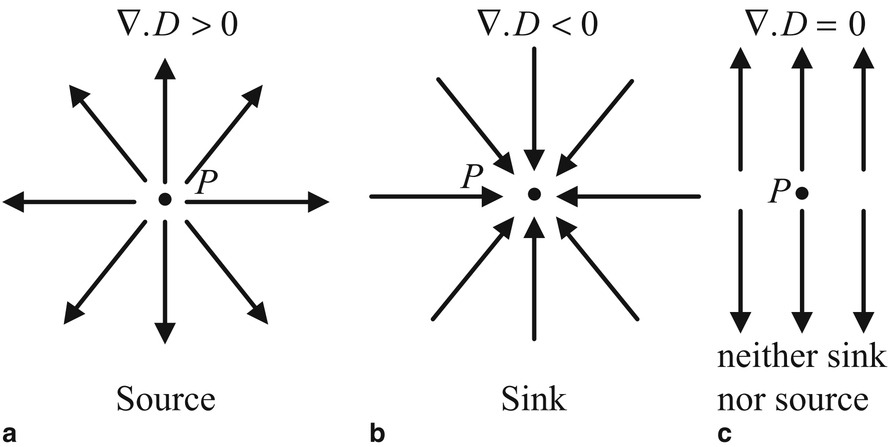


Fig. 2.26 Directions of flux lines. **a** Source. **b** Sink. **c** Neither sink nor source

Practice Problem 2.12 The expression of a scalar electric potential is given by $V = x^2 - 5yz$. Find the gradient of scalar potential at the point $P(-1.2, 2, 2.8)$.

Practice Problem 2.13 A scalar electric potential is given by $V(\rho, \phi, z) = z\cos\phi + 2\rho$. Determine the gradient of scalar potential at point $P(\rho = 1.2,\ \phi = 130°,\ z = -2)$.

2.12 Divergence of a Vector Field

The divergence of a vector **Ao** is abbreviated as div **A** or ∇.**A**. The divergence of a vector field is the outward flux per unit volume as the volume shrinks to zero. Mathematically, it can be expressed as

$$div\mathbf{A} = \nabla.\mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_s \mathbf{A}.d\mathbf{S}}{\Delta V}. \tag{2.216}$$

Figure 2.26 shows the direction of the flux lines. At source, the flux lines are coming out from the point P, which in turn increases volume. In this case, the divergence of a vector field is positive. At sink, the volume shrinks to zero as the flux lines come towards the point P. Therefore, the divergence of a vector field is negative. A differential volume element in the three directions is shown in Fig. 2.27. Consider the field component A_x is normal to the side $ABCD$. The net flux leaving through this side at $x = 0$ is

$$\mathbf{A}.d\mathbf{S} = -A_x \Delta y \Delta z. \tag{2.217}$$

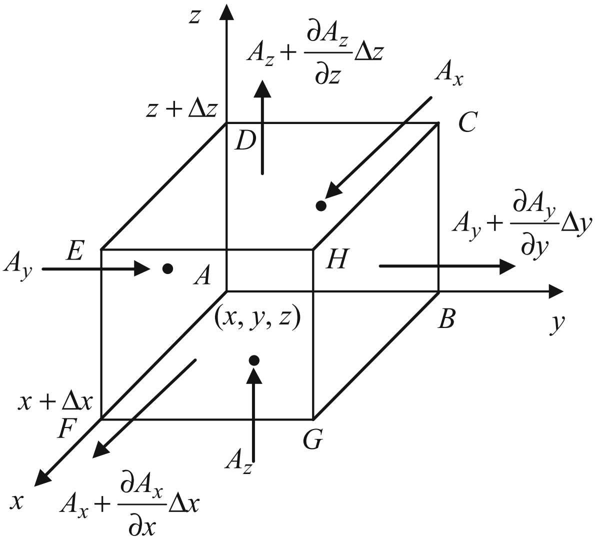


Fig. 2.27 Differential volume with field components

The field component $A_x + \frac{\partial A_x}{\partial x}\Delta x$ is normal to the face $FGHE$ and the net flux leaving through this side at $\Delta x = 0$ is

$$\mathbf{A}.d\mathbf{S} = \left(A_x + \frac{\partial A_x}{\partial x}\Delta x\right)\Delta y\Delta z. \tag{2.218}$$

The net flux leaving in the x-direction is

$$(\mathbf{A}.d\mathbf{S})_x = -A_x\Delta y\Delta z + \left(A_x + \frac{\partial A_x}{\partial x}\Delta x\right)\Delta y\Delta z, \tag{2.219}$$

$$(\mathbf{A}.d\mathbf{S})_x = \frac{\partial A_x}{\partial x}\Delta x\Delta y\Delta z. \tag{2.220}$$

The field component A_y is normal to the side $ADEF$. The net flux leaving through this side at $y = 0$ is

$$\mathbf{A}.d\mathbf{S} = -A_y\Delta x\Delta z. \tag{2.221}$$

The field component $A_y + \frac{\partial A_y}{\partial y}\Delta y$ is normal to the face $BCGH$ and the net flux leaving through this side at $\Delta y = 0$ is

$$\mathbf{A}.d\mathbf{S} = \left(A_y + \frac{\partial A_y}{\partial y}\Delta y\right)\Delta x\Delta z. \tag{2.222}$$

The net flux leaving in the y-direction is

$$(\mathbf{A}.d\mathbf{S})_y = -A_y\Delta x\Delta z + \left(A_y + \frac{\partial A_y}{\partial y}\Delta y\right)\Delta x\Delta z, \tag{2.223}$$

$$(\mathbf{A}.d\mathbf{S})_y = \frac{\partial A_y}{\partial y}\Delta x \Delta y \Delta z. \tag{2.224}$$

The field component A_z is normal to the side $ABGF$. The net flux leaving through this side at $z = 0$ is

$$\mathbf{A}.d\mathbf{S} = -A_z \Delta x \Delta y. \tag{2.225}$$

The field component $A_z + \frac{\partial A_z}{\partial z}\Delta z$ is normal to the face $ABGF$ and the net flux leaving through this side at $\Delta z = 0$ is

$$\mathbf{A}.d\mathbf{S} = \left(A_z + \frac{\partial A_z}{\partial z}\Delta z\right)\Delta x \Delta y. \tag{2.226}$$

The net flux leaving in the z-direction is

$$(\mathbf{A}.d\mathbf{S})_z = -A_z \Delta x \Delta y + \left(A_z + \frac{\partial A_z}{\partial z}\Delta z\right)\Delta x \Delta y, \tag{2.227}$$

$$(\mathbf{A}.d\mathbf{S})_z = \frac{\partial A_z}{\partial z}\Delta x \Delta y \Delta z. \tag{2.228}$$

Substituting Eqs. (2.220), (2.224) and (2.228) into Eq. (2.216) yields

$$div\mathbf{A} = \nabla.\mathbf{A} = \lim_{\Delta x \Delta y \Delta z \to 0} \frac{\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\Delta x \Delta y \Delta z}{\Delta x \Delta y \Delta z}, \tag{2.229}$$

$$div\mathbf{A} = \nabla.\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}. \tag{2.230}$$

To derive the divergence of a vector in cylindrical coordinates, Eqs. (2.211) and (2.212) can be written as

$$\frac{\partial}{\partial x} = \cos\phi \frac{\partial}{\partial \rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial \phi} \tag{2.231}$$

$$\frac{\partial}{\partial y} = \sin\phi \frac{\partial}{\partial \rho} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial \phi}. \tag{2.232}$$

Substituting Eqs. (2.231), (2.232), (2.121) and (2.122) into Eq. (2.230) yields

$$\nabla.\mathbf{A} = \left(\cos\phi \frac{\partial}{\partial \rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial \phi}\right)(A_\rho \cos\phi - A_\phi \sin\phi)$$
$$+ \left(\sin\phi \frac{\partial}{\partial \rho} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial \phi}\right)(A_\rho \sin\phi + A_\phi \cos\phi) + \frac{\partial A_z}{\partial z}, \tag{2.233}$$

$$\nabla.\mathbf{A} = \cos^2\phi \frac{\partial A_\rho}{\partial \rho} - \frac{\sin\phi\cos\phi}{\rho}\frac{\partial A_\rho}{\partial \phi} - \cos\phi\sin\phi \frac{\partial A_\phi}{\partial \rho} + \frac{\sin^2\phi}{\rho}\frac{\partial A_\phi}{\partial \phi}$$

$$+\sin^2\phi\frac{\partial A_\rho}{\partial\rho}+\frac{\sin\phi\cos\phi}{\rho}\frac{\partial A_\rho}{\partial\phi}+\cos\phi\sin\phi\frac{\partial A_\phi}{\partial\rho}+\frac{\cos^2\phi}{\rho}\frac{\partial A_\phi}{\partial\phi}+\frac{\partial A_z}{\partial z}, \tag{2.234}$$

$$\nabla.\mathbf{A}=\frac{\partial A_\rho}{\partial\rho}+\frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi}+\frac{\partial A_z}{\partial z}, \tag{2.235}$$

$$\nabla.\mathbf{A}=\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho}+\frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi}+\frac{\partial A_z}{\partial z}. \tag{2.236}$$

For spherical coordinates, it can be written as

$$\nabla.\mathbf{A}=\frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}. \tag{2.237}$$

The divergence theorem states that the integral of a divergence of any vector field over a volume is equal to the integral of that vector field over a closed surface. Mathematically, it can be written as

$$\int_V(\nabla.\mathbf{A})dV=\oint_S\mathbf{A}.d\mathbf{S}. \tag{2.238}$$

Here, the volume is bounded by the closed surface. In the case of special geometry, volume integrals are easier to evaluate than surface integrals. In this case, the divergence theorem is used.

Example 2.14 A vector in Cartesian coordinates is given by $\mathbf{A}=2yx\mathbf{a}_x-y^2\mathbf{a}_y+z\mathbf{a}_z$. Determine the div $\mathbf{A}$.

Solution The divergence of the vector $\mathbf{A}$ can be determined as

$$\nabla.\mathbf{A}=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}=2y-2y+1=1.$$

Example 2.15 A vector in cylindrical coordinates is given by $\mathbf{A}=\rho z\cos\phi\mathbf{a}_\rho+\rho^2z\sin\phi\mathbf{a}_\phi+2\rho z\sin^2\phi\mathbf{a}_z$.

Determine div $\mathbf{A}$ at the point $P(\rho=1,\ \phi=120°,\ z=2)$.

Solution The divergence of the vector $\mathbf{A}$ can be determined as

$$\nabla.\mathbf{A}=\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho}+\frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi}+\frac{\partial A_z}{\partial z},$$

$$\nabla.\mathbf{A}=\frac{1}{\rho}\frac{\partial(\rho^2z\cos\phi)}{\partial\rho}+\frac{1}{\rho}\frac{\partial(\rho^2z\sin\phi)}{\partial\phi}+\frac{\partial(2\rho z\sin^2\phi)}{\partial z},$$

$$\nabla.\mathbf{A}=\frac{z\cos\phi}{\rho}2\rho-\frac{1}{\rho}\rho^2z\cos\phi+2\rho\sin^2\phi,$$

$$\nabla.\mathbf{A}=2z\cos\phi-\rho z\cos\phi+2\rho\sin^2\phi,$$

$$\nabla.\mathbf{A}=4\cos120°-2\cos120°+2\sin^2120°=0.5.$$

Practice Problem 2.14 The expression of a vector in Cartesian coordinates is given by $\mathbf{A} = 2y^2x\mathbf{a}_x - zy^2\mathbf{a}_y + 2z\mathbf{a}_z$. Determine div $\mathbf{A}$ at the point $P(1, -2, 3)$.

Practice Problem 2.15 Vectors in the cylindrical and the spherical coordinates are given by $\mathbf{A} = 3\rho z^2 \cos\phi\mathbf{a}_\rho + 2\rho z \sin\phi\mathbf{a}_\phi - z\sin 2\phi\mathbf{a}_z$ and $\mathbf{B} = \sin\theta\mathbf{b}_r + 2r\cos\theta\mathbf{b}_\theta - \sin 2\phi\mathbf{b}_\phi$, respectively. Determine the div $\mathbf{A}$ and $\mathbf{B}$ at the points $P(\rho = 1,\ \phi = 125°,\ z = 1.2)$ and $Q(r = 0.5,\ \theta = 45°,\ \phi = 70°)$, respectively.

2.13 Curl of a Vector Field

The curl (rotation) of a vector $\mathbf{A}$ is abbreviated as curl $\mathbf{A}$ or $\nabla \times \mathbf{A}$. The net circulation (C) of a vector field $\mathbf{A}$ around the closed contour is defined as the line integral of $\mathbf{A}$. This can be expressed as

$$C = \oint_c \mathbf{A}.d\mathbf{l}. \tag{2.239}$$

The curl of a vector field is equal to the maximum circulation of a vector field per unit area as the area tends to zero. Mathematically, it can be expressed as

$$\nabla \times \mathbf{A} = \left(\lim_{\Delta s \to 0} \frac{\oint_L \mathbf{A}.d\mathbf{l}}{\Delta S} \right)_{\max} \mathbf{a}_n, \tag{2.240}$$

where

$\mathbf{a}_n$ is the unit vector normal to ΔS.

Consider Fig. 2.28a to find the expression of the curl of $\mathbf{A}$ in the $x - y$ plane. The sides of $ABCD$ are $AB = CD = \Delta x$ and $BC = AD = \Delta y$. The line integral around the complete path $ABCD$ is

$$\int_{ABCD} \mathbf{A}.d\mathbf{l} = A_x\Delta x + \left(A_y + \frac{\partial A_y}{\partial x}\Delta x \right)\Delta y - \left(A_x + \frac{\partial A_x}{\partial y}\Delta y \right)\Delta x - A_y\Delta y, \tag{2.241}$$

$$\int_{ABCD} \mathbf{A}.d\mathbf{l} = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)\Delta x\Delta y. \tag{2.242}$$

Substituting Eq. (2.242) and $\Delta S = \Delta x\Delta y$ into Eq. (2.240) and the z-component of the curl of $\mathbf{A}$ results in

$$(\Delta \times \mathbf{A})_z = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \tag{2.243}$$

Again, consider Fig. 2.28b and c to derive the expressions of the curl of $\mathbf{A}$ in the $y - z$ and $z - x$ planes, respectively.

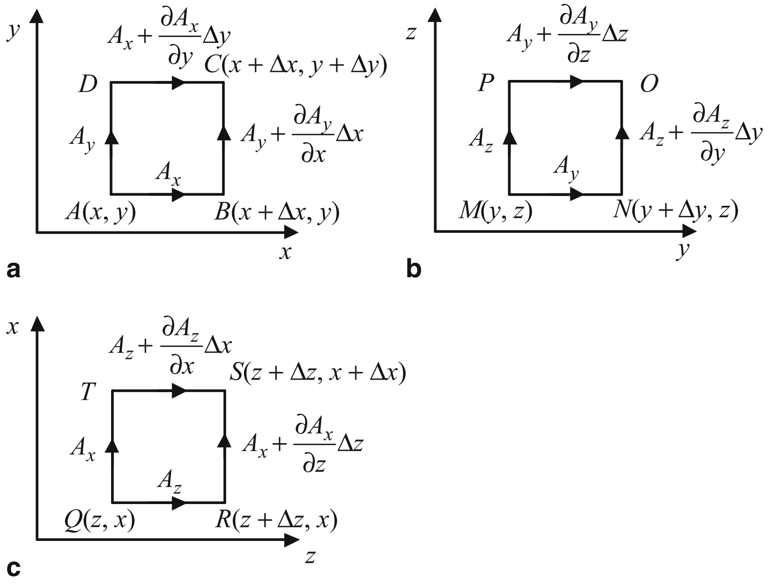


Fig. 2.28 (**a**) Path *ABCD* (**b**) Path *MNOP* (**c**) Path *QRST*

The line integral around the complete path $MNOP$ is

$$\int_{MNOP} \mathbf{A}.d\mathbf{l} = A_y\Delta y + \left(A_z + \frac{\partial A_z}{\partial y}\Delta y\right)\Delta z - \left(A_y + \frac{\partial A_y}{\partial z}\Delta z\right)\Delta y - A_z\Delta z, \tag{2.244}$$

$$\int_{MNOP} \mathbf{A}.d\mathbf{l} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\Delta y\Delta z. \tag{2.245}$$

Substituting Eq. (2.245) and $\Delta S = \Delta y\Delta z$ into Eq. (2.240) and the x-component of the curl **A** results in

$$(\Delta \times \mathbf{A})_x = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right). \tag{2.246}$$

The line integral around the complete path $QRST$ is

$$\int_{QRST} \mathbf{A}.d\mathbf{l} = A_z\Delta z + \left(A_x + \frac{\partial A_x}{\partial z}\Delta z\right)\Delta x - \left(A_z + \frac{\partial A_z}{\partial x}\Delta x\right)\Delta z - A_x\Delta x, \tag{2.247}$$

$$\int_{QRST} \mathbf{A}.d\mathbf{l} = \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\Delta x\Delta z. \tag{2.248}$$

Substituting Eq. (2.248) and $\Delta S = \Delta x \Delta z$ into Eq. (2.240) and the y-component of the curl $\mathbf{A}$ creates

$$(\Delta \times \mathbf{A})_y = \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right). \tag{2.249}$$

The curl of $\mathbf{A}$ in three dimensions can be written as

$$\nabla \times \mathbf{A} = (\nabla \times \mathbf{A})_x \mathbf{a}_x + (\nabla \times \mathbf{A})_y \mathbf{a}_y + (\nabla \times \mathbf{A})_z \mathbf{a}_z. \tag{2.250}$$

Substituting Eqs. (2.243), (2.246) and (2.249) into Eq. (2.250) yields

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \mathbf{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \mathbf{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \mathbf{a}_z. \tag{2.251}$$

In the matrix form, Eq. (2.251) can be written as

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}. \tag{2.252}$$

The curl of $\mathbf{A}$ in cylindrical and spherical coordinate systems can be written as

$$\nabla \times \mathbf{A} = \frac{1}{\rho} \begin{vmatrix} \mathbf{a}_\rho & \rho \mathbf{a}_\phi & \mathbf{a}_z \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}, \tag{2.253}$$

$$\nabla \times \mathbf{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \mathbf{a}_r & r \mathbf{a}_\theta & r \sin\theta \mathbf{a}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_r & r A_\theta & r \sin\theta A_\phi \end{vmatrix}. \tag{2.254}$$

The Stokes' theorem states that the surface integral of the curl of a vector field over an open surface is equal to the line integral of the vector along the closed contour bounding the surface. A surface with a closed boundary is shown in Fig. 2.29.

Mathematically, the Stokes' theorem can be written as

$$\int_S (\nabla \times \mathbf{A}).d\mathbf{s} = \oint_l \mathbf{A}.d\mathbf{l}. \tag{2.255}$$

Example 2.16 A vector in Cartesian coordinates is given by $\mathbf{A} = 2y^2 z \mathbf{a}_x + 3xz^2 \mathbf{a}_y + x^2 y^2 \mathbf{a}_z$. Determine the curl of $\mathbf{A}$ at the point $P(x = 1,\ y = 1.5,\ z = -2)$.

Solution The x component of the vector $\mathbf{A}$ and its partial differentiation with respect to y and z are

$$A_x = 2y^2 z,$$

$$\frac{\partial A_x}{\partial y} = 4yz, \frac{\partial A_x}{\partial z} = 2y^2.$$

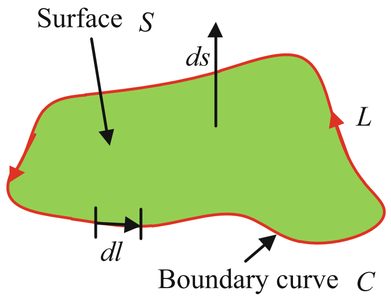


Fig. 2.29 Surface with a closed boundary

The y component of the vector **A** and its partial differentiation with respect to x and z are

$$A_y = 3xz^2,$$
$$\frac{\partial A_y}{\partial x} = 3z^2,$$
$$\frac{\partial A_y}{\partial z} = 6xz.$$

The z component of the vector **A** and its partial differentiation with respect to x and y are

$$A_z = x^2y^2,$$
$$\frac{\partial A_z}{\partial x} = 2xy^2, \frac{\partial A_z}{\partial y} = 2x^2y.$$
$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{a}_z,$$
$$\nabla \times \mathbf{A} = (2x^2y - 6xz)\mathbf{a}_x + (2y^2 - 2xy^2)\mathbf{a}_y + (3z^2 - 4yz)\mathbf{a}_z.$$

The curl of **A** at the point $P(x = 1,\ y = 1.5,\ z = -2)$ is

$$\nabla \times \mathbf{A} = 15\mathbf{a}_x + 0\mathbf{a}_y + 24\mathbf{a}_z.$$

Practice Problem 2.16 The expression of a vector in Cartesian coordinates is given by $\mathbf{A} = 2yz\mathbf{a}_x - xz^3\mathbf{a}_y + 3x^3y\mathbf{a}_z$. Determine the curl of **A** at the point $P(x = 0.8,\ y = 1.05,\ z = -2.2)$.

2.14 Two Important Vector Identities

The two vector identities are very important for studying electromagnetism, especially when it deals with potential functions. The first vector identity states that the curl of the gradient of any scalar field is identically zero. Mathematically, it can be written as

$$\nabla \times (\nabla V) \equiv 0. \tag{2.256}$$

Equation (2.256) can be verified by considering Cartesian coordinates as

$$\nabla \times (\nabla V) = \left(\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z\right) \times \left(\frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z\right), \tag{2.257}$$

$$\nabla \times (\nabla V) = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial V}{\partial x} & \dfrac{\partial V}{\partial y} & \dfrac{\partial V}{\partial z} \end{vmatrix}, \tag{2.258}$$

$$\nabla \times (\nabla V) = \left(\frac{\partial^2 V}{\partial y \partial z} - \frac{\partial^2 V}{\partial y \partial z}\right)\mathbf{a}_x + \left(\frac{\partial^2 V}{\partial x \partial z} - \frac{\partial^2 V}{\partial x \partial z}\right)\mathbf{a}_y + \left(\frac{\partial^2 V}{\partial x \partial y} - \frac{\partial^2 V}{\partial x \partial y}\right)\mathbf{a}_z = 0. \tag{2.259}$$

The second important vector identity states that the divergence of the curl of a vector field is identically zero. Mathematically, it can be expressed as

$$\nabla.(\nabla \times \mathbf{V}) \equiv 0. \tag{2.260}$$

Again consider Cartesian coordinates to verify Eq. (2.260). Considering the left-hand side of Eq. (2.260) provides

$$\nabla.(\nabla \times \mathbf{V}) = \nabla. \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}, \tag{2.261}$$

$$\nabla.(\nabla \times \mathbf{V}) = \nabla. \left[\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right)\mathbf{a}_y + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\mathbf{a}_z\right], \tag{2.262}$$

$$\nabla.(\nabla \times \mathbf{V}) = \frac{\partial}{\partial x}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right), \tag{2.263}$$

$$\nabla.(\nabla \times \mathbf{V}) = \frac{\partial^2 V_z}{\partial x \partial y} - \frac{\partial^2 V_y}{\partial x \partial z} + \frac{\partial^2 V_z}{\partial y \partial x} - \frac{\partial^2 V_x}{\partial y \partial z} + \frac{\partial^2 V_y}{\partial z \partial x} - \frac{\partial^2 V_x}{\partial z \partial y} = 0. \tag{2.264}$$

2.15 Exercise Problems

2.1 Three vectors are given by $\mathbf{A} = 3\mathbf{a}_x - 5\mathbf{a}_y + 5\mathbf{a}_z$, $\mathbf{B} = 2\mathbf{a}_x - 3\mathbf{a}_y + 4\mathbf{a}_z$ and $\mathbf{C} = 2\mathbf{a}_x - \mathbf{a}_y + 3\mathbf{a}_z$, respectively. Determine the magnitude of (1) $\mathbf{R}_a = \mathbf{A} + 2\mathbf{B}$ and (2) $\mathbf{R}_s = 3\mathbf{B} - \mathbf{C}$.

2.2 Calculate the unit vector which is parallel to the resultant (addition) of vectors $\mathbf{A} = \mathbf{a}_x + 3\mathbf{a}_y + 2\mathbf{a}_z$ and $\mathbf{B} = \mathbf{a}_x + 2\mathbf{a}_y - 3\mathbf{a}_z$.

2.3 Three vectors are given by $\mathbf{A} = \mathbf{a}_x - 2\mathbf{a}_y + 3\mathbf{a}_z$, $\mathbf{B} = 2\mathbf{a}_x - 4\mathbf{a}_y + 3\mathbf{a}_z$ and $\mathbf{C} = \mathbf{a}_x - 3\mathbf{a}_y + 2\mathbf{a}_z$, respectively. Determine the magnitude of (1) $\mathbf{R}_a = 3\mathbf{A} + 2\mathbf{B} + \mathbf{C}$ and (2) $\mathbf{R}_s = 3\mathbf{B} - \mathbf{C} - 2\mathbf{A}$.

2.4 Two vectors are given by $\mathbf{A} = \mathbf{a}_x - 3\mathbf{a}_y + 2\mathbf{a}_z$ and $\mathbf{B} = 2\mathbf{a}_x - 3\mathbf{a}_y + 4\mathbf{a}_z$, respectively. Determine the dot product of two vectors.

2.5 Two vectors are given by $\mathbf{A} = 4\mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z$ and $\mathbf{B} = \mathbf{a}_x - 4\mathbf{a}_y - 2\mathbf{a}_z$, respectively. Determine the angle between them.

2.6 Determine the projection of vector $\mathbf{A} = 4\mathbf{a}_x - 2\mathbf{a}_y - 3\mathbf{a}_z$ on the vector $\mathbf{B} = \mathbf{a}_x - 4\mathbf{a}_y - 2\mathbf{a}_z$.

2.7 The expressions of two vectors are given by $\mathbf{A} = p\mathbf{a}_x - 5\mathbf{a}_y - 2\mathbf{a}_z$ and $\mathbf{B} = 2\mathbf{a}_x + 4\mathbf{a}_y - 2\mathbf{a}_z$. Determine the value of p if the two vectors are in perpendicular directions.

2.8 Three vectors are given by the expressions $\mathbf{A} = 3\mathbf{a}_x + 2\mathbf{a}_y - 3\mathbf{a}_z$, $\mathbf{B} = \mathbf{a}_x + 2\mathbf{a}_y - 4\mathbf{a}_z$ and $\mathbf{C} = 3\mathbf{a}_x + 2\mathbf{a}_y - 5\mathbf{a}_z$. Determine the vectors $\mathbf{B} \times \mathbf{C}$ and $\mathbf{A} \bullet \mathbf{B} \times \mathbf{C}$.

2.9 Three vectors are given by the expressions $\mathbf{A} = \mathbf{a}_x + 3\mathbf{a}_y + 2\mathbf{a}_z$, $\mathbf{B} = 2\mathbf{a}_x + 3\mathbf{a}_y + 5\mathbf{a}_z$ and $\mathbf{C} = 3\mathbf{a}_x - 2\mathbf{a}_y + 3\mathbf{a}_z$. Calculate the vector $(\mathbf{A} + \mathbf{B}) \times (\mathbf{B} - \mathbf{C})$.

2.10 Two vectors are given by the expressions $\mathbf{A} = 2\mathbf{a}_x - 3\mathbf{a}_y + 2\mathbf{a}_z$ and $\mathbf{B} = p\mathbf{a}_x + q\mathbf{a}_y + 2\mathbf{a}_z$. Determine the value of p and q if two vectors are parallel.

2.11 Two vectors are moving from the origin to the points $P_1(2,4,7)$ and $P_2(1,2,3)$ as shown in Fig. 2.30. Write down the vector $\mathbf{R}_{12}$ in Cartesian coordinates and determine its distance.

2.12 Three points are given by $A(2,3,4)$, $B(-2,-1,2)$ and $C(1,3,5)$. Determine the (1) $\mathbf{R}_{AB}$, (2) $\mathbf{R}_{BC} and$ (3) $\mathbf{r}_C$.

2.13 The expression of a vector field is given by $\mathbf{A} = xy\mathbf{a}_x + (x^2 + yz)\mathbf{a}_y - (y^2 + 2z)\mathbf{a}_z$. Determine the (1) vector $\mathbf{A}$ at point $P(1,2,3)$, (2) unit vector $\mathbf{A}$ at point $Q(-1,2,5) and$ (3) unit vector directed from point $P(1,2,3)$ to point $Q(-1,2,5)$.

2.14 Two points in cylindrical coordinates are given by $P(\rho = 1, \phi = 65°, z = 2)$ and $Q(\rho = 1.4, \phi = 35°, z = 2.4)$. Determine the $\mathbf{R}_{PQ}$ in cylindrical coordinates and its unit vector.

2.15 Two vectors are given in cylindrical coordinates by $\mathbf{A} = 2\mathbf{a}_\rho - 3\mathbf{a}_\phi + \mathbf{a}_z$ and $\mathbf{B} = \mathbf{a}_\rho + 2\mathbf{a}_\phi - 3\mathbf{a}_z$. Determine the dot product and cross product of the two vectors, respectively.

2.16 A vector is given by $\mathbf{A} = \frac{x}{y}\mathbf{a}_y$. Convert this vector into cylindrical coordinates.

2.17 A vector is given by $\mathbf{A} = \frac{xz}{y}\mathbf{a}_x + \frac{y}{x}\mathbf{a}_y$. Convert this vector into cylindrical coordinates.

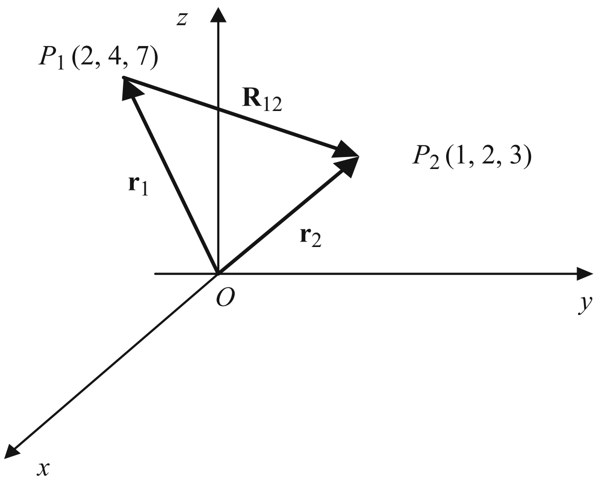


Fig. 2.30 Points in a space

2.18 A point in Cartesian coordinates is $Q(x = 2,\ y = 2.5,\ z = 1.2)$. Convert this point into cylindrical coordinates.

2.19 A vector in Cartesian coordinates is given by $\mathbf{A} = \frac{2}{y}\mathbf{a}_y$. Transform this vector into spherical coordinates.

2.20 A point in spherical coordinates is $P(r = 2,\ \theta = 120°,\ \phi = 230°)$. Convert this point into Cartesian and cylindrical coordinates.

2.21 A scalar electric potential is given by $V = xy + 2z^2$. Find the electric field $\mathbf{E}$ and its direction at point $P(-2, -3, 1)$.

2.22 The expression of a scalar electric potential is given by $V(\rho, \phi, z) = z^2 \cos\phi - \rho^2$. Determine the electric field $\mathbf{E}$ and its direction at point $P(\rho = 1,\ \phi = 120°,\ z = -1)$.

2.23 A vector in the Cartesian coordinates is given by $\mathbf{A} = yx\mathbf{a}_x - 2zy\mathbf{a}_y + z^2\mathbf{a}_z$. Determine the div $\mathbf{A}$.

2.24 The expression of a vector in cylindrical coordinates is given by $\mathbf{A} = \rho z\mathbf{a}_\rho - \cos\phi z\mathbf{a}_\phi + 2\rho z\mathbf{a}_z$. Find the div $\mathbf{A}$.

2.25 A vector in spherical coordinates is given by $\mathbf{A} = r\mathbf{a}_r + \sin\phi r^2\mathbf{a}_\theta + \sin\phi\mathbf{a}_\phi$. Calculate the div $\mathbf{A}$.

2.26 The expression of a vector in Cartesian coordinates is given as $\mathbf{A} = yx\mathbf{a}_x - zy\mathbf{a}_y + 2z\mathbf{a}_z$. Determine div $\mathbf{A}$ at point $P(-1, -1, 2)$.

2.27 A vector in cylindrical coordinates is given by $\mathbf{A} = \rho\mathbf{a}_\rho + \sin\phi\mathbf{a}_\phi + z\mathbf{a}_z$. Calculate div $\mathbf{A}$ at point $P(1, 85°, 2)$.

2.28 The expression of a vector in Cartesian coordinates is given by $\mathbf{A} = y^3z^2\mathbf{a}_x - (x + z^2)\mathbf{a}_y + 3(x^2 - y)\mathbf{a}_z$. Determine the curl of $\mathbf{A}$ at point $P(x = 1,\ y = 1,\ z = 1)$.

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Chapter 3
Electrostatic Field

3.1 Introduction

In Chap. 1, the basic definition of electromagnetic field parameters and their units were discussed. In Chap. 2, the fundamentals of vector algebra and orthogonal coordinate systems were discussed in detail. In a static electric field, the electric charges are always at rest and the electric field does not change with time, i.e., its magnitude is constant. The electric field due to time-invariant charges at rest is called the static electric field or the electrostatic field. Fundamental knowledge is required to design X-ray machines, lightning protection equipment, and other electrostatic devices. There are many applications of electrostatic fields. These applications are oscilloscopes, electrocardiograms (ECG), ink-jet printers, peripheral computer devices, paint spraying, electrochemical machinery, etc. To understand the basic of electrostatics, Coulomb's law, electric field intensity, Gauss' law, Ohm's law and energy, etc. will all be discussed in this chapter.

3.2 Coulomb's Law

The fundamentals of electrostatics are the outcome of an experiment that created Coulomb's law. In 1785, a French Army Engineers colonel performed a series of experiments to determine the force exerted between two small charge objects. According to his name, it is known as Coulomb's law. This law states that two small charges that exert a force on each other are directly proportional to the product of the magnitudes of the charges and inversely proportional to the square of the separation distance. This separation distance is large compared to the size of the charge bodies. Two point charges Q_1 and Q_2 with their separation distance is shown in Fig. 3.1. The electric force, $\mathbf{F}_{12}$ on a point charges Q_2 due to Q_1.

Mathematically, Coulomb's law can be written as

$$\mathbf{F}_{12} = k\frac{Q_1 Q_2}{{R_{12}}^2}\mathbf{a}_{12}, \tag{3.1}$$

Md. A. Salam, *Electromagnetic Field Theories for Engineering,*
DOI 10.1007/978-981-4585-66-8_3,

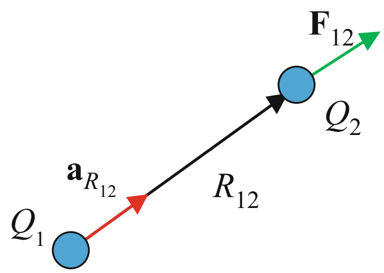


Fig. 3.1 Two point charges with separation distance

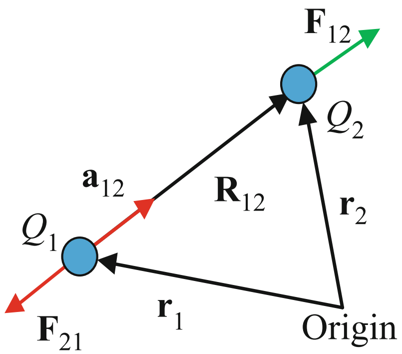


Fig. 3.2 Two point charges with separation distance

where

Q_1, Q_2 are the positive or negative charges in Coulomb (C),
R_{12} is the separation distance in meters,
$k = \frac{1}{4\pi\varepsilon_0}$ is the proportionality constant.

Like charges repel and unlike charges attract each other. Let the charges Q_1 and Q_2 be located from the origin as distances indicated by the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively, as shown in Fig. 3.2.

The electric force, $\mathbf{F}_{21}$ on a point charge Q_1 due to Q_2 can be written as

$$\mathbf{F}_{21} = k\frac{Q_1 Q_2}{{R_{21}}^2}\mathbf{a}_{21}. \tag{3.2}$$

The following vectors can be written as

$$\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1 \tag{3.3}$$

$$\mathbf{R}_{21} = \mathbf{r}_1 - \mathbf{r}_2. \tag{3.4}$$

The expressions of unit vectors can be written as

$$\mathbf{a}_{12} = \frac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{R_{12}} \tag{3.5}$$

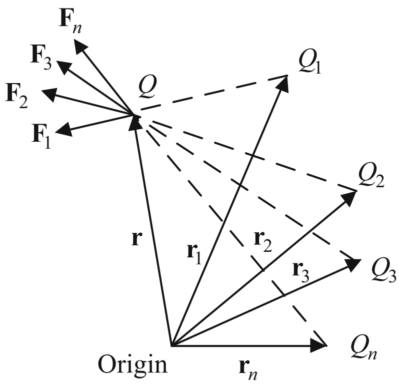


Fig. 3.3 The *n*-number charges with separation distance

$$\mathbf{a}_{21} = \frac{\mathbf{R}_{21}}{|\mathbf{R}_{21}|} = \frac{\mathbf{r}_1 - \mathbf{r}_2}{R_{21}}. \tag{3.6}$$

Equations (3.1) and (3.2) can be modified by substituting Eqs. (3.5) and (3.6) as

$$\mathbf{F}_{12} = -\mathbf{F}_{21} = \frac{1}{4\pi\varepsilon_0} Q_1 Q_2 \frac{(\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3}. \tag{3.7}$$

Figure 3.3 shows n number of charges located at n number of vectors. The superposition of the forces on the charge Q due to different point charges can be written as

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + - - - - + \mathbf{F}_N. \tag{3.8}$$

According to Eq. (3.7), Eq. (3.8) can be modified as

$$\mathbf{F} = \frac{QQ_1}{4\pi\varepsilon_0}\frac{(\mathbf{r} - \mathbf{r}_1)}{|\mathbf{r} - \mathbf{r}_1|^3} + \frac{QQ_2}{4\pi\varepsilon_0}\frac{(\mathbf{r} - \mathbf{r}_2)}{|\mathbf{r} - \mathbf{r}_2|^3} + \frac{QQ_3}{4\pi\varepsilon_0}\frac{(\mathbf{r} - \mathbf{r}_3)}{|\mathbf{r} - \mathbf{r}_3|^3} + - - - + \frac{QQ_n}{4\pi\varepsilon_0}\frac{(\mathbf{r} - \mathbf{r}_n)}{|\mathbf{r} - \mathbf{r}_n|^3}. \tag{3.9}$$

In general, Eq. (3.9) can be expressed as

$$\mathbf{F} = \frac{Q}{4\pi\varepsilon_0}\sum_{k=1}^{N}\frac{Q_k(\mathbf{r} - \mathbf{r}_k)}{|\mathbf{r} - \mathbf{r}_k|^3}. \tag{3.10}$$

Example 3.1 Point charges of 4×10^{-5}C and 6×10^{-6}C are located at points $M(1, 3, 4)$ and $N(4, 0, 3)$, respectively, in free space. Determine the electric force exerted on the second charge due to the first charge.

Solution The following vector can be written as

$$\mathbf{R}_{12} = (4 - 1)\mathbf{a}_x + (0 - 3)\mathbf{a}_y + (3 - 4)\mathbf{a}_z$$
$$\mathbf{R}_{12} = 3\mathbf{a}_x - 3\mathbf{a}_y - \mathbf{a}_z$$
$$|\mathbf{R}_{12}| = R_{12} = \sqrt{3^2 + (-3)^2 + 1} = 4.36$$

The force can be determined as

$$\mathbf{F}_{12} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{R_{12}{}^2}\mathbf{a}_{12} = \frac{4 \times 10^{-5} \times 6 \times 10^{-6}}{4\pi \times 8.854 \times 10^{-12} \times 19}\frac{3\mathbf{a}_x - 3\mathbf{a}_y - \mathbf{a}_z}{4.36}$$

$$\mathbf{F}_{12} = 0.078\mathbf{a}_x - 0.078\mathbf{a}_y - 0.026\mathbf{a}_z N.$$

Practice Problem 3.1 Point charges of 6×10^{-6}C and 9×10^{-6}C are located at points $M(1, 2, 3)$ and $N(2, 4, 6)$,respectively, in free space. Determine the (a) $\mathbf{R}_{12}$, (b) R_{12}, (c) $\mathbf{a}_{12}$, and (d) force exerted on the first charge due to the second charge.

3.3 Electric Field Intensity

The electric force exerted on each other of the point charges has been discussed in the previous section. There is existence of an electric field in the vicinity of those point charges. Consider a test charge, Q_t, is moving towards the charge Q_1. According to Coulomb's law, the force exerted on the test charge is

$$\mathbf{F}_t = \frac{Q_1 Q_t}{4\pi\varepsilon_0 R_{1t}{}^2}\mathbf{a}_{1t}. \tag{3.11}$$

Equation (3.11) can be modified as

$$\frac{\mathbf{F}_t}{Q_t} = \frac{Q_1}{4\pi\varepsilon_0 R_{1t}{}^2}\mathbf{a}_{1t}. \tag{3.12}$$

Equation (3.12) represents the expression of electric field intensity. In general, the electric field intensity is defined as the force per unit charge when a small stationary test charge is placed in this region. The unit of electric field intensity is V/m or N/C. Mathematically, electric field intensity can be expressed as

$$\mathbf{E} = \lim_{Q_t \to 0}\frac{\mathbf{F}_t}{Q_t}. \tag{3.13}$$

In general, electric field intensity can be expressed as

$$\mathbf{E} = \frac{\mathbf{F}_t}{Q_t}. \tag{3.14}$$

Substituting Eq. (3.12) into Eq. (3.14) yields the general equation by omitting the unnecessary subscripts,

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\mathbf{a}_R. \tag{3.15}$$

Electric field intensity due to point charges, Q_1 at $\mathbf{r}_1$ and Q_2 at $\mathbf{r}_2$ can be written as

$$\mathbf{E} = \frac{Q_1}{4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}_1|^2}\mathbf{a}_1 + \frac{Q_2}{4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}_2|^2}\mathbf{a}_2. \tag{3.16}$$

Electric field intensity due to n point charges is

$$\mathbf{E} = \sum_{k=1}^{N}\frac{Q_k}{4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}_k|^2}\mathbf{a}_k. \tag{3.17}$$

If the point charge is placed at the center of the spherical coordinates, then the electric field intensity becomes

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\mathbf{a}_r. \tag{3.18}$$

In scalar, Eq. (3.18) can be expressed as

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}. \tag{3.19}$$

In general, Eq. (3.14) can be written as

$$\mathbf{F} = Q\mathbf{E}. \tag{3.20}$$

Example 3.2 A point charge of 4×10^{-10}C is located at point $M(1,2,3)$. Determine the electric field intensity at point $N(2,0,1)$ in free space. All dimensions are in meters.

Solution The position vector from origin to the field point $N(2,0,1)$ is

$$\mathbf{r} = (2-0)\mathbf{a}_x + (0-0)\mathbf{a}_y + (1-0)\mathbf{a}_z = 2\mathbf{a}_x + \mathbf{a}_z.$$

The position vector from origin to point charge $M(1,2,3)$ is

$$\mathbf{r}_1 = (1-0)\mathbf{a}_x + (2-0)\mathbf{a}_y + (3-0)\mathbf{a}_z = \mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z$$
$$\mathbf{r}-\mathbf{r}_1 = \mathbf{a}_x - 2\mathbf{a}_y - 2\mathbf{a}_z$$
$$|\mathbf{r}-\mathbf{r}_1| = \sqrt{1^2+2^2+2^2}=3$$

The electric field intensity at the point is

$$\mathbf{E} = \frac{Q_1(\mathbf{r}-\mathbf{r}_1)}{4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}_1|^3} = \frac{4\times10^{-10}(\mathbf{a}_x - 2\mathbf{a}_y - 2\mathbf{a}_z)}{4\pi\times8.854\times10^{-12}\times27}$$

$$\mathbf{E} = 0.133\mathbf{a}_x - 0.266\mathbf{a}_y - 0.266\mathbf{a}_z\,\text{V/m}$$

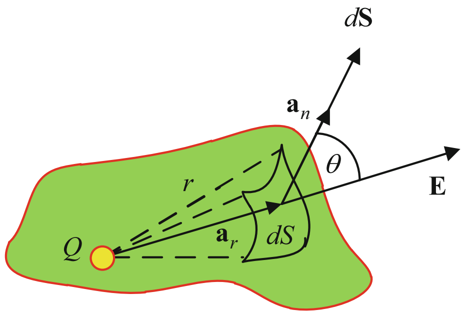


Fig. 3.4 Point charge with closed surface

Practice Problem 3.2 A point charge of -5×10^{-9}C is located at point $M(1, 0, 2)$. Determine the electric field intensity at point $N(2, 1, 3)$ in free space. All dimensions are in meters.

3.4 Gauss' Law

Gauss' law is one of the fundamental laws of electromagnetic engineering. Johann Karl Friedrich Gauss (1777–1855), a great German mathematician, generalized Faraday's experiment and created the statement, "Total electric flux passing any closed surface is equal to the total charge enclosed by that surface, divided by ε_0." Let us consider a point charge Q in free space is enclosed by the surface as shown in Fig. 3.4. The total flux is $\mathbf{E}$ through an arbitrary positioned surface element of area dS, whose surface area vector is $d\mathbf{S}$. The elementary flux can be written as

$$d\psi_E = \mathbf{E}.d\mathbf{S}. \tag{3.21}$$

The area vector can be written as

$$d\mathbf{S} = dS\mathbf{a}_n, \tag{3.22}$$

where $\mathbf{a}_n$ is the unit vector normal to the surface. According to the rules of dot product, Eq. (3.21) can be expressed as

$$d\psi_E = EdS\cos\theta. \tag{3.23}$$

Substituting Eq. (3.19) into Eq. (3.23) yields

$$d\psi_E = \frac{Q}{4\pi\varepsilon_0 r^2}dS\cos\theta = \frac{Q}{4\pi\varepsilon_0}\frac{dS\cos\theta}{r^2}. \tag{3.24}$$

By definition, the element of solid angle is

$$d\Omega = \frac{dS\cos\theta}{r^2}. \tag{3.25}$$

Substituting Eq. (3.25) into Eq. (3.24) provides

$$d\psi_E = \frac{Q}{4\pi\varepsilon_0} d\Omega. \tag{3.26}$$

Integrating Eq. (3.26) over the surface yields

$$\oint_S d\psi_E = \oint_S \mathbf{E}.d\mathbf{S} = \frac{Q}{4\pi\varepsilon_0} \oint_S d\Omega. \tag{3.27}$$

Over the surface, the value of the solid angle will be the total surface area of a unit sphere, namely

$$d\Omega = 4\pi(1)^2 = 4\pi. \tag{3.28}$$

Substituting Eq. (3.28) into Eq. (3.27) yields

$$\psi_E = \oint_S \mathbf{E}.d\mathbf{S} = \frac{Q}{4\pi\varepsilon_0} \times 4\pi \tag{3.29}$$

$$\oint_S \varepsilon_0 \mathbf{E}.d\mathbf{S} = Q \tag{3.30}$$

$$\oint_S \mathbf{E}.d\mathbf{S} = \frac{Q}{\varepsilon_0}. \tag{3.31}$$

From Eq. (3.31), Gauss' law states that the total outward flux of the electric field over any closed surface in free space is equal to the total charge enclosed by that surface divided by ε_0.

But the electric flux density is defined as

$$\mathbf{D} = \varepsilon_0 \mathbf{E}. \tag{3.32}$$

Substituting Eq. (3.32) into Eq. (3.30) provides

$$\oint_S \mathbf{D}.d\mathbf{S} = Q. \tag{3.33}$$

The surface element vector always involves the differentials of two coordinates, such as $dxdy$, $\rho d\phi\, d\rho$, and $r^2 \sin\theta d\theta d\phi$, and the integral will be a double integral. For N point charges, Eq. (3.32) can be written as

$$\oint_S \mathbf{E}_1.d\mathbf{S} + \oint_S \mathbf{E}_2.d\mathbf{S} + \ldots\ldots + \oint_S \mathbf{E}_N.d\mathbf{S} = \frac{Q_1}{\varepsilon_0} + \frac{Q_2}{\varepsilon_0} + \ldots + \frac{Q_N}{\varepsilon_0}. \tag{3.34}$$

For several point charges, the charge enclosed is

$$Q = \sum_{i=1}^{N} Q_i. \tag{3.35}$$

For a line charge, the expression is

$$Q = \int \rho_L dL. \tag{3.36}$$

For a surface charge, the expression is

$$Q = \int_S \rho_S dS. \tag{3.37}$$

For a volume charge, the expression is

$$Q = \int_{\text{vol}} \rho_v dv. \tag{3.38}$$

Gauss' law in terms of charge distribution may be written by substituting Eq. (3.38) into Eq. (3.33) as

$$\oint_S \text{D}.d\text{S} = \int_{\text{vol}} \rho_v dv. \tag{3.39}$$

Applying the divergence theorem to Gauss' law, the following equation can be written as

$$\oint_S \mathbf{D}.d\mathbf{S} = \int_{\mathbf{vol}} (\nabla.\mathbf{D})dv. \tag{3.40}$$

From Eqs. (3.39) and (3.40), the following relation can be derived:

$$\rho_v = \nabla.\mathbf{D}. \tag{3.41}$$

Equation (3.41) is known as the first Maxwell's equation and point form of Gauss' law. Equations (3.40) and (3.41) are known as integral and differential forms of Gauss' law. Equation (3.41) can be modified for free space as

$$\nabla.\mathbf{E} = \frac{\rho}{\varepsilon_0}. \tag{3.42}$$

Example 3.3 The electric flux density is defined as $\mathbf{D} = x^2\mathbf{a}_x + 2y\mathbf{a}_y + 3z^2\mathbf{a}_z$. Determine the $\oint_S \mathbf{D}.d\mathbf{S}$, where S is the surface of a rectangular box, and the limits are $x = 0,\ x = 1, y = 0,\ y = 2, z = 0,\ z = 3$.

Solution The following integration can be determined as:

$$
\begin{aligned}
\oint_S \mathbf{D}.d\mathbf{S} &= \int_{vol} \nabla.\mathbf{D}dv \\
&= \int_{vol} \left(\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z\right).\left(x^2\mathbf{a}_x + 2y\mathbf{a}_y + 3z^2\mathbf{a}_z\right)dv \\
&= \int_{vol} (2x + 2 + 6z)dxdydz \\
&= \iint dxdy \int_0^3 (2x + 2 + 6z)dz \\
&= \iint dxdy \left[2xz + 2z + 3z^2\right]_0^3 \\
&= \int dx \int_0^2 (6x + 33)dy \\
&= \int dx\, [6xy + 33y]_0^3 \\
&= \int_0^1 (12x + 66)dx \\
&= \left[6x^2 + 66x\right]_0^1 = 72.
\end{aligned}
$$

Example 3.4 Use Gauss' theorem to determine $\oint_S \mathbf{D}.d\mathbf{S}$, where the electric flux density is $\mathbf{D} = x^2\mathbf{a}_x + 2y^2\mathbf{a}_y + z\mathbf{a}_z$ and the surface is surrounded by the region of $x^2 + y^2 = 9, z = 0$ and $z = 2$.

Solution The following integration can be determined as

$$
\begin{aligned}
\oint_S \mathbf{D}.d\mathbf{S} &= \int_{vol} \nabla.\mathbf{D}dv \\
&= \int_{vol} \left(\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z\right).\left(x^2\mathbf{a}_x + 2y^2\mathbf{a}_y + z\mathbf{a}_z\right)dv \\
&= \int\int\int (2x + 4y + z)dxdydz \\
&= \iint dxdy \int_0^2 (2x + 4y + z)dz \\
&= \iint dxdy \left[2xz + 4yz + \frac{1}{2}z^2\right]_0^2 \\
&= \int\int (4x + 8y + 2)dxdy.
\end{aligned}
$$

Let $x = r\cos\theta, y = r\sin\theta$, and $dS = dxdy = rd\theta dr$. Then the above integration becomes

$$= 4\int_0^{2\pi} d\theta \int_0^3 (r\cos\theta + 2r\sin\theta + \frac{1}{2})rdr$$

$$= 4\int_0^{2\pi} \left[\frac{r^3\cos\theta}{3} + \frac{2r^3\sin\theta}{3} + \frac{r^2}{2}\right]_0^3 d\theta$$

$$= 4\int_0^{2\pi} \left(9\cos\theta + 18\sin\theta + \frac{9}{2}\right) d\theta$$

$$= 36[\sin\theta]_0^{2\pi} - 72[\cos\theta]_0^{2\pi} + 18[\theta]_o^{2\pi} = 36\pi.$$

Practice Problem 3.3 The electric flux density is defined as $\mathbf{D} = x^3\mathbf{a}_x + xy^2\mathbf{a}_y + yz^2\mathbf{a}_z$. Determine the $\oint_S \mathbf{D}.d\mathbf{S}$, where S is the surface bounded by the region of limits, which are $x = 0,\ x = 2, y = 0,\ y = 1, z = 0,\ z = 3$.

Practice Problem 3.4 Use Gauss' theorem to determine $\oint_S \mathbf{D}.d\mathbf{S}$, where the electric flux density is $\mathbf{D} = x^2\mathbf{a}_x - y^2\mathbf{a}_y - z\mathbf{a}_z$ and the surface is surrounded by the region of $x^2 + y^2 = 4, z = 0$ and $z = 1$.

3.5 Electric Field of Continuous Charge Distribution

The electric field intensity due to a continuous distribution of charge can be obtained by integrating a single element of charge. From Eq. (3.17), the general expression of electric field can be written as

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\mathbf{a}_r. \tag{3.43}$$

Consider a point charge that is located on the line, surface, and volume. For a differential length, surface, and volume, the differential charges are

$$dQ = \rho_l dl \tag{3.44}$$

$$dQ = \rho_s ds \tag{3.45}$$

$$dQ = \rho_v dv. \tag{3.46}$$

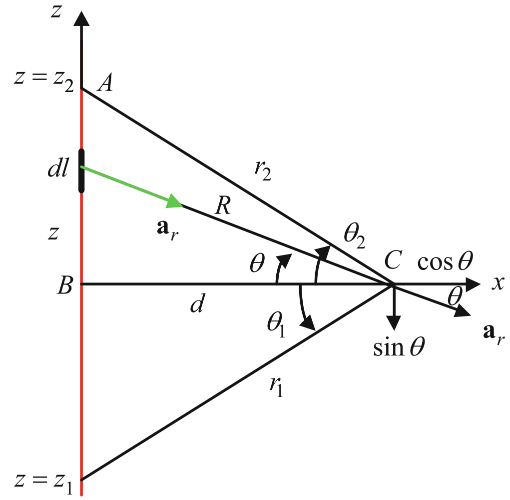
In differential form, Eq. (3.43) can be written as

$$d\mathbf{E} = \frac{dQ}{4\pi\varepsilon_0 R^2}\mathbf{a}_r. \tag{3.47}$$

Substituting Eq. (3.44) into Eq. (3.47) yields

$$d\mathbf{E} = \frac{\rho_l dl}{4\pi\varepsilon_0 R^2}\mathbf{a}_r. \tag{3.48}$$

Fig. 3.5 Straight line with a uniform charge distribution



Integrating Eq. (3.48) yields

$$E = \int d\mathbf{E} = \int_l \frac{\rho_l dl}{4\pi\varepsilon_0}\frac{\mathbf{a}_r}{R^2}. \tag{3.49}$$

Since $\mathbf{a}_r = \frac{\mathbf{R}}{R}$, Eq. (3.49) becomes

$$E = \frac{1}{4\pi\varepsilon_0}\int_l \rho_l \frac{\mathbf{R}}{R^3} dl. \tag{3.50}$$

If the charge is distributed over the surface and volume, the expressions for electric field are

$$E = \frac{1}{4\pi\varepsilon_0}\int_s \rho_s \frac{\mathbf{R}}{R^3} ds \tag{3.51}$$

$$E = \frac{1}{4\pi\varepsilon_0}\int_v \rho_v \frac{\mathbf{R}}{R^3} dv. \tag{3.52}$$

Example 3.5 The total charge of Q is distributed uniformly over a straight line of length of l. Determine the expression for the electric field.

Solution The charge is located at a distance d from the center of the line as shown in Fig. 3.5. Here, the angles are $\theta_1 < 0$ and $\theta_2 > 0$. Consider a differential length dl of the straight line. From Fig. 3.5, the following expressions can be written as:

$$R^2 = z^2 + d^2 \tag{3.53}$$

$$dl = dz \tag{3.54}$$

$$\mathbf{a}_r = \cos\theta \mathbf{a}_x - \sin\theta \mathbf{a}_z \tag{3.55}$$

$$\tan\theta = \frac{z}{d} \tag{3.56}$$

$$z = d\tan\theta \tag{3.57}$$

$$\cos\theta = \frac{d}{R}. \tag{3.58}$$

Differentiating Eq. (3.57) with respect to θ provides

$$dz = d\sec^2\theta d\theta \tag{3.59}$$

$$dz = \frac{d}{\cos^2\theta} d\theta. \tag{3.60}$$

Substituting Eq. (3.58) into Eq. (3.60) yields

$$dz = \frac{d}{\frac{d^2}{R^2}} d\theta \tag{3.61}$$

$$dz = \frac{R^2}{d} d\theta \tag{3.62}$$

$$\frac{dz}{R^2} = \frac{1}{d} d\theta. \tag{3.63}$$

Substituting Eqs. (3.55) and (3.63) into Eq. (3.48) and then integrating provides

$$\mathbf{E} = \int d\mathbf{E} = \frac{\rho_l}{4\pi\varepsilon_0 d} \int_{\theta_1}^{\theta_2} (\cos\theta \mathbf{a}_x - \sin\theta \mathbf{a}_z) d\theta \tag{3.64}$$

$$\mathbf{E} = \frac{\rho_l}{4\pi\varepsilon_0 d}[\sin\theta_2 - \sin\theta_1]\mathbf{a}_x + \frac{\rho_l}{4\pi\varepsilon_0 d}[\cos\theta_2 - \cos\theta_1]\mathbf{a}_z. \tag{3.65}$$

If the straight line is extended from negative infinity to positive infinity, the angles will be $\theta_1 \to -\frac{\pi}{2}$ and $\theta_2 \to \frac{\pi}{2}$, respectively, and $\mathbf{a}_x = \mathbf{a}_r$. Equation (3.65) can be modified as

$$\mathbf{E} = \frac{\rho_l}{4\pi\varepsilon_0 d}(1+1)\mathbf{a}_r \tag{3.66}$$

$$\mathbf{E} = \frac{\rho_l}{2\pi\varepsilon_0 d}\mathbf{a}_r. \tag{3.67}$$

The electric flux density can be written as

$$\mathbf{D} = \frac{\rho_l}{2\pi d}\mathbf{a}_r. \tag{3.68}$$

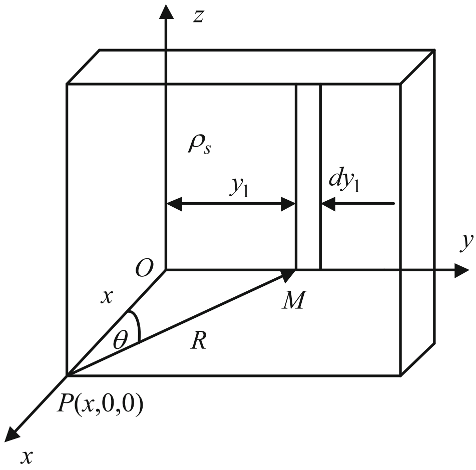


Fig. 3.6 Charge of an infinite sheet

Practice Problem 3.5 An infinitely long straight line charge distributed uniformly along the z-axis contains air, the density of which is ρ. Determine the electric field intensity.

3.6 Electric Field Due to an Infinite Sheet Charge

Consider an infinite sheet of charge, which is placed in the yz plane and it is in symmetry on the respective axis (Fig. 3.6). The sheet has a uniform charge density of $\rho_s \mathrm{C/m^2}$. The sheet is divided into a differential width strip, dy_1. Then the charge per unit length is

$$\rho_L = \rho_s dy_1. \tag{3.69}$$

The distance between the point P and the sheet charge is

$$R = \sqrt{x^2 + {y_1}^2}. \tag{3.70}$$

The field components due to y and z axes are cancelled out and only the field component due to x-axis is present. In this case, the unit vector is

$$\mathbf{a}_r = \cos\theta \mathbf{a}_x. \tag{3.71}$$

Substituting Eqs. (3.69) and (3.71) into Eq. (3.67) provides

$$dE_x = \frac{\rho_s dy_1}{2\pi\varepsilon R}\cos\theta. \tag{3.72}$$

From OPMtriangle, the following equation can be written as

$$\cos\theta = \frac{x}{R} = \frac{x}{\sqrt{x^2 + {y_1}^2}}. \tag{3.73}$$

Substituting Eqs. (3.70) and (3.73) into Eq. (3.72) yields

$$dE_x = \frac{\rho_s x dy_1}{2\pi\varepsilon(x^2 + {y_1}^2)}. \tag{3.74}$$

Integrating Eq. (3.74) from $-\infty$ to ∞ gives

$$E_x = \frac{\rho_s}{2\pi\varepsilon}\int_{-\infty}^{\infty}\frac{xdy_1}{(x^2 + {y_1}^2)} \tag{3.75}$$

$$E_x = \frac{\rho_s}{2\pi\varepsilon}\left[\tan^{-1}\frac{y_1}{x}\right]_{-\infty}^{\infty} \tag{3.76}$$

$$E_x = \frac{\rho_s}{2\pi\varepsilon}\left[\tan^{-1}\frac{\infty}{x} - \tan^{-1}\frac{-\infty}{x}\right] \tag{3.77}$$

$$E_x = \frac{\rho_s}{2\pi\varepsilon}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \frac{\rho_s}{2\varepsilon}. \tag{3.78}$$

If point P is chosen on the negative side of the x-axis, then the electric field is

$$E_x = -\frac{\rho_s}{2\varepsilon}. \tag{3.79}$$

In general, the expression of the electric field along with the unit vector is

$$\mathbf{E} = \frac{\rho_s}{2\varepsilon}\mathbf{a}_N. \tag{3.80}$$

3.7 Electric Potential

Electric potential is defined as the work done by moving a point charge from one point to another point in an electric field. Suppose a force **F** is acting to move a point charge Q in the electric field. The expression for a force due to a point charge Q in the electric field is

$$\mathbf{F} = Q\mathbf{E}. \tag{3.81}$$

The work done for an elementary vector distance, $d\mathbf{l}$ is

$$dW = \mathbf{F}.d\mathbf{l} = Fdl\cos\theta. \tag{3.82}$$

Consider a point charge moves from point b to point a as shown in Fig. 3.7. In this case, point b is the initial point and point a is the final point. Then, the total work done can be determined as

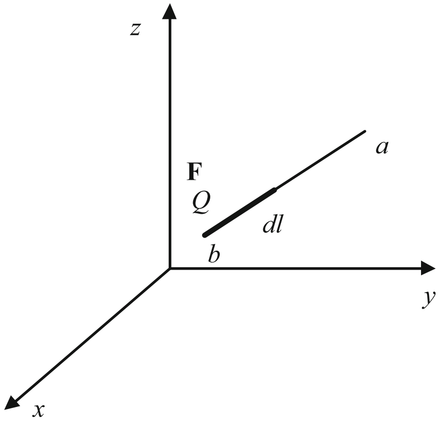


Fig. 3.7 Point charge movement through a small distance

$$W = \int dW = \int_b^a \mathbf{F}.d\mathbf{l}. \tag{3.83}$$

Substituting Eq. (3.81) into Eq. (3.83) yields

$$W = \int_b^a -Q\mathbf{E}.d\mathbf{l}. \tag{3.84}$$

Here, the negative sign indicates that the work is done by an external force. The potential difference between points a and b is defined as the work done per unit charge and it may be expressed as

$$V_{ab} = \frac{W}{Q}. \tag{3.85}$$

Equation (3.85) can be written as

$$W = QV_{ab}. \tag{3.86}$$

Substitute Eq. (3.86) into Eq. (3.84) provides

$$V_{ab} = V_a - V_b = -\int_b^a \mathbf{E}.d\mathbf{l}. \tag{3.87}$$

From Eq. (3.87), the following points can be summarized as:

- $V_{ab} < 0$ means loss in potential energy and work done by an electric field, E.
- $V_{ab} > 0$ means gain in potential energy and work done by an external force.
- V_{ab} is path independent.
- V_{ab} is measured in J/C or volts (V).

The differential length in terms of Cartesian, cylindrical, and spherical coordinates can be expressed as

$$d\mathbf{l} = dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z \tag{3.88}$$

$$d\mathbf{l} = d\rho\mathbf{a}_\rho + \rho d\phi\mathbf{a}_\phi + dz\mathbf{a}_z \tag{3.89}$$

$$d\mathbf{l} = dr\mathbf{a}_r + rd\theta\mathbf{a}_\theta + r\sin\theta d\phi\mathbf{a}_\phi. \tag{3.90}$$

Suppose a charge is moving from point $b(\rho_b, \phi_b, z_b)$ to point $a(\rho_a, \phi_a, z_a)$ along the direction of ρ. Then, the work done can be determined by substituting Eqs. (3.67) and (3.89) into Eq. (3.84) as

$$W = \int_b^a -Q\frac{\rho_L}{2\pi\varepsilon_0\rho}\mathbf{a}_\rho.(d\rho\mathbf{a}_\rho + \rho d\phi\mathbf{a}_\phi + dz\mathbf{a}_z). \tag{3.91}$$

$$W = \int_b^a -Q\frac{\rho_L}{2\pi\varepsilon_0\rho}\mathbf{a}_\rho.d\rho\mathbf{a}_\rho = \int_b^a -Q\frac{\rho_L}{2\pi\varepsilon_0\rho}d\rho \tag{3.92}$$

$$W = Q\frac{\rho_L}{2\pi\varepsilon_0}\ln\frac{b}{a}. \tag{3.93}$$

The potential difference can be determined by substituting Eq. (3.93) into Eq. (3.85) as

$$V_{ab} = \frac{\rho_L}{2\pi\varepsilon_0}\ln\frac{b}{a}. \tag{3.94}$$

Again, consider a charge that moves from point $b(r_b, \theta_b, \phi_b)$ to point $a(r_a, \theta_a, \phi_a)$ along the direction of r. Then the work done can be determined by substituting Eqs. (3.18) and (3.90) into Eq. (3.84) as

$$W = \int_{r_b}^{r_a} -\frac{Q}{4\pi\varepsilon_0 r^2}\mathbf{a}_r.(dr\mathbf{a}_r + rd\theta\mathbf{a}_\theta + r\sin\theta d\phi\mathbf{a}_\phi) \tag{3.95}$$

$$W = \int_{r_b}^{r_a} -\frac{Q}{4\pi\varepsilon_0 r^2}\mathbf{a}_r.rdr\mathbf{a}_r = \int_{r_b}^{r_a} -\frac{Q}{4\pi\varepsilon_0 r^2}dr \tag{3.96}$$

$$W = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right). \tag{3.97}$$

The potential difference can be determined by substituting Eq. (3.97) into Eq. (3.85) as

$$V_{ab} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right) \tag{3.98}$$

$$V_{ab} = \frac{Q}{4\pi\varepsilon_0 r_a} - \frac{Q}{4\pi\varepsilon_0 r_b} = V_a - V_b \tag{3.99}$$

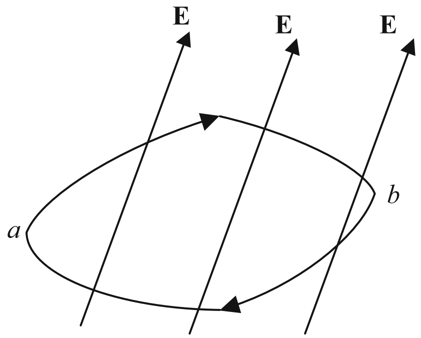


Fig. 3.8 Closed path with electric field

where V_a and V_b are the potentials at point a and b, respectively. Equation (3.99) might be used for the analysis of the capacitor.

According to Eq. (3.99), the following equation can be written as

$$V_{ab} = -(V_b - V_a) = -V_{ba} \tag{3.100}$$

$$V_{ab} + V_{ba} = 0. \tag{3.101}$$

From Eqs. (3.87) and (3.101), the following equation can be written as:

$$V_{ab} + V_{ba} = \oint \mathbf{E}.d\mathbf{l} = 0 \tag{3.102}$$

$$\oint_L \mathbf{E}.d\mathbf{l} = 0. \tag{3.103}$$

Equation (3.103) states that the line integral of electric field around a closed path as shown in Fig. 3.8 is equal to zero. This condition is known as irrotational or conservative field.

Applying Stock's theorem to Eq. (3.103) provides

$$\oint_L \mathbf{E}.d\mathbf{l} = \int_S (\nabla \times \mathbf{E}).d\mathbf{S} = 0 \tag{3.104}$$

$$\nabla \times \mathbf{E} = 0. \tag{3.105}$$

Example 3.6 A nonuniform electric field is given by $\mathbf{E} = x\mathbf{a}_x + y\mathbf{a}_y + 2z\mathbf{a}_z$ V/m. A charge of 3C is transferred from point $A(1,2,3)$ to point $B(2,4,5)$ along the differential length of the line. Consider the differential length in Cartesian coordinates and determine the work done.

Solution The following dot product can be determined as

$$\mathbf{E}.d\mathbf{l} = (x\mathbf{a}_x + y\mathbf{a}_y + 2z\mathbf{a}_z).(dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z)$$

$$\mathbf{E}.d\mathbf{l} = xdx + ydy + 2zdz.$$

The work done can be determined as

$$W = -3\int_{1}^{2} xdx - 3\int_{2}^{4} ydy - 6\int_{3}^{5} zdz$$

$$W = -\frac{3}{2}(4-1) - \frac{3}{2}(16-4) - 6(25-9) = -118.5\text{J}.$$

Example 3.7 A nonuniform electric field is given by $\mathbf{E} = y\mathbf{a}_x + x\mathbf{a}_y + 2y\mathbf{a}_z$V/m. A charge of 2C is transferred from point $A(1,0,3)$ to point $B(2,1,3)$ along the straight line from point A to point B. Calculate the work done.

Solution The following dot product can be determined as

$$\mathbf{E}.d\mathbf{l} = (y\mathbf{a}_x + x\mathbf{a}_y + 2y\mathbf{a}_z).(dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z)$$

$$\mathbf{E}.d\mathbf{l} = ydx + xdy + 2ydz.$$

The equation of the lines can be determined as

$$\frac{x-2}{2-1} = \frac{y-1}{1}$$

$$x = y + 1$$

$$\frac{z-3}{3-3} = \frac{y-1}{1}$$

$$z = 3.$$

The work done can be determined as

$$W = -2\int_{1}^{2} ydx - 2\int_{0}^{1} xdy - 4\int_{3}^{3} ydz$$

$$W = -2\int_{1}^{2} (x-1)dx - 2\int_{0}^{1} (y+1)dy$$

$$W = -(4-1) + 2(2-1) - (1) - 2(1) = -4\text{J}.$$

Practice Problem 3.6 A charge of 4C is transferred from point $A(1,0,0)$ to point $B(2,1,2)$ through an electric field whose expression is given by $\mathbf{E} = 2y\mathbf{a}_x + x\mathbf{a}_y$V/m. The charge is transferred along the path $y = 2x - 1$ and $z = 2$. Determine the work done.

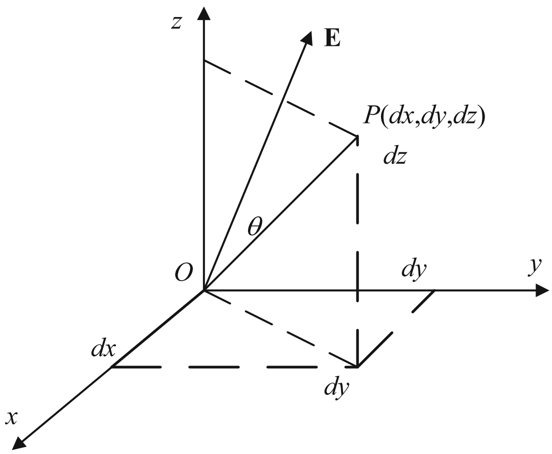


Fig. 3.9 Electric field with a specific point

Practice Problem 3.7 A charge of 2C is transferred from point $A(1, 0, 1)$ to point $B(0.8, 1, 1.5)$ through an electric field whose expression is given by $\mathbf{E} = 3y\mathbf{a}_x$V/m. The charge is transferred along the path whose equations are $y = x - 2$ and $z = 1$. Determine the work done.

3.8 Derivation of Electric Field

The electric field can be expressed in terms of scalar potential, i.e., $\text{E} = -\nabla V$. The electric field is working at point O as shown in Fig. 3.9. Consider a point $P(dx, dy, dz)$ at an infinitesimal distance dl. The work is required to move a unit charge from point O to point P. In this case, the expression of voltage is

$$V = -\int \mathbf{E}.d\mathbf{l} \tag{3.106}$$

$$dV = -\mathbf{E}.d\mathbf{l}. \tag{3.107}$$

According to the vector dot product rules, Eq. (3.107) can be modified as

$$dV = -Edl\cos\alpha. \tag{3.108}$$

From Fig. 3.9, the expression dl can be written as

$$d\mathbf{l} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z. \tag{3.109}$$

The electric field in Cartesian coordinates

$$\mathbf{E} = E_x\mathbf{a}_x + E_y\mathbf{a}_y + E_z\mathbf{a}_z. \tag{3.110}$$

Substituting Eqs. (3.109) and (3.110) into Eq. (3.107) yields

$$dV = -(E_x\mathbf{a}_x + E_y\mathbf{a}_y + E_z\mathbf{a}_z).(dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z) \tag{3.111}$$

$$dV = -(E_x dx + E_y dy + E_z dz). \tag{3.112}$$

Using differential calculus to express total differentiation of voltage into partial differentiation as

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz. \tag{3.113}$$

The gradient of potential in Cartesian coordinates can be written as

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z. \tag{3.114}$$

Then the following relation can be written as

$$\nabla V.d\mathbf{l} = \left(\frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z\right).(dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z) \tag{3.115}$$

$$\nabla V.d\mathbf{l} = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz. \tag{3.116}$$

Substituting Eq. (3.113) into Eq. (3.116) yields

$$\nabla V.d\mathbf{l} = dV. \tag{3.117}$$

Substituting Eq. (3.117) into Eq. (3.107) yields

$$\nabla V.d\mathbf{l} = -\mathbf{E}.d\mathbf{l} \tag{3.118}$$

$$(\mathbf{E} + \nabla V).d\mathbf{l} = 0. \tag{3.119}$$

The length dl cannot be zero, and then the following equation is:

$$(\mathbf{E} + \nabla V) = 0 \tag{3.120}$$

$$\mathbf{E} = -\nabla V. \tag{3.121}$$

Equation (3.121) states that electric field intensity is the negative of the gradient of potential.

If $\alpha = 0°$, then the derivative of potential difference is

$$E = -\frac{dV}{dl}. \tag{3.122}$$

If $\alpha = 180°$, then the derivative of potential difference is

$$\left.\frac{dV}{dl}\right|_{\max} = E. \tag{3.123}$$

The gradients of potentials in cylindrical and spherical coordinates are

$$\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z \tag{3.124}$$

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi. \tag{3.125}$$

Example 3.8 A potential in Cartesian coordinates is given as $V = xy^2+3z^2$ and the point is $P(1, -2, 3)$. Determine the (a) numerical value of the voltage, (b) electric field, (c) direction of electric field, and (d) volume charge density.

Solution

a. The numerical value of the voltage is

$$V_P = 1(-2)^2+3(3)^2 = 31\text{V}.$$

b. The electric field intensity can be determined as

$$\mathbf{E} = -\left(\frac{\partial(xy^2+3z^2)}{\partial x}\mathbf{a}_x + \frac{\partial(xy^2+3z^2)}{\partial y}\mathbf{a}_y + \frac{\partial(xy^2+3z^2)}{\partial z}\mathbf{a}_z\right)$$

$$\mathbf{E} = -y^2\mathbf{a}_x - 2xy\mathbf{a}_y - 6z\mathbf{a}_z$$

$$\mathbf{E}_P = -(-2)^2\mathbf{a}_x - 2(1)(-2)\mathbf{a}_y - 6(3)\mathbf{a}_z$$

$$\mathbf{E}_P = -4\mathbf{a}_x + 4\mathbf{a}_y - 18\mathbf{a}_z\text{V/m}.$$

c. The direction of electric field can be determined as

$$\mathbf{a}_P = \frac{-4\mathbf{a}_x + 4\mathbf{a}_y - 18\mathbf{a}_z}{\sqrt{(-4)^2 + 4^2 + (-18)^2}}$$

$$\mathbf{a}_P = -0.21\mathbf{a}_x + 0.21\mathbf{a}_y - 0.95\mathbf{a}_z.$$

d. The volume charge density can be determined as

$$\mathbf{D} = \varepsilon_0\mathbf{E} = -8.854y^2\mathbf{a}_x - 17.71xy\mathbf{a}_y - 53.12z\mathbf{a}_z pC/m^3$$

$$\rho_v = \nabla.\mathbf{D} = \left(\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z\right).(-8.854y^2\mathbf{a}_x - 17.71xy\mathbf{a}_y - 53.12z\mathbf{a}_z)$$

$$\rho_v = \nabla.\mathbf{D} = -8.854\frac{\partial}{\partial x}(y^2) - 17.71x\frac{\partial}{\partial y}(y) - 53.12\frac{\partial}{\partial z}(z)$$

$$\rho_v = -17.71x - 53.12pC/m^3.$$

At point $P(1, -2, 3)$, the numerical value of the volume charge density is

$$\rho_v = -17.71 - 53.12 = -70.83pC/m^3.$$

Practice Problem 3.8 A potential in cylindrical coordinates is given as $V = 15\rho z^2 \sin\phi$ and the point is $P(\rho = 2\text{m}, \phi = 60°, z = 1\text{m})$. Calculate the (a) numerical value of the voltage, (b) electric field, (c) electric flux density, and (d) volume charge density.

3.9 Line Integral of Irrotational Field

The expression of the electric field in terms of grad of voltage is considered to calculate the line integral of irrotational field. This equation is

$$\mathbf{E} = -\nabla V. \tag{3.126}$$

The line integral of irrotational field can be derived by integrating Eq. (3.126) as

$$\int_b^a \mathbf{E}.d\mathbf{l} = \int_b^a -\nabla V.d\mathbf{l}. \tag{3.127}$$

Substituting Eqs. (3.88) and (3.122) into Eq. (3.127) provides

$$\int_b^a \mathbf{E}.d\mathbf{l} = \int_b^a -\left(\frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z\right).(dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z) \tag{3.128}$$

$$\int_b^a \mathbf{E}.d\mathbf{l} = -\int_b^a \left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right). \tag{3.129}$$

In partial differentiation, the following relation can be written as

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz. \tag{3.130}$$

Substituting Eq. (3.130) into Eq. (3.129) yields

$$\int_b^a \mathbf{E}.d\mathbf{l} = -\int_b^a dV = V(b) - V(a). \tag{3.131}$$

From Eq. (3.131), it is seen that the irrotational field equals the difference between voltages at the points of the path and not dependent on the path from *a* and *b*.

3.10 Potential Due to a Point Charge

The electric scalar potential needs to be determined due to a point charge in free space. The point charge is placed at the origin as shown in Fig. 3.10. The electric potential of a point at a distance *r* from the point charge at the infinite distance can be determined as

Fig. 3.10 Charge at origin

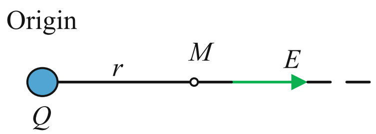
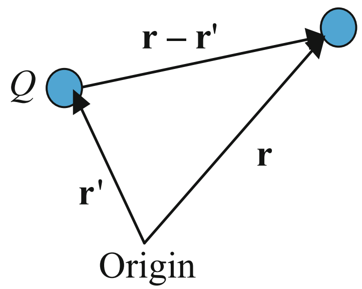


Fig. 3.11 Charge at a distance



$$V = -\int_{\infty}^{r} \mathbf{E}.d\mathbf{l}. \tag{3.132}$$

Substituting Eqs. (3.18) and (3.90) into Eq. (3.132) yields

$$V = -\int_{\infty}^{r} \frac{Q}{4\pi\varepsilon_0 r^2}\mathbf{a}_r.dr\mathbf{a}_r \tag{3.133}$$

$$V = -\frac{Q}{4\pi\varepsilon_0}\int_{\infty}^{r} \frac{1}{r^2}dr \tag{3.134}$$

$$V = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r}\right]_{\infty}^{r} \tag{3.135}$$

$$V = \frac{Q}{4\pi\varepsilon_0 r}. \tag{3.136}$$

The charge Q is placed at a distance $\mathbf{r}'$ from the origin as shown in Fig. 3.11. Then the expression of voltage can be written as

$$V = \frac{Q}{4\pi\varepsilon_0 |\mathbf{r} - \mathbf{r}'|} \tag{3.137}$$

The potential due to n number of point charges can be expressed as

$$V = \frac{Q_1}{4\pi\varepsilon_0 |\mathbf{r} - \mathbf{r}_1|} + \frac{Q_2}{4\pi\varepsilon_0 |\mathbf{r} - \mathbf{r}_2|} + - - - + \frac{Q_n}{4\pi\varepsilon_0 |\mathbf{r} - \mathbf{r}_n|}. \tag{3.138}$$

$$V = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{n} \frac{Q_k}{|\mathbf{r} - \mathbf{r}_k|}. \tag{3.139}$$

Based on Eq. (3.139), the potential due to line, surface, and volume charges can be expressed as

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int_L \frac{\rho_L(r')}{|\mathbf{r}-\mathbf{r}'|}dl' \tag{3.140}$$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\rho_S(r')}{|\mathbf{r}-\mathbf{r}'|}dS' \tag{3.141}$$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int_v \frac{\rho_v(r')}{|\mathbf{r}-\mathbf{r}'|}dv'. \tag{3.142}$$

Example 3.9 Determine the potential at point (1,2, 3) when two point charges 2μC and 3μC are located at points (0.5,1, 2) and (− 1.2,1.5, −2), respectively. Consider zero potential at infinity.

Solution The following distances can be determined as

$$|\mathbf{r}-\mathbf{r}_1| = |(1,2,3)-(0.5,1,2)| = |(0.5,1,1)| = \sqrt{\frac{9}{4}} = 1.5$$

$$|\mathbf{r}-\mathbf{r}_2| = |(1,2,3)-(-1.2,1.5,-2)| = |(2.2,0.5,5)| = \sqrt{30.09} = 5.49.$$

The potential at point (1,2, 3) is determined as

$$V(r) = \frac{Q_1}{4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}_1|} + \frac{Q_2}{4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}_2|}$$

$$V(1,2,3) = \frac{10^{-6}}{4\pi\times 8.854\times 10^{-12}}\left(\frac{2}{1.5}+\frac{3}{5.49}\right) = 16.89\text{kV}.$$

Practice Problem 3.9 Three point charges -1.5μC,32μC, and 4.5μC are located at points (1.2, −1.5,2), (− 1.4, −1.5,2.5), and (− 1.6, −2.5, −2.8), respectively. Determine the potential at point (2,2.5,5) by considering zero potential at infinity.

3.11 Electric Dipole

An electric dipole is often known as a dipole. Basically, a dipole is two equal and opposite point charges separated by a small distance compared to a specific point where the electric potential will be calculated. The point P is located at distances r_1 and r_2 from the positive and negative charges, respectively, as shown in Fig. 3.12. The potential at point P due to the dipole is

$$V = \frac{Q}{4\pi\varepsilon_0 r_1} - \frac{Q}{4\pi\varepsilon_0 r_2} \tag{3.143}$$

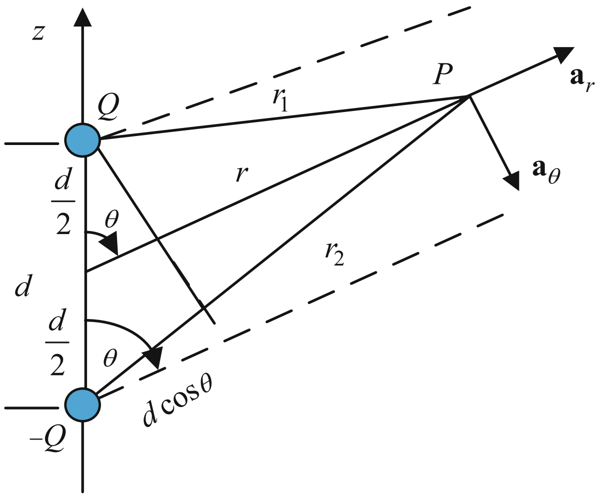


Fig. 3.12 Equal and opposite point charge

$$V = \frac{Q}{4\pi\varepsilon_0}\frac{r_2 - r_1}{r_2 r_1}. \tag{3.144}$$

If $r >> d$, then the following approximation can be written as

$$r_2 - r_1 = d\cos\theta \tag{3.145}$$

$$r_2 r_1 = r^2. \tag{3.146}$$

Substituting Eqs. (3.145) and (3.146) into Eq. (3.144) yields

$$V = \frac{Q}{4\pi\varepsilon_0}\frac{d\cos\theta}{r^2} \tag{3.147}$$

The dipole moment is defined as

$$P = Qd. \tag{3.148}$$

In vector form, the dipole moment is

$$\mathbf{P} = Q\mathbf{d}. \tag{3.149}$$

The unit of dipole moment is C m. Substituting Eq. (3.148) into Eq. (3.147) provides

$$V = \frac{P\cos\theta}{4\pi\varepsilon_0 r^2}. \tag{3.150}$$

Since $d\cos\theta = \mathbf{d}.\mathbf{a}_r$, and using Eq. (3.149), Eq. (3.147) can be modified as

$$V = \frac{\mathbf{P}.\mathbf{a}_r}{4\pi\varepsilon_0 r^2}. \tag{3.151}$$

In spherical coordinates, the expression of electric field without ϕ component can be written as

$$\mathbf{E} = -\nabla V = -\left(\frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta\right). \tag{3.152}$$

Substituting Eq. (3.150) into Eq. (3.152) provides

$$\mathbf{E} = \frac{2P\cos\theta}{4\pi\varepsilon_0 r^3}\mathbf{a}_r + \frac{P\sin\theta}{4\pi\varepsilon_0 r^3}\mathbf{a}_\theta. \tag{3.153}$$

$$\mathbf{E}=\frac{P}{4\pi\varepsilon_0 r^3}(2\cos\theta\mathbf{a}_r + \sin\theta\mathbf{a}_\theta) \tag{3.154}$$

From Eq. (3.151), the generalized expression of potential can be written as

$$V = \frac{\mathbf{P}}{4\pi\varepsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}, \tag{3.155}$$

where $\mathbf{r}'$ indicates the center of the dipole.

Example 3.10 A moment of an electric field is located in free space whose expression is given as $\mathbf{P} = 2\mathbf{a}_x + 1.5\mathbf{a}_y + 3\mathbf{a}_z nCm$. Determine the voltage at the point $M(2,1,5)$.

Solution The unit vector can be determined as

$$a_r = \frac{2\mathbf{a}_x + 1\mathbf{a}_y + 5\mathbf{a}_z}{\sqrt{2^2+1^2+5^2}} = \frac{2\mathbf{a}_x + 1\mathbf{a}_y + 5\mathbf{a}_z}{\sqrt{30}}$$

The voltage can be determined as

$$V = \frac{\mathrm{P.a}_r}{4\pi\varepsilon_0 r^2} = \frac{(2\mathrm{a}_x + 1.5\mathrm{a}_y + 3\mathrm{a}_z).(2\mathrm{a}_x + \mathrm{a}_y + 5\mathrm{a}_z)}{4\pi \times 8.854\times 10^{-12}\times 30\times\sqrt{30}} \times 10^{-9}$$

$$V = \frac{(4+1.5+15)}{4\pi \times 8.854\times 10^{-12}\times 30\times\sqrt{30}} \times 10^{-9} = 1.21\mathrm{V}.$$

Example 3.11 A dipole moment $\mathbf{P} = 3\mathbf{a}_x - 2.5\mathbf{a}_y + 5\mathbf{a}_z nCm$ is located at the origin in free space. Determine the voltage at the point $M(\rho = 1.5, \phi = 125°, z = 0.6)$.

Solution The Cartesian components can be determined as

$$x = \rho\cos\phi = 1.5\cos 125° = -0.86$$

$$y = \rho\sin\phi = 1.5\sin 125° = 1.228$$

$$z = 0.6$$

$$a_r = \frac{-0.86\mathbf{a}_x + 1.228\mathbf{a}_y + 3.6\mathbf{a}_z}{\sqrt{0.86^2 + 1.228^2 + 3.6^2}} = \frac{-0.866\mathbf{a}_x + 1.228\mathbf{a}_y + 3.6\mathbf{a}_z}{\sqrt{15.21}}.$$

The voltage can be determined as

$$V = \frac{\text{P.a}_r}{4\pi\varepsilon_0 r^2} = \frac{(3\text{a}_x - 2.5\text{a}_y + 5\text{a}_z).(-0.866\text{a}_x + 1.228\text{a}_y + 3.6\text{a}_z)}{4\pi \times 8.854 \times 10^{-12} \times 15.21 \times \sqrt{15.21}} \times 10^{-9}$$

$$V = \frac{(-2.58 - 3.07 + 18)}{4\pi \times 8.854 \times 10^{-12} \times 15.21 \times \sqrt{15.21}} \times 10^{-9} = 1.87\text{V}.$$

Practice Problem 3.10 An electric dipole moment is located in free space whose expression is given as $\mathbf{P} = 2\mathbf{a}_x - 3\mathbf{a}_y + 4\mathbf{a}_z nCm$. Determine the voltage at point $M(2.5{,}3.8{,}1.5)$.

Practice Problem 3.11 A dipole moment $\mathbf{P} = 6.5\mathbf{a}_z nCm$ is located at the origin in free space. Determine the voltage and electric field at point $M(r = 2, \theta = 30°, \phi = 0°)$.

3.12 Materials for Static Electric Field

According to electrical property, materials are classified into three categories, namely conductor, semiconductor, and insulator. A conductor has enough charge conduction property in the presence of free electrons. The electrons of an insulator atom are attached strongly with the outer orbits. These electrons cannot be detached from the orbits with the application of external electric fields. The electrical properties of semiconductors fall in between conductors and insulators (10^{-3}S/mto 1S/m). A constitutive parameter is used to characterize the macroscopic electrical property of a material. This constitutive parameter is known as conductivity and it is represented by σ. The unit of conductivity is S/m. The reciprocal of conductivity is called resistivity and is generally represented by ρ. When there is no charge inside the conductor, then $\rho = 0$. As a result, the value of the electric field is zero. An interface between a conductor and free space is shown in Fig. 3.13. The sides ad and bc are parallel to the interface. Let, $ad = bc = \Delta l$ and $ab = cd = \Delta h = 0$. Apply Stokes' theorem and the following equation can be written as

$$\oint_c \mathbf{E}.d\mathbf{l} = 0. \tag{3.156}$$

$$\oint_{abcd} \mathbf{E}.d\mathbf{l} = E_t \Delta l = 0 \tag{3.157}$$

$$E_t = 0. \tag{3.158}$$

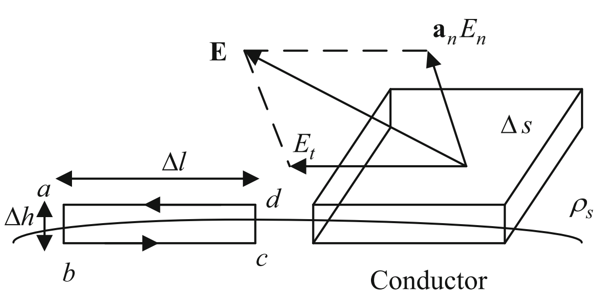


Fig. 3.13 Conductor and free space with an interface

From Eq. (3.158), it is concluded that the tangential component of an electric field along the conductor surface is zero.

Consider the rectangular box in such a way that the top part is in the free space and the bottom part is in the conductor region whereby $\mathbf{E} = 0$. Apply Gauss' law to this box and the equation is

$$\oint_s \mathbf{E}.d\mathbf{S} = E_n \Delta s = \frac{\rho_s \Delta s}{\varepsilon_0}. \tag{3.159}$$

$$E_n = \frac{\rho_s}{\varepsilon_0}. \tag{3.160}$$

3.13 Dielectric Polarization

Dielectric materials are polarized when an electric field is applied to them. As a result, electric flux density becomes greater than under the free space condition. In this condition, the divergence postulates can be modified as

$$\nabla.\mathbf{E} = \frac{1}{\varepsilon_0}(\rho_v + \rho_{pv}), \tag{3.161}$$

where

ρ_v is the volume charge density of free charge,
ρ_{pv} is the polarization volume charge density.

The polarization is defined as the dipole moment per unit volume and its unit is $\mathrm{C/m^2}$. Mathematically, it can be expressed as

$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{N} Q_k \mathbf{d}_k}{\Delta v}. \tag{3.162}$$

Consider a cubic dielectric material as shown in Fig. 3.14. Apply an electric field to an incremental surface Δs, which in turn produces a dipole moment, $\mathbf{P} = Q\mathbf{d}$ in

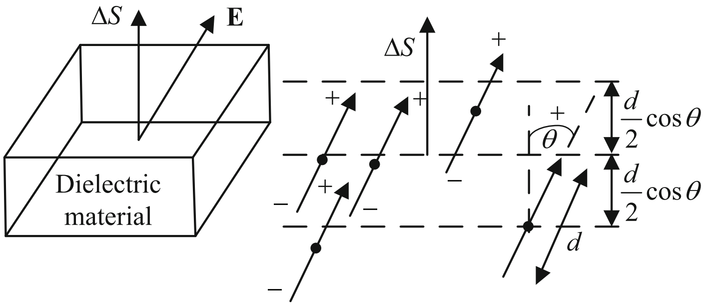


Fig. 3.14 Cubic dielectric material with field

each molecule. This dipole moment makes an angle θ with the Δs. For n molecules, the net total charge which crosses the upward direction is

$$\Delta Q_b = nQd\cos\theta\Delta S. \tag{3.163}$$

Equation (3.163) can be written in vector form as

$$\Delta Q_b = nQ\mathbf{d}.\Delta\mathbf{S}. \tag{3.164}$$

In differential form, Eq. (3.164) can be written as

$$dQ_b = nQ\mathbf{d}.d\mathbf{S}. \tag{3.165}$$

Now substituting the expression of dipole moment into Eq. (3.165) provides

$$dQ_b = \mathbf{P}.d\mathbf{S}. \tag{3.166}$$

The net charge within the closed surface is obtained by integrating Eq. (3.166) as

$$Q_b = -\oint_s \mathbf{P}.d\mathbf{S}. \tag{3.167}$$

Here, the negative sign indicates the outward direction of the bound charge. The total charge within the closed surface consists of the bound charge and the free charge. Then, this expression can be expressed as

$$Q_T = Q_b + Q. \tag{3.168}$$

According to Gauss' law, the expression of total charge in terms of electric field can be expressed as

$$Q_T = \oint_s \varepsilon_0\mathbf{E}.d\mathbf{S}. \tag{3.169}$$

Substituting Eqs. (3.167) and (3.169) into Eq. (3.168) yields

$$\oint_s \varepsilon_0 \mathbf{E}.d\mathbf{S} = -\oint_s \mathbf{P}.d\mathbf{S} + Q \tag{3.170}$$

$$Q = \oint_s (\varepsilon_0 \mathbf{E} + \mathbf{P}).d\mathbf{S} \tag{3.171}$$

$$Q = \oint_s \mathbf{D}.d\mathbf{S}, \tag{3.172}$$

where the expression of the term **D** in more general form is

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}. \tag{3.173}$$

According to the differential form of Gauss' law, the following equation can be written as:

$$\rho_{pv} = -\nabla.\mathbf{P}. \tag{3.174}$$

Substituting Eq. (3.174) into Eq. (3.161) provides

$$\nabla.\mathbf{E} = \frac{1}{\varepsilon_0}(\rho_v - \nabla.\mathbf{P}) \tag{3.175}$$

$$\rho_v = \nabla.(\varepsilon_0 \mathbf{E} + \mathbf{P}) = \nabla.\mathbf{D}, \tag{3.176}$$

where the expression of electric flux density is

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}.$$

From Eq. (3.173), it is seen that the electric flux density is equal to the sum of the dipole moment and electric field intensity.

3.14 Dielectric Material Characteristics

The relationship between the electric field (**E**) and the dipole moment or polarization vector (**P**) is used to represent polarization properties of dielectric materials. In certain materials, the directions of the electric field (**E**) and dipole moment (**P**) are not the same. In isotropic and linear materials, the polarization is directly proportional to the electric field and the proportionality constant does not depend on the field. The following relation can be written as:

$$\mathbf{P} = \chi_e \varepsilon_0 \mathbf{E}, \tag{3.177}$$

where

χ_e (chi) is the electric susceptibility of the material.

Substituting Eq. (3.177) into Eq. (3.173) provides

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \chi_e\varepsilon_0\mathbf{E} \tag{3.178}$$

$$\mathbf{D} = \varepsilon_0(1 + \chi_e)\mathbf{E} \tag{3.179}$$

$$\mathbf{D} = \varepsilon_0\varepsilon_r\mathbf{E} = \varepsilon\mathbf{E}. \tag{3.180}$$

In Cartesian coordinates, Eq. (3.180) can be written as

$$D_x = \varepsilon_{xx}E_x + \varepsilon_{xy}E_y + \varepsilon_{xz}E_z \tag{3.181}$$

$$D_y = \varepsilon_{yx}E_x + \varepsilon_{yy}E_y + \varepsilon_{yz}E_z \tag{3.182}$$

$$D_z = \varepsilon_{zx}E_x + \varepsilon_{zy}E_y + \varepsilon_{zz}E_z, \tag{3.183}$$

where, the relative permittivity can be defined as

$$\varepsilon_r = 1 + \chi_e. \tag{3.184}$$

In matrix (3×3) form, Eqs. (3.181)–(3.183) can be expressed as

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}. \tag{3.185}$$

Electrons of the molecules will come out completely, if a strong electric field is applied to it. As a result, electrons will accelerate and collide with molecular lattice structure, which causes permanent damage to the material. This avalanche effect is observed at the material surface. In this condition, the material will become conducting and a large current will flow. This phenomenon is known as dielectric breakdown. The maximum electric field at which the dielectric material can withstand without breakdown is known as dielectric strength of the material. The dielectric constant and dielectric strength of some materials are given in Table 3.1.

3.15 Dielectric Boundary Conditions

Consider two rules for determining the boundary conditions for electrostatic fields. These are the total net work done along a closed path in a static field, **E** is zero, i.e., **E** is a conservative (irrotational) field and net electric flux leaving from a closed surface equals the total charge enclosed by that surface. Consider two dielectric regions as shown in Fig. 3.15 to find out the tangential and normal boundary conditions. Here,

Table 3.1 Dielectric constant and strength

	Material	Dielectric constant	Dielectric strength (MV/m)
1	Glass	4–10	30
2	Mineral oil	2.3	15
3	Rubber	2.3–4	25
4	Paper	2–4	15
5	Polycarbonate	2.3	70
6	Air	1	3
7	Aluminum	8.8	12
8	Sea water	80	80
9	Porcelain	5–6.5	11

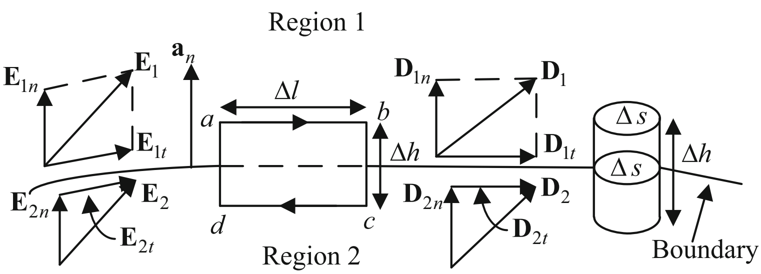


Fig. 3.15 Two regions with boundary

$bc = ad = \Delta h = 0$; therefore, the contribution of these paths on line integral of electric field is zero. The line integral of other paths is

$$\oint_{abcd} \mathbf{E}.d\mathbf{l} = \mathbf{E}_1.(\Delta\mathbf{l}) + \mathbf{E}_1.(-\Delta\mathbf{l}) = 0 \tag{3.186}$$

$$E_{1t}\Delta l - E_{2t}\Delta l = 0 \tag{3.187}$$

$$E_{1t} = E_{2t}. \tag{3.188}$$

Equation (3.188) indicates that the tangential components are the same or $\mathbf{E}_t$ is continuous across the boundary. Consider the relationship $\mathbf{D} = \varepsilon\mathbf{E}$, and Eq. (3.188) can be modified as

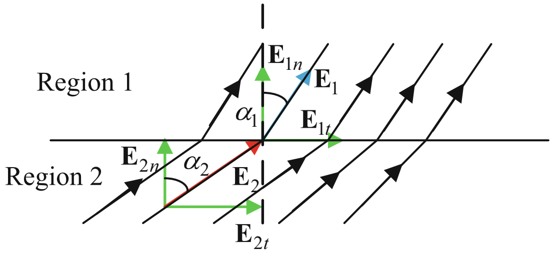
$$\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}. \tag{3.189}$$

To find the normal boundary condition, applying Gauss' law to a small cylinder provides

$$\oint_s \mathbf{D}.d\mathbf{s} = Q_{en} \cong D_{1n}\Delta S - D_{2n}\Delta S = \rho_s \Delta S \tag{3.190}$$

$$D_{1n} - D_{2n} = \rho_s \tag{3.191}$$

Fig. 3.16 Two perfect dielectric with refraction of



$$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s. \tag{3.192}$$

In the absence of charge, $\rho_s = 0$, Eqs. (3.191) and (3.192) can be modified as

$$D_{1n} = D_{2n} \tag{3.193}$$

$$\frac{E_{1n}}{\varepsilon_2} = \frac{E_{2n}}{\varepsilon_1}. \tag{3.194}$$

From Eq. (3.183), it is concluded that the normal components of **D** for two regions are same or $\mathbf{D}_n$ is continuous across the boundary free of charge.

3.16 Refraction of Electric Field at Dielectric Boundary

Consider two perfect dielectric regions as shown in Fig. 3.16 to find the law of refraction of the electric field lines at the dielectric-dielectric boundary. Let $\mathbf{E}_1$ make an angle α_1 with the normal component E_{1n}. Let $\mathbf{E}_2$ make an angle α_2 with the normal component E_{2n}. From Fig. 3.16, the following equations can be written as:

$$\tan\alpha_1 = \frac{E_{1t}}{E_{1n}} \tag{3.195}$$

$$\tan\alpha_2 = \frac{E_{2t}}{E_{2n}}. \tag{3.196}$$

Dividing Eq. (3.195) by Eq. (3.196) provides

$$\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{E_{1t}}{E_{2t}} \times \frac{E_{2n}}{E_{1n}}. \tag{3.197}$$

Substituting Eqs. (3.188) and (3.191) into Eq. (3.197) yields

$$\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{\varepsilon_1 E_{1n}}{\varepsilon_2} \times \frac{1}{E_{1n}} \tag{3.198}$$

$$\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{\varepsilon_1}{\varepsilon_2}. \tag{3.199}$$

Equation (3.199) is known as law of refraction of the electric field at a free of charge boundary.

Again, from Fig. 3.16, the following equations can be written as

$$E_{1t} = E_1 \sin \alpha_1 \tag{3.200}$$

$$E_{2t} = E_2 \sin \alpha_2 \tag{3.201}$$

$$E_{1n} = E_1 \cos \alpha_1 \tag{3.202}$$

$$E_{2n} = E_2 \cos \alpha_2. \tag{3.203}$$

Substituting Eqs. (3.200) and (3.201) into Eq. (3.188) provides

$$E_1 \sin \alpha_1 = E_2 \sin \alpha_2 \tag{3.204}$$

Substituting Eqs. (3.202) and (3.203) into Eq. (3.194) provides

$$\frac{E_1 \cos \alpha_1}{\varepsilon_2} = \frac{E_2 \cos \alpha_2}{\varepsilon_1}. \tag{3.205}$$

The magnitude of $\mathbf{E}_2$ can be determined as

$$E_2 = \sqrt{{E_{2t}}^2 + {E_{2n}}^2}. \tag{3.206}$$

Substituting Eqs. (3.201) and (3.203) into Eq. (3.206) provides

$$E_2 = \sqrt{(E_2 \sin \alpha_2)^2 + (E_2 \cos \alpha_2)^2}. \tag{3.207}$$

Substituting Eqs. (3.203) and (3.205) into Eq. (3.207) yields

$$E_2 = \sqrt{(E_1 \sin \alpha_1)^2 + \left(\frac{\varepsilon_1 E_1 \cos \alpha_1}{\varepsilon_2}\right)^2} \tag{3.208}$$

$$E_2 = E_1 \sqrt{\sin^2 \alpha_1 + \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2 \cos^2 \alpha_1}. \tag{3.209}$$

Example 3.12 The parameters of the first and second regions are $z > 0, \varepsilon_{r1} = 2$, and $z < 0, \varepsilon_{r2} = 4$, respectively. The electric field for the first region is $\mathbf{E}_1 = 3\mathbf{a}_x + 5\mathbf{a}_y + 2\mathbf{a}_z \text{kV/m}$. Determine the (a) tangential electric field, $\mathbf{E}_{2t}$, (b) normal electric field, $\mathbf{E}_{2n}$, and (c) angles of $\mathbf{E}_1$ and $\mathbf{E}_2$ with an interface.

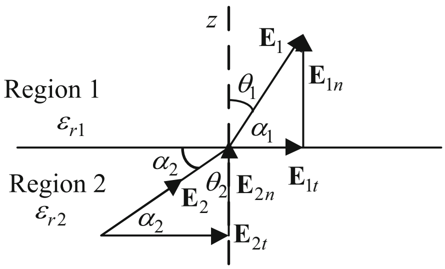


Fig. 3.17 Two regions meet on the plane

Solution Figure 3.17 is considered to solve this problem.

a. The normal component of the first region is

$$\mathbf{E}_{1n} = \mathbf{E}_1 . \mathbf{a}_z$$

$$\mathbf{E}_{1n} = (3\mathbf{a}_x + 5\mathbf{a}_y + 2\mathbf{a}_z). \mathbf{a}_z = 2$$

$$\mathbf{E}_{1n} = 2\mathbf{a}_z.$$

The tangential component of the first region can be determined by

$$\mathbf{E}_{1t} = \mathbf{E}_1 - \mathbf{E}_{1n} = 3\mathbf{a}_x + 5\mathbf{a}_y + 2\mathbf{a}_z - 2\mathbf{a}_z = 3\mathbf{a}_x + 5\mathbf{a}_y.$$

The tangential component of the second region is

$$\mathbf{E}_{2t} = \mathbf{E}_{1t} = 3\mathbf{a}_x + 5\mathbf{a}_y.$$

b. For free of charge, the following equation can be written as:

$$\mathbf{D}_{2n} = \mathbf{D}_{1n}$$

$$\varepsilon_{r2}\mathbf{E}_{2n} = \varepsilon_{r1}\mathbf{E}_{1n}$$

$$\mathbf{E}_{2n} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}\mathbf{E}_{1n} = \frac{2}{4}(2\mathbf{a}_z) = \mathbf{a}_z.$$

c. The following angles can be determined as:

$$\tan\theta_1 = \frac{\mathbf{E}_{1t}}{\mathbf{E}_{1n}} = \frac{\sqrt{3^2 + 5^2}}{2}$$

$$\theta_1 = 71.07°$$

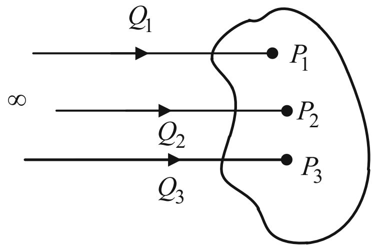


Fig. 3.18 Point charges moving towards fixed points

$$\tan\theta_2 = \frac{\mathbf{E}_{2t}}{\mathbf{E}_{2n}} = \frac{\sqrt{3^2+5^2}}{1}$$

$$\theta_2 = 80.27°.$$

The angle of $\mathbf{E}_1$ with an interface is

$$\alpha_1 = 90° - \theta_1 = 90° - 71.07° = 18.93°.$$

The angle of $\mathbf{E}_2$ with an interface is

$$\alpha_2 = 90° - \theta_2 = 90° - 80.27° = 9.73°.$$

Practice Problem 3.12 The parameters of the first and second regions are $z > 0, \varepsilon_{r1} = 3$, and $z < 0, \varepsilon_{r2} = 7$, respectively. The electric field for the first region is $\mathbf{E}_1 = 5\mathbf{a}_x - 8\mathbf{a}_y - 3\mathbf{a}_z$kV/m. Determine the (a) tangential electric field, $\mathbf{E}_{2t}$, (b) normal electric field, $\mathbf{E}_{2n}$, (c) $\mathbf{E}_2$, and (d) angles of $\mathbf{E}_1$and $\mathbf{E}_2$with an interface.

3.17 Electrostatic Energy

Consider three points P_1, P_2, and P_3 in a charge-free space as shown in Fig. 3.18. The point charges Q_1, Q_2, and Q_3 are brought from infinity to those points, respectively. Net work done in positioning charges Q_1, Q_2, and Q_3 at the points P_1, P_2, and P_3 starting with Q_1 is

$$W_E = W_1 + W_2 + W_3. \tag{3.210}$$

No work is done in bringing point charge Q_1 from infinity to point P_1. Then Eq. (3.210) can be modified as

$$W_E = 0 + W_2 + W_3 \tag{3.211}$$

$$W_E = 0 + Q_2 V_{21} + Q_3(V_{31} + V_{32}), \quad (3.212)$$

where

V_{21} is the potential at point P_2 due to Q_1,
V_{31} is the potential at point P_3 due to Q_1,
V_{32} is the potential at point P_3 due to Q_2.

Again, placing the point charges in the reverse order, the expression of net work done is

$$W_E = W_3 + W_2 + W_1 \quad (3.213)$$

$$W_E = 0 + W_2 + W_1 \quad (3.214)$$

$$W_E = 0 + Q_2 V_{23} + Q_1(V_{12} + V_{13}), \quad (3.215)$$

where

V_{23} is the potential at point P_2 due to Q_3,
V_{13} is the potential at point P_1 due to Q_3,
V_{12} is the potential at point P_1 due to Q_2.

Adding Eqs. (3.212) and (3.215) yields

$$2W_E = Q_1(V_{12} + V_{13}) + Q_2(V_{21} + V_{23}) + Q_3(V_{31} + V_{32}) \quad (3.216)$$

$$2W_E = Q_1 V_1 + Q_2 V_2 + Q_3 V_3 \quad (3.217)$$

$$2W_E = Q_1 V_1 + Q_2 V_2 + Q_3 V_3 \quad (3.218)$$

$$W_E = \frac{1}{2}(Q_1 V_1 + Q_2 V_2 + Q_3 V_3). \quad (3.219)$$

In general, the work done can be expressed as

$$W_E = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k. \quad (3.220)$$

For line, surface, and volume charges, the following equations can be written as

$$W_E = \frac{1}{2}\int_L \rho_L V dl. \quad (3.221)$$

$$W_E = \frac{1}{2}\int_S \rho_S V dS \quad (3.222)$$

$$W_E = \frac{1}{2}\int_v \rho_v V dv. \quad (3.223)$$

Substituting $\rho_v = \nabla.\mathbf{D}$ into Eq. (3.222) yields

$$W_E = \frac{1}{2}\int_v (\nabla.\mathbf{D})V dv. \tag{3.224}$$

The following vector identity can be written as

$$\nabla.V\mathbf{D} = \mathbf{D}.\nabla V + (\nabla.\mathbf{D})V \tag{3.225}$$

$$(\nabla.\mathbf{D})V = \nabla.V\mathbf{D} - \mathbf{D}.\nabla V. \tag{3.226}$$

Substituting Eq. (3.226) into Eq. (3.224) yields

$$W_E = \frac{1}{2}\int_v (\nabla.V\mathbf{D} - \mathbf{D}.\nabla V)dv. \tag{3.227}$$

Applying divergence theorem to the first term of Eq. (3.227) provides

$$W_E = \frac{1}{2}\oint_S \nabla\mathbf{D}ds - \frac{1}{2}\int_v \mathbf{D}.(\nabla V)dv. \tag{3.228}$$

For closed surfaces, the surface integral is zero. Then, Eq. (3.228) can be modified as

$$W_E = -\frac{1}{2}\int_v \mathbf{D}.(\nabla V)dv. \tag{3.229}$$

Substituting the expression $\mathbf{E} = -\nabla V$ into Eq. (3.229) yields

$$W_E = \frac{1}{2}\int_v \mathbf{D}.\mathbf{E}dv \tag{3.230}$$

$$W_E = \frac{1}{2}\int_v \varepsilon_0 E^2 dv. \tag{3.231}$$

From Eq. (3.231), the electrostatic energy density can be written as

$$w_E = \frac{dW_E}{dv} = \frac{1}{2}\varepsilon_0 E^2 \text{ J/m}^3. \tag{3.232}$$

3.18 Exercise Problems

3.1 Point charges of 3×10^{-5}C and 4.5×10^{-5}C are located at points $M(1, -2, 3)$ and $N(2, 3, 5)$, respectively, in free space. Determine the electric force exerted on the second charge due to first charge.

3.2 Point charges of 3.5×10^{-4}C and -6.4×10^{-5}C are located at points $M(1, -2, 3.5)$, and $N(1.5, 3, 5)$, respectively, in free space. Determine the (a) $\mathbf{R}_{12}$, (b) R_{12}, (c) $\mathbf{a}_{12}$, and (d) force exerted on the first charge due to the second charge.

3.3 A point charge of 4×10^{-10}C is located at point $M(1, 2, 3)$. Determine the electric field intensity at point $N(2.5, 3.2, 5.4)$ in free space. All dimensions are in meters.

3.4 A point charge of 8.5×10^{-8}C is located at point $M(2, 4, 6)$. Determine the electric field intensity at point $N(4, -3, 7)$ in free space. All dimensions are in meters.

3.5 The electric flux density is defined as $\mathbf{D} = 2x\mathbf{a}_x + 3y\mathbf{a}_y - 5z\mathbf{a}_z$. Determine the $\oint_S \mathbf{D}.d\mathbf{S}$, where S is the surface of a rectangular box and the limits are $x = 0,\ x = 2, y = 0,\ y = 1, z = 0,\ z = 4$.

3.6 Use Gauss' theorem to determine $\oint_S \mathbf{D}.d\mathbf{S}$, where the electric flux density is $\mathbf{D} = 2x^2\mathbf{a}_x - 4y^2\mathbf{a}_y + 3z\mathbf{a}_z$ and the surface is surrounded by the region of $x^2 + y^2 = 4, z = 0$ and $z = 1$.

3.7 The electric flux density is defined as $\mathbf{D} = 3x^2\mathbf{a}_x + 2xy\mathbf{a}_y + yz\mathbf{a}_z$. Determine the $\oint_S \mathbf{D}.d\mathbf{S}$, where S is the surface bounded by the region of limits $x = 0,\ x = 1, y = 0,\ y = 2, z = 0,\ z = 3$.

3.8 Use Gauss' theorem to determine $\oint_S \mathbf{D}.d\mathbf{S}$, where the electric flux density is $\mathbf{D} = 3x^2\mathbf{a}_x - 5y^2\mathbf{a}_y - z^2\mathbf{a}_z$ and the surface is surrounded by the region of $x^2 + y^2 = 16, z = 0$, and $z = 2$.

3.9 A electric field is given by $\mathbf{E} = 2y\mathbf{a}_x + x\mathbf{a}_y + y\mathbf{a}_z$V/m. A charge of 4C is transferred from point $A(1, 2, 3)$ to point $B(2, 3, 1)$ along the differential length of the line. Consider the differential length in Cartesian coordinates and determine the work done.

3.10 The expression of a nonuniform electric field is given as $\mathbf{E} = 3x\mathbf{a}_x + 2y\mathbf{a}_y - z\mathbf{a}_z$V/m. A charge of 5C is transferred from point $A(1, 2, 4)$ to point $B(2, 4, 6)$ along the straight line from point A to point B. Calculate the work done.

3.11 A charge of 2C is transferred from point $A(1, 2, 3)$ to point $B(2, 3, 5)$ through an electric field whose expression is given by $\mathbf{E} = 2y\mathbf{a}_x$V/m. The charge is transferred along the path $y = x + 1$ and $z = 2y - 1$. Determine the work done.

3.12 A charge of 6C is transferred from point $A(1, 3, 2)$ to point $B(4, 2, 5)$ through an electric field whose expression is given by $\mathbf{E} = 5y\mathbf{a}_x + x\mathbf{a}_z$V/m. The charge is transferred along the path whose Eqs. are $y = 2x + 5$ and $x = z - 1$. Determine the work done.

3.13 A potential in Cartesian coordinates is given as $V = 2xy^3 - 3z$ and the point is P(3,2, 1). Determine the (a) numerical value of the voltage, (b) electric field, (c) direction of the electric field, and (d) volume charge density.

3.14 A potential in cylindrical coordinates is given as $V = 2\rho \sin\phi + z^2$ and the point is $P(\rho = 1\text{m}, \phi = 50°, z = 2\text{m})$. Calculate the (a) numerical value of the voltage, (b) electric field, (c) electric flux density, and (d) volume charge density.

3.15 A potential in spherical coordinates is given as $V = 5r^2 \sin\theta + \cos\phi$ and the point is $P(r = 2\text{m}, \theta = 60°, \phi = 150°)$. Determine the (a) numerical value of the voltage, (b) electric field, (c) electric flux density, and (d) volume charge density.

3.16 Calculate the potential at point (1,3, 2) when two points charges 2.5μC and 3.5μC are located at points (0.5,2, 1) and (-1, -1.5,2), respectively. Consider zero potential at infinity.

3.17 Three points charges -2.5μC,4μC, and 6.5μC are located at points (2.2, -2.5,1), ($-$ 3.4, -2.5,1.5),and ($-$ 2.6, -3.5, -4.8), respectively. Determine the potential at point (3,4.5,6) by considering zero potential at infinity.

3.18 A moment of an electric field is located in free space whose expression is given as $\mathbf{P} = 5\mathbf{a}_x + 4\mathbf{a}_y - 2\mathbf{a}_z nCm$. Determine the voltage at point $M(3,2,4)$.

3.19 A dipole moment $\mathbf{P} = -5\mathbf{a}_x + 3\mathbf{a}_y + 7\mathbf{a}_z nCm$ is located at the origin in free space. Determine the voltage at point $M(\rho = 2, \phi = 140°, z = 1)$.

3.20 A dipole moment $\mathbf{P} = 2\mathbf{a}_y + 3\mathbf{a}_z nCm$ is located at the origin in free space. Determine the voltage and electric field at point $M(r = 3, \theta = 40°, \phi = 120°)$.

3.21 The parameters of the first and second regions are $z > 0, \varepsilon_{r1} = 2.5$, and $z < 0, \varepsilon_{r2} = 4.5$, respectively. The electric field for the first region is $\mathbf{E}_1 = 2\mathbf{a}_x + 3\mathbf{a}_y + \mathbf{a}_z$kV/m. Determine the (a) tangential electric field, E_{2t}, (b) normal electric field, E_{2n}, and (c) angles of $\mathbf{E}_1$ and $\mathbf{E}_2$with an interface.

3.22 The parameters of the first and second regions are $z > 0, \varepsilon_{r1} = 5$, and $z < 0, \varepsilon_{r2} = 7$, respectively. The electric field for the first region is $\mathbf{E}_1 = 3\mathbf{a}_x + 5\mathbf{a}_y - 2\mathbf{a}_z$kV/m. Determine the (a) tangential electric field, E_{2t}, (b) normal electric field, E_{2n}, (c) $\mathbf{E}_2$, and (d) angles of $\mathbf{E}_1$and $\mathbf{E}_2$with an interface.

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Chapter 4
Poisson's and Laplace's Equations

4.1 Introduction

In the previous few chapters, the electric field has been determined using either Gauss' law or Coulomb's law. In initial condition, charge distribution or electrostatic potential should be known to apply those laws. There are many practical problems where the charge distribution is not known for every place. There is some complex geometry in high voltage engineering equipment, namely insulators, bushing, surge arrestors, etc. In that case, it is difficult to use Gauss' law to find their electrostatic potential and electric field intensity distributions. The method of images can be used if the conducting bodies have a boundary with simple geometry. Therefore, some differential equations need to be solved to find the voltage and field distribution around the conductor and air interface of the simple and complex geometry of the electrical engineering equipment. In this chapter, Poisson's equation, Laplace's equation, uniqueness theorem, and the solution of Laplace's equation will be discussed.

4.2 Derivation of Poisson's and Laplace's Equations

The relationship between the electric field and electrostatic potential is required to derive Poisson's equation. This equation is

$$\mathbf{E} = -\nabla V. \tag{4.1}$$

Taking the divergence of both sides of Eq. (4.1) yields

$$\nabla \cdot \mathbf{E} = -\nabla \cdot \nabla V. \tag{4.2}$$

Substituting the expression of $\mathbf{E} = \frac{\mathbf{D}}{\varepsilon}$ into Eq. (4.2) provides

$$\nabla \cdot \frac{\mathbf{D}}{\varepsilon} = -\nabla \cdot \nabla V, \tag{4.3}$$

$$\frac{1}{\varepsilon}\nabla \cdot \mathbf{D} = -\nabla \cdot \nabla V. \tag{4.4}$$

Md. A. Salam, *Electromagnetic Field Theories for Engineering*,
DOI 10.1007/978-981-4585-66-8_4, © Springer Science+Business Media Singapore 2014

Substituting the differential form of Gauss' law, $\rho_v = \nabla \cdot \mathbf{D}$ into Eq. (4.4) provides

$$\nabla \cdot \nabla V = -\frac{\rho_v}{\varepsilon}. \tag{4.5}$$

According to the rules of vector dot product, the operator $\nabla \cdot \nabla$can be written as

$$\nabla \cdot \nabla = \left(\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z\right) \cdot \left(\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z\right), \tag{4.6}$$

$$\nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla^2. \tag{4.7}$$

Substituting Eq. (4.7) into Eq. (4.5) yields

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}. \tag{4.8}$$

Equation (4.8) is known as Poisson's equation. If the region contains no free charge, i.e. $\rho_v = 0$, then Eq. (4.8) can be modified as

$$\nabla^2 V = 0. \tag{4.9}$$

Equation (4.9) is known as Laplace's equation. The Laplace's equation in Cartesian, cylindrical, and spherical coordinates can be expressed as

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0, \tag{4.10}$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0, \tag{4.11}$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0. \tag{4.12}$$

Equations (4.10), (4.11), and (4.12) are normally used to a specific configuration for valid boundary conditions.

Example 4.1 The electric potential in Cartesian coordinates is given by $V(x, y, z) = 2x^2y + 3z^2$. Determine the (a) numerical value of the voltage at point $P(1,3,2)$, (b) the electric field, and (c) verify the Laplace's equation.

Solution (a) The numerical value of the potential can be determined as

$$V(x, y, z) = 2(1)^2 3 + 3(2)^2 = 18\text{V}.$$

(b) The expression of the electric field can be determined as

$$\mathbf{E} = -\left(\frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z\right),$$

$$\frac{\partial V}{\partial x} = \frac{\partial(2x^2y + 3z^2)}{\partial x} = 4xy,$$

$$\frac{\partial V}{\partial y} = \frac{\partial(2x^2y + 3z^2)}{\partial y} = 2x^2,$$

$$\frac{\partial V}{\partial z} = \frac{\partial(2x^2y + 3z^2)}{\partial z} = 6z,$$

$$\mathbf{E} = -\left(4xy\mathbf{a}_x + 2x^2\mathbf{a}_y + 6z\mathbf{a}_z\right) \text{ V/m}$$

The electric field at point $P(1,3,2)$ is

$$\mathbf{E} = -(12\mathbf{a}_x + 2\mathbf{a}_y + 12\mathbf{a}_z)\text{V/m}.$$

(c) The derivatives of the respective products are

$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}(2x^2y + 3z^2) = 4xy,$$
$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}(4xy) = 4y,$$
$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}(2x^2y + 3z^2) = 2x^2,$$
$$\frac{\partial^2 V}{\partial y^2} = \frac{\partial}{\partial y}(2x^2) = 0,$$
$$\frac{\partial V}{\partial z} = \frac{\partial}{\partial z}(2x^2y + 3z^2) = 6z,$$
$$\frac{\partial^2 V}{\partial z^2} = \frac{\partial}{\partial z}(6z) = 6,$$
$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 4y + 6 = 4 \times 3 + 6 = 18.$$

Thus, it does not satisfy Laplace's equation.

Practice problem 4.1 The expression of electric potential in cylindrical coordinates is given as $V(\rho, \phi, z) = 3\rho^2 z \sin\phi$. Determine the (a) numerical value of the voltage at point $P(\rho = 1, \phi = 30°, z = 3)$, (b) the electric field at point $P(\rho = 1, \phi = 30°, z = 3)$, and (c) verify the Laplace's equation.

4.3 Uniqueness Theorem

Each electrostatic object has its own boundary and this boundary is known as boundary potential. The solution of a quadratic equation must be unique if it satisfies its related boundary conditions. Therefore, any solution of Laplace's equation that satisfies the boundary conditions is known as the uniqueness theorem.

Consider a finite volume v is bounded by the closed surface s. To prove the uniqueness theorem, we assume that there are two solutions of Laplace's equation. These solutions are

$$\nabla^2 V_1 = 0, \tag{4.13}$$

$$\nabla^2 V_2 = 0. \tag{4.14}$$

Subtracting Eq. (4.14) from Eq. (4.13) provides

$$\nabla^2 (V_1 - V_2) = 0. \tag{4.15}$$

The potential at the boundary of the surface must be identical and it can be expressed as

$$V|_b = V_1|_b = V_2|_b. \tag{4.16}$$

The following vector identity is used to verify the uniqueness theorem:

$$f \cdot \mathbf{A} = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f). \tag{4.17}$$

In this case, consider f is a scalar function and $\mathbf{A}$ is a vector function. Then, the following functions can be defined as:

$$f = V_1 - V_2, \tag{4.18}$$

$$\mathbf{A} = \nabla(V_1 - V_2). \tag{4.19}$$

Substituting Eqs. (4.21) and (4.22) into Eq. (4.20) yields

$$(V_1 - V_2) \cdot \nabla(V_1 - V_2) = (V_1 - V_2)[\nabla \cdot \nabla(V_1 - V_2)] + \nabla(V_1 - V_2) \cdot \nabla(V_1 - V_2), \tag{4.20}$$

$$(V_1 - V_2) \cdot \nabla(V_1 - V_2) = (V_1 - V_2)[\nabla^2(V_1 - V_2)] + [\nabla(V_1 - V_2)]^2. \tag{4.21}$$

Integrating Eq. (4.21) over the volume v yields

$$\int_v (V_1 - V_2) \cdot \nabla(V_1 - V_2) dv = \int_v (V_1 - V_2)[\nabla^2(V_1 - V_2)] dv + \int_v [\nabla(V_1 - V_2)]^2 dv. \tag{4.22}$$

Applying the divergence theorem to replace the volume integral in the left side of the Eq. (4.22) provides

$$\int_v (V_1 - V_2) \cdot \nabla(V_1 - V_2) dv = \oint_s [(V_1 - V_2)]_b [\nabla(V_1 - V_2)]_b \cdot d\mathbf{S}. \tag{4.23}$$

Substituting Eq. (4.23) into Eq. (4.22) yields

$$\oint_s [(V_1 - V_2)]_b [\nabla(V_1 - V_2)]_b \cdot d\mathbf{S} = \int_v (V_1 - V_2)[\nabla^2(V_1 - V_2)] dv + \int_v [\nabla(V_1 - V_2)]^2 dv. \tag{4.24}$$

By hypothesis, the first and second integrals of Eq. (4.24) are equal to zero, and Eq. (4.24) can be modified as

$$\int_v [\nabla(V_1 - V_2)]^2 dv = 0, \tag{4.25}$$

$$\nabla(V_1 - V_2) = 0. \tag{4.26}$$

If the gradient of $V_1 - V_2$ is zero everywhere in the closed surface, then $V_1 - V_2$ does not change with any coordinates. Then, Eq. (4.26) can be represented as the following equation:

$$V_1 - V_2 = \text{constant}. \tag{4.27}$$

The constant of Eq. (4.27) can be determined by considering a specific point on the boundary of the object. If the constant is zero at the specific point, then Eq. (4.27) becomes

$$V_1 = V_2. \tag{4.28}$$

Equation (4.32) normally provides two identical solutions. The uniqueness theorem can also be applied to the Poisson's equation as

$$\nabla^2 V_1 = -\frac{\rho_v}{\varepsilon}, \tag{4.29}$$

$$\nabla^2 V_2 = -\frac{\rho_v}{\varepsilon}. \tag{4.30}$$

Subtracting Eq. (4.34) from Eq. (4.33) yields

$$\nabla^2(V_1 - V_1) = -\frac{\rho_v}{\varepsilon} + \frac{\rho_v}{\varepsilon} = 0. \tag{4.31}$$

The solution of Eq. (4.35) can be obtained by considering proper boundary conditions.

4.4 Solutions of Laplace's Equation

Direct integration and differentiation methods are used to solve Laplace's equation. The solutions of Laplace's equation for one dimension, two dimensions, and three dimensions are discussed below in detail.

4.4.1 One-Dimension Solution

In one-dimension solution, let us consider the potential V varies only in x-direction. Then, the Laplace's equation can be written as

$$\frac{\partial^2 V}{\partial x^2} = 0. \tag{4.32}$$

The partial derivative of Eq. (4.32) can be represented by an ordinary differential equation and it can be expressed as

$$\frac{d^2 V}{dx^2} = 0. \tag{4.33}$$

Integrating Eq. (4.33) provides

$$\frac{dV}{dx} = C. \tag{4.34}$$

Again, integrating Eq. (4.34) yields

$$\int dV = C \int dx, \tag{4.35}$$

$$V = Cx + D. \tag{4.36}$$

Equation (4.36) is the solution of Laplace's equation in the x-direction, and C and D are the integrating constants and these can be determined by the appropriate boundary conditions.

Consider the potential V in cylindrical coordinates varies only in the ρ-direction. Then, the Laplace's equation can be written as

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) = 0. \tag{4.37}$$

The partial derivative of Eq. (4.37) can be represented by an ordinary differential equation and it can be expressed as

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dV}{d\rho}\right) = 0, \tag{4.38}$$

$$\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right) = 0. \tag{4.39}$$

Integrating Eq. (4.39) yields

$$\rho\frac{dV}{d\rho} = A. \tag{4.40}$$

$$dV = A\frac{d\rho}{\rho}. \tag{4.41}$$

Integrating Eq. (4.41) provides

$$V = A\ln\rho + B. \tag{4.42}$$

Equation (4.42) is the solution of Laplace's equation in the ρ-direction.

Consider that the potential V in spherical coordinates varies only in the r-direction. Then, the Laplace's equation can be written as

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0. \tag{4.43}$$

The partial derivative of Eq. (4.43) can be represented by an ordinary differential equation and it can be expressed as

$$\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0. \tag{4.44}$$

Integrating Eq. (4.44) yields

$$r^2\frac{dV}{dr} = k_1, \tag{4.45}$$

$$dV = k_1 r^{-2} dr. \tag{4.46}$$

Again integrating Eq. (4.46) yields

$$V = k_2 - k_1\frac{1}{r}. \tag{4.47}$$

Equation (4.47) is the solution of Laplace's equation in the r-direction.

4.4.2 Two-Dimension Solution

In two-dimension solution, let us consider the potential V varies only in the x and y directions. Then, the Laplace's equation in rectangular form can be written as

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0. \tag{4.48}$$

The partial derivative of Eq. (4.48) can be represented by an ordinary differential equation and it can be expressed as

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} = 0. \tag{4.49}$$

Consider the general solution of Eq. (4.49) is

$$V(x, y) = X(x)Y(y). \tag{4.50}$$

Substituting Eq. (4.50) into Eq. (4.49) yields

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0. \tag{4.51}$$

Dividing Eq. (4.51) by XY yields

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0. \tag{4.52}$$

It is seen that the first part of Eq. (4.52) is a function of x and equal to a constant. Similarly, the second part is a function of y and equal to a constant. The following equations can be written as:

$$\frac{1}{X}\frac{d^2X}{dx^2} = {A_1}^2, \tag{4.53}$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = {B_1}^2. \tag{4.54}$$

Equation (4.52) is then modified to

$${A_1}^2 + {B_1}^2 = 0, \tag{4.55}$$

$${A_1}^2 = -{B_1}^2. \tag{4.56}$$

Equation (4.53) can be rearranged as

$$\frac{d^2X}{dx^2} - X{A_1}^2 = 0. \tag{4.57}$$

Considering $\frac{d}{dx} = D$, then Eq. (4.57) can be modified to

$$D^2 - {A_1}^2 = 0, \tag{4.58}$$

$$D = \pm A_1, \tag{4.59}$$

$$DX = \pm A_1 X, \tag{4.60}$$

$$\frac{dX}{dx} = A_1 X, \tag{4.61}$$

$$\frac{dX}{X} = A_1 dx. \tag{4.62}$$

Integrating Eq. (4.61) yields

$$\ln X = A_1 x + k, \tag{4.63}$$

$$X = e^k e^{A_1 x} = k_3 e^{A_1 x}, \tag{4.64}$$

where $k_3 = e^k$.

Similarly, the other solution is

$$X = k_4 e^{-A_1 x}. \tag{4.65}$$

In general, the solution is

$$X(x) = k_3 e^{A_1 x} + k_4 e^{-A_1 x}. \tag{4.66}$$

Since $\cosh A_1 x = \frac{e^{A_1 x} + e^{-A_1 x}}{2}$ and $\sinh A_1 x = \frac{e^{A_1 x} - e^{-A_1 x}}{2}$, the following equations can be written as:

$$e^{A_1 x} = \cosh A_1 x + \sinh A_1 x, \tag{4.67}$$

$$e^{-A_1 x} = \cosh A_1 x - \sinh A_1 x. \tag{4.68}$$

The solution of Eq. (4.57) is

$$X(x) = k_1 \cosh A_1 x + k_2 \sinh A_1 x, \tag{4.69}$$

where $k_1 = k_3 + k_4$ and $k_2 = k_3 - k_4$.

Equation (4.54) can be rearranged as

$$\frac{d^2 Y}{dy^2} = {B_1}^2 Y. \tag{4.70}$$

Substituting Eq. (4.56) into Eq. (4.70) yields

$$\frac{d^2 Y}{dy^2} = -{A_1}^2 Y, \tag{4.71}$$

$$D^2 = -{A_1}^2, \tag{4.72}$$

$$D = \pm j A_1. \tag{4.73}$$

Therefore, the solution of Eq. (4.71) is

$$Y(y) = c_3 e^{jA_1 y} + c_4 e^{-jA_1 y}. \tag{4.74}$$

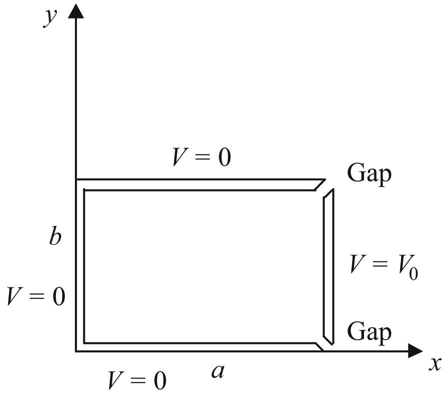


Fig. 4.1 A rectangular conducting object

Since $\cos A_1 y = \frac{e^{jA_1y}+e^{-jA_1y}}{2}$ and $\sin A_1 y = \frac{e^{jA_1y}-e^{-jA_1y}}{j2}$, the following equations can be written as:

$$e^{jA_1y} = \cos A_1 y + j \sin A_1 y, \tag{4.75}$$

$$e^{-jA_1y} = \cos A_1 y - j \sin A_1 y. \tag{4.76}$$

Equation (4.74) can be modified as

$$Y(y) = k_5 \cos A_1 y + k_6 \sin A_1 y, \tag{4.77}$$

where $k_5 = c_3 + c_4$band $k_6 = c_3 - jc_4$.

The two-dimension solution is

$$V(x, y) = (k_1 \cosh A_1 x + k_2 \sinh A_1 x)(k_5 \cos A_1 y + k_6 \sin A_1 y). \tag{4.78}$$

Consider Fig. 4.1 to determine the constants k_1, k_2, k_5, and k_6. The boundary conditions of Fig. 4.1 are

$$V = 0 \text{ at } x = 0,$$
$$V = V_0 \text{ at } x = a,$$
$$V = 0 \text{ at } y = 0,$$
$$V = 0 \text{ at } y = b.$$

Applying the third boundary condition ($V = 0$ at $y = 0$) to Eq. (4.78) provides

$$0 = (k_1 \cosh A_1 x + k_2 \sinh A_1 x)(k_5 + 0), \tag{4.79}$$

$$k_5 = 0. \tag{4.80}$$

Substituting Eq. (4.80) into Eq. (4.78) yields

$$V(x, y) = (k_1 \cosh A_1 x + k_2 \sinh A_1 x)(k_6 \sin A_1 y). \tag{4.81}$$

Applying the fourth boundary condition ($V = 0$ at $y = 0$) to Eq. (4.78) provides

$$0 = \sin A_1 b, \tag{4.82}$$

$$\sin m\pi = \sin A_1 b, \tag{4.83}$$

where $m = 0,1,2...$

$$A_1 = \frac{m\pi}{b}. \tag{4.84}$$

Substituting Eq. (4.84) into Eq. (4.81) yields

$$V(x, y) = \left(k_1 \cosh \frac{m\pi}{b} x + k_2 \sinh \frac{m\pi}{b} x\right) \left(k_6 \sin \frac{m\pi}{b} y\right). \tag{4.85}$$

Applying the first boundary condition ($V = 0$ at $x = 0$) to Eq. (4.85) provides

$$0 = (k_1 + 0) \left(k_6 \sin \frac{m\pi}{b} y\right), \tag{4.86}$$

$$k_1 = 0. \tag{4.87}$$

Substituting Eq. (4.87) into Eq. (4.85) yields

$$V(x, y) = k_2 k_6 \sinh\left(\frac{m\pi}{b} x\right) \sin\left(\frac{m\pi}{b} y\right), \tag{4.88}$$

$$V(x, y) = k \sinh\left(\frac{m\pi}{b} x\right) \sin\left(\frac{m\pi}{b} y\right), \tag{4.89}$$

where $k = k_2 k_6$

Again, applying the boundary condition $V = V_0$ at $x = a$ into Eq. (4.89) provides

$$V(a, y) = V_0 = k \sinh\left(\frac{m\pi}{b} a\right) \sin\left(\frac{m\pi}{b} y\right). \tag{4.90}$$

For an infinite series, Eq. (4.89) can be written as

$$V(x, y) = \sum_{m=1}^{\infty} k \sinh\left(\frac{m\pi}{b} x\right) \sin\left(\frac{m\pi}{b} y\right), \tag{4.91}$$

$$V_0 = \sum_{m=1}^{\infty} k \sinh\left(\frac{m\pi}{b} a\right) \sin\left(\frac{m\pi}{b} y\right). \tag{4.92}$$

Multiplying both sides by $\sin\left(\frac{n\pi y}{b}\right)$of Eq. (4.92) and integrating over $0 < y < b$ yields

$$\int_0^b V_0 \sin\left(\frac{n\pi}{b} y\right) dy = \sum_{m=1}^{\infty} k \sinh\left(\frac{m\pi}{b} a\right) \int_0^b \sin\left(\frac{m\pi}{b} y\right) \sin\left(\frac{n\pi}{b} y\right) dy. \tag{4.93}$$

The orthogonal product rule is

$$\int_0^b \sin(my)\sin(ny)\,dy = \begin{vmatrix} 0 & m \neq n \\ \frac{1}{2} & m = n \end{vmatrix}.$$

Applying the rules of Eq. (4.4.2) into Eq. (4.93) yields

$$\int_0^b V_0 \sin\left(\frac{m\pi}{b}y\right) dy = \sum_{m=1}^{\infty} k \sinh\left(\frac{m\pi}{b}a\right) \int_0^b \sin^2\left(\frac{m\pi}{b}y\right) dy, \tag{4.94}$$

$$\int_0^b V_0 \sin\left(\frac{m\pi}{b}y\right) dy = k \sinh\left(\frac{m\pi}{b}a\right) \frac{1}{2} \int_0^b \left(1 - \cos\frac{m\pi}{b}y\right) dy, \tag{4.95}$$

$$-V_0 \frac{b}{m\pi}\left[\cos\left(\frac{m\pi}{b}y\right)\right]_0^b = k\frac{b}{2}\sinh\left(\frac{m\pi}{b}a\right), \tag{4.96}$$

$$k \sinh\left(\frac{m\pi}{b}a\right) = \frac{2V_0}{m\pi}[1 - \cos m\pi], \tag{4.97}$$

$$k \sinh\left(\frac{m\pi}{b}a\right) = \begin{vmatrix} \frac{4V_0}{m\pi} & m = 1,3,5\ldots \\ 0 & m = 2,4,6\ldots \end{vmatrix}, \tag{4.98}$$

$$k = \frac{4V_0}{m\pi \sinh\left(\frac{m\pi}{b}a\right)} \qquad \text{for } m = \text{odd}, \tag{4.99}$$

$$k = 0 \qquad \text{for } m = \text{even}. \tag{4.100}$$

Substituting Eq. (4.99) into Eq. (4.91) gives the complete solution as

$$V(x, y) = \frac{4V_0}{\pi} \sum_{m=1,3,5\ldots}^{\infty} \frac{\sinh\left(\frac{m\pi}{b}x\right)\sin\left(\frac{m\pi}{b}y\right)}{m \sinh\left(\frac{m\pi}{b}a\right)}. \tag{4.101}$$

Example 4.2 The boundary conditions of region 1 and region 2 are defined as $y = 0$, $V = 0$ and $y = l$, $V = V_0$, respectively. Use Laplace's equation to determine the expression of voltage.

Solution For region 1, the Laplace's equation is

$$\frac{\partial^2 V_1}{\partial y^2} = 0.$$

The solution is

$$V_1 = Ay + B.$$

If $y = 0$, $V = 0$, then $B = 0$.

For region 2, the Laplace's equation is

$$\frac{\partial^2 V_2}{\partial y^2} = 0.$$

The solution is

$$V_2 = Ay + B.$$

If $y = l$, $V = V_0$, then $A = \frac{V_0}{l}$.

Let $V_1 = V_2 = V$, then the expression of voltage is

$$V = \frac{V_0 y}{l}.$$

Example 4.3 Determine the potential of a rectangular trough of infinite length. Considering $a = b = 1\text{m}$, $V_0 = 100\text{V}$, $x = \frac{a}{2}$ and $y = \frac{b}{2}$. Also, find the electric field intensity.

Solution The voltage can be determined as,

$$V\left(\frac{a}{2}, \frac{b}{2}\right) = \frac{400}{\pi}\left[\frac{\sinh\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)}{\sinh(\pi)} + \frac{\sinh\left(\frac{3\pi}{2}\right)\sin\left(\frac{3\pi}{2}\right)}{3\sinh(3\pi)} + \frac{\sinh\left(\frac{5\pi}{2}\right)\sin\left(\frac{5\pi}{2}\right)}{5\sinh(5\pi)}\right]$$

$$V\left(\frac{a}{2}, \frac{b}{2}\right) = \frac{400}{\pi}[0.1992 - 0.00299 + 0.0000776] = 25\text{V}.$$

The electric field intensity can be determined as

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial x}\mathbf{a}_x - \frac{\partial V}{\partial y}\mathbf{a}_y,$$

$$\mathbf{E} = -\frac{400}{b}\left[\left\{\frac{\cosh\left(\frac{m\pi}{b}x\right)\sin\left(\frac{m\pi}{b}y\right)}{\sinh\left(\frac{m\pi}{b}a\right)}\mathbf{a}_x\right\} + \left\{\frac{\sinh\left(\frac{m\pi}{b}x\right)\cos\left(\frac{m\pi}{b}y\right)}{\sinh\left(\frac{m\pi}{b}a\right)}\mathbf{a}_y\right\}\right],$$

$$\mathbf{E} = -\frac{400}{b}\left[(0.217 - 0.0089 + 0.00038)\mathbf{a}_x + 0\mathbf{a}_y\right],$$

$$\mathbf{E} = -83.39\mathbf{a}_x\,\text{V/m}.$$

Practice problem 4.2 The potential is a function of ϕ in cylindrical coordinates of the radial planes. The boundary conditions of these planes are $V = 0$ at $\phi = 0$ and $V = V_0$ at $\phi = \gamma$. Determine the potential and electric field intensity.

Practice problem 4.3 Find the potential of a rectangular trough of infinite length. Consider that $a = 2b = 4\text{m}$, $V_0 = 100\text{V}$, $x = a$, and $y = \frac{b}{2}$.

4.5 Solution of Laplace's Equation in Cylindrical Coordinates

In cylindrical coordinates, Laplace's equation for electrostatic potential, V, is given as

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{4.102}$$

The general solution of Eq. (4.102) is

$$V(\rho, \phi, z) = R(\rho)\Phi(\phi)Z(z). \tag{4.103}$$

Substituting Eq. (4.103) into Eq. (4.102) yields

$$\Phi Z\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + RZ\frac{1}{\rho^2}\frac{d^2\Phi}{d\phi^2} + R\Phi\frac{d^2 Z}{dz^2} = 0. \tag{4.104}$$

Dividing Eq. (4.102) by the term $R\Phi Z$provides

$$\frac{1}{R\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\Phi\rho^2}\frac{d^2\Phi}{d\phi^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = 0, \tag{4.105}$$

$$\frac{1}{R\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\Phi\rho^2}\frac{d^2\Phi}{d\phi^2} = -\frac{1}{Z}\frac{d^2 Z}{dz^2}. \tag{4.106}$$

The right-side term of Eq. (4.106) is only a function of z and then it can be defined as

$$-\frac{1}{Z}\frac{d^2 Z}{dz^2} = -k^2, \tag{4.107}$$

$$\frac{d^2 Z}{dz^2} + zk^2 = 0. \tag{4.108}$$

The solution of Eq. (4.108) is

$$Z(z) = Ae^{kz} + Be^{-kz}, \tag{4.109}$$

$$Z(z) = A_z \cosh(kz) + B_z \sinh(kz). \tag{4.110}$$

Substituting Eq. (4.107) into Eq. (4.106) provides

$$\frac{1}{R\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\Phi\rho^2}\frac{d^2\Phi}{d\phi^2} = -k^2, \tag{4.111}$$

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \rho^2 k^2 = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}. \tag{4.112}$$

Again the right side of Eq. (4.112) is a function ϕ and it is represented by m^2. Then, the following equation can be written as:

$$-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = m^2, \tag{4.113}$$

$$\frac{d^2\Phi}{d\phi^2} + \Phi m^2 = 0. \tag{4.114}$$

The solution of Eq. (4.114) is

$$\Phi(\phi) = Ae^{jm\phi} + Be^{-jm\phi}, \tag{4.115}$$

$$\Phi(\phi) = A_\phi \cos(m\phi) + B_\phi \sin(m\phi). \tag{4.116}$$

Equation (4.112) can be modified as

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \rho^2k^2 - m^2 = 0, \tag{4.117}$$

$$\rho\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + (\rho^2k^2 - m^2)R = 0, \tag{4.118}$$

$$\rho^2\frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho} + (\rho^2k^2 - m^2)R = 0, \tag{4.119}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2}\right)R = 0. \tag{4.120}$$

The solution of Eq. (4.120) is

$$R = B_1 J_n(m\rho) + B_2 N_n(m\rho), \tag{4.121}$$

where,

$J_n(m\rho)$ is the Bessel function of the first kind of order n with argument $m\rho$ and
$N_n(m\rho)$ is the Bessel function of the second kind of order n with argument $m\rho$.

4.6 Solutions of Poisson's Equation

In Cartesian coordinates, consider that Poisson's equation varies in the x-direction only. Then for one dimension, Poisson's equation reduces as

$$\frac{d^2V}{dx^2} = -\frac{\rho}{\varepsilon}. \tag{4.122}$$

Integrating Eq. (4.122) yields

$$\frac{dV}{dx} = -\frac{\rho}{\varepsilon}x + A. \tag{4.123}$$

Again integrating Eq. (4.124) yields

$$V = -\frac{\rho}{2\varepsilon}x^2 + Ax + B. \tag{4.124}$$

Consider the boundary conditions $V = V_1$ at $x = x_1$ and $V = V_2$ at $x = x_2$. Then integrating constants can be determined as

$$V_1 = -\frac{\rho}{2\varepsilon}{x_1}^2 + Ax_1 + B, \tag{4.125}$$

$$V_2 = -\frac{\rho}{2\varepsilon}{x_2}^2 + Ax_2 + B. \tag{4.126}$$

Subtracting Eq. (4.126) from Eq. (4.125) yields

$$V_1 - V_2 = \frac{\rho}{2\varepsilon}{x_2}^2 - \frac{\rho}{2\varepsilon}{x_1}^2 + A(x_1 - x_2), \tag{4.127}$$

$$A(x_1 - x_2) = (V_1 - V_2) + \frac{\rho}{2\varepsilon}(x_1 + x_2)(x_1 - x_2), \tag{4.128}$$

$$A = \frac{V_1 - V_2}{x_1 - x_2} + \frac{\rho}{2\varepsilon}(x_1 + x_2). \tag{4.129}$$

Substituting Eq. (4.129) into Eq. (4.125) yields

$$V_1 = -\frac{\rho}{2\varepsilon}{x_1}^2 + \left[\frac{V_1 - V_2}{x_1 - x_2} + \frac{\rho}{2\varepsilon}(x_1 + x_2)\right]x_1 + B, \tag{4.130}$$

$$B = V_1 - \frac{V_1 - V_2}{x_1 - x_2}x_1 - \frac{\rho}{2\varepsilon}{x_1}^2 + \frac{\rho}{2\varepsilon}{x_1}^2 - \frac{\rho}{2\varepsilon}x_1x_2, \tag{4.131}$$

$$B = \frac{V_2x_1 - V_1x_2}{x_1 - x_2} - \frac{\rho}{2\varepsilon}x_1x_2. \tag{4.132}$$

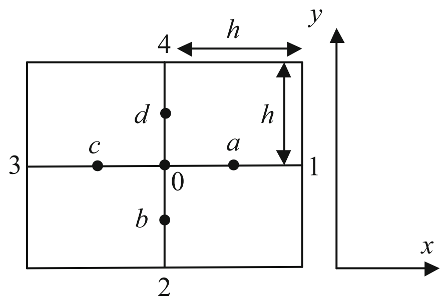
Substituting Eqs. (4.129) and (4.132) into Eq. (4.124) yields

$$V = -\frac{\rho}{2\varepsilon}x^2 + \left[\frac{V_1 - V_2}{x_1 - x_2} + \frac{\rho}{2\varepsilon}(x_1 + x_2)\right]x + \frac{V_2x_1 - V_1x_2}{x_1 - x_2} - \frac{\rho}{2\varepsilon}x_1x_2. \tag{4.133}$$

4.7 Numerical Solution of Laplace's Equation

There are few numerical methods that are normally used to find electric potential and field distribution of a specific object in the area of electrical engineering. These are the finite difference method (FDM), the finite element method (FEM), and the boundary element method. The available commercial software in this area is developed based on Laplace's and Poisson's equations. In the FDM, the selected object is divided

Fig. 4.2 A square mesh object



into forward and backward directions with an equal length. For example, a two-dimensional square mesh object is shown in Fig. 4.2. Consider that the length of each side is h and potentials of the points 0, 1, 2, 3, and 4 are V_0, V_1, V_2, V_3, and V_4, respectively. In this case, the voltage does not vary in the z-direction. Therefore, the Laplace's equation in two dimensions is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0. \tag{4.134}$$

For the x-axis, the voltage derivative in the forward direction is

$$\left.\frac{\partial V}{\partial x}\right|_a = \frac{V_1 - V_0}{h}. \tag{4.135}$$

In the backward direction, the voltage derivative is

$$\left.\frac{\partial V}{\partial x}\right|_c = \frac{V_0 - V_3}{h}. \tag{4.136}$$

According to the rules of derivative calculus, the following equation is:

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_0 = \frac{\left.\frac{\partial V}{\partial x}\right|_a - \left.\frac{\partial V}{\partial x}\right|_c}{h}. \tag{4.137}$$

Substituting Eqs. (4.135) and (4.136) into Eq. (4.137) yields

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_0 = \frac{V_1 - V_0 - V_0 + V_3}{h^2}. \tag{4.138}$$

For the y-axis, the voltage derivative in the forward direction is

$$\left.\frac{\partial V}{\partial y}\right|_d = \frac{V_4 - V_0}{h}. \tag{4.139}$$

The voltage derivative in the backward direction is

$$\left.\frac{\partial V}{\partial y}\right|_b = \frac{V_0 - V_2}{h}. \tag{4.140}$$

Again applying the rules of derivative calculus, the following equation is yielded:

$$\left.\frac{\partial^2 V}{\partial y^2}\right|_0 = \frac{\left.\frac{\partial V}{\partial y}\right|_d - \left.\frac{\partial V}{\partial y}\right|_b}{h}. \tag{4.141}$$

Substituting Eqs. (4.139) and (4.140) into Eq. (4.141) yields

$$\left.\frac{\partial^2 V}{\partial y^2}\right|_0 = \frac{V_4 - V_0 - V_0 + V_2}{h^2}. \tag{4.142}$$

Substituting Eqs. (4.138) and (4.142) into Eq. (4.134) yields

$$\frac{V_1 - V_0 - V_0 + V_3}{h^2} + \frac{V_4 - V_0 - V_0 + V_2}{h^2} = 0, \tag{4.143}$$

$$4V_0 = V_1 + V_2 + V_3 + V_4, \tag{4.144}$$

$$V_0 = \frac{V_1 + V_2 + V_3 + V_4}{4}. \tag{4.145}$$

The value of the potential V_0 can be found if the potentials at the corners of the mesh are known.

The FEM is another numerical technique to solve two-dimensional Laplace's equations. In 1943, the FEM was first developed by R. Courant to obtain the approximate solution of a complex object. Initially, this technique was applied in mechanical and civil engineering to study respective parameters. Later on, the FEM started being used in electrical engineering to find the flux, potential, and electric field distributions around and inside an object. The selected object is discretized either by triangular or rectangular elements. According to P. P. Silvester and R. L. Ferrari, the approximate solution of potential for the whole region is

$$V(x, y) = \sum_{e=1}^{N} V_e(x, y), \tag{4.146}$$

where e represents the number of the element and N is the total number of triangular elements. The polynomial approximation for V_e within a single element is

$$V_e(x, y) = a + bx + cy. \tag{4.147}$$

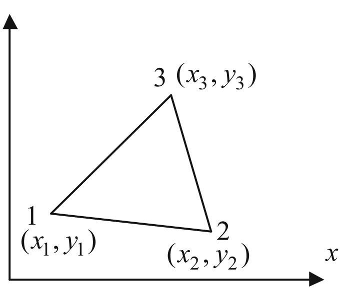
The electric field within the element is

$$\mathbf{E}_e = -\nabla V_e. \tag{4.148}$$

Substituting Eq. (4.147) into Eq. (4.148) yields

$$\mathbf{E}_e = -(b\mathbf{a}_x + c\mathbf{a}_y). \tag{4.149}$$

Fig. 4.3 A triangular element



The potentials V_{e1}, V_{e2}, and V_{e3} at nodes 1, 2, and 3 of the triangular element as shown in Fig. 4.3 is

$$V_{e1} = a + bx_1 + cy_1, \tag{4.150}$$

$$V_{e2} = a + bx_2 + cy_2, \tag{4.151}$$

$$V_{e3} = a + bx_3 + cy_3. \tag{4.152}$$

Equations (4.150–4.152) can be arranged in the matrix format as

$$\begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}. \tag{4.153}$$

The coefficients a, b, and c can be determined from Eq. (4.153) as

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix}, \tag{4.154}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix}, \tag{4.155}$$

where, A is the area of the element and it can be written as

$$A = \frac{1}{2} \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}. \tag{4.156}$$

Substituting Eq. (4.155) into Eq. (4.147) yields

$$V_e(x, y) = \begin{bmatrix} 1 & x & y \end{bmatrix} \frac{1}{2A} \begin{bmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix}, \tag{4.157}$$

$$V_e(x, y) = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix}, \tag{4.158}$$

$$V_e(x, y) = \sum_{i=1}^{3} \alpha_i(x, y) V_{ei}. \tag{4.159}$$

Here α_1, α_2, and α_3 are the shape functions of the element and their expressions from the Eq. (4.157) can be written as

$$\alpha_1 = \frac{1}{2A}\left[(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x + (x_3 - x_2)y\right], \tag{4.160}$$

$$\alpha_2 = \frac{1}{2A}\left[(x_3 y_1 - x_1 y_3) + (y_3 - y_1)x + (x_1 - x_3)y\right], \tag{4.161}$$

$$\alpha_3 = \frac{1}{2A}\left[(x_1 y_2 - x_2 y_1) + (y_1 - y_2)x + (x_2 - x_1)y\right]. \tag{4.162}$$

The energy per unit length of the single element is

$$W_e = \frac{1}{2}\int_s \varepsilon \left|\mathbf{E}\right|^2 dS. \tag{4.163}$$

Substituting $\mathbf{E} = -\nabla V_e$ into Eq. (4.163) yields

$$W_e = \frac{1}{2}\int_s \varepsilon \left|\nabla V_e\right|^2 dS. \tag{4.164}$$

Equation (4.159) can be modified as

$$\nabla V_e = \sum_{i=1}^{3} V_{ei} \nabla \alpha_i. \tag{4.165}$$

Substituting Eq. (4.165) into Eq. (4.164) provides

$$W_e = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} \varepsilon V_{ei} V_{ej} \left[\int_s \nabla\alpha_i . \nabla\alpha_j dS\right], \tag{4.166}$$

$$W_e = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} \varepsilon V_{ei} V_{ej} C_{ij}, \tag{4.167}$$

where the expression of the following equation is:

$$C_{ij} = \int_s \nabla\alpha_i . \nabla\alpha_j dS. \tag{4.168}$$

Equation (4.167) can be written in the matrix format as

$$W_e = \frac{1}{2}\varepsilon[V_e]^T[C][V_e], \tag{4.169}$$

where the element coefficient matrix and potentials are

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}, \tag{4.170}$$

$$[V_e] = \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix}. \tag{4.171}$$

The total energy for whole elements can be determined as

$$W = \sum_{e=1}^{N} W_e = \frac{1}{2}\varepsilon[V]^T[C][V]. \tag{4.172}$$

The Laplace's equation is satisfied when the total energy in the region is a minimum. It can be expressed as

$$\frac{\partial W}{\partial V_k} = 0 \quad k = 1, 2, 3...n. \tag{4.173}$$

For free and prescribed potentials, Eq. (4.172) can be written as

$$W = \frac{1}{2}\varepsilon \begin{bmatrix} V_f & V_p \end{bmatrix} \begin{bmatrix} C_{ff} & C_{fp} \\ C_{pf} & C_{pp} \end{bmatrix} \begin{bmatrix} V_f \\ V_p \end{bmatrix}. \tag{4.174}$$

Applying Eq. (4.173) to Eq. (4.174), i.e. differentiating with respect to V_f yields

$$C_{ff}V_f + C_{fp}V_p = 0, \tag{4.175}$$

$$[C_{ff}][V_f] = -[C_{fp}][V_p], \tag{4.176}$$

$$[A][V] = [B], \tag{4.177}$$

where the following equations are:

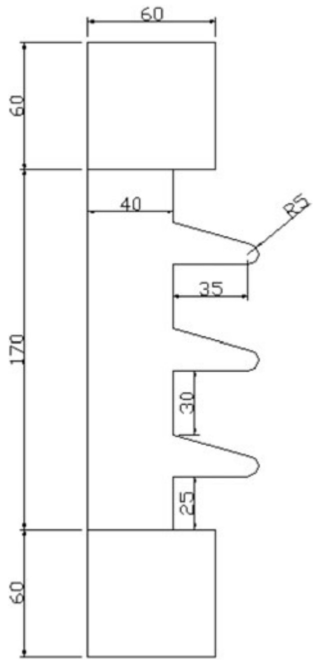
$$[A] = [C_{ff}], \tag{4.178}$$

$$[V] = [V_f]. \tag{4.179}$$

Equation (4.177) can be rearranged as

$$[V] = [A]^{-1}[B]. \tag{4.180}$$

Fig. 4.4 Axi-symmetric line-post insulator



Therefore, the potential can be determined from Eq. (4.180) if other parameters are known.

There are few commercial softwares about FEM. Electromagnetic engineering software SLIM is one of them. ALSTOM Engineering Research Center (ERC) has developed a commercial standard finite element analysis software package, SLIM 3.6.2, which can solve a wide range of electromagnetic field problems over a spectrum of frequencies from DC to GHz of any complex geometry by considering boundary conditions and material properties. SLIM is a professionally integrated software which provides facilities for the generation of finite element mesh, solution of electric, magnetic, and thermal fields and the post processing of results of the geometry. In this software, FEA is used and the whole domain of interest is divided into smaller triangular elements. Therefore, it is important to know the changes in the field distribution around an outdoor insulator caused by surface pollution of different nature and severity. Here, the electric fields and voltage distributions of line-post insulator have been studied using the electromagnetic engineering software, SLIM 3.6.2. The geometry of the line-post insulator is shown in Fig. 4.4 where the dimension of the insulator is in millimetre (mm).

The following are taken into consideration during simulation:

- Porcelain is used as the material of the insulator.
- The insulator is surrounded by air.
- High voltage is applied at the top and zero voltage is applied at the bottom of the insulator.

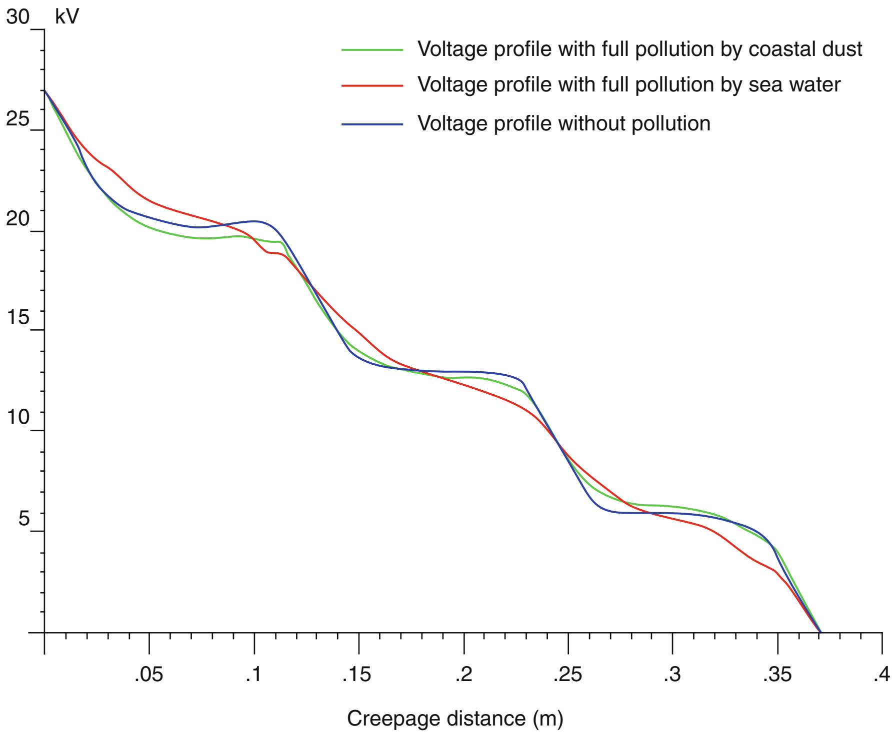


Fig. 4.5 Voltage distribution of a line-post insulator

- The applied voltage is sinusoidal with 50 Hz.
- The insulator is considered as axi-symmetric with two dielectrics.

Due to the geometric symmetry, the simulation works have finished using the two-dimensional Laplace's equation. The geometry of the insulator is drawn by the AutoCAD software and saved as dxf format. Then this geometry is imported to the SLIM 3.6.2 software platform for two-dimensional calculation. Initially, the upper part of the insulator geometry is defined by high voltage and the lower part is defined by zero voltage. The insulator is also surrounded by air. The insulator including air is discretized with approximately 14,902 triangular elements and 7,681 nodes. The input data are the permittivity of the porcelain, permittivity of the pollution material, and the boundary conditions. The permittivity of the seawater is considered 80, whereas the permittivity of the coastal dust is considered 10. The electrostatic analysis is performed for 26.94 kV (Max.) per phase. In the post-processing module, the path is defined from top to bottom of the insulator, i.e. between the air–porcelain interfaces. The voltages along the creepage distance are calculated under clean, full pollution with seawater, and coastal dust conditions. The voltage distributions under those conditions are plotted versus creepage distance which is shown in Fig. 4.5.

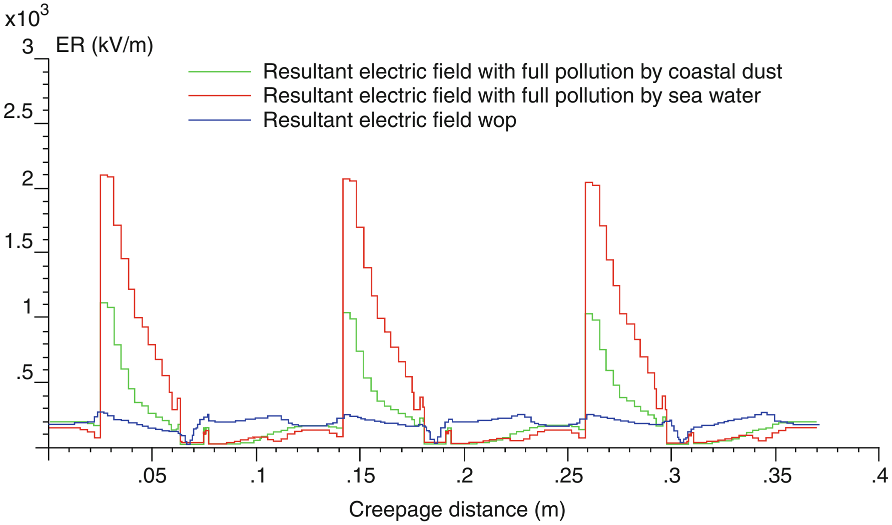


Fig. 4.6 Resultant electric field along shed–air interface of a line-post insulator

From Fig. 4.5, it is observed that the voltage distribution in some areas of the insulator with seawater is higher than the voltage distribution under clean and even coastal dust conditions. These areas can be categorized as the upper surface to the rim of the insulator shed.

The highest pollution is normally accumulated in this area of the insulator. In the practical field, less pollution is normally accumulated in the lower areas of the shed which does not get washed by normal rain, and slowly it forms into a thick non-uniform pollution layer. The non-uniform pollution layer increases the surface roughness, which in turn increases wetness. This thick pollution wetness layer provides a more conductive path with the help of a mist or a light rain. In this region, the voltage profile without pollution is higher than the voltage profile with full pollution by seawater and coastal dusts. The resultant, normal, and tangential electric fields are calculated along shed–air interface of the insulator as shown in Figs. 4.6–4.8, respectively. From Figs. 4.6 to 4.8, it is observed that the normal electric field is dominating the resultant electric field.

From Fig. 4.6, it can also be seen that the resultant electric field distribution due to seawater is higher than the clean and coastal dust pollution from the upper surface to the rim of the insulator. The resultant electric field distributions due to pollution by seawater and coastal dust are capacitive to resistive. However, under wet conditions, especially when the insulator is polluted, the field distribution may worsen and surface discharges may be possible. The field distribution due to clean condition is fully resistive throughout the surface.

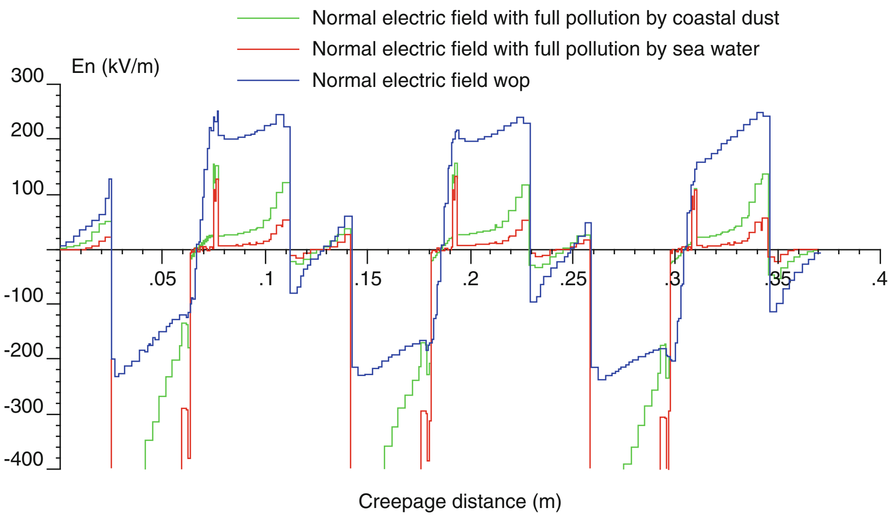


Fig. 4.7 Normal electric field along shed–air interface of a line-post insulator

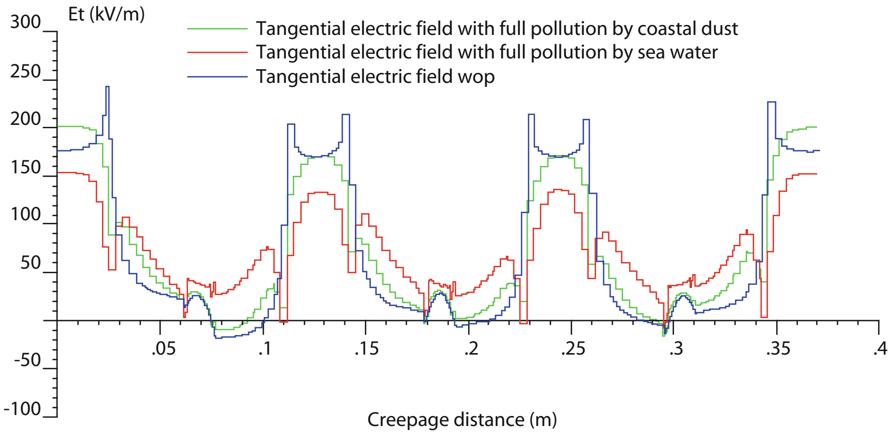


Fig. 4.8 Tangential electric field shed–air interface of line-post insulator

4.8 Exercise Problems

4.1 The expression of electric potential in Cartesian coordinates is $V(x, y, z) = x^2y - z^2 + 8$. Determine the (a) numerical value of the voltage at point $P(1, -1,2)$, (b) the electric field, and (c) verify the Laplace's equation.

4.2 The electric potential in Cartesian coordinates is given by $V(x, y, z) = e^x - e^{-y} + z^2$. Determine the (a) numerical value of the voltage at point $P(1,1, -2)$, (b) the electric field atpoint $P(1,1, -2)$, and (c) verify the Laplace's equation.

4.3 The expression of electric potential in cylindrical coordinates is given as $V(\rho, \phi, z) = \rho^2 z \cos\phi$. Determine the (a) numerical value of the voltage at point $P(\rho = -1, \phi = 45°, z = 5)$, (b) electric field at point $P(\rho = -1, \phi = 45°, z = 5)$, and (c) verify the Laplace's equation.

4.4 The electric potential in spherical coordinates is given by $V(r, \theta, \phi) = 5r^2 \sin\theta \cos\phi$. Determine the (a) numerical value of the voltage at point $P(r = 1, \theta = 40°, \phi = 120°)$, (b) the electric field at point $P(r = 1, \theta = 40°, \phi = 120°)$, and (c) verify the Laplace's equation.

4.5 In Cartesian coordinates, the volume charge density is $\rho_v = -1.6 \times 10^{-11} \varepsilon_0 x$ C/m^{3}in the free space. Consider $V = 0$ at $x = 0$ and $V = 4$Vat $x = 2$m. Determine the electric potential and field at $x = 5$m.

4.6 The charge density in cylindrical coordinates is $\rho_v = \frac{25}{\rho}$ pC/m^3. Consider $V = 0$ at $\rho = 2$mand $V = 120$Vat $\rho = 5$m. Calculate the electric potential and field at $\rho = 6$m.

4.7 The concentric spherical shells with radii of $r = 1$m and $r = 2$mcontain the potentials of $V = 0$ and $V = 80$V, respectively. Find the potential and electric field.

4.8 Determine the potential of a rectangular trough of infinite length. Consider $a = b = 1$m, $V_0 = 50$V, $x = \frac{3a}{2}$, and $y = \frac{b}{2}$.

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Chapter 5
Electric Currents

5.1 Introduction

Electric current mainly depends on the movement of charge. Different types of currents are considered in electromagnetic field engineering. These are conduction currents, convection currents, electrolytic currents, displacement currents, etc. Conduction current is generated in the conductor by the motion of electric charges. The actual migration of the positive and the negative ions generates electrolytic current. The motion of electric charges in a vacuum or fluids generates the convection current. In this chapter, current, current density, resistance, capacitance, the continuity equation, etc. will be discussed.

5.2 Current and Current Density

The existence of current and current density is very important to identify the behavior of the electric field in the conductor. Current is defined as the flow of free electrons through a conductor. In an alternative way, the rate of change of charge through a given area is known as current. Current is symbolized by the letter I and its unit is ampere (A). Mathematically, the expression of current is,

$$I = \frac{dQ}{dt}. \tag{5.1}$$

The general definition of current density is the current per unit area. The current density is symbolized by the letter $\mathbf{J}$ and its unit is ampere per square meter (A/m^2). Consider an incremental current ΔI that is crossing an incremental surface ΔS. This incremental current is normal to the incremental surface as shown in Fig. 5.1.

The expression of the normal current density can be written as

$$J_n = \frac{\Delta I}{\Delta S}, \tag{5.2}$$

$$\Delta I = J_n \Delta S. \tag{5.3}$$

Md. A. Salam, *Electromagnetic Field Theories for Engineering*,
DOI 10.1007/978-981-4585-66-8_5,

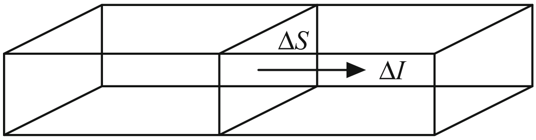

Fig. 5.1 Incremental current crossing an incremental surface

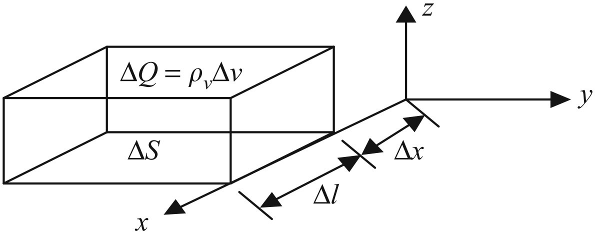

Fig. 5.2 Charge with an incremental distance

In vector format, Eq. (5.3) can be written as

$$\Delta I = \mathbf{J}.\Delta\mathbf{S}. \tag{5.4}$$

The total current I flowing through the surface S can be obtained by integrating Eq. (5.4) as

$$I = \oint_S \mathbf{J}.d\mathbf{S}. \tag{5.5}$$

Again, consider that an incremental charge ΔQ is moved to the Δx distance from the yz plane as shown in Fig. 5.2.

The expression of the incremental charge can be written as

$$\Delta Q = \rho_v \Delta v = \rho_v \Delta S \Delta x. \tag{5.6}$$

The expression of the incremental current can be written as

$$\Delta I = \frac{\Delta Q}{\Delta t}. \tag{5.7}$$

Substituting Eq. (5.6) into Eq. (5.7) yields

$$\Delta I = \frac{\rho_v \Delta S \Delta x}{\Delta t}, \tag{5.8}$$

$$\Delta I = \rho_v \Delta S v_x, \tag{5.9}$$

where v_x represents the x component of the velocity.

From Eq. (5.9), the current density in the x-direction is

$$J_x = \frac{\Delta I}{\Delta S} = \rho_v v_x. \tag{5.10}$$

In general, the current density in vector format is

$$\mathbf{J} = \rho_v \mathbf{v} \tag{5.11}$$

From Eq. (5.11), it is seen that the convection current density is equal to the product of volume charge density and velocity.

Example 5.1 The current density in cylindrical coordinates is given as $\mathbf{J} = 5\rho z \mathbf{a}_\rho + 2z\cos\phi \mathbf{a}_\phi$ A/m^2. Determine (a) the current density at point ($\rho = 2,\ \phi = 35°,\ z = 0.8$) and (b) the total current flowing through the cylindrical surface $\rho = 2,\ 0 < \phi < 2\pi,\ 0.8 < z < 3.5$.

Solution

(a) The value of the current density at point ($\rho = 2,\ \phi = 35°,\ z = 0.8$) is

$$\mathbf{J} = 5 \times 2 \times 0.8\mathbf{a}_\rho + 2 \times 2\cos 45°\mathbf{a}_\phi,$$

$$\mathbf{J} = 8\mathbf{a}_\rho + 2.83\mathbf{a}_\phi \text{ A/m}^2.$$

(b) The differential surface for the cylindrical coordinates is

$$d\mathbf{S} = \rho d\phi dz \mathbf{a}_\rho$$

$$d\rho dz \mathbf{a}_\phi$$

$$\rho d\rho d\phi \mathbf{a}_z.$$

The total current can be determined as

$$I = \int_0^{2\pi} \int_{0.8}^{3.5} (5\rho z \mathbf{a}_\rho + 2z\cos\phi \mathbf{a}_\phi).\rho d\phi dz \mathbf{a}_\rho,$$

$$I = 5\rho^2 \int_0^{2\pi} d\phi \int_{0.8}^{3.5} z dz,$$

$$I = \frac{5\rho^2}{2}(3.5^2 - 0.8^2) \int_0^{2\pi} d\phi,$$

$$I = \frac{5 \times 2^2}{2}(3.5^2 - 0.8^2) \times 2\pi = 729.48\text{A}.$$

Example 5.2 The current density in spherical coordinates is given as $\mathbf{J} = \frac{3}{r}\cos\left(\frac{\phi}{3}\right)\mathbf{a}_r + 2r\cos\theta \mathbf{a}_\theta$ A/m^2. Determine (a) the current density at point ($r = 2$, $\theta = 25°,\ \phi = 134°$) and (b) the total current flowing through the spherical surface $0 < \theta < \frac{\pi}{2},\ 0 < \phi < 2\pi$.

Solution

(a) The value of the current density at point ($r = 2,\ \theta = 25°,\ \phi = 134°$) is

$$\mathbf{J} = \frac{3}{2}\cos\frac{134°}{3}\mathbf{a}_r + 2 \times 2\cos 25°\mathbf{a}_\theta,$$

$$\mathbf{J} = 1.07\mathbf{a}_r + 3.63\mathbf{a}_\theta \text{ A/m}^2.$$

(b) The differential surface for the spherical coordinates is

$$
\begin{aligned}
d\mathbf{S} =& r^2 \sin\theta d\theta d\phi \mathbf{a}_r \\
& r\sin\theta dr d\phi \mathbf{a}_\theta \\
& r dr d\theta \mathbf{a}_\phi .
\end{aligned}
$$

The total current can be determined as

$$
\begin{aligned}
I &= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}} \left(\frac{3}{r}\cos\frac{\phi}{3}\mathbf{a}_r + 2r\cos\theta\mathbf{a}_\theta\right).r^2\sin\theta d\theta d\phi \mathbf{a}_r, \\
I &= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}} 3r\cos\left(\frac{\phi}{3}\right)\sin\theta d\theta d\phi, \\
I &= -3\times 2\left(\cos\frac{\pi}{2}-1\right)\int_{\phi=0}^{2\pi}\cos\left(\frac{\phi}{3}\right)d\phi, \\
I &= -3\times 3\times 2\left(\cos\frac{\pi}{2}-1\right)\left(\sin\frac{2\pi}{3}-0\right) = 15.59\text{A}.
\end{aligned}
$$

Practice problem 5.1 The current density is given by $\mathbf{J} = z\mathbf{a}_\rho + 2\rho\sin\phi\mathbf{a}_\phi$ A/m^2. Calculate the (a) current density at point ($\rho = 1,\ \phi = 55°,\ z = 2$) and (b) the total current flowing through the cylindrical surface $\rho = 1,\ 0 < \phi < 2\pi,\ 2 < z < 4$.

Practice problem 5.2 The expression of current density in spherical coordinates is given by $\mathbf{J} = 3\cos\left(\frac{\phi}{3}\right)\mathbf{a}_r + 2r\cos\theta\mathbf{a}_\theta$ A/m^2. Find the (a) current density at point ($r = 1,\ \theta = 40°,\ \phi = 125°$) and (b) the total current flowing through the cylindrical surface $0 < \theta < \frac{\pi}{2},\ 0 < \phi < 2\pi$.

5.3 Conductivity and Resistance

The current density is directly related to the density (ρ_v) of conduction electrons and the drift velocity ($\mathbf{v}_d$). This relationship can be expressed as

$$\mathbf{J} = \rho_v \mathbf{v}_d. \tag{5.12}$$

The drift velocity is linearly proportional to the electric field vector and it can be expressed as

$$\mathbf{v}_d = -\mu_e \mathbf{E} \tag{5.13}$$

where μ_e the mobility of electrons is in the given material and its unit is (m^2/Vs).

The current density in terms of concentration of the charge carriers ($N\nu$)and the drift velocity is

$$\mathbf{J} = N_v(-e)\mathbf{v}_d. \tag{5.14}$$

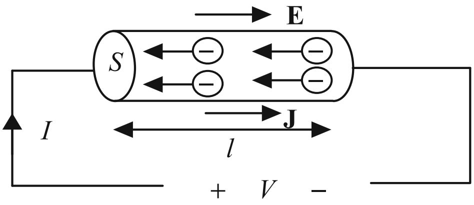

Fig. 5.3 Homogenous material with a fixed cross-section

Substituting Eq. (5.13) into Eq. (5.14) yields

$$\mathbf{J} = N_v(-e)(-\mu_e \mathbf{E}) = N_v e \mu_e \mathbf{E}, \tag{5.15}$$

$$\mathbf{J} = \sigma \mathbf{E} \tag{5.16}$$

where σ is the proportionality constant and it is represented as

$$\sigma = N_v e \mu_e. \tag{5.17}$$

This proportionality constant is a macroscopic parameter of the medium which is known as conductivity. The conductivity represents the ability of a material to conduct the current. The unit of conductivity is S/m. The reciprocal of the conductivity is known as resistivity and it is symbolized by ρ. The unit of the resistivity is $\Omega - \mathrm{m}$.

Consider a homogeneous material of length l and a uniform cross-sectional area S as shown in Fig. 5.3. In this case, both the current density and electric field are in the direction of the current flow. From Eq. (5.5), the expression of the total current can be derived as

$$I = JS, \tag{5.18}$$

$$J = \frac{I}{S}. \tag{5.19}$$

The total voltage between two ends of the conduction material can be determined as

$$V = \oint_l \mathbf{E}.d\mathbf{l}, \tag{5.20}$$

$$V = El, \tag{5.21}$$

$$E = \frac{V}{l}. \tag{5.22}$$

Substituting Eqs. (5.19) and (5.22) into the scalar format of Eq. (5.16) provides

$$\frac{I}{S} = \sigma \frac{V}{l}, \tag{5.23}$$

$$\frac{l}{\sigma S} = \frac{V}{I}, \tag{5.24}$$

$$R = \frac{l}{\sigma S}. \tag{5.25}$$

From Eq. (5.25), it is concluded that the resistance is directly proportional to the length and inversely proportional to the conductivity and the area.

Example 5.3 The radius of a 1.5 km-long wire is 2 mm. Find the dc resistance if the wire is made of copper and the conductivity of copper wire is $\sigma = 5.8 \times 10^7$ S/m.

Solution The value of the area is

$$S = \pi r^2 = \pi (2 \times 10^{-3})^2 = 1.26 \times 10^{-5}\,\mathrm{m}^2.$$

The value of the resistance can be calculated as

$$R = \frac{l}{\sigma S} = \frac{1.5 \times 1000}{5.8 \times 10^7 \times 1.26 \times 10^{-5}} = 2.05\,\Omega.$$

Practice problem 5.3 The radius of a 2.5 km-long aluminum wire is 1.5 mm. Determine the dc resistance if the conductivity of the aluminum wire is $\sigma = 3.54 \times 10^7$ S/m.

5.4 Power and Joule's Law

The work is done when a charge moves within the conductor. Consider a charge Q moves at a velocity $\mathbf{u}$ by an electric field $\mathbf{E}$ to a distance Δl. In this case, the expression of the work done is

$$\Delta W = Q\mathbf{E}.\mathbf{u}. \tag{5.26}$$

The power is defined as the rate of receiving or delivering energy or force times velocity and it can be expressed as

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t}. \tag{5.27}$$

Substituting Eq. (5.26) into Eq. (5.27) yields

$$P = Q\mathbf{E}.\mathbf{u}. \tag{5.28}$$

But, the total charge is represented as

$$Q = \oint_v \rho_v dv. \tag{5.29}$$

Substituting Eq. (5.29) into Eq. (5.28), then the total power delivered to all charge carriers in a volume dv is

$$P = \oint_v \rho_v dv \mathbf{E}.\mathbf{u}, \tag{5.30}$$

$$P = \oint_v \mathbf{E}.\rho_v \mathbf{u} dv. \tag{5.31}$$

Substituting Eq. (5.12) into Eq. (5.31) provides

$$P = \oint_v \mathbf{E}.\mathbf{J} dv. \tag{5.32}$$

Equation (5.32) is known as Joule's law. For a constant cross-section conductor, the expression of the volume is

$$dv = dSdl. \tag{5.33}$$

Substituting Eq. (5.33) into Eq. (5.32) yields

$$P = \int_l \mathbf{E}.d\mathbf{l} \int_s \mathbf{J}.d\mathbf{S}. \tag{5.34}$$

Substituting the expressions of the voltage $\left(V = \int_l \mathbf{E}.d\mathbf{l}\right)$ and the current $\left(I = \int_s \mathbf{J}.d\mathbf{S}\right)$ into Eq. (5.34) yields

$$P = VI = I^2 R. \tag{5.35}$$

Power density is defined as the total power per volume and it can be expressed as

$$w_p = \frac{dP}{dv}. \tag{5.36}$$

From Eqs. (5.32, 5.36), and (5.16) the expression of power density can be written as

$$w_p = \mathbf{E}.\mathbf{J} = \sigma |\mathbf{E}|^2. \tag{5.37}$$

From Eq. (5.37), it is shown that the power density is equal to the product of the conductivity and square of the magnitude of the electric field.

5.5 Continuity Equation

Consider a closed surface as shown in Fig. 5.4, where a wire carries a current I_{in} in the surface and the current I_{out} goes out from the surface. From the basic definition, the expression of current that enter to the closed surface is

$$I_{in} = \frac{dQ}{dt}. \tag{5.38}$$

The expression of current leaving from the closed surface is

$$I_{out} = -I_{in} = \frac{-dQ}{dt}. \tag{5.39}$$

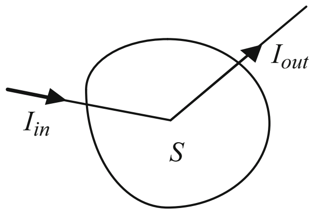

Fig. 5.4 Closed surface with currents

The total outward current through the closed surface is

$$I_{out} = \oint_S \mathbf{J}.d\mathbf{S}. \tag{5.40}$$

Substituting Eq. (5.39) into Eq. (5.40) yields

$$\oint_S \mathbf{J}.d\mathbf{S} = \frac{-dQ}{dt}. \tag{5.41}$$

Substituting Eq. (5.29) into Eq. (5.41) yields

$$\oint_S \mathbf{J}.d\mathbf{S} = \frac{-d}{dt}\int_v \rho dv. \tag{5.42}$$

Applying divergence theorem to change the surface integral into a volume integral provides

$$\oint_S \mathbf{J}.d\mathbf{S} = \int_v (\nabla.\mathbf{J})dv \tag{5.43}$$

Substituting Eq. (5.43) into Eq. (5.42) yields

$$\int_v (\nabla.\mathbf{J})dv = \frac{-d}{dt}\int_v \rho dv. \tag{5.44}$$

For incremental volume, Eq. (5.44) is modified as

$$(\nabla.\mathbf{J})\Delta v = \frac{-d\rho}{dt}\Delta v, \tag{5.45}$$

$$\nabla.\mathbf{J} = \frac{-d\rho}{dt}. \tag{5.46}$$

Equation (5.46) is known as a continuity of the current equation or simply continuity equation. For a steady current, the charge density is constant and Eq. (5.46) can be modified as

$$\nabla.\mathbf{J} = 0. \tag{5.47}$$

From Eq. (5.47), it is concluded that the total charge leaving the volume is equal to the total charge entering it.

Substituting Eq. (5.16) into Eq. (5.46) yields

$$\nabla.\sigma\mathbf{E} = \frac{-d\rho}{dt}. \tag{5.48}$$

Substituting the differential form of Gauss' law ($\nabla.\mathbf{D} = \rho$) into Eq. (5.48) provides

$$\frac{\sigma\rho}{\varepsilon} = \frac{-d\rho}{dt}, \tag{5.49}$$

$$\frac{d\rho}{dt} + \frac{\sigma\rho}{\varepsilon} = 0. \tag{5.50}$$

The general solution of Eq. (5.50) is

$$\rho = ke^{-(\frac{\sigma}{\varepsilon})t}. \tag{5.51}$$

Substituting the initial condition (at $t=0$, $\rho=\rho_0$) to Eq. (5.51) yields

$$\rho_0 = k. \tag{5.52}$$

Again, substituting Eq. (5.51) into Eq. (5.51) and the final solution can be written as

$$\rho = \rho_0 e^{-(\frac{\sigma}{\varepsilon})t} \tag{5.53}$$

where ρ_0 is the initial charge density at $t=0$. Equation (5.53) shows that there is a decay of charge density at each point with a time constant of $T_r = \frac{\varepsilon}{\sigma}$. This time constant is known as relaxation time. The relaxation time constant is defined as the time required for decaying ρ to $\frac{1}{e} = 36.78\%$ of its initial value.

Example 5.4 The initial volume charge density and the time constant of a system are found to be 150 C/m^3 and $T_r = 0.2$s respectively. Determine the value of the volume charge density at $t=0.1$s.

Solution The value of the volume charge density can be determined as

$$\rho = \rho_0 e^{-\frac{t}{T_r}} = 150e^{-\frac{0.1}{0.2}} = 90.98\,\text{C/m}^3.$$

Practice problem 5.4 The initial and final volume charge densities are found to be 120 C/m^3 and 100 C/m^3, respectively. Calculate the time constant if $t=0.2$ s.

5.6 Current Density Boundary Conditions

The current density vector changes both in magnitude and direction, when the current enters from one region to another region as shown in Fig. 5.5. According to the electric field and electric flux density, some boundary conditions can be derived for

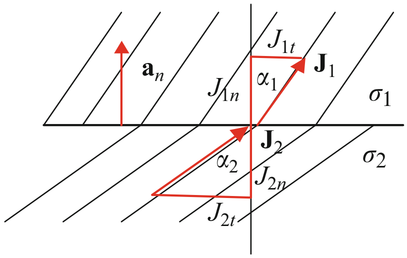

Fig. 5.5 Refraction of steady current lines at two conductors interface

the current density. From Eq. (5.46), the normal components of current vector can be written as

$$\mathbf{a}_n.\mathbf{J}_1 - \mathbf{a}_n.\mathbf{J}_2 = \frac{-d\rho}{dt}. \tag{5.54}$$

For dc (steady) current $\frac{d\rho}{dt} = 0$, then Eq. (5.54) becomes

$$\mathbf{a}_n.\mathbf{J}_1 - \mathbf{a}_n.\mathbf{J}_2 = 0, \tag{5.55}$$

$$J_{1n} = J_{2n}. \tag{5.56}$$

The differential form of the generalized Gauss' law is

$$\rho = \nabla.\mathbf{D}, \tag{5.57}$$

$$\rho = \nabla.\varepsilon\mathbf{E}. \tag{5.58}$$

Substituting Eq. (5.16) into Eq. (5.58) provides

$$\rho = \nabla.\frac{\varepsilon}{\sigma}\mathbf{J}. \tag{5.59}$$

The following rules from the vector identity can be written as:

$$\nabla.(x\mathbf{A}) = \mathbf{A}.\nabla x + x\nabla.\mathbf{A}. \tag{5.60}$$

According to Eq. (5.60), Eq. (5.59) can be expanded as

$$\rho = \mathbf{J}.\nabla\frac{\varepsilon}{\sigma} + \frac{\varepsilon}{\sigma}\nabla.\mathbf{J}. \tag{5.61}$$

Substituting Eq. (5.47) into (5.61) provides

$$\rho = \mathbf{J}.\nabla\frac{\varepsilon}{\sigma}. \tag{5.62}$$

From Eq. (5.62), the surface charge density in between mediums 1 and 2 can be written as

$$\rho_s = \mathbf{a}_n.\mathbf{D}_1 - \mathbf{a}_n.\mathbf{D}_2 = \frac{\varepsilon_1}{\sigma_1}\mathbf{a}_n.\mathbf{J}_1 - \frac{\varepsilon_2}{\sigma_2}\mathbf{a}_n.\mathbf{J}_2, \tag{5.63}$$

$$\rho_s = \left(\frac{\varepsilon_1}{\sigma_1} - \frac{\varepsilon_2}{\sigma_2}\right)\mathbf{J}_n. \tag{5.64}$$

The curl of the electric field is

$$\nabla \times \mathbf{E} = 0. \tag{5.65}$$

Substituting Eq. (5.16) into Eq. (5.65) yields

$$\nabla \times \frac{\mathbf{J}}{\sigma} = 0. \tag{5.66}$$

For the two mediums, Eq. (5.66) can be written as

$$\frac{J_{1t}}{\sigma_1} - \frac{J_{2t}}{\sigma_2} = 0, \tag{5.67}$$

$$\frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}. \tag{5.68}$$

From Eq. (5.68), it is concluded that the ratio of tangential components of the current density at the interface of two conductors is equal to the ratio of their conductivities.

From Fig. 5.5, the following equations can be written as:

$$\cos\alpha_1 = \frac{J_{1n}}{J_1}, \tag{5.69}$$

$$\cos\alpha_2 = \frac{J_{2n}}{J_2}, \tag{5.70}$$

$$\sin\alpha_1 = \frac{J_{1t}}{J_1}, \tag{5.71}$$

$$\sin\alpha_2 = \frac{J_{2t}}{J_2}. \tag{5.72}$$

Substituting Eqs. (5.69) and (5.70) into Eq. (5.56) yields

$$J_1 \cos\alpha_1 = J_2 \cos\alpha_2. \tag{5.73}$$

Substituting Eqs. (5.71) and (5.72) into Eq. (5.68) provides

$$\frac{J_1 \sin\alpha_1}{J_2 \sin\alpha_2} = \frac{\sigma_1}{\sigma_2}, \tag{5.74}$$

$$\sigma_2 J_1 \sin\alpha_1 = \sigma_1 J_2 \sin\alpha_2. \tag{5.75}$$

Dividing Eq. (5.75) by Eq. (5.73) yields

$$\sigma_2 \tan\alpha_1 = \sigma_1 \tan\alpha_2, \tag{5.76}$$

$$\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{\sigma_1}{\sigma_2}. \tag{5.77}$$

The magnitude of current density at the second conductor is

$$J_2 = \sqrt{{J_{2t}}^2 + {J_{2n}}^2}. \tag{5.78}$$

Substituting Eqs. (5.71) and (5.72) into Eq. (5.78) yields

$$J_2 = \sqrt{(J_2 \sin\alpha_2)^2 + (J_2 \cos\alpha_2)^2}. \tag{5.79}$$

Substituting Eqs. (5.73) and (5.75) into Eq. (5.79) provides

$$J_2 = \sqrt{\left(\frac{\sigma_2}{\sigma_1} J_1 \sin\alpha_1\right)^2 + (J_1 \cos\alpha_1)^2}, \tag{5.80}$$

$$J_2 = J_1\sqrt{\left(\frac{\sigma_2}{\sigma_1} \sin\alpha_1\right)^2 + (\cos\alpha_1)^2}. \tag{5.81}$$

In the presence of the steady current, the tangential electric fields are continuous across the boundary, in between two lossy dielectrics. In this case, it can be written as

$$E_{1t} = E_{2t}. \tag{5.82}$$

Equation (5.56) can be re-arranged as

$$\sigma_1 E_{1n} = \sigma_2 E_{2n}, \tag{5.83}$$

$$E_{1n} = \frac{\sigma_2}{\sigma_1} E_{2n}. \tag{5.84}$$

The surface charge density across two mediums is

$$\rho_s = D_{1n} - D_{2n} = \varepsilon_1 E_{1n} - \varepsilon_2 E_{2n}. \tag{5.85}$$

A surface charge density exists at the interface if the term $\left(\frac{\varepsilon_1}{\sigma_1} - \frac{\varepsilon_2}{\sigma_2}\right)$ of Eq. (5.64) is equal to zero. Then this relation can be expressed as

$$\frac{\varepsilon_1}{\sigma_1} - \frac{\varepsilon_2}{\sigma_2} = 0, \tag{5.86}$$

$$\frac{\sigma_2}{\sigma_1} = \frac{\varepsilon_2}{\varepsilon_1}. \tag{5.87}$$

Substituting Eq. (5.87) into Eq. (5.84) yields

$$E_{1n} = \frac{\varepsilon_2}{\varepsilon_1} E_{2n}. \tag{5.88}$$

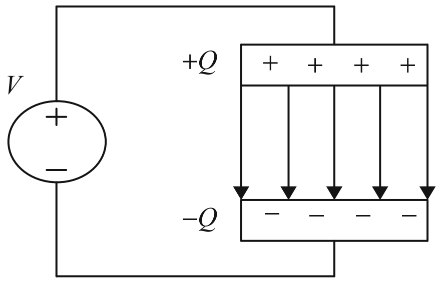

Fig. 5.6 Parallel plates capacitor

Substituting Eq. (5.88) into Eq. (5.85) yields

$$\rho_s = \left(\varepsilon_1 \frac{\sigma_2}{\sigma_1} - \varepsilon_2\right) E_{2n} = \left(\varepsilon_1 - \frac{\sigma_1}{\sigma_2}\varepsilon_2\right) E_{1n}. \tag{5.89}$$

If the medium 2 is holding more conductor than the medium 1, then $\sigma_2 > \sigma_1$. So, in this case $\frac{\sigma_1}{\sigma_2} = 0$. Therefore, Eq. (5.89) becomes

$$\rho_s = \varepsilon_1 E_{1n} = D_{1n}. \tag{5.90}$$

From Eq. (5.90), it is observed that the normal electric flux density of the medium 1 is equal to the surface charge density.

5.7 Capacitance

A capacitor is formed when two conducting plates are placed very close to each other. It is represented by the letter C and its unit is Farad (F). The ability of a capacitor to store charge on its plates is known as capacitance. Two perfectly conducting plates are connected in parallel as shown in Fig. 5.6. A voltage source is connected in between the plates, making the upper plate higher in potential than the lower plate. As a result, an electric field is set up in the downward direction and the charge is transferred between the plates which establish positive and negative charges on the upper and lower plates, respectively. A capacitor has a capacitance, even when no voltage is applied to it or when it is free of charges of the conductor.

Experimentally, it is found that the capacitor is directly proportional to the electric potential across it. Mathematically, it can be expressed as

$$Q \infty V, \tag{5.91}$$

$$Q = CV, \tag{5.92}$$

$$C = \frac{Q}{V} \tag{5.93}$$

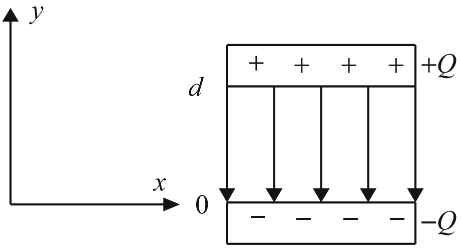

Fig. 5.7 Parallel plates capacitor with a separation distance

where

C is the capacitance in F,
Q is the charge in C,
V is the voltage across the capacitor in V.

5.8 Parallel Plate Capacitor

Consider a parallel plate capacitor as shown in Fig. 5.7. The separation distance between the plates is d and the area of each plate is S. The space in between the plates is filled by a dielectric whose permittivity is ε. The positive and negative charges are placed on the upper and lower plates, respectively. The expression of surface charge density can be written as

$$\rho_s = \frac{Q}{S}. \tag{5.94}$$

According to Eq. (5.85), when $\mathbf{D}_2 = 0$, the following equation can be written as:

$$\mathbf{E} = -\frac{\rho_s}{\varepsilon}\mathbf{a}_y. \tag{5.95}$$

The expression of the voltage can be determined as

$$V = -\int_0^d \mathbf{E}.d\mathbf{l}. \tag{5.96}$$

Substituting Eq. (5.95) and the expression of differential length in Cartesian coordinates into Eq. (5.96) provides

$$V = -\int_0^d \left(-\frac{\rho_s}{\varepsilon}\mathbf{a}_y\right).(dy\mathbf{a}_y), \tag{5.97}$$

$$V = \int_0^d \frac{\rho_s}{\varepsilon}dy, \tag{5.98}$$

$$V = \frac{\rho_s}{\varepsilon}d. \tag{5.99}$$

Substituting Eq. (5.94) into Eq. (5.99) yields

$$V = \frac{Q}{S\varepsilon} d. \tag{5.100}$$

From Eqs. (5.93) and (5.100), the final expression for the two parallel plate capacitors can be derived as

$$C = \frac{Q}{\frac{Q}{A\varepsilon} d}, \tag{5.101}$$

$$C = \frac{\varepsilon A}{d}. \tag{5.102}$$

Again, consider a total charge of Q is transferred from the positive to the negative plates. Then the potential difference between the plates can be written as

$$V = \frac{dW}{dq}. \tag{5.103}$$

Substituting Eq. (5.93) into Eq. (5.103) yields

$$\frac{q}{C} = \frac{dW}{dq}, \tag{5.104}$$

$$dW = \frac{q}{C} dq. \tag{5.105}$$

Integrating Eq. (5.105) from 0 to Q yields

$$W = \int_0^Q \frac{q}{C}\, dq = \frac{1}{2} \frac{Q^2}{C}. \tag{5.106}$$

Again, substituting Eq. (5.93) into Eq. (5.106) yields

$$W = \frac{Q^2}{2C} = \frac{CV^2}{2} = \frac{QV}{2}. \tag{5.107}$$

The electric field in between the plates is uniform, so the potential is

$$V = Ed. \tag{5.108}$$

Substituting Eqs. (5.102) and (5.108) into Eq. (5.107) yields

$$W = \frac{\varepsilon A}{d} \frac{E^2 d^2}{2}, \tag{5.109}$$

$$W = \frac{\varepsilon E^2 A d}{2}. \tag{5.110}$$

In Eq. (5.109), the product Ad is the volume of the field in between the plates. Then, the energy density or energy per unit volume w is

$$w = \frac{1}{2} \varepsilon E^2. \tag{5.111}$$

From Eq. (5.111), it is seen that the energy density is equal to the half of the permittivity and square of the electric field.

Example 5.5 The dielectric thickness between the plates of a capacitor is 4 mm and its relative permittivity is 5. The area of the plate is 25 m^2. The electric field strength of dielectric medium is 300 V/mm Determine the (a) capacitance and (b) the total charge on each plate.

Solution

(a) The value of the capacitance is

$$C = \frac{\varepsilon_0 \varepsilon_r S}{d} = \frac{8.854 \times 10^{-12} \times 5 \times 25}{4 \times 10^{-3}} = 276.69 \times 10^{-9}\text{F},$$

$$C = 0.276\ \mu\text{F}.$$

(b) The voltage across the plates is

$$V = E \times d = 300 \times 4 = 1200\,\text{V}.$$

The value of the charge is

$$Q = CV = 0.664 \times 10^{-6} \times 800 = 531.2\,\mu\text{C}.$$

Practice problem 5.5 The parallel plates of a capacitor are separated by a 3 mm solid dielectric medium and the area of the plate is 3 m^2. The relative permittivity of the dielectric is 5. The dielectric medium has an electric field strength of 360 V/mm. Calculate the (a) capacitance and (b) the total charge on each plate.

5.9 Determination of Resistance

The concept and definition of resistance for a uniform cross-section has already been discussed in Sect. 5.3. For a nonuniform cross-section, the resistance of the conductor can be obtained from the following equation:

$$R = \frac{V}{I} = \frac{-\int_L \mathbf{E}.d\mathbf{l}}{\oint_S \mathbf{J}.d\mathbf{S}} = \frac{-\int_L \mathbf{E}.d\mathbf{l}}{\oint_S \sigma \mathbf{E}.d\mathbf{S}}. \tag{5.112}$$

In terms of electric field, the expression of the capacitance from Eq. (5.93) can be expressed as

$$C = \frac{Q}{V} = \frac{\oint_S \mathbf{D}.d\mathbf{S}}{-\int_L \mathbf{E}.d\mathbf{l}}, \tag{5.113}$$

$$C = \frac{\oint_S \varepsilon \mathbf{E}.d\mathbf{S}}{-\int_L \mathbf{E}.d\mathbf{l}}. \tag{5.114}$$

Multiplying Eqs. (5.112) and (5.114) yields

$$RC = \frac{-\int_L \mathbf{E}.d\mathbf{l}}{\oint_S \sigma \mathbf{E}.d\mathbf{S}} \times \frac{\oint_S \varepsilon \mathbf{E}.d\mathbf{S}}{-\int_L \mathbf{E}.d\mathbf{l}}, \tag{5.115}$$

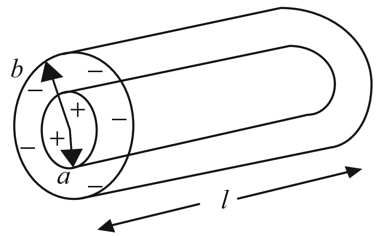

Fig. 5.8 Schematic of coaxial cable

$$RC = \frac{\varepsilon}{\sigma}. \tag{5.116}$$

Equation (5.116) exists if the medium is homogeneous. The resistance can be determined directly from Eq. (5.116) if the capacitance between two conductors is known.

5.10 Coaxial Capacitor

Consider a coaxial cable with a length of l as shown in Fig. 5.8. The inner and outer radii of the cable are a and b. The respective charges are $+Q$ and $-Q$. The space between the conductors is filled with a homogeneous dielectric whose permittivity is ε. Then the total charge can be determined as

$$Q = \int \mathbf{D}.d\mathbf{S}, \tag{5.117}$$

$$Q = \varepsilon \int \mathbf{E}.d\mathbf{S}. \tag{5.118}$$

The electric field intensity is normal to the cylindrical Gaussian surface and it can be represented as

$$\mathbf{E} = E_\rho \mathbf{a}_\rho. \tag{5.119}$$

The term $\int dS$ is the surface area of the Gaussian surface and it can be represented as

$$\int d\mathbf{S} = 2\pi\rho l. \tag{5.120}$$

Here, l is the arbitrary length of the cylinder. Substituting Eqs. (5.119) and (5.120) into Eq. (5.118) yields

$$Q = \varepsilon E_\rho 2\pi\rho l, \tag{5.121}$$

$$E_\rho = \frac{Q}{2\varepsilon\pi\rho l}. \tag{5.122}$$

Again, substituting Eq. (5.122) into Eq. (5.119) yields

$$\mathbf{E} = \frac{Q}{2\varepsilon\pi\rho l}\mathbf{a}_\rho. \tag{5.123}$$

The voltage difference between the two conductors can be determined as

$$V = -\int_b^a \mathbf{E}.d\mathbf{l}. \tag{5.124}$$

Substituting Eq. (5.123) and differential length in cylindrical coordinates into Eq. (5.124) yields

$$V = -\int_b^a \frac{Q}{2\varepsilon\pi\rho l}\mathbf{a}_\rho.(d\rho\mathbf{a}_\rho), \tag{5.125}$$

$$V = -\frac{Q}{2\varepsilon\pi l}[\ln\rho]_b^a. \tag{5.126}$$

$$V = -\frac{Q}{2\varepsilon\pi l}\ln\frac{a}{b}, \tag{5.127}$$

$$V = \frac{Q}{2\varepsilon\pi l}\ln\frac{b}{a}. \tag{5.128}$$

Substituting Eq. (5.128) into Eq. (5.93) yields

$$C = \frac{2\varepsilon\pi l}{\ln\frac{b}{a}}. \tag{5.129}$$

Equation (5.129) represents the expression of capacitance for coaxial cable.

5.11 Spherical Capacitor

Consider two spheres whose radii are a and b. The radius a is greater than the radius b. The inner and outer spheres contain the $+Q$ and $-Q$ charges, respectively. Apply Gauss' law to an arbitrary Gaussian spherical surface whose radius is r and $a < r < b$. Then the total charge can be determined as

$$Q = \varepsilon\int \mathbf{E}.d\mathbf{S}. \tag{5.130}$$

The electric field intensity is normal to the spherical Gaussian surface and it can be represented as

$$\mathbf{E} = E_r\mathbf{a}_r. \tag{5.131}$$

The term $\int dS$ is the surface area of the Gaussian surface and it can be represented as

$$\int dS = 4\pi r^2. \tag{5.132}$$

Substituting Eqs. (5.131) and (5.132) into Eq. (5.130) yields

$$Q = \varepsilon E_r 4\pi r^2, \tag{5.133}$$

$$E_r = \frac{Q}{4\varepsilon\pi r^2}. \tag{5.134}$$

Again, substituting Eq. (5.134) into Eq. (5.131) yields

$$\mathbf{E} = \frac{Q}{4\varepsilon\pi r^2}\mathbf{a}_r. \tag{5.135}$$

Substituting Eq. (5.135) and differential length in cylindrical coordinates into Eq. (5.124) yields

$$V = -\int_b^a \frac{Q}{4\varepsilon\pi r^2}\mathbf{a}_r.(dr\mathbf{a}_r), \tag{5.136}$$

$$V = -\frac{Q}{2\varepsilon\pi l}[\ln\rho]_b^a, \tag{5.137}$$

$$V = -\frac{Q}{2\varepsilon\pi l}\ln\frac{a}{b}, \tag{5.138}$$

$$V = \frac{Q}{2\varepsilon\pi l}\ln\frac{b}{a}. \tag{5.139}$$

Substituting Eq. (5.128) into Eq. (5.93) yields

$$C = \frac{2\varepsilon\pi l}{\ln\frac{b}{a}}. \tag{5.140}$$

Equation (5.140) represents the expression of capacitance for two spheres.

Example 5.6 The inner and outer radii of a 2 km cable are 3 mm and 5 mm, respectively. Determine the value of the capacitance.

Solution The value of the capacitance can be determined as

$$C = \frac{2\pi\varepsilon l}{\ln\frac{b}{a}} = \frac{2\pi \times 8.854 \times 10^{-12} \times 2 \times 10^3}{\ln\frac{5}{3}} = 0.218\,\mu\text{F}.$$

Practice problem 5.6 The outer radius of a 5 km cable is 5 mm. Calculate the value of the inner radius of the cable if the value of the capacitance is 0.3 μF.

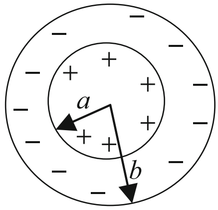

Fig. 5.9 Schematic of spheres

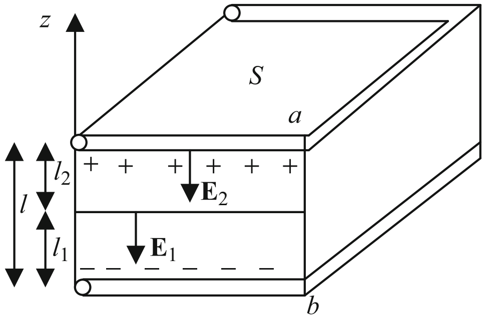

Fig. 5.10 Capacitor with two dielectric slabs

5.12 Parallel Plate Capacitor with Two Dielectric Slabs

Consider conductors a and b are separated by two dielectric slabs as shown in Fig. 5.10. The lengths of the dielectric slabs are l_1 and l_2, respectively. The electric fields for two dielectric slabs in the z-direction can be written as

$$\mathbf{E}_1 = -\frac{\rho_s}{\varepsilon_1}\mathbf{a}_z, \tag{5.141}$$

$$\mathbf{E}_2 = -\frac{\rho_s}{\varepsilon_2}\mathbf{a}_z. \tag{5.142}$$

The voltage difference between two conductors can be determined as

$$V = -\int \mathbf{E}.d\mathbf{l} = -\left[\int_0^{l_1} \mathbf{E}_1.d\mathbf{l} + \int_{l_1}^{l_1+l_1} \mathbf{E}_2.d\mathbf{l}\right]. \tag{5.143}$$

Substituting Eqs. (5.141, 5.142) and the differential length in z-component into Eq. (5.143) yields

$$V = \int_0^{l_1} \frac{\rho_s}{\varepsilon_1}\mathbf{a}_z.dz\mathbf{a}_z + \int_{l_1}^{l_1+l_1} \frac{\rho_s}{\varepsilon_2}\mathbf{a}_z.dz\mathbf{a}_z. \tag{5.144}$$

$$V = \int_0^{l_1} \frac{\rho_s}{\varepsilon_1}dz + \int_{l_1}^{l_1+l_1} \frac{\rho_s}{\varepsilon_2}dz, \tag{5.145}$$

$$V = \frac{\rho_s}{\varepsilon_1}l_1 + \frac{\rho_s}{\varepsilon_2}(l_1 + l_2 - l_1), \tag{5.146}$$

$$V = \frac{\rho_s}{\varepsilon_1} l_1 + \frac{\rho_s}{\varepsilon_2} l_2. \tag{5.147}$$

Substituting Eq. (5.147) into Eq. (5.93) yields

$$C = \frac{\rho_s S}{\rho_s \left[\frac{l_1}{\varepsilon_1} + \frac{l_2}{\varepsilon_2} \right]}, \tag{5.148}$$

$$C = \frac{1}{\frac{l_1}{\varepsilon_1 S} + \frac{l_2}{\varepsilon_2 S}}, \tag{5.149}$$

$$C = \frac{1}{\frac{l_1}{\varepsilon_1 S} + \frac{l_2}{\varepsilon_2 S}}, \tag{5.150}$$

$$C = \frac{1}{\frac{1}{\frac{\varepsilon_1 S}{l_1}} + \frac{1}{\frac{\varepsilon_2 S}{l_2}}}. \tag{5.151}$$

According to the expression of capacitance, the terms $\frac{\varepsilon_1 S}{l_1}$ and $\frac{\varepsilon_2 S}{l_2}$ can be defined as the capacitance of the lower and upper dielectric slabs. Then, Eq. (5.151) can be modified as

$$C = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}, \tag{5.152}$$

$$C = \frac{C_1 C_2}{C_1 + C_2}. \tag{5.153}$$

Equation (5.153) represents the equivalent capacitance when two capacitors are in the series. The expression of capacitance for the two capacitors in parallel is

$$C = C_1 + C_2. \tag{5.154}$$

The derivation of Eq. (5.154) can be found in any circuit analysis book.

Example 5.7 The length of the first and the second dielectric slabs are 3 mm and 5 mm, respectively. The permittivity of the dielectric slabs are $\varepsilon_1 = 2$ and $\varepsilon_2 = 4$. A voltage of 100 V is applied across the parallel plates and the area of the plate is 0.56×10^{-6} m^2 Determine the (a) value of the capacitance, (b) Q, (c) D, (d) E_1, E_2, and (e) V_1, V_2.

Solution

(a) The value of the capacitance can be determined as

$$C = \frac{S}{\frac{l_1}{\varepsilon_1} + \frac{l_2}{\varepsilon_2}} = \frac{8.854 \times 10^{-12} \times 0.56 \times 10^{-6}}{\frac{0.003}{2} + \frac{0.005}{4}} = 1803 \times 10^{-18}\text{F}.$$

(b) The value of the charge is

$$Q = CV = 0.0018 \times 100 = 0.18\,\text{pC}.$$

(c) The value of the electric flux density can be determined as

$$D = \frac{Q}{S} = \frac{0.18 \times 10^{-12}}{0.56 \times 10^{-6}} = 0.321 \times 10^{-6}\,\text{C/m}^2.$$

(d) The value of the electric field intensities are

$$E_1 = \frac{D}{\varepsilon_1} = \frac{0.321 \times 10^{-6}}{2 \times 8.854 \times 10^{-12}} = 18127.4\,\text{V/m},$$

$$E_2 = \frac{D}{\varepsilon_2} = \frac{0.321 \times 10^{-6}}{4 \times 8.854 \times 10^{-12}} = 9063.7\,\text{V/m}.$$

(e) The value of the voltages can be determined as

$$V_1 = E_1 l_1 = 18127.4 \times 003 = 54.38\,\text{V},$$

$$V_2 = E_2 l_2 = 9063.7 \times 0.005 = 45.32\,\text{V}.$$

Practice problem 5.7 The length of two dielectric slabs are $l_1 = 2$ mm and $l_2 = 3.5$ mm. The permittivity of the dielectric slabs are $\varepsilon_1 = 3$ and $\varepsilon_2 = 5$. The 80 Vsource is connected across the parallel plates and the area of the plate is 0.026×10^{-7} m^2. Calculate the (a) value of the capacitance (b) Q, (c) D, (d) E_1, E_2, and (e) V_1, V_2.

5.13 Exercise Problems

5.1 The current density in Cartesian coordinates is given as $\mathbf{J} = 3yz^2\mathbf{a}_x + 5xy\mathbf{a}_z$ A/m^2. Determine the (a) current density at point $(x = 1,\ y = 1.5,\ z = 2)$ and (b) the total current flowing through the Cartesian surface $0 < x < 2,\ 1 < y < 3$.

5.2 The expression of the current density in Cartesian coordinates is given as $\mathbf{J} = 2x^2yz\mathbf{a}_x + 5z\mathbf{a}_y + 3y\mathbf{a}_z$ A/m^2. Calculate the (a) current density at point $(x = 1,\ y = 2,\ z = 2)$ and (b) the total current flowing through the Cartesian surface $x = 3,\ 1 < y < 3,\ 2 < z < 5$.

5.3 In cylindrical coordinates, the current density is given as $\mathbf{J} = \rho^2 z\mathbf{a}_\rho - 2z\sin\phi\mathbf{a}_\phi$ A/m^2. Determine the (a) current density at point $(\rho = 1,\ \phi = 45°,\ z = 1.5)$ and (b) the total current flowing through the cylindrical surface $\rho = 2,\ 0 < \phi < 2\pi,\ 1.5 < z < 4$.

5.4 The current density in spherical coordinates is given by $\mathbf{J} = -3\cos\left(\frac{\phi}{4}\right)\mathbf{a}_r + \frac{4}{r}\sin\theta\mathbf{a}_\theta$ A/m^2. Find the (a) current density at point $(r = 1,\ \theta = 30°,\ \phi = 125°)$ and (b) the total current flowing through the spherical surface $0 < \theta < 40°,\ 0 < \phi < 2\pi$.

5.5 The radius of a 4.5 km-long wire is 3 mm. Calculate the dc resistance if the wire is made of copper. Consider that the conductivity of the copper wire is $\sigma = 5.8 \times 10^7$ S/m.

5.6 The radius of a 8.5 km-long aluminum wire is 2 mm. Determine the dc resistance if the conductivity of the aluminum wire is $\sigma = 3.54 \times 10^7$ S/m.

5.7 The initial volume charge density and the time constant of a system are found to be 100 C/m^3 and $T_r = 0.11$s, respectively. Find the value of the volume charge density at $t = 0.15$s.

5.8 The initial and final volume charge densities are found to be 90 C/m^3 and 65 C/m^3, respectively. Determine the time constant if $t = 0.11$s.

5.9 The dielectric thickness between the plates of a capacitor is 5 mm and its relative permittivity is 8. The electric field strength of dielectric medium is 5 V/mm and the area of the plate is 15 m^2. Calculate the (a) capacitance and (b) the total charge on each plate.

5.10 The inner and outer radii of a 3.5 km cable are 1.5 and 4.5 mm, respectively. Determine the value of the capacitance.

5.11 The inner radius of a 2.5 km cable is 2.5 mm. Calculate the value of the outer radius of the cable if the value of the capacitance is 1.15 μF.

5.12 The lengths of the two dielectric slabs are 2.5 mm and 6.5 mm. The permittivity of the dielectric slabs are $\varepsilon_1 = 3$ and $\varepsilon_2 = 6$. The 100 V is connected across the parallel plates and the area of the plate is 0.0124×10^{-6} m^2. Determine the (a) value of the capacitance (b) Q, (c) D, (d) E_1, E_2, and (e) V_1, V_2.

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Chapter 6
Static Magnetic Field

6.1 Introduction

Magnetic fields are very important in the field of electrical engineering. Magnetic fields basically originate from currents and exert forces on currents, whereas electric fields originate from charges and exert forces on charges. There are many applications of magnetic fields. These applications include motors, transformers, microphones, telephone bell ringers, memory stores, magnetic contactors, magnetically levitated (Maglev) high speed vehicles, etc. An electric field in a medium exists due to the presence of a static charge. For a linear and isotropic medium, the relationship between electric field and electric flux density is $\mathbf{D} = \varepsilon\mathbf{E}$, where ε is the permittivity. A magnetic field exists due to a charge moving at a constant velocity. For new installations of transmission lines, electric and magnetic fields need to be measured by the relevant equipment and the latest commercial software. The electric field is defined as the force per unit charge, and the electric force due to a test charge, q, placed in an electric field is

$$\mathbf{F}_e = q\mathbf{E}. \tag{6.1}$$

The magnetic force due to a test charge, q, placed in a magnetic field is

$$\mathbf{F}_m = q\mathbf{v} \times \mathbf{B}. \tag{6.2}$$

The total force due to electric and magnetic forces is

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m. \tag{6.3}$$

Substituting Eqs. (6.1) and (6.2) into Eq. (6.3) provides

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}, \tag{6.4}$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{6.5}$$

Equation (6.5) is known as the Lorentz's force equation, and it is the first postulate of the magnetic field. The Lorentz force is the force acting on a charge, q, moving with a velocity, v, in the presence of electric and magnetic fields. In this chapter, Lorentz's force, magnetic flux density, Biot–Savart's law, Ampere's circuital law, vector magnetic potential, etc. will be discussed.

Md. A. Salam, *Electromagnetic Field Theories for Engineering,*
DOI 10.1007/978-981-4585-66-8_6, © Springer Science+Business Media Singapore 2014

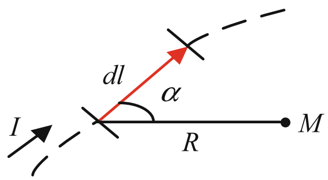


Fig. 6.1 Magnetic field due to a current

6.2 Magnetic Flux Density

The magnetic flux density is related to the magnetic field intensity in the free space. This relation can be expressed as

$$\mathbf{B} = \mu_0 \mathbf{H}, \tag{6.6}$$

where
  B is the magnetic flux density in Wb/m^2(T),
  H is the magnetic field intensity in (A/m) and
  $\mu_0 = 4\pi \times 10^{-7}$ is the permeability in free space in H/m.

The magnetic flux is defined as the amount of magnetic flux lines passing through a surface (area), **S**, near the magnet. Alternatively, it can be defined as the component of the magnetic field perpendicular to an area multiplied by the side of that area. It is represented by the symbol $\phi(\psi_m)$ and its unit is Wb. Consider Fig. 6.1, where the magnetic fields or flux lines are perpendicular to the area. The magnetic flux density is defined as the flux per unit area and it can be expressed as

$$B = \frac{\psi_m}{A}, \tag{6.7}$$

$$\psi_m = BA. \tag{6.8}$$

If B is not uniform over the specific area, then Eq. (6.8) can be modified as

$$\psi_m = \oint_s BA \cos\theta dS. \tag{6.9}$$

In dot product, Eq. (6.9) can be written as,

$$\psi_m = \oint_s \mathbf{B}.d\mathbf{S}. \tag{6.10}$$

In a magnet, the North and South poles are attached to each other. Therefore, there is no source where the magnetic flux lines starts or finish. Over a closed surface, Eq. (6.10) can be modified as

$$\oint_s \mathbf{B}.d\mathbf{S} = 0. \tag{6.11}$$

Applying divergence theorem to Eq. (6.11) yields

$$\oint_v \nabla.\mathbf{B}dv = \oint_s B.d\mathbf{S} = 0. \tag{6.12}$$

In general, the following equation can be written as

$$\nabla.\mathbf{B} = 0. \tag{6.13}$$

Equation (6.13) is the fourth Maxwell's equation for the static field. The curl of the magnetic flux density can be written as

$$\nabla \times \mathbf{B} = \mu_0\mathbf{J}. \tag{6.14}$$

Integrating both sides of Eq. (6.14) for an open surface and applying Stokes theorem provides

$$\int_s (\nabla \times \mathbf{B}).d\mathbf{S} = \mu_0 \int_s \mathbf{J}.d\mathbf{S}, \tag{6.15}$$

$$\oint_c \mathbf{B}.d\mathbf{l} = \mu_0 I_{en}, \tag{6.16}$$

where

c is the contour surrounding the surface S and

I_{en} is the total current in the closed surface.

6.3 Biot–Savart's Law

Two French scientists, Jean Baptiste Biot and Felix Savart, worked together to develop the relationship between magnetic fields and currents. According to their names, this law is known as the Biot–Savart law. The differential magnetic field at any point is proportional to the product of the current, differential length and the sine of the angle between the element and the line joining to that point. It is inversely proportional to the square of the distance between the element and the point. Let us consider a current that is flowing through a differential length as shown in Fig. 6.1. According to the statement, the following relations can be written as:

$$dH \infty I, \tag{6.17}$$

$$dH \infty dl, \tag{6.18}$$

$$dH \infty \sin\alpha, \tag{6.19}$$

$$dH \infty \frac{1}{R^2}. \tag{6.20}$$

Combining Eqs. from (6.17) to (6.20) yields

$$dH \infty \frac{Idl \sin\alpha}{R^2}, \tag{6.21}$$

$$dH = k\frac{Idl \sin\alpha}{R^2}. \tag{6.22}$$

In SI units, the value of the proportionality constant is

$$k = \frac{1}{4\pi}. \tag{6.23}$$

Substituting Eq. (6.23) into Eq. (6.22) yields

$$dH = \frac{Idl \sin\alpha}{4\pi R^2}. \tag{6.24}$$

According to Eq. (6.24), the differential magnetic flux density can be written as

$$dB = \mu\frac{Idl \sin\alpha}{4\pi R^2}. \tag{6.25}$$

The total magnetic flux density at the point M is

$$B = \frac{\mu I}{4\pi}\int \frac{\sin\alpha}{R^2} dl. \tag{6.26}$$

According to the rules of cross product, Eq. (6.24) can be expressed as

$$d\mathbf{H} = \frac{I d\mathbf{l} \times \mathbf{a}_r}{4\pi R^2}, \tag{6.27}$$

where
- B is the magnetic flux density in T,
- H is the magnetic field intensity in A/m,
- μ is the permeability of the medium in H/m,
- I is the current in the conductor in A,
- R is the distance from element to point M in m,
- α is the angle measured in the clockwise direction
- $R = |\mathbf{R}|$,
- $\mathbf{a}_r = \frac{\mathbf{R}}{|\mathbf{R}|}$.

Equations from (6.24) to (6.27) are the statements of the Biot–Savart law.

6.4 Magnetic Field of a Long Straight Conductor

A long straight conductor is shown in Fig. 6.2. The magnetic field will be calculated at point P for the semi-infinite and the infinite length of the conductor. The vector can be written as

$$\mathbf{R} = \rho\mathbf{a}_\rho + z\mathbf{a}_z. \tag{6.28}$$

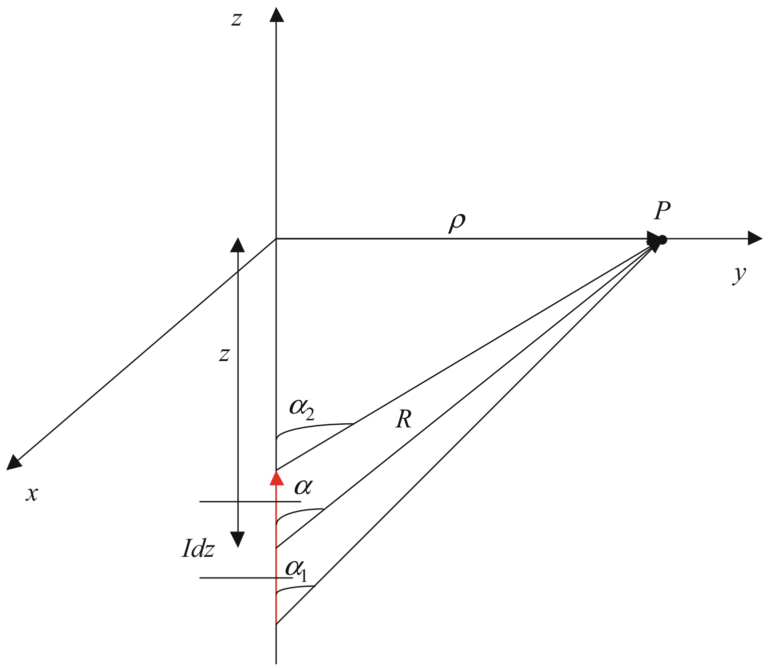


Fig. 6.2 A long straight conductor

The magnitude of the vector is

$$|\mathbf{R}| = \sqrt{\rho^2 + z^2}. \tag{6.29}$$

From Fig. 6.2, the following equation can be written as

$$\tan\alpha = \frac{\rho}{z}, \tag{6.30}$$

$$z = \rho\cot\alpha, \tag{6.31}$$

$$dz = -\rho\cos ec^2\alpha d\alpha. \tag{6.32}$$

The differential length in the z-direction is

$$d\mathbf{l} = dz\mathbf{a}_z. \tag{6.33}$$

Then Eq. (6.27) can be modified as

$$d\mathbf{H} = \frac{Idz\mathbf{a}_z \times (\rho\mathbf{a}_\rho + z\mathbf{a}_z)}{4\pi(\rho^2 + z^2)^{\frac{3}{2}}}, \tag{6.34}$$

$$d\mathbf{H} = \frac{I\rho dz}{4\pi(\rho^2 + z^2)^{\frac{3}{2}}}\mathbf{a}_\phi, \tag{6.35}$$

$$\mathbf{H} = \int \frac{Irdz}{4\pi(\rho^2 + z^2)^{\frac{3}{2}}} \mathbf{a}_\phi. \tag{6.36}$$

The previous equation can be written as:

$$(\rho^2 + z^2)^{\frac{3}{2}} = (\rho^2 + \rho^2 \cot^2\alpha)^{\frac{3}{2}} = \rho^3 \cos ec^3 \alpha. \tag{6.37}$$

Substituting Eqs. (6.32) and (6.37) into Eq. (6.36) yields

$$\mathbf{H} = \frac{I}{4\pi} \int_{\alpha_1}^{\alpha_2} \frac{\rho(-\rho \cos ec^2\alpha) d\alpha}{\rho^3 \cos ec^3 \alpha} \mathbf{a}_\phi, \tag{6.38}$$

$$\mathbf{H} = \frac{-I}{4\pi\rho} \mathbf{a}_\phi \int_{\alpha_1}^{\alpha_2} \sin\alpha d\alpha, \tag{6.39}$$

$$\mathbf{H} = \frac{-I}{4\pi\rho} \mathbf{a}_\phi [-\cos\alpha]_{\alpha_1}^{\alpha_2} = \frac{I}{4\pi\rho} \mathbf{a}_\phi (\cos\alpha_2 - \cos\alpha_1). \tag{6.40}$$

For the semi-infinite length of the conductor, the following conditions are considered:

$$\alpha_1 = 90^\circ \quad \text{and} \quad \alpha_2 = 0. \tag{6.41}$$

Substituting Eq. (6.41) into Eq. (6.40) yields

$$\mathbf{H} = \frac{I}{4\pi\rho} \mathbf{a}_\phi. \tag{6.42}$$

For the infinite length of the conductor, the following conditions are considered:

$$\alpha_1 = 180^\circ \quad \text{and} \quad \alpha_2 = 0. \tag{6.43}$$

Substituting Eq. (6.43) into Eq. (6.40) yields

$$\mathbf{H} = \frac{I}{4\pi\rho} \mathbf{a}_\phi (1 + 1), \tag{6.44}$$

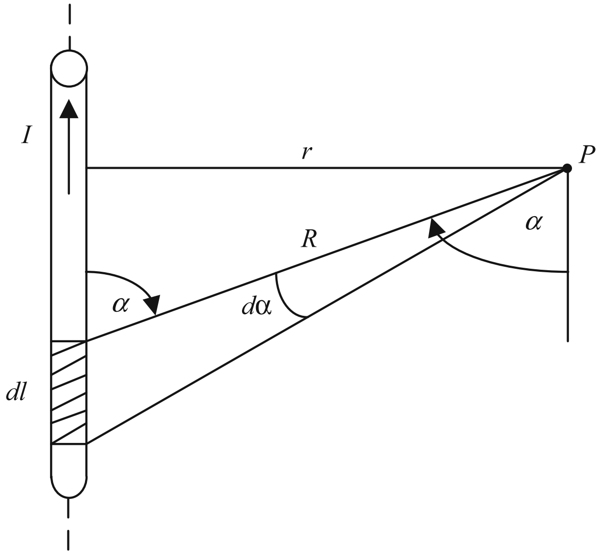
$$\mathbf{H} = \frac{I}{2\pi\rho} \mathbf{a}_\phi. \tag{6.45}$$

From Eq. (6.45), it is concluded that the magnetic field intensity is directly related to the current and inversely related to ρ.

Alternative method: An infinite conductor carries a current, I, as shown in Fig. 6.3. Here, P is the point at a distance, r, from the conductor where the magnetic flux density is to be calculated. The dl is the differential length of the conductor and α is the angle between the conductor and the line, R. From Fig. 6.3, the following equations can be written as:

$$\sin\alpha = \frac{r}{R}, \tag{6.46}$$

Fig. 6.3 Infinite straight conductor



$$\frac{\sin\alpha}{r} = \frac{1}{R}, \tag{6.47}$$

$$dl \sin\alpha = R d\alpha. \tag{6.48}$$

Substituting Eqs. (6.48) and (6.47) into Eq. (6.24) yields

$$B = \frac{\mu I}{4\pi r}\int \sin\alpha d\alpha. \tag{6.49}$$

The integration limits are set from $\alpha = 0$ to $\alpha = \pi$ to calculate the magnetic flux density over the whole length of the conductor. Then Eq. (6.49) becomes

$$B = \frac{\mu I}{4\pi r}\int_0^{\pi} \sin\alpha d\alpha, \tag{6.50}$$

$$B = \frac{\mu I}{4\pi r}[-\cos\alpha]_0^{\pi} = -\frac{\mu I}{4\pi r}(\cos\pi - 1), \tag{6.51}$$

$$B = \frac{\mu I}{2\pi r}. \tag{6.52}$$

From Eq. (6.52), it is observed that the magnetic flux density is directly proportional to the current and inversely proportional to the radial distance.

6.5 Ampere's Circuital Law

In electrostatic field, Gauss's law has been used to simplify the symmetrical charge distribution cases. A special theory is required to simplify the symmetrical current distribution cases in the magnetostatic field. This theory is known as Ampere's circuital law. This law states that the line integral of the static magnetic field intensity around a closed path is equal to the total current enclosed by that path. Integrating Eq. (6.52) for a differential length of dl yields

$$\oint \mathbf{B}.d\mathbf{l} = \frac{\mu I}{2\pi r} \oint dl. \tag{6.53}$$

Substituting the circumferential length $\oint dl = 2\pi r$ into Eq. (6.53) yields

$$\oint \mathbf{B}.d\mathbf{l} = \frac{\mu I}{2\pi r} 2\pi r, \tag{6.54}$$

$$\oint \mathbf{B}.d\mathbf{l} = \mu I. \tag{6.55}$$

Again, substituting $\mathbf{B} = \mu\mathbf{H}$ into Eq. (6.55) provides

$$\oint \mu\mathbf{H}.d\mathbf{l} = \mu I, \tag{6.56}$$

$$\oint \mathbf{H}.d\mathbf{l} = I. \tag{6.57}$$

For a closed loop, introducing the suffix l and Eq. (6.57) becomes

$$\oint_l \mathbf{H}.d\mathbf{l} = I. \tag{6.58}$$

Applying Stokes theorem to the left hand side of Eq. (6.58) provides

$$\int_s (\nabla \times \mathbf{H}).ds = \oint_l \mathbf{H}.d\mathbf{l}, \tag{6.59}$$

$$\int_s \mathbf{J}.d\mathbf{s} = \oint_l \mathbf{H}.d\mathbf{l}, \tag{6.60}$$

where the expression of the current density is

$$\mathbf{J} = \nabla \times \mathbf{H}. \tag{6.61}$$

Equation (6.61) is known as the third Maxwell's equation. It is also known as the differential or point form of Ampere's law.

Substituting Eq. (6.60) into Eq. (6.57) yields

$$I = \int_s \mathbf{J}.d\mathbf{s} = \oint_l \mathbf{H}.d\mathbf{l}. \tag{6.62}$$

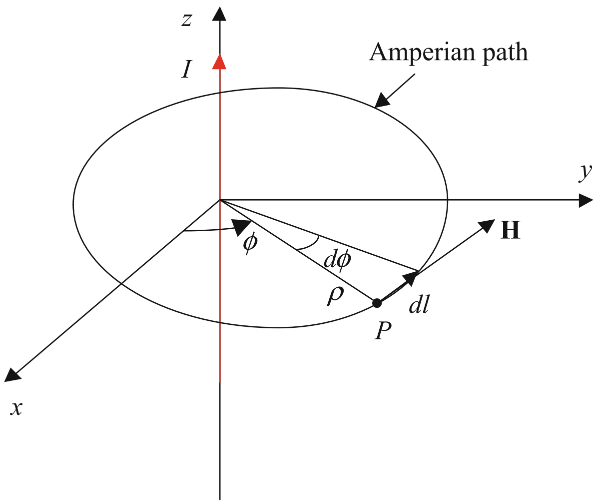


Fig. 6.4 Long straight conductor with Amperian path

Equation (6.62) is known as the generalized integral form of Ampere's law. The magnetic field intensity is either tangential ($\mathbf{H}.d\mathbf{l} = H_t dl$) or normal ($\mathbf{H}.d\mathbf{l} = H_n dl = 0$) at each point of the closed path. The total magnetic field intensity is the sum of the tangential and normal components and it can be written as

$$\mathbf{H} = H_t + H_n. \tag{6.63}$$

6.6 Ampere's Circuital Law in a Long Straight Conductor

A very long straight conductor carries a current of I as shown in Fig. 6.4. The point P is at a distance ρ from the conductor and the magnetic field intensity will be determined at this point. The path that is passing through point P is known as Amperian path. The following relations can be written as:

$$\mathbf{H} = H_\phi \mathbf{a}_\phi. \tag{6.64}$$

$$d\mathbf{l} = \cancel{d\rho \mathbf{a}_\rho} + \rho d\phi \mathbf{a}_\phi + \cancel{dz \mathbf{a}_z}. \tag{6.65}$$

Substituting Eqs. (6.64) and (6.65) into Eq. (6.62) yields

$$I = \int_{\phi=0}^{2\pi} H_\phi \mathbf{a}_\phi . \rho d\phi \mathbf{a}_\phi, \tag{6.66}$$

$$I = \rho H_\phi \int_{\phi=0}^{2\pi} d\phi = \rho H_\phi 2\pi, \tag{6.67}$$

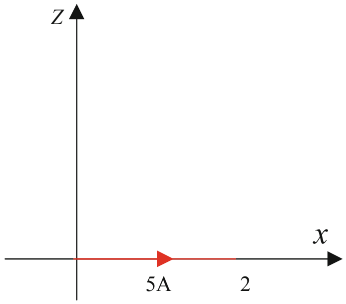


Fig. 6.5 Conductor in horizontal position

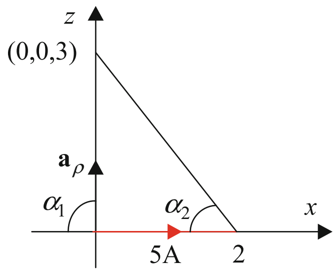


Fig. 6.6 Conductor with unit vectors

$$H_\phi = \frac{I}{2\pi\rho}. \tag{6.68}$$

Substituting Eq. (6.68) into Eq. (6.64) yields

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi. \tag{6.69}$$

Equation (6.69) is the same as Eq. (6.45) for an infinite length of the conductor.

The unit vector $\mathbf{a}_\phi$ can be determined by the following equation:

$$\mathbf{a}_\phi = \mathbf{a}_l \times \mathbf{a}_\rho, \tag{6.70}$$

where

$\mathbf{a}_l$ is the unit vector in the direction of the line current and

$\mathbf{a}_\rho$ is the unit vector in the direction of the perpendicular line, that is, from the line current to the point where magnetic field is to be determined.

Example 6.1 A conductor is laid out horizontally carrying a current of 5 A as shown in Fig. 6.5. Determine the magnetic field intensity at point $(0, 0, 3)$.

Solution From Fig. 6.6, the following angles can be determined as

$$\alpha_1 = 90^\circ,\ \cos\alpha_1 = 0,\ \cos\alpha_2 = {}^2\!/\!\sqrt{13}.$$

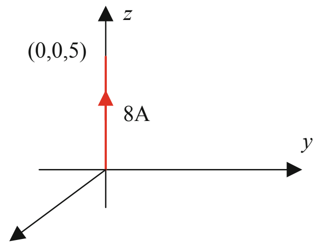


Fig. 6.7 Conductor in vertical position

The magnetic field intensity can be determined as

$$\mathbf{H} = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\mathbf{a}_\phi,$$

$$\mathbf{H} = \frac{5}{4\pi \times 3} \times \frac{2}{\sqrt{13}}\mathbf{a}_\phi.$$

From Fig. 6.6, the following relations can be written as

$$\mathbf{a}_\rho = \mathbf{a}_z,$$
$$\mathbf{a}_l = \mathbf{a}_x.$$

Then, the following unit vector can be determined as

$$\mathbf{a}_\phi = \mathbf{a}_x \times \mathbf{a}_z = -\mathbf{a}_y.$$

Finally, the value of the magnetic field is

$$\mathbf{H} = -\frac{5}{4\pi \times 3} \times \frac{2}{\sqrt{13}}\mathbf{a}_y = 73.56 \text{ mA/m}.$$

Practice Problem 6.1 A conductor is laid out in the z-axis carrying a current of 8 A as shown in Fig. 6.7. Determine the magnetic field intensity at point $(0, 2, 0)$.

6.7 Infinite Sheet of Current

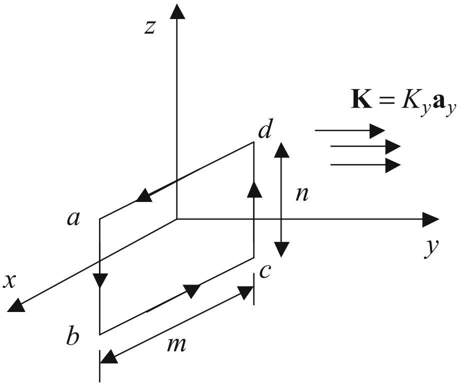
An infinite sheet of current is shown in Fig. 6.8. The current in the sheet is flowing in the positive y-direction and it is located in the $z = 0$ plane. Consider that the sheet has a uniform surface current density and it can be written as

$$\mathbf{K} = K_y\mathbf{a}_y. \tag{6.71}$$

Applying Ampere's law to the closed $a - b - c - d$ path will be given as

$$\oint \mathbf{H}.d\mathbf{l} = I = K_y m. \tag{6.72}$$

Fig. 6.8 An infinite sheet



The magnetic field intensity of the infinite sheet for different conditions of z can be written as

$$\mathbf{H} = H_0\mathbf{a}_x \quad \text{for } z > 0 \tag{6.73}$$

$$\mathbf{H} = -H_0\mathbf{a}_x \quad \text{for } z < 0 \tag{6.74}$$

Again, applying the Ampere's law in the closed path provides

$$\oint \mathbf{H}.d\mathbf{l} = \left[\int_a^b + \int_b^c + \int_c^d + \int_d^a\right]\mathbf{H}.d\mathbf{l}, \tag{6.75}$$

$$\oint \mathbf{H}.d\mathbf{l} = 0(-n) + (-H_0)(-m) + 0(a) + (H_0)(m) = 2H_0m. \tag{6.76}$$

Substituting Eq. (6.76) into Eq. (6.72) yields

$$K_ym = 2H_0m, \tag{6.77}$$

$$H_0 = \frac{1}{2}K_y. \tag{6.78}$$

Substituting Eq. (6.78) into Eqs. (6.73) and (6.74) yields

$$\mathbf{H} = \frac{1}{2}K_y\mathbf{a}_x \quad \text{for } z > 0, \tag{6.79}$$

$$\mathbf{H} = -\frac{1}{2}K_y\mathbf{a}_x \quad \text{for } z < 0. \tag{6.80}$$

In general, the relationship between the magnetic field intensity and infinity sheet current density is

$$\mathbf{H} = \frac{1}{2}K \times \mathbf{a}_n, \tag{6.81}$$

where $\mathbf{a}_n$ is the unity vector which is normal to the sheet.

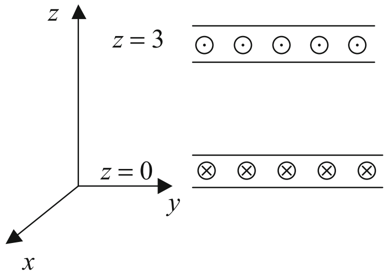


Fig. 6.9 An infinite sheet

Example 6.2 The current densities of the planes $z = 0$ and $z = 3$ are given as $\mathbf{K} = -5\mathbf{a}_x$ A/m and $\mathbf{K} = 5\mathbf{a}_x$ A/m, respectively. Determine the magnetic field at point $(1, 2, 1)$.

Solution The parallel infinite current sheets are shown in Fig. 6.9. The planes follow the condition $0 < z = 1 < 3$. The magnetic field intensity of the following angles can be determined as

$$\mathbf{H}_0 = \frac{1}{2}\mathbf{K} \times \mathbf{a}_n = \frac{1}{2}(-5\mathbf{a}_x) \times \mathbf{a}_z = 2.5\mathbf{a}_y.$$

The planes are within the condition $z = 3 > 1 > 0$. The magnetic field intensity of the following angles can be determined as

$$\mathbf{H}_3 = \frac{1}{2}\mathbf{K} \times \mathbf{a}_n = \frac{1}{2}(5\mathbf{a}_x) \times (-\mathbf{a}_z) = 2.5\mathbf{a}_y.$$

Total magnetic field intensity is

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_3 = 5\mathbf{a}_y \text{ A/m}.$$

Practice Problem 6.2 The current densities of the planes $z = 0$ and $z = 5$ are given as $\mathbf{K} = -20\mathbf{a}_x$ A/m and $\mathbf{K} = 20\mathbf{a}_x$ A/m, respectively. Determine the magnetic field at point $(1, 2, 4)$.

6.8 Curl of a Magnetic Field

A closed small rectangle whose incremental lengths Δz and Δy are shown in Fig. 6.10. The line integral of $\mathbf{H}$ around the path $1 - 2 - 3 - 4$ is equal to four times of $\mathbf{H}.\Delta\mathbf{l}$ on each side of the rectangle. For side $1 - 2$, the following relation is:

$$(\mathbf{H}.\Delta\mathbf{l})_{1-2} = (H_y)_{1-2}\Delta y. \tag{6.82}$$

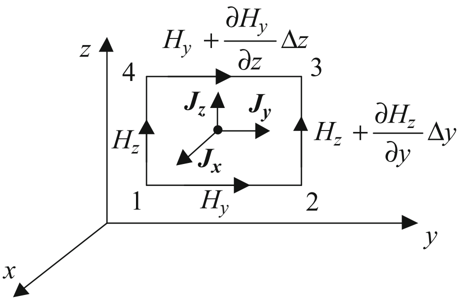


Fig. 6.10 A small rectangle with current densities

The magnetic field intensity for side 1–2 is

$$(H_y)_{1-2} = H_y. \tag{6.83}$$

Substituting Eq. (6.83) into Eq. (6.82) yields

$$(\mathbf{H}.\Delta\mathbf{l})_{1-2} = H_y\Delta y. \tag{6.84}$$

For side $2-3$, the following equation can be written as

$$(\mathbf{H}.\Delta\mathbf{l})_{2-3} = (H_z)_{2-3}(\Delta z). \tag{6.85}$$

The magnetic field intensity for side $2-3$ is

$$(H_z)_{2-3} = H_z + \frac{\partial H_z}{\partial y}\Delta y. \tag{6.86}$$

Substituting Eq. (6.86) into Eq. (6.85) yields

$$(\mathbf{H}.\Delta\mathbf{l})_{2-3} = H_z\Delta z + \frac{\partial H_z}{\partial y}\Delta z\Delta y. \tag{6.87}$$

For side $3-4$, the relation is

$$(\mathbf{H}.\Delta\mathbf{l})_{3-4} = (H_y)_{4-3}(-\Delta y). \tag{6.88}$$

The magnetic field intensity for side $4-3$ is

$$(H_y)_{4-3} = H_y + \frac{\partial Hy}{\partial z}\Delta z. \tag{6.89}$$

Substituting Eq. (6.89) into Eq. (6.88) provides

$$(\mathbf{H}.\Delta\mathbf{l})_{3-4} = -H_y\Delta y - \frac{\partial H_z}{\partial z}\Delta y\Delta z. \tag{6.90}$$

For side $4-1$, the following relation is:

$$(\mathbf{H}.\Delta\mathbf{l})_{4-1} = (H_z)_{1-4}(-\Delta z). \tag{6.91}$$

The magnetic field intensity for side 1 – 4 is

$$(H_z)_{1-4} = H_z. \tag{6.92}$$

Substituting Eq. (6.92) into Eq. (6.91) provides

$$(\mathbf{H}.\Delta\mathbf{l})_{4-1} = -H_z\Delta z. \tag{6.93}$$

Adding Eqs. (6.84), (6.87), (6.90) and (6.93) yields

$$\oint_{l(x)} \mathbf{H}.d\mathbf{l} = H_y\Delta y + H_z\Delta z + \frac{\partial H_z}{\partial y}\Delta y\Delta z - H_y\Delta y - \frac{\partial H_y}{\partial z}\Delta y\Delta z - H_z\Delta z, \tag{6.94}$$

$$\oint_{l(x)} \mathbf{H}.d\mathbf{l} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\Delta y\Delta z. \tag{6.95}$$

Dividing Eq. (6.95) by $\Delta y\Delta z$ provides

$$\frac{\oint_{l(x)} \mathbf{H}.d\mathbf{l}}{\Delta y\Delta z} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) = \frac{I}{\Delta y\Delta z}, \tag{6.96}$$

$$\lim_{\Delta y\Delta z\to 0}\frac{\oint_{l(x)} \mathbf{H}.d\mathbf{l}}{\Delta y\Delta z} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) = \lim_{\Delta y\Delta z\to 0}\left(\frac{I}{\Delta y\Delta z}\right), \tag{6.97}$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x. \tag{6.98}$$

Similarly, the components of the current density in the y and z directions are

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \lim_{\Delta z\Delta x\to 0}\frac{\oint_{l(y)} \mathbf{H}.d\mathbf{l}}{\Delta z\Delta x} = J_y, \tag{6.99}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \lim_{\Delta x\Delta y\to 0}\frac{\oint_{l(z)} \mathbf{H}.d\mathbf{l}}{\Delta x\Delta y} = J_z. \tag{6.100}$$

From Eqs. (6.98), (6.99) and (6.100), the rectangular form of the curl can be written as

$$\nabla\times\mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z = \mathbf{J}. \tag{6.101}$$

In determinant format, Eq. (6.101) can be written as

$$\nabla\times\mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}. \tag{6.102}$$

From Eq. (6.101), the point form of Ampere's circuital law can be written as

$$\mathbf{J} = \nabla \times \text{H}. \tag{6.103}$$

Equation (6.103) is the second of Maxwell's four equations.

The curl expansions for the cylindrical coordinate is

$$\nabla \times \mathbf{H} = \frac{1}{\rho} \begin{vmatrix} \mathbf{a}_\rho & \rho\mathbf{a}_\phi & \mathbf{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ H_\rho & \rho H_\phi & H_z \end{vmatrix}, \tag{6.104}$$

$$\nabla \times \mathbf{H} = \left(\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right)\mathbf{a}_\rho + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right)\mathbf{a}_\phi + \frac{1}{\rho}\left(\frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{\partial H_\rho}{\partial \phi}\right)\mathbf{a}_z. \tag{6.105}$$

The curl expansions for the spherical coordinates can be written as

$$\nabla \times \mathbf{H} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \mathbf{a}_r & r\mathbf{a}_\theta & r\sin\theta\,\mathbf{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ H_r & rH_\theta & r\sin\theta\, H_\phi \end{vmatrix}, \tag{6.106}$$

$$\nabla \times \mathbf{H} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}\left(H_\phi \sin\theta\right) - \frac{\partial H_\theta}{\partial \phi}\right]\mathbf{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial H_r}{\partial \phi} - \frac{\partial(rH_\phi)}{\partial r}\right]$$
$$\mathbf{a}_\theta + \frac{1}{r}\left[\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta}\right]\mathbf{a}_\phi. \tag{6.107}$$

Example 6.3 The magnetic field intensity in Cartesian coordinates is given as $\mathbf{H} = x^2y^2z^2\mathbf{a}_x + xyz\mathbf{a}_y - 2x^3y^3z\mathbf{a}_z$ A/m. Determine the value of the current density at point $P(1,1,1)$.

Solution The following components can be written as:

$$H_x = x^2y^2z^2,$$
$$\frac{\partial H_x}{\partial y} = 2yz^2x^2,$$
$$\frac{\partial H_x}{\partial z} = 2zy^2x^2,$$
$$H_y = xyz,$$
$$\frac{\partial H_y}{\partial x} = yz,$$
$$\frac{\partial H_y}{\partial z} = xy,$$

$$H_z = -2x^3y^3z,$$
$$\frac{\partial H_z}{\partial x} = -6x^2y^3z,$$
$$\frac{\partial H_z}{\partial y} = -6x^3y^2z.$$

The current density can be determined as

$$\nabla \times \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z = \mathbf{J},$$
$$\mathbf{J} = (-6x^3y^2z - xy)\mathbf{a}_x + (2x^2y^2z + 6x^2y^3z)\mathbf{a}_y + (yz - 2yz^2x^2)\mathbf{a}_z.$$

At point $P(1,1,1)$, the value of the current density is

$$\mathbf{J} = -7\mathbf{a}_x + 8\mathbf{a}_y - \mathbf{a}_z \text{ A/m}^2.$$

Example 6.4 The magnetic field intensity in the cylindrical coordinates is given by $\mathbf{H} = \frac{1}{\rho}\sin(0.3\phi)\mathbf{a}_\rho + \sin z\mathbf{a}_\phi + \cos\phi\mathbf{a}_z$ A/m. Determine the value of the current density at point $P(1.2, 80°, 0.3)$.

Solution The following components can be written as:

$$\frac{\partial H_z}{\partial \phi} = -\sin\phi,$$
$$\frac{\partial H_z}{\partial \rho} = 0,$$
$$\frac{\partial H_\phi}{\partial z} = \cos z,$$
$$\frac{\partial(\rho H_\phi)}{\partial \rho} = \frac{\partial}{\partial \rho}(\rho \sin z) = \sin z,$$
$$\frac{\partial H_\rho}{\partial z} = 0,$$
$$\frac{\partial H_\rho}{\partial \phi} = \frac{0.3}{\rho}\cos 0.3\phi.$$

The curl of the magnetic field is

$$\nabla \times \mathbf{H} = \left(-\frac{1}{\rho}\sin\phi - \cos z\right)\mathbf{a}_\rho + (0 - 0)\,\mathbf{a}_\phi + \frac{1}{\rho}\left(\sin z - \frac{0.3}{1.2}\cos 0.3\phi\right)\mathbf{a}_z.$$

The value of the current density at point $P(1.2, 80°, 0.3)$ is

$$\mathbf{J} = \left(-\frac{1}{1.2}\sin 80° - \cos 0.3\right)\mathbf{a}_\rho + \frac{1}{1.2}\left(\sin 0.3 - \frac{0.3}{1.2}\cos 24°\right)\mathbf{a}_z,$$

$$\mathbf{J} = (-0.8206 - 0.9553)\,\mathbf{a}_\rho + \frac{1}{1.2}\,(0.2955 - 0.2283)\,\mathbf{a}_z,$$

$$\mathbf{J} = -1.78\mathbf{a}_\rho + 0.056\mathbf{a}_z \text{ A/m}^2.$$

Example 6.5 The magnetic field intensity in the spherical coordinates is given by $\mathbf{H} = r\sin\phi\mathbf{a}_r + \sin\phi\mathbf{a}_\theta + 2r\mathbf{a}_\phi$ A/m. Determine the value of the current density at point $P(2, 40°, 60°)$.

Solution The following components can be written as

$$\frac{\partial}{\partial\theta}(\sin\theta H_\phi) = \frac{\partial}{\partial\theta}(2r\sin\theta) = 2r\cos\theta,$$

$$\frac{\partial}{\partial r}(rH_\phi) = \frac{\partial}{\partial r}(r2r) = 4r,$$

$$\frac{\partial H_\theta}{\partial\phi} = \frac{\partial}{\partial\phi}(\cos\phi) = -\sin\phi,$$

$$\frac{\partial}{\partial r}(rH_\theta) = \frac{\partial}{\partial r}(r\cos\phi) = \cos\phi,$$

$$\frac{\partial H_r}{\partial\phi} = \frac{\partial}{\partial\phi}(r\sin\phi) = r\cos\phi,$$

$$\frac{\partial H_r}{\partial\theta} = \frac{\partial}{\partial\theta}(r\sin\phi) = 0,$$

$$\nabla\times\mathbf{H} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(H_\phi\sin\theta) - \frac{\partial H_\theta}{\partial\phi}\right]\mathbf{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial H_r}{\partial\phi} - \frac{\partial(rH_\phi)}{\partial r}\right]\mathbf{a}_\theta$$
$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(rH_\theta) - \frac{\partial H_r}{\partial\theta}\right]\mathbf{a}_\phi,$$

$$\mathbf{J} = \frac{1}{r\sin\theta}[2r\cos\theta + \sin\phi]\mathbf{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}r\cos\phi - 4r\right]\mathbf{a}_\theta$$
$$+ \frac{1}{r}[\cos\phi - 0]\mathbf{a}_\phi.$$

At point $P(2, 40°, 60°)$, the value of the current density is

$$\mathbf{J} = \frac{1}{2\sin 40°}[2r\cos 40° + \sin 60°]\mathbf{a}_r$$
$$+ \frac{1}{2}\left[\frac{1}{\sin 40°}2\cos 60° - 4\times 2\right]\mathbf{a}_\theta + \frac{1}{2}[\cos 60° - 0]\mathbf{a}_\phi,$$

$$\mathbf{J} = 3.057\mathbf{a}_r - 3.22\mathbf{a}_\theta + 0.25\mathbf{a}_\phi \text{ A/m}^2.$$

Practice Problem 6.3 The magnetic field intensity in the Cartesian coordinates is given by $\mathbf{H} = yz\mathbf{a}_x + (z^2 + x^2)\mathbf{a}_y + 5x^2y\mathbf{a}_z$ A/m. Determine the value of the current density at point $P(1,1,1)$.

Practice Problem 6.4 The expression of the magnetic field intensity in the cylindrical coordinates is given by $\mathbf{H} = z\cos(0.2\phi)\mathbf{a}_\rho + \sin(0.1z)\mathbf{a}_\phi + \rho\sin\phi\mathbf{a}_z$ A/m. Calculate the value of the current density at point $P(1.5, 35°, 2.3)$.

Practice Problem 6.5 The magnetic field intensity in the spherical coordinates is given by $\mathbf{H} = r\cos\phi\mathbf{a}_r + \sin\phi\mathbf{a}_\theta + r^2\mathbf{a}_\phi$ A/m. Calculate the value of the current density at point $P(1, 60°, 90°)$.

6.9 Scalar and Vector Magnetic Potential

In an electrostatic field, an electric field is normally determined by taking the gradient of the voltage. This relation is given by $\mathbf{E} = -\nabla\mathbf{V}$. The magnetic field intensity is defined as the negative gradient of magnetomotive force or the scalar magnetic potential, i.e. $\mathbf{H} = -\nabla V_m$ It has been seen that the divergence of the curl of any vector field is identically zero, i.e.

$$\nabla.\nabla \times \mathbf{A} = 0, \tag{6.108}$$

where **A** is the magnetic vector potential and its SI unit is Wb/m or Tm.

Equating Eqs. (6.13) and (6.108) yields

$$\nabla.\mathbf{B} = \nabla.\nabla \times \mathbf{A}, \tag{6.109}$$

$$\mathbf{B} = \nabla \times \text{A}. \tag{6.110}$$

Substituting Eq. (6.110) into Eq. (6.14) yields

$$\nabla \times \nabla \times \mathbf{A} = \mu_0\mathbf{J}. \tag{6.111}$$

The following vector identity can be written as:

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla.\mathbf{A}) - \nabla^2\mathbf{A}, \tag{6.112}$$

$$\nabla^2\mathbf{A} = \nabla(\nabla.\mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}). \tag{6.113}$$

Substituting Eq. (6.111) into Eq. (6.113) provides

$$\nabla^2\mathbf{A} = \nabla(\nabla.\mathbf{A}) - \mu_0\mathbf{J}. \tag{6.114}$$

According to Eq. (6.13), the following relation can be written as

$$\nabla.\mathbf{A} = 0. \tag{6.115}$$

Substituting Eq. (6.115) into Eq. (6.114) yields

$$\nabla^2\mathbf{A} = -\mu_0\mathbf{J} \tag{6.116}$$

Equation (6.116) is known as Poisson's equation for the vector magnetic potential and the current density. In Cartesian coordinates, the three scalar equations can be written as

$$\nabla^2 A_x = -\mu_0 J_x, \tag{6.117}$$

$$\nabla^2 A_y = -\mu_0 J_y, \tag{6.118}$$

$$\nabla^2 A_z = -\mu_0 J_z. \tag{6.119}$$

Mathematically, Eqs. (6.117)–(6.119) are also the same as the Poisson's equation. In free space, the Poisson's equation is

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}. \tag{6.120}$$

The particular solution for V of Eq. (6.120) is

$$V = \frac{1}{4\pi\varepsilon_0}\int_{v'} \frac{\rho}{R} dv'. \tag{6.121}$$

According to Eq. (6.121), the solutions for the components of **A** are,

$$A_x = \frac{1}{4\pi}\int_{v'} \frac{\mu_0 J_x}{R} dv', \tag{6.122}$$

$$A_y = \frac{1}{4\pi}\int_{v'} \frac{\mu_0 J_y}{R} dv', \tag{6.123}$$

$$A_z = \frac{1}{4\pi}\int_{v'} \frac{\mu_0 J_z}{R} dv'. \tag{6.124}$$

Combining Eqs. (6.122)–(6.124) yields

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{v'} \frac{\mathbf{J}}{R} dv'. \tag{6.125}$$

Example 6.6 The magnetic vector potential in a given region is $\mathbf{A} = e^{-mx}\sin\alpha y\,\mathbf{a}_z$ Wb/m. Determine the magnetic flux density.

Solution The previous equations from the magnetic vector potential can be written as:

$$A_x = 0, A_y = 0, A_z = e^{-mx}\sin\alpha y,$$

The curl of a magnetic vector potential is

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix},$$

$$\mathbf{B} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{a}_x - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\mathbf{a}_y + \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{a}_z,$$

$$\mathbf{B} = \frac{\partial A_z}{\partial y}\mathbf{a}_x - \frac{\partial A_z}{\partial x}\mathbf{a}_y,$$

$$\mathbf{B} = \frac{\partial}{\partial y}(e^{-mx}\sin\alpha y)\mathbf{a}_x - \frac{\partial}{\partial x}(e^{-mx}\sin\alpha y)\mathbf{a}_y,$$

$$\mathbf{B} = \alpha e^{-mx}\cos\alpha y\mathbf{a}_x + me^{-mx}\sin\alpha y\mathbf{a}_x \text{ Wb/m}^2.$$

Practice Problem 6.6 The magnetic vector potential for a specific region is given by $\mathbf{A} = zx\mathbf{a}_x + yz\mathbf{a}_y$ Wb/m. Calculate the magnetic flux density.

6.10 Magnetization

In the electric field, the polarizations of materials have already been discussed to understand electric dipole moment. The magnetization is also very important to understand magnetic dipole moment. The magnetization of a magnetic field is defined as the magnetic dipole moment per unit volume. It is represented by the letter **M** and its unit is A/m. The magnetic dipole moment with a small volume Δv is shown in Fig. 6.11. Let us consider N atoms within a small volume Δv The magnetic dipole moment for the kth atom is

$$\mathbf{M} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{N} \mathbf{m}_k}{\Delta v}. \tag{6.126}$$

The magnetization or magnetic dipole moment is normally produced by the bond current. This bond current is generated along the surface of the magnetized material due to inner small current loops. According to the point form of Ampere's law, the following equation can be written as:

$$\nabla \times \mathbf{M} = \mathbf{J}_b. \tag{6.127}$$

In this case, the total current density is the combination of bond current and free current densities. Therefore, the total current density can be written as

$$\mathbf{J} = \mathbf{J}_b + \mathbf{J}_f. \tag{6.128}$$

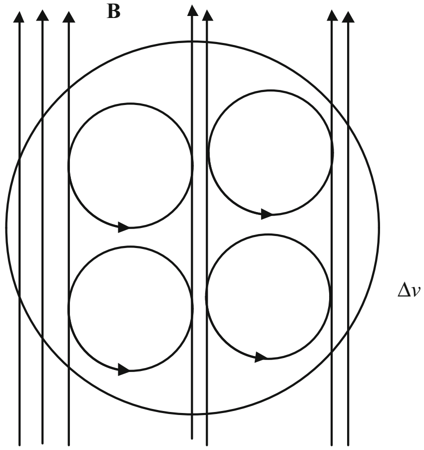


Fig. 6.11 An overhead view of magnetized material with small current loops

But it has already been studied that the current density is equal to the curl of magnetic field intensity and it can be written as

$$\mathbf{J} = \nabla \times \mathbf{H}. \tag{6.129}$$

Substituting the expression $\mathbf{H} = \dfrac{\mathbf{B}}{\mu_0}$ and Eq. (6.129) into Eq. (6.128) yields

$$\nabla \times \frac{\mathbf{B}}{\mu_0} = \mathbf{J}_b + \mathbf{J}_f. \tag{6.130}$$

Again, substituting Eq. (6.127) into Eq. (6.130) provides

$$\nabla \times \frac{\mathbf{B}}{\mu_0} = \nabla \times \mathbf{M} + \mathbf{J}_f, \tag{6.131}$$

$$\nabla \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = \mathbf{J}_f. \tag{6.132}$$

Comparing Eqs. (6.129) and (6.132) yields

$$\frac{\mathbf{B}}{\mu_0} - \mathbf{M} = \mathbf{H}, \tag{6.133}$$

$$\mathbf{B} = \mu_0(\mathbf{M} + \mathbf{H}). \tag{6.134}$$

But the magnetization is directly proportional to the magnetic field intensity for a linear magnetic material. In this case, it can be written as

$$\mathbf{M} = \chi_m \mathbf{H}. \tag{6.135}$$

where χ_m is the magnetic susceptibility.

Substituting Eq. (6.135) into Eq. (6.134) yields

$$\mathbf{B} = \mu_0(1 + \chi_m)\mathbf{H}. \tag{6.136}$$

Again, the general expression of magnetic flux density is

$$\mathbf{B} = \mu_0\mu_r\mathbf{H}. \tag{6.137}$$

Comparing Eqs. (6.136) and (6.137), the following expression can be written as

$$\mu_r = 1 + \chi_m. \tag{6.138}$$

In general, the expression of relative permeability is

$$\mu_r = 1 + \chi_m = \frac{\mu}{\mu_0}, \tag{6.139}$$

where $\mu = \mu_0\mu_r$ is the absolute permeability and is measured in H/m.

Based on the susceptibility, magnetic materials are classified as diamagnetic, paramagnetic and ferromagnetic. These are discussed below:

Diamagnetic Materials Materials that have a very small negative susceptibility ($\chi_m \leq 0$) and $\mu_r \leq 1$ are known as diamagnetic materials. Copper, gold, diamond, silicon, bismuth and silver are examples of diamagnetic materials. These materials are weakly affected by a magnetic field and hold linear properties. In the absence of an external magnetic field, these materials do not retain magnetic fields.

Paramagnetic Materials Materials that have a very small positive susceptibility ($\chi_m > 0$) and $\mu_r \geq 0$ are known as paramagnetic materials. Potassium, oxygen and yttrium oxide are examples of paramagnetic materials. These materials are slightly attracted by the magnetic field and do not retain this property after removing the external field.

Ferromagnetic Materials Materials that have a very large positive susceptibility ($\chi_m \gg 0$) and $\mu_r \gg 1$are known as ferromagnetic materials. Nickel oxide (NiO), ferrous sulfide (FeS), cobalt chloride ($CoCl_2$) are examples of ferromagnetic materials. These materials are strongly magnetized by an external magnetic field and retain this property when the external field is removed. These materials are nonlinear.

Example 6.7 The magnetic flux density of a ferrite material is found to be 0.08 T. Determine the (a) susceptibility, (b) magnetic field intensity and (c) magnetization. Consider that the material operates in linear mode and $\mu_r = 40$.

Solution (a) The value of the susceptibility is

$$\chi_m = \mu_r - 1 = 40 - 1 = 39.$$

(b) The value of the magnetic field intensity is

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0\mu_r} = \frac{0.08}{4\pi \times 10^{-7} \times 40} = 1591.55\,\text{A/m}.$$

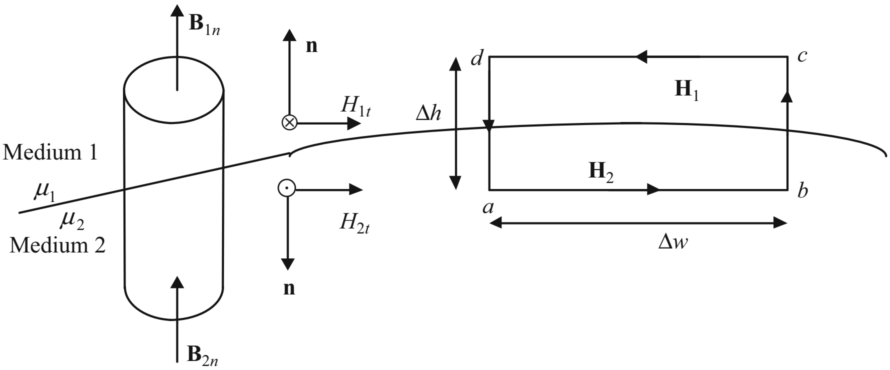


Fig. 6.12 A small box with two media

(c) The magnetization can be determined as

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}),$$

$$\mathbf{M} = \frac{\mathbf{B}}{\mu_0} - \mathbf{H} = \frac{0.08}{4\pi \times 10^{-7}} - 1591.55 = 62070.43\ \text{A/m}.$$

Practice Problem 6.7 The permeability and the magnetic field intensity of a magnetic material are given as 1.8×10^{-6} H/m and 125 A/m. Determine the value of the magnetization.

6.11 Magnetic Field Boundary Conditions

Figure 6.12 represents a small box of length Δw and height Δh. This box is separated by an imaginary line. This type of boundary of two media is used to derive the relations between normal and tangential components of the magnetic flux density. Consider that B_{1n} is the outward normal component of **B** from the medium 1 and that B_{2n} is the inward normal component of **B** to the medium 2. According to Gauss's law, the total magnetic flux over a closed surface equals zero. Then applying Gauss's law to the box as shown in Fig. 6.12, the following relation can be written as:

$$\oint_{abcd} \mathbf{B}.\, d\mathbf{S} = 0, \tag{6.140}$$

$$B_{n1}(\Delta w) + B_{n2}(-\Delta w) = 0, \tag{6.141}$$

$$B_{n1} - B_{n2} = 0, \tag{6.142}$$

$$B_{n1} = B_{n2}. \tag{6.143}$$

From Eq. (6.143), it is concluded that the normal component of the magnetic flux density is continuous across the boundary between two media.

For the linear medium, the following relations for two media can be written as:

$$\mathbf{B}_1 = \mu_1 \mathbf{H}_1, \tag{6.144}$$

$$\mathbf{B}_2 = \mu_2 \mathbf{H}_2. \tag{6.145}$$

Substituting Eqs. (6.144) and (6.145) into Eq. (6.143) yields

$$\mu_1 H_{n1} = \mu_2 H_{n2}, \tag{6.146}$$

$$H_{n1} = \frac{\mu_2}{\mu_1} H_{n2}. \tag{6.147}$$

From Eq. (6.147), it is concluded that the normal component of **H** is discontinuous by the ratio $\frac{\mu_2}{\mu_1}$.

Again, consider that H_{1t} and H_{2t} are the tangential components of **H** for medium 1 and medium 2, respectively. Applying Ampere's circuital law ($\oint \mathbf{H}.d\mathbf{l} = I$) to the closed surface yields

$$H_{1t}\Delta w - H_{2t}\Delta w = I, \tag{6.148}$$

$$H_{1t} - H_{2t} = \frac{I}{\Delta w} = K, \tag{6.149}$$

where K is the linear current density at the boundary. In terms of magnetic flux density, Eq. (6.149) can be modified as

$$\frac{B_{1t}}{\mu_1} - \frac{B_{2t}}{\mu_2} = K. \tag{6.150}$$

If there is no current density at the boundary, then Eqs. (6.149) and (6.150) can be modified as

$$H_{1t} = H_{2t}, \tag{6.151}$$

$$B_{1t} = \frac{\mu_1}{\mu_2} B_{2t}. \tag{6.152}$$

From Eq. (6.151), it is concluded that the tangential components of **H** are continuous across the boundary without a current density. The tangential components of **B** are discontinuous at the boundary by the ratio $\frac{\mu_1}{\mu_2}$ as shown in Eq. (6.152).

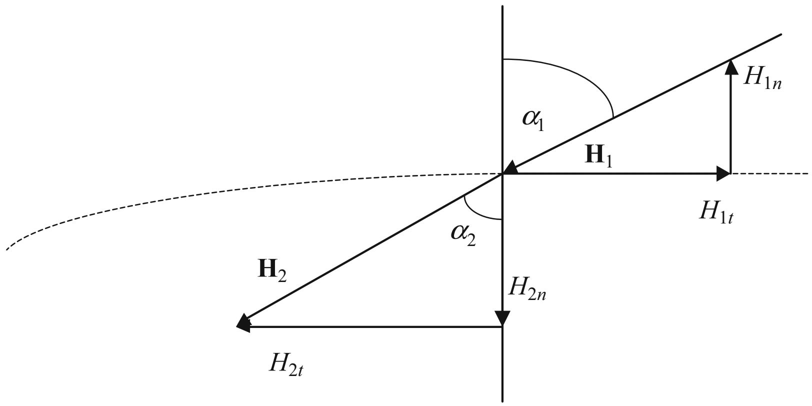


Fig. 6.13 A schematic for two media

6.12 Magnetic Field of Two Media

Consider μ_1 and μ_2 are the permeability for medium 1 and medium 2. Again consider that the magnetic field intensity of the medium 1, H_1 and the medium 2, H_2 makes an angle, respectively, of α_1 and α_2 with the normal as shown in Fig. 6.13. For medium 1, the following equations can be written as:

$$\cos(90^\circ - \alpha_1) = \frac{H_{1t}}{H_1}, \quad (6.153)$$

$$H_{1t} = H_1 \sin\alpha_1, \quad (6.154)$$

$$\sin(90^\circ - \alpha_1) = \frac{H_{1n}}{H_1}, \quad (6.155)$$

$$H_{1n} = H_1 \cos\alpha_1. \quad (6.156)$$

Similarly, for medium 2, the following equations can be written as:

$$\sin\alpha_2 = \frac{H_{2t}}{H_2}, \quad (6.157)$$

$$H_{2t} = H_2 \sin\alpha_2, \quad (6.158)$$

$$\cos\alpha_2 = \frac{H_{2n}}{H_2}, \quad (6.159)$$

$$H_{2n} = H_2 \cos\alpha_2. \quad (6.160)$$

Substituting Eqs. (6.154) and (6.158) into Eq. (6.151) yields

$$H_1 \sin\alpha_1 = H_2 \sin\alpha_2. \quad (6.161)$$

Again, substituting Eqs. (6.156) and (6.160) into Eq. (6.146) provides

$$\mu_1 H_1 \cos\alpha_1 = \mu_2 H_2 \cos\alpha_2. \tag{6.162}$$

Dividing Eq. (6.161) by Eq. (6.162) yields

$$\frac{\tan\alpha_1}{\mu_1} = \frac{\tan\alpha_2}{\mu_2}, \tag{6.163}$$

$$\alpha_2 = \tan^{-1}\left(\tan\alpha_1 \frac{\mu_2}{\mu_1}\right). \tag{6.164}$$

Equation (6.164) represents the refraction property of the magnetic field. The magnitude of the magnetic field intensity in medium 2 can be determined as

$$H_2 = \sqrt{{H_{2t}}^2 + {H_{2n}}^2}. \tag{6.165}$$

Substituting Eqs. (6.158) and (6.160) into Eq. (6.165) provides

$$H_2 = \sqrt{(H_2 \cos\alpha_2)^2 + (H_2 \sin\alpha_2)^2}. \tag{6.166}$$

Again, substituting Eqs. (6.161) and (6.162) into Eq. (6.166) yields

$$H_2 = \sqrt{(H_1 \sin\alpha_1)^2 + \left(\frac{\mu_1}{\mu_2} H_1 \cos\alpha_1\right)^2}, \tag{6.167}$$

$$H_2 = H_1 \sqrt{\sin^2\alpha_1 \left(\frac{\mu_1}{\mu_2} \cos\alpha_1\right)^2}. \tag{6.168}$$

6.13 Magnetic Circuit

The circuit followed by the magnetic flux is known as magnetic circuit, and the circuit followed by the electric current is known as the electric circuit. The electric and magnetic potentials between any two given points can be written as

$$V_{eab} = \int_a^b \mathrm{E}.d\mathrm{L}, \tag{6.169}$$

$$V_{mab} = \int_a^b \mathbf{H}.d\mathbf{L}. \tag{6.170}$$

The expressions for the current and the magnetic flux can be written as

$$I = \oint_S \mathbf{J}.d\mathbf{S}, \tag{6.171}$$

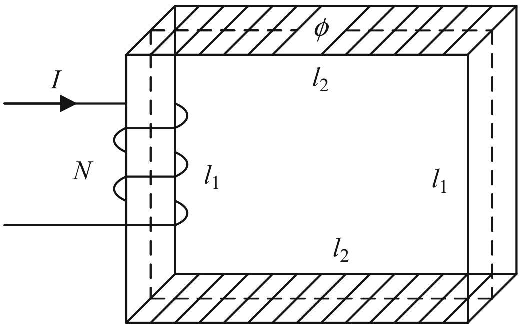


Fig. 6.14 Magnetic circuit with two materials

$$\phi = \oint_S \mathbf{B}.d\mathbf{S}. \tag{6.172}$$

The resistance with length l and a cross-sectional area of A is given as

$$R = \frac{l}{\sigma A}. \tag{6.173}$$

The reluctance of a magnetic circuit can be written as

$$\Re = \frac{l}{\mu A}, \tag{6.174}$$

where σ and μ are the conductivity and permeability of the material. The resistance and reluctance can also be defined in an alternative way as

$$R = \frac{V}{I} \tag{6.175}$$

$$\Re = \frac{\Im}{\phi}. \tag{6.176}$$

6.14 Series Magnetic Circuit

The same magnetic flux flows within a different materials series magnetic circuit. Figure 6.14 shows a series magnetic circuit with two different materials, whose mean length are l_1 and l_2, respectively. The respective cross-sectional areas are A_1 and A_2. The absolute permeability of the two materials are μ_1 and μ_2, respectively. The left side of the core is wound by N turns and it carries a current of I ampere. The equivalent circuit of the series magnetic circuit is shown in Fig. 6.15.

Applying the magnetic potential drop around the circuit yields

$$NI = H_1 l_1 + H_2 l_2. \tag{6.177}$$

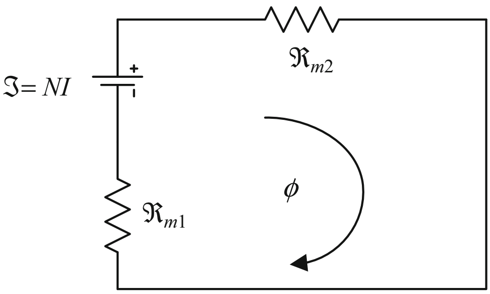


Fig. 6.15 Equivalent series magnetic circuit

Substituting $B = \mu H$ in Eq. (6.177) provides

$$NI = \frac{B_1}{\mu_1} l_1 + \frac{B_2}{\mu_2} l_2. \tag{6.178}$$

Rearranging Eq. (6.178) in the following way:

$$NI = B_1 A_1 \frac{l_1}{\mu_1 A_1} + B_2 A_2 \frac{l_2}{\mu_2 A_2}. \tag{6.179}$$

and substituting the expressions of the reluctance, $\Re = \dfrac{l}{\mu A}$ and the flux $\phi = B_1 A_1 = B_2 A_2$ in Eq. (6.179) yields

$$\Im = \phi \Re_{m1} + \phi \Re_{m2} = \phi(\Re_{m1} + \Re_{m2}), \tag{6.180}$$

$$\Im = \phi \Re_m, \tag{6.181}$$

where the expression of the total reluctance is

$$\Re_m = (\Re_{m1} + \Re_{m2}). \tag{6.182}$$

Example 6.8 A series magnetic circuit with the required dimensions is shown in Fig. 6.16. Determine the flux, flux density and field intensity if the relative permeability of a magnetic material is 850.

Solution The value of magnetomotive force is

$$\Im = NI = 300 \times 0.3 = 90 \text{ At}.$$

The value of the mean length of the path is

$$l_c = (0.20 - 0.02)4 = 0.72 \text{ m}.$$

The value of the cross-sectional area is

$$A = 0.02 \times 0.02 = 0.0004 \text{ m}^2.$$

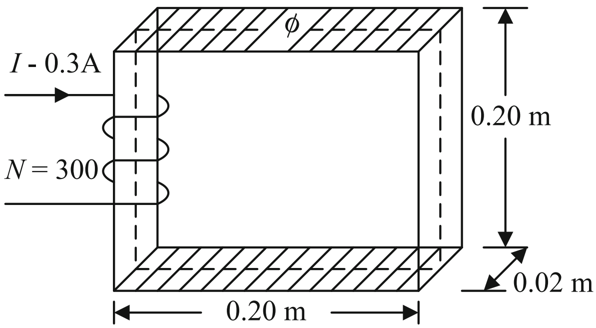


Fig. 6.16 Series magnetic circuit with a coil

The value of the reluctance is

$$\Re_m = \frac{l_c}{\mu_0 \mu_r A} = \frac{0.72}{4\pi \times 10^{-7} \times 850 \times 0.0004} = 1.69 \times 10^6 \text{ At/Wb}.$$

The value of the flux is

$$\phi = \frac{\Im}{\Re_m} = \frac{90}{1.69 \times 10^6} = 53.25 \times 10^{-6} \text{ Wb}.$$

The value of the magnetic flux density is

$$B = \frac{\phi}{A} = \frac{53.25 \times 10^{-6}}{0.0004} = 0.0025 \text{ Wb/m}^2.$$

The value of the magnetic field intensity is

$$H = \frac{B}{\mu_0 \mu_r} = \frac{0.0025}{4\pi \times 10^{-7} \times 850} = 2936.21 \text{ A/m}.$$

Practice Problem 6.8 The length, width and height of a magnetic circuit as shown in Fig. 6.16 are changed to 0.3 m, 0.015 m and 0.3 m, respectively. The relative permeability of a magnetic material is 900. Find the flux and the magnetic field intensity.

6.15 Parallel Magnetic Circuit

The circuit is said to be a parallel magnetic circuit when the main flux is divided into two or more branches in the magnetic core. Figure 6.17 shows a parallel magnetic circuit, where the magnetomotive force is connected to the left limb of the circuit. This magnetomotive force generates the flux, ϕ, and this flux is divided into ϕ_1 and ϕ_2, respectively, to the middle and right limbs whose lengths are l_1 and l_2.

Consider H, H_1 and H_2 are the magnetic field intensity of the left, middle and right limbs of the parallel magnetic, respectively. The equivalent circuit of a parallel

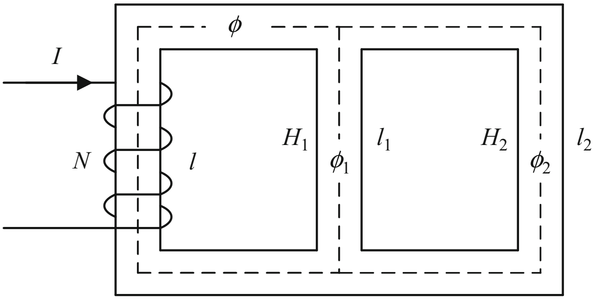


Fig. 6.17 Parallel magnetic circuit

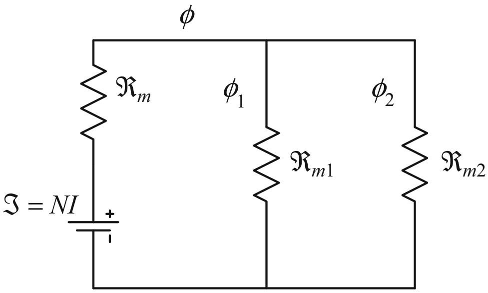


Fig. 6.18 Equivalent parallel magnetic circuit

magnetic circuit is shown in Fig. 6.18, and the magnetic potential drop around the equivalent circuit is

$$\Im - Hl = H_1 l_1 = H_2 l_2. \tag{6.183}$$

The expression of the total flux in this circuit is

$$\phi = \phi_1 + \phi_2. \tag{6.184}$$

From the basic definition of reluctance, the following relation can be written as

$$Hl = \phi \Re_m. \tag{6.185}$$

Using Eqs. (6.183) and (6.185) yields

$$\Im - \Re_m \phi = \Re_{m1} \phi_1 = \Re_{m2} \phi_2. \tag{6.186}$$

Equation (6.186) can be modified as

$$\Im = \Re_m \phi + \Re_{m1} \phi_1 = \Re_m \phi + \Re_{m2} \phi_2. \tag{6.187}$$

The total reluctance of $\Re_{m1}$ and $\Re_{m2}$ is

$$\Re_{mt} = \frac{\Re_{m1} \Re_{m2}}{\Re_{m1} + \Re_{m2}}. \tag{6.188}$$

Equation (6.188) is similar to the parallel electric circuit.

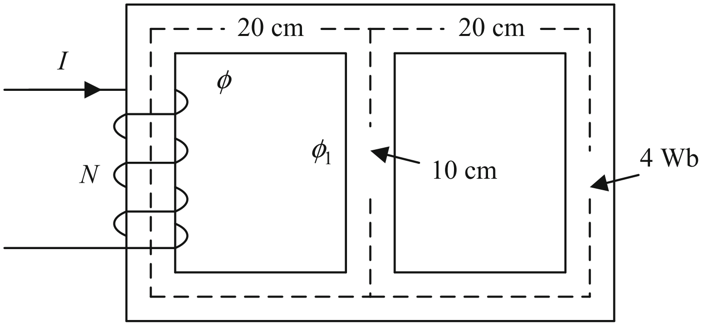


Fig. 6.19 Parallel magnetic circuit

Example 6.9 A parallel magnetic circuit is shown in Fig. 6.19 and its cross-sectional area is $9\,\mathrm{m}^2$. The 400 turns is wound on the left limb and the flux in the right limb is found to be 4 Wb. Assume the relative permeability is 500 and calculate the value of the current.

Solution The magnetic potential drop in both the middle and the right limb will be the same. This relation is

$$\phi_1 \Re_{m1} = \phi_2 \Re_{m2},$$

$$\phi_1 \frac{l_1}{\mu A} = \phi_2 \frac{l_2}{\mu A},$$

$$\phi_1 \frac{10}{\mu A} = 4 \frac{20}{\mu A},$$

$$\phi_1 = 8 \text{ Wb}$$

The value of the total flux is

$$\phi = 8 + 4 = 12 \text{ Wb}.$$

The flux density in the left limb is

$$B_1 = \frac{\phi}{A} = \frac{12}{9} = 1.33 \text{ Wb/m}^2.$$

The magnetic field intensity is

$$H_1 = \frac{B_1}{\mu_0 \mu_r} = \frac{1.33}{4\pi \times 10^{-7} \times 500} = 1591.55 \text{ A/m}.$$

The magnetomotive force in this case is

$$\Im_1 = H_1 l = 1591.55 \times 0.20 = 318.31 \text{ At}.$$

The magnetic flux density in the right limb is

$$B_2 = \frac{\phi_2}{A} = \frac{4}{9} = 0.44 \text{ Wb/m}^2.$$

The magnetic field intensity is

$$H_2 = \frac{B_2}{\mu_0 \mu_r} = \frac{0.44}{4\pi \times 10^{-7} \times 500} = 700.28 \text{ A/m}.$$

The magnetomotive force in this case is

$$\Im_2 = H_2 l = 700.28 \times 0.20 = 140.06 \text{ At}.$$

The value of the total magnetomotive force is

$$\Im_t = NI = 1591.55 + 140.06 = 1731.61 \text{ At}.$$

The value of the current is

$$I = \frac{1731.61}{400} = 4.33 \text{ A}.$$

Practice Problem 6.9 Figure 6.19 shows a parallel magnetic circuit and its cross-sectional area is 20 m^2. The right limb having the flux 6 Wb and the left limb is wound with N number of turns. Calculate the value of N if 20 A current flows in the coil and the relative permeability is 700.

6.16 Magnetic Circuit with Air Gap

An air gap is defined as the gap between two parts of a magnetic material. The air gap is normally filled by a nonmagnetic material. In the case of a three-phase induction motor, the rotor moves freely within a given small air gap. The flux crosses directly from one bar to the other at the middle of an air gap, whereas it bends outward and reduces the magnetic flux density at the edges of the air gap as shown in Fig. 6.20. This phenomenon is known as fringing. Figure 6.21 shows a magnetic circuit with a small air gap. Consider the mean length, permeability and cross-sectional area of the core are l_c, μ and A_c, respectively. The above parameters for the air gap are represented, respectively, by l_{ag}, μ_0 and A_{ag}.

The core and the air gap are connected in series in a composite circuit. As a result, the same flux will flow through this circuit and the equivalent circuit can be considered as a series magnetic circuit, which is shown in Fig. 6.22. The magnetic flux density for the core can be written as

$$B_c = \frac{\phi}{A_c}. \tag{6.189}$$

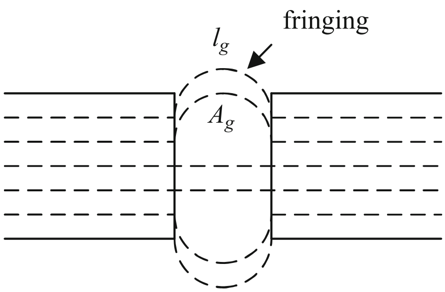


Fig. 6.20 Two magnetic bars with an air gap

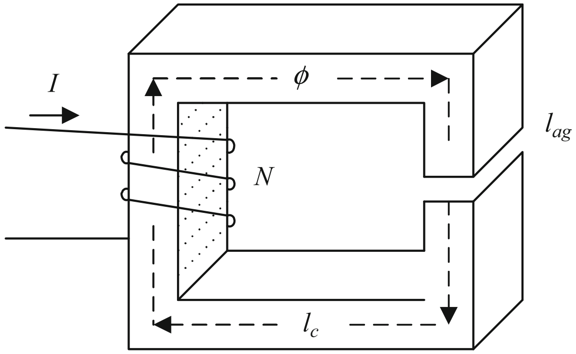


Fig. 6.21 Magnetic circuit with an air gap

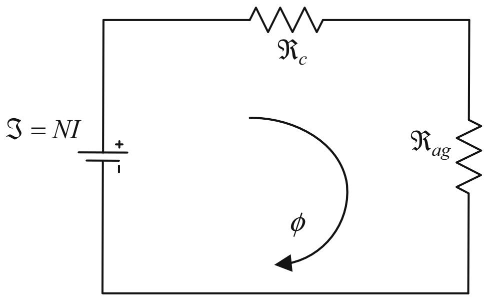


Fig. 6.22 Equivalent circuit

The magnetic flux density for the air gap is

$$B_{ag} = \frac{\phi}{A_{ag}}. \tag{6.190}$$

The total magnetomotive force is

$$\Im_t = \Im_c + \Im_{ag} = H_c l_c + H_{ag} l_{ag}. \tag{6.191}$$

Substituting the expression of $H = \dfrac{B}{\mu}$ into Eq. (6.191) provides

$$\Im_t = \frac{B_c}{\mu} l_c + \frac{B_{ag}}{\mu_0} l_{ag}. \tag{6.192}$$

Rearranging Eq. (6.192) yields

$$\Im_t = \frac{\phi}{\mu A_c} l_c + \frac{\phi}{\mu_0 A_{ag}} l_{ag}, \tag{6.193}$$

$$\Im_t = \phi \left(\frac{l_c}{\mu A_c} + \frac{l_{ag}}{\mu_0 A_{ag}} \right), \tag{6.194}$$

$$\Im_t = \phi(\Re_c + \Re_{ag}), \tag{6.195}$$

$$\Im_t = \phi \Re_t, \tag{6.196}$$

where the total reluctance, reluctance for the core and the air gap are

$$\Re_t = \Re_c + \Re_{ag}, \tag{6.197}$$

$$\Re_c = \frac{l_c}{\mu A_c}. \tag{6.198}$$

$$\Re_{ag} = \frac{l_{ag}}{\mu_0 A_{ag}}. \tag{6.199}$$

The reluctance of the core will be very small if the permeability of the core is infinity. As a result, it can be neglected for the analysis and the total magnetomotive will then be expressed as

$$\Im_t = \phi \ \Re_{ag}. \tag{6.200}$$

Example 6.10 The mean length and cross-sectional area of an iron ring are 35 cm and 15 cm^2 respectively. There is an air gap in the iron ring whose length is 0.5 mm. The ring is wound by 250 turns and carries a current of 2 A which produces a flux of 0.5 mWb. Find the reluctance and the relative permeability of the iron ring.

Solution The value of the magnetic flux density is

$$B = \frac{\phi}{A} = \frac{0.5 \times 10^{-3}}{15 \times 10^{-4}} = 0.33 \text{ Wb/m}^2.$$

The value of the total magnetomotive force is

$$\Im = 250 \times 2 = 500 \text{ At}.$$

The value of the magnetizing force is

$$H = \frac{B}{\mu_0} = \frac{0.33}{4\pi \times 10^{-7}} = 262.61 \times 10^3 \text{ A/m}.$$

The value of the magnetomotive force is

$$\Im_{ag} = Hl = 262.61 \times 10^3 \times 0.5 \times 10^{-3} = 131.31 \text{ At}.$$

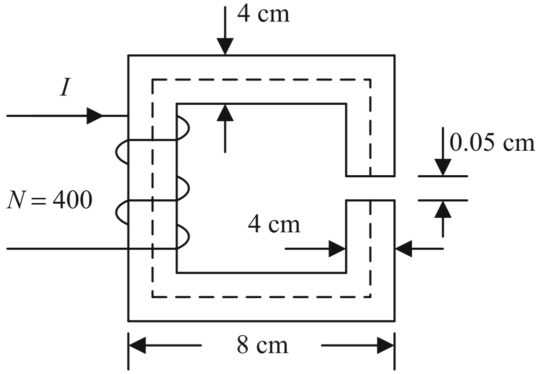


Fig. 6.23 Magnetic circuit with an air gap

The magnetomotive force for the iron ring is,

$$\Im_i = \Im - \Im_{ag} = 500 - 131.31 = 368.69 \text{ At}.$$

The value of the magnetic field intensity for the iron part is

$$H_i = \frac{\Im_i}{l} = \frac{368.69}{0.35} = 1053.40 \text{ A/m}.$$

The value of the relative permeability of the iron is

$$\mu_r = \frac{B}{\mu_0 H_i} = \frac{0.33}{4\pi \times 10^{-7} \times 1053.40} = 249.29.$$

Example 6.11 Figure 6.23 shows a magnetic circuit with an air gap, and the cross-sectional area of the core is 2 cm^2. The core is made of iron whose relative permeability is 4,000. If the total flux of the circuit is 0.5 mWb, find the current in the coil.

Solution For an air gap:

$$\Re_{mg} = \frac{l}{\mu A} = \frac{0.0005}{4\pi \times 10^{-7} \times 2 \times 10^{-4}} = 1.98 \times 10^6 \text{ At/Wb}$$

For the iron part: the mean length is

$$l = 4[8 - 4 - 4 + (2 + 2)] - 0.05 = 15.95 \text{ cm},$$

$$\mu = \mu_0 \mu_r = 4\pi \times 10^{-7} \times 4000 = 0.0050.$$

The reluctance of the iron is

$$\Re_{mi} = \frac{l}{\mu A} = \frac{0.16}{0.0050 \times 2 \times 10^{-4}} = 1.6 \times 10^5 \text{ At/Wb}.$$

The value of the total reluctance is

$$\Re_{mt} = \Re_{mg} + \Re_{mi} = 2.15 \times 10^6 \text{ At/Wb}.$$

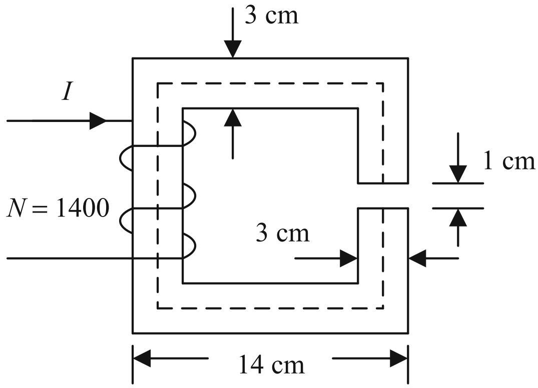


Fig. 6.24 Magnetic circuit with an air gap and a coil

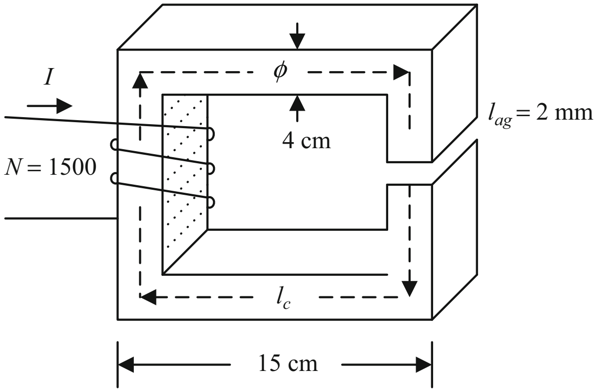


Fig. 6.25 Magnetic circuit with a small air gap

The value of the current can be determined as

$$\phi = \frac{\Im}{\Re_{mt}},$$

$$0.5 \times 10^{-3} = \frac{400 \times I}{2.15 \times 10^6},$$

$$I = 2.69\text{A}.$$

Practice Problem 6.10 Figure 6.24 shows a magnetic circuit with an air gap and a coil wound by 1,400 turns. The cross-sectional area of the core is 4 cm^2. The core is made of iron and the relative permeability of the core is 4,000. Consider the total flux of the circuit is 1.5 Wb. Find the current in the coil.

Practice Problem 6.11 A magnetic circuit with an air gap is shown in Fig. 6.25. The cross-sectional area of the core is 3 cm^2 and it is wound by 1,500 turns. The core is made of iron and the relative permeability of the core is 4,000. Calculate the current in the coil if the total flux of the circuit is 4.5 mWb.

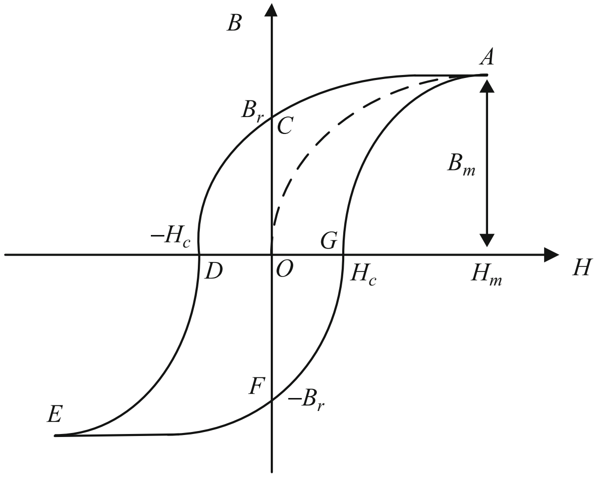


Fig. 6.26 Schematic of hysteresis curve

6.17 Hysteresis Curve

The relationship between the magnetic flux density (B) and magnetic field intensity (H) is known as hysteresis curve. The hysteresis phenomena occur in magnetic and ferromagnetic materials, in which a lag occurs between the application and the removal of the field. The hysteresis curve is obtained by plotting the magnetic flux density in the abscissa or x-axis and the magnetic field intensity in the ordinate, or y-axis. To explain the hysteresis curve, consider the following equation:

$$H = \frac{NI}{l}. \tag{6.201}$$

If the current in the coil is zero, then the value of $H = 0$ and the curve will start from the zero point of the axes. If we increase the value of the current from zero to some value, then the value of H will increase as shown in Fig. 6.26. If the magnetic field intensity H is increased until saturation due to the increasing current, then the curve will reach up to the maximum value of the material. After that if the magnetic field intensity increases, the value of the magnetic flux density will not increase, meaning that the material has reached the saturation region. Now, if the magnetic field intensity is slowly decreased to some value, the magnetic flux density also decreases. This decreasing value is higher than the previous one and the curve will move through another path. If the magnetic field intensity is reduced to zero, then it is seen that the magnetic core sustains some flux density (OC). This remaining flux density is known as the retentivity of the material. Again, if the magnetic field intensity increases in the reverse direction, then the magnetic flux density will vanish at point D. This value (OD) of the magnetic field is required to demagnetize the material. This amount of magnetic field intensity is known as coercive force. If the core magnetizes in the same direction, the magnetic flux density will be developed in the opposite direction and it will increase slowly with increasing the value of magnetic field intensity.

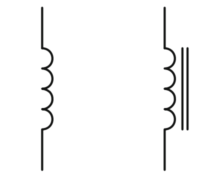


Fig. 6.27 Inductor symbols

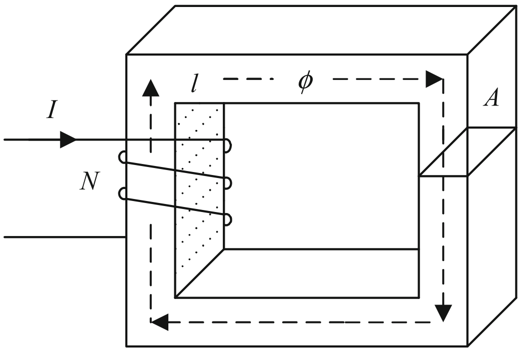


Fig. 6.28 A simple magnetic circuit

The value of the magnetic flux density will decrease if the value of the magnetic field intensity is reduced. The complete hysteresis curve will be obtained if this process is continued. The hysteresis curve normally varies in shape from one material to another material.

6.18 Inductance and Mutual Inductance

The property of a coil that opposes any change of current flowing through the coil is known as inductance. A current will flow in a coil when it is connected across the source. The flux will be associated due to this current. Therefore, the inductance is defined as the flux linkage per unit current. Mathematically, it can be expressed as

$$L = \frac{\psi}{i}. \tag{6.202}$$

The SI unit of inductance is Henry (H) and the symbol of inductors is shown in Fig. 6.27. Figure 6.28 shows a magnetic circuit, which contains N number of turns. The length and cross-sectional area of the core are l and A, respectively.

The current i in the core creates the flux ϕ and Eq. (6.202) can be modified as

$$L = \frac{N\phi}{i}. \tag{6.203}$$

According to the Faraday's laws of electromagnetic induction, the voltage induced across the coil can be written as

$$e = \frac{d\psi}{dt} \tag{6.204}$$

Substituting Eq. (6.202) into Eq. (6.204) yields

$$e = \frac{d(Li)}{dt} = L\frac{di}{dt}. \tag{6.205}$$

Again, substituting $\psi = N\phi$ into Eq. (6.204) provides

$$e = \frac{d(N\phi)}{dt} = N\frac{d\phi}{dt}. \tag{6.206}$$

Rearranging Eq. (6.206) yields

$$e = N\frac{d\phi}{di}\frac{di}{dt}. \tag{6.207}$$

Comparing Eqs. (6.205) and (6.207) provides

$$e = L\frac{di}{dt}, \tag{6.208}$$

where the expression of inductance is

$$L = N\frac{d\phi}{di}. \tag{6.209}$$

According to definition, the expression of flux is

$$\phi = \frac{\Im}{\Re}. \tag{6.210}$$

Substituting the expressions of magnetomotive force and reluctance into Eq. (6.210) yields

$$\phi = \frac{Ni}{\frac{l}{\mu A}} = \frac{N\mu Ai}{l}. \tag{6.211}$$

Differentiating Eq. (6.211) with respect to i provides

$$\frac{d\phi}{di} = \frac{N\mu A}{l}. \tag{6.212}$$

Substituting Eq. (6.212) into Eq. (6.209) yields

$$L = \frac{N^2\mu A}{l} = \frac{N^2}{\Re}. \tag{6.213}$$

The magnetic energy stored in an inductor is derived as

$$W_m = \frac{1}{2}LI^2. \tag{6.214}$$

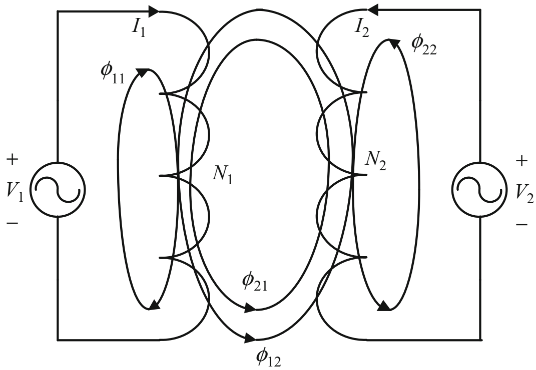


Fig. 6.29 Mutual fluxes of a magnetic circuit

From energy point of view, the expression of inductance from Eq. (6.214) can be written as

$$L = \frac{2W_m}{I^2}. \tag{6.215}$$

Again, consider that the magnetic circuit contains two coils, which are closed to each other. The self-inductances of the coils can be represented as $L_{11}(L_1)$ and $L_{22}(L_2)$, respectively. The mutual flux ϕ_{12} is the flux passing to the first circuit due to the current I_2 in the second circuit. Similarly, the mutual flux ϕ_{21} is the flux passing to the second circuit due to the current I_1 in the first circuit. The self and mutual fluxes are shown in Fig. 6.29.

The magnetic flux density $\mathbf{B}_1$ is created due to the current I_1 in the first circuit, whose area is S_1.

Some portion of the flux due to $\mathbf{B}_1$ will link to the second circuit whose area is bounded by S_2 and the expression of the mutual flux is

$$\phi_{12} = \oint_{S_2} \mathbf{B}_1.d\mathbf{S}_2. \tag{6.216}$$

The magnetic flux density $\mathbf{B}_2$ is created due to the current I_2 in the second circuit, whose area is S_2. Some portion of the flux due to $\mathbf{B}_2$ will link to the second circuit whose area is bounded by S_1 and the expression of the mutual flux is

$$\phi_{21} = \oint_{S_1} \mathbf{B}_2.d\mathbf{S}_1 \tag{6.217}$$

The mutual flux ϕ_{12} is directly proportional to the current I_1 and it can be expressed as

$$\phi_{12} \infty I_1, \tag{6.218}$$

$$\phi_{12} = L_{12} I_1, \tag{6.219}$$

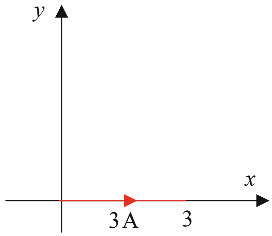


Fig. 6.30 Conductor is in horizontal position

where L_{12} is the proportionality constant and it is known as mutual inductance. In general, the mutual inductance is represented by M. If the secondary circuit contains N_2 number of turns, then the mutual inductance between the first and secondary circuits is

$$M_{12} = \frac{N_2\phi_{12}}{I_1}. \tag{6.220}$$

Substituting Eq. (6.216) into Eq. (6.220) yields

$$M_{12} = \frac{N_2}{I_1}\oint_{S_2} \mathrm{B}_1.d\mathrm{S}_2. \tag{6.221}$$

Similarly

$$M_{21} = \frac{N_1}{I_2}\oint_{S_1} \mathrm{B}_2.d\mathrm{S}_1. \tag{6.222}$$

Example 6.12 An air-core coil having a length of 15 cm and a cross-sectional area is 3 cm^2. Determine the inductance of the coil if the coil is wound by 75 turns.

Solution The value of the inductance can be calculated as,

$$L = \frac{N^2\mu A}{l} = \frac{75^2 \times 1 \times 4\pi \times 10^{-7} \times 3 \times 10^{-4}}{0.15} = 14.14\mu\mathrm{H}$$

Practice Problem 6.12 A circular air core having the length of 240 mm and the diameter of 4 mm. If the core is wound by 130 turns, then determine the inductance of the core.

6.19 Exercise Problems

6.1 A conductor is laid out horizontally carrying a current of 3 A as shown in Fig. 6.30. Determine the magnetic field intensity at point (0, 0, 4).

6.2 A conductor is laid in the z-axis carrying a current of 4 A as shown in Fig. 6.31. Determine the magnetic field intensity at point (0, 2, 0).

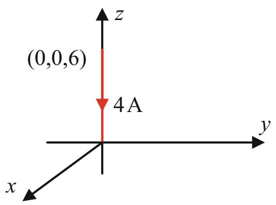


Fig. 6.31 Conductor is in vertical position

6.3 The expression of the magnetic field intensity in Cartesian coordinates is given by $\mathbf{H} = xy^3z^2\mathbf{a}_x + x^2yz\mathbf{a}_y - 2x^2y^3z^2\mathbf{a}_z$ A/m. Determine the value of the current density at point $P(2, -1, -3)$.

6.4 The expression of the magnetic field intensity in Cartesian coordinates is given by $\mathbf{H} = y^2z\mathbf{a}_x + 3(z^2 + x^2)\mathbf{a}_y + 2xy\mathbf{a}_z$ A/m. Calculate the value of the current density at point $P(-1,2,1)$.

6.5 The expression of the magnetic field intensity in cylindrical coordinates is given by $\mathbf{H} = \dfrac{1}{\rho^2}\sin(0.2\phi)\mathbf{a}_\rho + \rho\cos z\mathbf{a}_\phi + \sin\phi\mathbf{a}_z$ A/m. Determine the value of the current density at point $P(1, 130°, 0.5)$.

6.6 The expression of the magnetic field intensity in cylindrical coordinates is given by $\mathbf{H} = z^2\cos(0.4\phi)\mathbf{a}_\rho + \dfrac{1}{\rho}\sin(0.1z)\mathbf{a}_\phi + \rho\sin\phi\mathbf{a}_z$ A/m. Calculate the value of the current density at point $P(1, 30°, 1.2)$.

6.7 The expression of the magnetic field intensity in spherical coordinates is given by $\mathbf{H} = r^2\mathbf{a}_r + \sin\theta\mathbf{a}_\theta + \frac{2}{r}\sin\phi\mathbf{a}_\phi$ A/m. Determine the value of the current density at point $P(1.5, 70°, 100°)$.

6.8 The magnetic vector potential in a given region is $\mathbf{A} = e^{-px}\cos qy\,\mathbf{a}_z$ Wb/m. Determine the magnetic flux density.

6.9 The magnetic vector potential for a specific region is given by $\mathbf{A} = 3z^2x\mathbf{a}_x - 4y^2z\mathbf{a}_y$ Wb/m. Calculate the magnetic flux density.

6.10 The magnetic flux density of a ferrite material is found to be 0.04 T. Determine the (1) susceptibility, (2) magnetic field intensity and (3) magnetization. Consider the material operates in linear mode and $\mu_r = 35$.

6.11 The permeability and the magnetic field intensity of a magnetic material are given by 1.3×10^{-6} H/m and 195 At/m. Determine the value of the magnetization.

6.12 Calculate the flux, flux density and field intensity of the series magnetic circuit shown in Fig. 6.32. The length, width and height are 0.2 m 0.03 m and 0.2 m respectively and the relative permeability of the magnetic material is 548.

6.13 Fig. 6.33 shows a parallel magnetic circuit whose cross-sectional area is 6 m^2. The turns wound on the left limb are 200, and the flux in the right limb is 4 Wb. Assume the relative permeability is 350 and calculate the value of the current.

6.14 Determine the total reluctance of the magnetic circuit with an air gap as shown in Fig. 6.34. The cross-sectional area of the core is 3 cm^2 and it is wound by 300 turns. The core is made of iron and the relative permeability of the core is 3,000. If the total flux of the circuit is 0.05 mWb then determine the current in the coil.

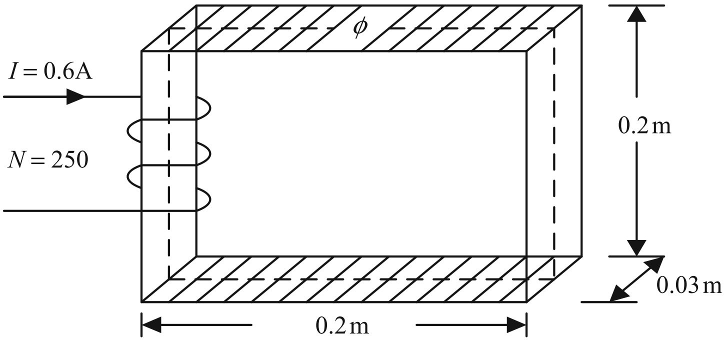


Fig. 6.32 Series magnetic circuit

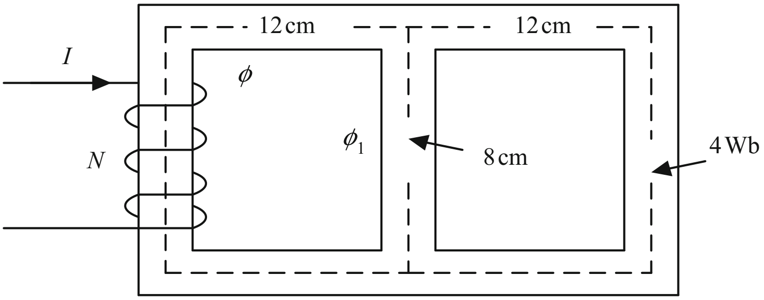
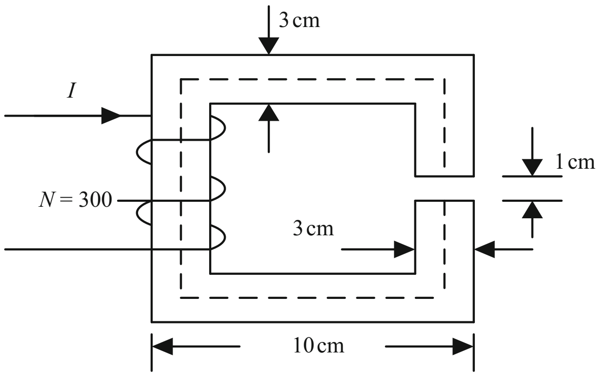


Fig. 6.33 Parallel magnetic circuit

Fig. 6.34 Magnetic circuit with an air gap



6.15 The length of an air-core coil is 35 cm and its cross-sectional area is 6 cm^2. The coil is wound by 125 turns. Calculate the inductance of the coil.

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Chapter 7
Time-Varying Fields

7.1 Introduction

The static electric and magnetic fields have already been discussed in previous chapters. These static fields are not dependent on each other. However, the time-varying electric and magnetic fields are dependent on each other. In other words, a time-varying electric field is produced by a time-varying magnetic field and a time-varying magnetic field is produced by a time-varying electric field. The first concept was experimentally introduced by Michael Faraday and the second was theoretically introduced by James Clerk Maxwell. In this chapter, Faraday's law, conduction current, displacement current, motional voltage, Maxwell's equation, transformers, time-varying potentials, field of series circuits and time-harmonic fields will be discussed.

7.2 Faraday's Law

In 1820, Oersted introduced that a steady electric current produces a magnetic field. In 1831, Michael Faraday in London and Joseph Henry in New York jointly discovered that a time-varying magnetic field can produce electric current. An electromotive force (emf) is induced either by a conductor moving in a magnetic field or by changing the magnetic field. Then, the Faraday's law can be written as

$$emf = -\frac{d\phi}{dt}. \tag{7.1}$$

If the conductor has N-turn, then Eq. (7.1) can be modified as

$$emf = -N\frac{d\phi}{dt}. \tag{7.2}$$

The electromotive force in any closed path can be written as

$$emf = \oint_c \mathbf{E}.d\mathbf{l}. \tag{7.3}$$

Md. A. Salam, *Electromagnetic Field Theories for Engineering*,
DOI 10.1007/978-981-4585-66-8_7, © Springer Science+Business Media Singapore 2014

The magnetic flux can be expressed as

$$\phi = \int_S \mathbf{B}.d\mathbf{S}. \tag{7.4}$$

Combining Eqs. (7.1), (7.3) and (7.4) yields

$$emf = \oint_c \mathbf{E}.d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}.d\mathbf{S}. \tag{7.5}$$

For a stationary circuit or path, the differentiation of (7.5) is

$$emf = \oint_c \mathbf{E}.d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t}.d\mathbf{S}. \tag{7.6}$$

Applying Stokes theorem to the closed line integral of Eq. (7.6) yields

$$\int_S (\nabla \times \mathbf{E}).d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t}.d\mathbf{S}, \tag{7.7}$$

$$(\nabla \times E).d\mathbf{S} = \frac{\partial \mathbf{B}}{\partial t}.d\mathbf{S}, \tag{7.8}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{7.9}$$

Equation (7.9) represents the point form or differential form of Maxwell's equation. From Eq. (7.9), it can be stated that the curl of the electric field is equal to the negative rate of change of magnetic field. If the magnetic field does not change with time, then Eqs. (7.6) and (7.9) can be reduced as

$$\oint_c \mathbf{E}.d\mathbf{l} = 0, \tag{7.10}$$

$$\nabla \times \mathbf{E} = 0. \tag{7.11}$$

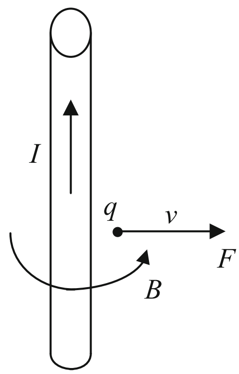
From Eqs. (7.10) and (7.11), it is concluded that the line integration and curl of electric fields are equal to zero.

7.3 Motional Voltage

A straight conductor carries a current, I, which creates the magnetic field B. A charge of proton or electron is placed in a magnetic field and moves at a velocity v as shown in Fig. 7.1. The Lorentz force can be written as

$$\mathbf{F}_m = q(\mathbf{v} \times \mathbf{B}), \tag{7.12}$$

Fig. 7.1 Straight conductor with a charge of proton or electron



$$\frac{\mathbf{F}_m}{q} = \mathbf{v} \times \mathbf{B}. \tag{7.13}$$

The motional voltage $\mathbf{E}_m$ is defined as

$$\mathbf{E}_m = \frac{\mathbf{F}_m}{q} = \mathbf{v} \times \mathbf{B}. \tag{7.14}$$

The induced *emf* is

$$emf = \oint \mathbf{E}_m.d\mathbf{l}. \tag{7.15}$$

Substituting Eq. (7.14) into Eq. (7.15) yields

$$emf = \oint (\mathbf{v} \times \mathbf{B}).d\mathbf{l}. \tag{7.16}$$

In a straight conductor, the wires, **v** and **B** are mutually perpendicular. Then, Eq. (7.16) can be modified as

$$emf = El = Blv. \tag{7.17}$$

From Eq. (7.17), it is seen that the *emf* is directly proportional to the magnetic flux density, length and velocity of the conductor.

Example 7.1 A conducting bar slides over two conducting rails as shown in Fig. 7.2. The bar moves from its initial position to a specific position at a distance of 5 cm. The vertical distance between two conducting rails is 3 cm. Consider the magnetic field $\mathbf{B} = 0.6\cos 1000t\mathbf{a}_z\text{mW/m}^2$. Determine the induced voltage.

Solution The rate of change of magnetic flux density is

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{\partial}{\partial t}(0.6\cos 1000t)\mathbf{a}_z = 600\sin 1000t\mathbf{a}_z.$$

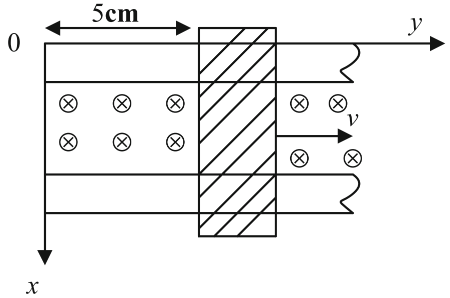


Fig. 7.2 Magnetic bar with rails

The value of the induced voltage is

$$V_{ind} = -\int_s \frac{\partial \mathbf{B}}{\partial t}.d\mathbf{S} = \int_{y=0}^{0.05}\int_{x=0}^{0.03} 600 \sin 1000t\mathbf{a}_z.dxdy\mathbf{a}_z,$$

$$V_{ind} = 600(0.05)(0.03)\sin 1000t = 0.9 \sin 1000t \text{ V}.$$

Practice Problem 7.1 A 10 cm conducting bar slides over two conducting rails at velocity $v = 10\mathbf{a}_y$m/s and the magnetic field is $\mathbf{B} = 2\mathbf{a}_z$Wb/m^2. Determine the induced voltage in the bar.

7.4 Maxwell's Equations

There are four equations normally known as Maxwell's equations. The integral and differential forms of Maxwell's equations are mentioned below:

$$\oint_c \mathbf{H}.d\mathbf{l} = \int_s \mathbf{J}.d\mathbf{S}, \nabla \times \mathbf{H} = \mathbf{J}, \tag{7.18}$$

$$\oint_s \mathbf{D}.\mathbf{S} = \int_v \rho dv, \rho = \nabla.\mathbf{D}, \tag{7.19}$$

$$\oint_s \mathbf{B}.d\mathbf{S} = 0, \nabla.\mathbf{B} = 0, \tag{7.20}$$

$$\oint_c \mathbf{E}.d\mathbf{l} = -\int_s \frac{\partial \mathbf{B}}{\partial t}.d\mathbf{S}, \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{7.21}$$

7.5 Conduction and Displacement Currents

A voltage source is connected in the circuit as shown in Fig. 7.3, where a resistor is in parallel with a capacitor. The current will flow in the resistor and capacitor. The current in the resistor is different than the current in the capacitor. The current in the

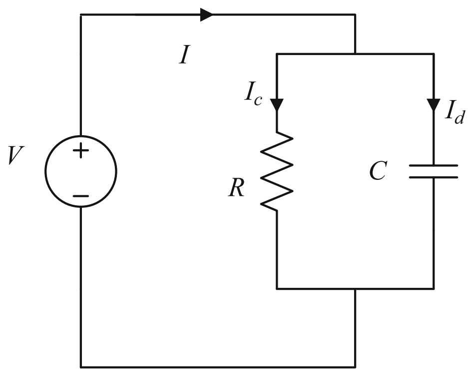


Fig. 7.3 Parallel circuit with source

resistor is continuous with respect to time and it is known as conduction current. The expressions for the conduction current and current density are

$$I_c = \frac{V}{R}, \tag{7.22}$$

$$\mathbf{J}_c = \frac{I_c}{A}. \tag{7.23}$$

Substituting the expression of $R = \frac{\rho l}{A} = \frac{l}{\sigma A}$ and Eq. (7.22) into Eq. (7.23) yields

$$\mathbf{J}_c = \frac{V\sigma A}{lA} = \sigma \mathbf{E}, \tag{7.24}$$

where A is the area of the capacitor plates, σ is the conductivity and $\mathbf{E}$ is the electric field.

The current will flow through the capacitor when the voltage across it changes. Therefore, the current in the capacitor is discontinuous and it is known as displacement current. The expression of displacement current is

$$I_d = \frac{dq}{dt} = C\frac{dV}{dt}. \tag{7.25}$$

Substituting the expressions of $C = \frac{\varepsilon A}{d}$ and $V = Ed$ into Eq. (7.25) yields

$$I_d = \frac{\varepsilon A}{d}\frac{d(Ed)}{dt}, \tag{7.26}$$

$$I_d = \frac{\varepsilon A}{d}d\frac{d(\mathbf{E})}{dt}, \tag{7.27}$$

$$\frac{I_d}{A} = \frac{d(\varepsilon \mathbf{E})}{dt}, \tag{7.28}$$

$$\mathbf{J}_d = \frac{d\mathbf{D}}{dt}, \tag{7.29}$$

where $\mathbf{J}_d$ is the conduction current density.

Total current density is

$$\mathbf{J} = \mathbf{J}_c + \mathbf{J}_d. \tag{7.30}$$

The concept of displacement current is introduced by James Clerk Maxwell. The conduction current is absent in the empty space and the displacement current is responsible for the magnetic field.

Example 7.2 The voltage source of $V = 20 \sin 314t$ V is applied across the parallel plates of a capacitor. The area of the plates and separation distance between the plates are 6 cm^2 and 2 mm, respectively. Determine the displacement current and density if $\varepsilon = 3\varepsilon_0$.

Solution The value of the displacement current is

$$I_d = C\frac{dV}{dt} = \frac{\varepsilon A}{d}\frac{dV}{dt} = \frac{3 \times 8.854 \times 10^{-12} \times 6 \times 10^{-4}}{0.002} \times 20 \times 314 \cos 314t,$$

$$I_d = 5 \times 10^{-8} \cos 314t \text{ A}.$$

The value of the current density is

$$J_d = \frac{I_d}{A} = \frac{5 \times 10^{-8} \cos 314t}{6 \times 10^{-6}} = 8.33 \times 10^{-3} \cos 314t \text{ A/m}^2.$$

Practice Problem 7.2 The voltage source of $V = 300 \sin 10^4 t$ V is connected across the parallel plates of a capacitor. The area of the plates and separation distance between the plates are 2 cm^2 and 3 mm, respectively. Determine the displacement current density if $\varepsilon = 1.5\varepsilon_0$.

7.6 Maxwell's Equation from Ampere's Law

The statement of Ampere's law is that the line integral of a magnetic field around a closed path is equal to the current enclosed by that path. Mathematically, this law is

$$\oint_l \mathbf{H}.d\mathbf{l} = I. \tag{7.31}$$

The expression of a current through any surface is

$$I = \int_s \mathbf{J}.d\mathbf{S}. \tag{7.32}$$

Substituting Eq. (7.32) into Eq. (7.31) provides

$$\oint_l \mathbf{H}.d\mathbf{l} = \int_s \mathbf{J}.d\mathbf{S}. \tag{7.33}$$

Substituting Eq. (7.30) into Eq. (7.33) yields

$$\oint_l \mathbf{H}.d\mathbf{l} = \int_s (\mathbf{J}_c + \mathbf{J}_d).d\mathbf{S}. \tag{7.34}$$

Again, substituting Eqs. (7.24) and (7.29) into Eq. (7.34) provides

$$\oint_l \mathbf{H}.d\mathbf{l} = \int_s \left(\sigma\mathbf{E} + \frac{\partial \mathbf{D}}{\partial t}\right).d\mathbf{S}. \tag{7.35}$$

Applying Stokes theorem to the left hand side of Eq. (7.35) gives

$$\int_s (\nabla \times \mathbf{H}).d\mathbf{S} = \int_s \left(\sigma\mathbf{E} + \frac{\partial \mathbf{D}}{\partial t}\right).d\mathbf{S}, \tag{7.36}$$

$$\int_s (\nabla \times \mathbf{H}).d\mathbf{S} = \int_s \sigma\mathbf{E}.d\mathbf{S} + \int_s \frac{\partial \mathbf{D}}{\partial t}.d\mathbf{S}, \tag{7.37}$$

$$(\nabla \times \mathbf{H}).d\mathbf{S} = \mathbf{J}_c.d\mathbf{S} + \frac{\partial \mathbf{D}}{\partial t}.d\mathbf{S}, \tag{7.38}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \tag{7.39}$$

where $\mathbf{J} = \mathbf{J}_c$. Equation (7.39) is the differential form of Maxwell's equation. If the electric field varies sinusoidally with time (i.e. $\mathbf{E} = \mathbf{E}_0 \sin \omega t$), then the conduction and displacement current densities are

$$\mathbf{J} = \sigma\mathbf{E} = \sigma\mathbf{E}_0 \sin \omega t, \tag{7.40}$$

$$\mathbf{J}_d = \varepsilon\frac{\partial \mathbf{E}}{\partial t} = \varepsilon\omega\mathbf{E}_0 \cos \omega t. \tag{7.41}$$

The ratio of the magnitude of conduction to displacement current density is

$$\frac{|\mathbf{J}|}{|\mathbf{J}_d|} = \frac{\sigma}{\omega\varepsilon}. \tag{7.42}$$

At $\omega t = 0°$, the conduction current density is zero, whereas the displacement current density is maximum. At $\omega t = 90°$, the conduction current density is maximum, whereas the displacement current density is zero.

Example 7.3 The expression for magnetic field intensity is given as $\mathbf{H} = 5\cos(10^6 t - \alpha x)\mathbf{a}_z$ A/m. Use Maxwell's equation to determine expressions for **B**, **D**, **E**. Consider the permittivity $\varepsilon = 1.5 \times 10^{-6}$ F/m.

Solution The components of magnetic field intensity are

$$H_x = H_y = 0,$$

$$H_z = 5\cos(10^6 t - \alpha x).$$

The curl of the magnetic field is

$$\nabla \times \mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix},$$

$$\nabla \times \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_z - \left(\frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z,$$

$$\nabla \times \mathbf{H} = -\frac{\partial H_z}{\partial x}\mathbf{a}_y,$$

$$\nabla \times \mathbf{H} = -\frac{\partial(5\cos(10^6 t - \alpha x))}{\partial x}\mathbf{a}_y = 5\alpha \sin(10^6 t - \alpha x)\mathbf{a}_y = -\frac{\partial \mathbf{D}}{\partial t},$$

$$\mathbf{D} = \int 5\alpha \sin(10^6 t - \alpha x)\mathbf{a}_y dt = \frac{5\alpha}{10^6}\cos(10^6 t - \alpha x)\mathbf{a}_y \text{ C/m}^2.$$

The electric field intensity can be determined as

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon} = \frac{5\alpha}{10^6 \times 1.5 \times 10^{-6}}\cos(10^6 t - \alpha x)\mathbf{a}_y = 3.33\cos(10^6 t - \alpha x)\mathbf{a}_y \text{ V/m}.$$

The components of the electric field are

$$E_x = E_z = 0,$$

$$E_y = 3.33\cos(10^6 t - \alpha x).$$

The curl of the electric field is

$$\nabla \times \mathbf{E} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix},$$

$$\nabla \times \mathbf{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\mathbf{a}_z - \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right)\mathbf{a}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\mathbf{a}_z,$$

$$\nabla \times \mathbf{E} = \frac{\partial E_y}{\partial x}\mathbf{a}_z,$$

$$\nabla \times \mathbf{E} = -\frac{\partial(3.33\alpha\cos(10^6 t - \alpha x))}{\partial x}\mathbf{a}_y = 3.33\alpha^2 \sin(10^6 t - \alpha x)\mathbf{a}_z = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\mathbf{B} = -\int 3.33\alpha^2 \sin(10^6 t - \alpha x)\mathbf{a}_z dt = \frac{3.33\alpha^2}{10^6}\cos(10^6 t - \alpha x)\mathbf{a}_z \text{ Wb/m}^2.$$

Practice Problem 7.3 The expression for magnetic field intensity is given as $\mathbf{H} = 2\sin(10^{12}t - \alpha x)\mathbf{a}_y$ A/m. Use Maxwell's equation to determine expressions for **B**, **D**, **E**. Consider the permittivity $\varepsilon = 1.5 \times 10^{-6}$ F/m.

7.7 Transformer

The transformer is used to transfer voltage from one circuit to another circuit without change of frequency. The current, voltage and the number of turns in the primary and secondary circuits are I_1, V_1, N_1 and I_2, V_2, N_2, respectively. The primary circuit of the transformer is connected to an alternating source and the secondary is an open circuit. The current flows in the primary circuit, which will create the flux. The flux moves from the primary circuit to the secondary circuit, known as mutual flux and the expression of this flux is

$$\phi_m = \frac{\Im_1}{\Re}. \tag{7.43}$$

Substituting the expressions of magnetomotive force ($\Im_1 = N_1 I_1$) and reluctance ($\Re = \frac{l}{\mu A}$) to Eq. (7.43) yields

$$\phi_m = \frac{N_1 I_1}{\frac{l}{\mu A}} = \frac{N_1 I_1}{l}\mu A, \tag{7.44}$$

where

A is the cross-sectional area of the core,
l is the mean length of the core and
$\Re$ is the total reluctance.

The voltage in the secondary winding is

$$V_2 = N_2 \frac{d\phi_m}{dt}. \tag{7.45}$$

Substituting Eq. (7.44) into Eq. (7.45) yields

$$V_2 = \frac{\mu A N_1 N_2}{l}\frac{dI_1}{dt}, \tag{7.46}$$

$$V_2 = M\frac{dI_1}{dt}, \tag{7.47}$$

where the value of the mutual inductance is

$$M = \frac{\mu A N_1 N_2}{l}. \tag{7.48}$$

Consider that the expression of complex current is

$$I_1 = I_m e^{j\omega t}. \tag{7.49}$$

The rate of change of the current is

$$\frac{dI_1}{dt} = \omega I_m e^{j\omega t}. \tag{7.50}$$

The magnitude of the rate of change of the current is

$$\left|\frac{dI_1}{dt}\right| = \omega \left|I_1\right|. \tag{7.51}$$

Substituting Eq. (7.51) into Eq. (7.47) yields

$$V_2 = M\omega I_1. \tag{7.52}$$

The primary current is solely responsible to generate the mutual flux, ϕ_m. If the winding resistance is neglected, then the flux links to the primary and secondary windings. Under this condition, the induced voltages in the primary and secondary windings are

$$V_1 = \omega N_1 \phi_m, \tag{7.53}$$

$$V_2 = \omega N_2 \phi_m, \tag{7.54}$$

where V_1 and V_2 are the maximum value of the voltages in the primary and secondary windings, respectively.

In general, the root mean square (rms) value of the voltage is

$$V_{rms} = \frac{V_m}{\sqrt{2}} = \frac{2\pi f N\phi_m}{\sqrt{2}}. \tag{7.55}$$

Dividing Eq. (7.53) by Eq. (7.54) provides

$$\frac{V_1}{V_2} = \frac{N_1}{N_2}. \tag{7.56}$$

The ratio of either the primary to the secondary voltages or turns is known as turns ratio. Under no-load condition, the magnetomotive forces in both the windings are same. In this case, the expression can be written as

$$\Im_1 = \Im_2, \tag{7.57}$$

$$N_1 I_1 = N_2 I_2, \tag{7.58}$$

$$\frac{N_1}{N_2} = \frac{I_2}{I_1}. \tag{7.59}$$

The turns ratio is represented by a, then Eqs. (7.56) and (7.59) can be expressed as

$$a = \frac{V_1}{V_2} = \frac{N_1}{N_2} = \frac{I_2}{I_1}. \tag{7.60}$$

Example 7.4 A transformer having a uniform cross section of 6 cm^2 is connected to the 220 V, 50 Hz source. Determine the induced voltage in the secondary coil which contains 120 turns and if the magnetic flux density is 0.5 T.

Solution The value of the magnetic flux is

$$\phi_m = BA = 0.5 \times 6 \times 10^{-4} = 3 \times 10^{-4} \text{ Wb}.$$

The induced voltage in the secondary coil is

$$V_2 = \omega N_2 \phi_m = 2\pi \times 50 \times 120 \times 3 \times 10^{-4} = 11.31 \text{ V}.$$

Practice Problem 7.4 A transformer having a uniform cross section of 4 cm^2 is connected to the 120 V, 50 Hz source. Determine the secondary turns and if the magnetic flux density is 0.9 T and the induce voltage is 30 V.

7.8 Time-Varying Potentials

The time-varying potential is usually known as retarded potential. This potential is used to determine electromagnetic field radiation near high voltage transmission lines and substation. For static electromagnetic fields, the scalar electric field and the vector magnetic potential are expressed as

$$V = \int_v \frac{\rho_v dv}{4\pi\varepsilon R}, \tag{7.61}$$

$$\mathbf{A} = \int_v \frac{\mu \mathbf{J} dv}{4\pi R}. \tag{7.62}$$

The magnetic flux density can be expressed as the curl of a vector potential. This can be expressed as

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{7.63}$$

The curl of an electric field is not zero. Therefore, the relation $\mathbf{E} = -\nabla V$ is not adequate to determine time-varying fields. An additional parameter is required to derive time-varying fields. The electric field with an additional parameter is

$$\mathbf{E} = -\nabla V + \mathbf{G}. \tag{7.64}$$

Taking the curl of the Eq. (7.64) yields

$$\nabla \times \mathbf{E} = -\nabla \times (\nabla V) + \nabla \times \mathbf{G}, \tag{7.65}$$

$$\nabla \times \mathbf{E} = 0 + \nabla \times \mathbf{G}. \tag{7.66}$$

According to Maxwell's equation, Eq. (7.66) can be expressed as

$$\nabla \times \mathbf{E} = \nabla \times \mathbf{G} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{7.67}$$

Substituting Eq. (7.63) into Eq. (7.67) yields

$$\nabla \times \mathbf{G} = -\frac{\partial(\nabla \times \mathbf{A})}{\partial t}, \tag{7.68}$$

$$\nabla \times \mathbf{G} = -\nabla \times \frac{\partial \mathbf{A}}{\partial t}, \tag{7.69}$$

$$\mathbf{G} = -\frac{\partial \mathbf{A}}{\partial t}. \tag{7.70}$$

Substituting Eq. (7.70) into Eq. (7.64) provides

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}. \tag{7.71}$$

The Maxwell's equation (7.39) can be modified as

$$\nabla \times \frac{\mathbf{B}}{\mu} = \mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t}. \tag{7.72}$$

Substituting Eqs. (7.63) and (7.71) into Eq. (7.72) yields

$$\nabla \times \frac{1}{\mu}(\nabla \times \mathbf{A}) = \mathbf{J} + \varepsilon \frac{\partial}{\partial t}\left[-\nabla V - \frac{\partial \mathbf{A}}{\partial t}\right], \tag{7.73}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \mathbf{J}\mu + \mu\varepsilon \frac{\partial}{\partial t}\left[-\nabla V - \frac{\partial \mathbf{A}}{\partial t}\right]. \tag{7.74}$$

Applying vector rules to the left-hand side of Eq. (7.74) provides

$$\nabla(\nabla.\mathbf{A}) - \nabla^2 \mathbf{A} = \mathbf{J}\mu - \mu\varepsilon\nabla\frac{\partial V}{\partial t} - \mu\varepsilon\frac{\partial^2 \mathbf{A}}{\partial t^2}. \tag{7.75}$$

From Eq. (7.75), the following identity can be written as

$$\nabla.\mathbf{A} = -\mu\varepsilon\frac{\partial V}{\partial t}, \tag{7.76}$$

$$\nabla^2 \mathbf{A} = -\mathbf{J}\mu + \mu\varepsilon\frac{\partial^2 \mathbf{A}}{\partial t^2}. \tag{7.77}$$

The Maxwell's equation (7.19) can be modified as

$$\nabla.\varepsilon\mathbf{E} = \rho_v. \tag{7.78}$$

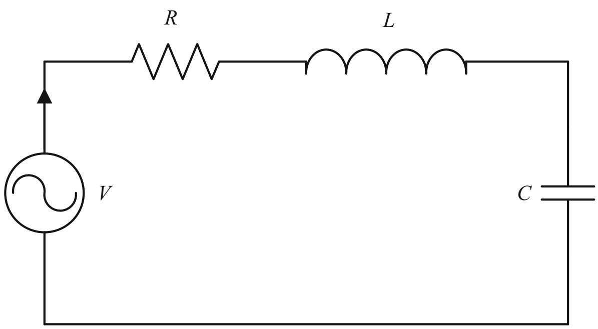


Fig. 7.4 RLC series circuit

Substituting Eq. (7.71) into Eq. (7.78) yields

$$\nabla.\left(-\nabla V - \frac{\partial \mathbf{A}}{\partial t}\right) = \frac{\rho_v}{\varepsilon}, \tag{7.79}$$

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla.\mathbf{A}) = -\frac{\rho_v}{\varepsilon}. \tag{7.80}$$

Substituting Eq. (7.76) into Eq. (7.80) yields

$$\nabla^2 V + \frac{\partial}{\partial t}\left(-\mu\varepsilon\frac{\partial V}{\partial t}\right) = -\frac{\rho_v}{\varepsilon}, \tag{7.81}$$

$$\nabla^2 V - \mu\varepsilon\frac{\partial^2 V}{\partial t^2} = -\frac{\rho_v}{\varepsilon}, \tag{7.82}$$

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} + \mu\varepsilon\frac{\partial^2 V}{\partial t^2}. \tag{7.83}$$

Equations (7.80) and (7.83) are related to the wave equations which will be analysed further in Chap. 9.

7.9 Field of a Series Circuit

Figure 7.4 shows an RLC series circuit with a voltage source. Based on the circuit theory, the Kirchhoff's voltage law (KVL) equation of the circuit is

$$V = RI + L\frac{dI}{dt} + \frac{1}{C}\int I dt. \tag{7.84}$$

The resistive drop can be expressed as

$$RI = \frac{l}{\sigma A}I. \tag{7.85}$$

Substituting the expression of current density ($J = \frac{I}{A}$) into Eq. (7.85) provides

$$RI = \frac{J}{\sigma} l. \tag{7.86}$$

In terms of closed-loop integration, Eq. (7.86) can be expressed as

$$RI = \oint \frac{\mathbf{J}}{\sigma} . d\mathbf{l}. \tag{7.87}$$

The inductive drop can be expressed as

$$L\frac{dI}{dt} = \frac{d(LI)}{dt} = \frac{d\phi}{dt}. \tag{7.88}$$

Substituting the expression of flux or flux linkage ($\phi = \int_s \mathbf{B}.d\mathbf{S}$) into Eq. (7.88) provides

$$L\frac{dI}{dt} = \frac{d}{dt}\int_s \mathbf{B}.d\mathbf{S}. \tag{7.89}$$

Substituting Eq. (7.63) into Eq. (7.89) yields

$$L\frac{dI}{dt} = \frac{d}{dt}\int_s (\nabla \times \mathbf{A}).d\mathbf{S}. \tag{7.90}$$

Applying Stokes theorem to Eq. (7.90) provides

$$L\frac{dI}{dt} = \frac{d}{dt}\oint \mathbf{A}.d\mathbf{l}, \tag{7.91}$$

$$L\frac{dI}{dt} = \oint \frac{\partial \mathbf{A}}{\partial t}.d\mathbf{l}. \tag{7.92}$$

The expression of current is

$$I = \frac{dQ}{dt}, \tag{7.93}$$

$$dQ = Idt, \tag{7.94}$$

$$Q = \int Idt. \tag{7.95}$$

The capacitive drop can be expressed as

$$\frac{1}{C}\int Idt = \frac{Q}{C}. \tag{7.96}$$

Substituting the expression of capacitance into Eq. (7.96) yields

$$\frac{1}{C}\int Idt = \frac{Q}{\frac{\varepsilon A}{d}} = \frac{Qd}{\varepsilon A}. \tag{7.97}$$

Again, substituting the electric flux density ($D = \frac{Q}{A}$) into Eq. (7.97) provides

$$\frac{1}{C}\int Idt = \frac{Q}{\frac{\varepsilon A}{d}} = \frac{Dd}{\varepsilon}, \tag{7.98}$$

$$\frac{1}{C}\int Idt = Ed. \tag{7.99}$$

Substituting the expression of $E = -\nabla V$ into Eq. (7.99) yields

$$\frac{1}{C}\int Idt = -\nabla Vd, \tag{7.100}$$

$$\frac{1}{C}\int Idt = -\oint \nabla V.d\mathbf{l}. \tag{7.101}$$

Substituting Eqs. (7.87), (7.92) and (7.101) into Eq. (7.84) provides the final expression of the field of a series

$$\oint \mathbf{E}.d\mathbf{l} = \oint \frac{\mathbf{J}}{\sigma}.d\mathbf{l} + \oint \frac{\partial \mathbf{A}}{\partial t}.d\mathbf{l} - \oint \nabla V.d\mathbf{l}. \tag{7.102}$$

7.10 Time-Harmonic Fields

The basic concept of phasors has already been discussed in the electrical AC circuit course. This concept is needed to be reviewed here before applying it in electromagnetic fields. The phasor is a complex term having both magnitude and phase angle. It is often known as a vector due to its magnitude and phase angle. Any phasor, m, in rectangular and polar forms can be represented as

$$m = a + jb = n\underline{|\theta}, \tag{7.103}$$

where
$n = \sqrt{a^2 + b^2}$ is the magnitude,
$\theta = \tan^{-1}\left(\frac{b}{a}\right)$ is the phase angle.
The phasor in exponential and sinusoidal forms can be represented as

$$m = ne^{j\theta} = n(\cos\theta + j\sin\theta). \tag{7.104}$$

Again, consider any two phasors to discuss phasor addition, subtraction, multiplication, division, power and conjugate. These phasors are

$$m_1 = a_1 + jb_1 = n_1\underline{|\theta}_1, \tag{7.105}$$

$$m_2 = a_2 + jb_2 = n_2\underline{|\theta}_2. \tag{7.106}$$

The phasor addition and subtraction are

$$m_1 + m_2 = (a_1 + a_2) + j(b_1 + b_2), \tag{7.107}$$

$$m_1 - m_2 = (a_1 - a_2) + j(b_1 - b_2). \tag{7.108}$$

The phasor multiplication and division are

$$m_1 m_2 = n_1 \underline{|\theta_1} \times n_2 \underline{|\theta_2} = n_1 n_2 \underline{|\theta_1 + \theta_2}, \tag{7.109}$$

$$\frac{m_1}{m_2} = \frac{n_1}{n_2} \frac{\underline{|\theta_1}}{\underline{|\theta_2}} = \frac{n_1}{n_2} \underline{|\theta_1 - \theta_2}. \tag{7.110}$$

The power of any phasor is

$$m^p = (n\underline{|\theta})^p = n^p \underline{|p\theta}. \tag{7.111}$$

The conjugate of any phasor is

$$m^* = (n\underline{|\theta})^* = n\underline{|-\theta}. \tag{7.112}$$

A time-harmonic field is defined as any field whose time variation is sinusoidal. Consider the angle θ in terms of time element for a detailed analysis. The expression of the angle is

$$\theta = \omega t + \phi. \tag{7.113}$$

The following analysis can be carried out using circuit theory as

$$ne^{i\theta} = n^{j(\omega t+\phi)} = n\cos(\omega t + \phi) + jn\sin(\omega t + \phi). \tag{7.114}$$

The real and imaginary parts of Eq. (7.114) are $n\cos(\omega t + \phi)$ and $n\sin(\omega t + \phi)$, respectively. Consider the phasor form of the instantaneous current $I(t)$ is $\mathbf{I}_s$ and the expression of this current is

$$\mathbf{I}_s = I_0 e^{j\phi}. \tag{7.115}$$

Again, consider the instantaneous value of the current is

$$I(t) = I_0 \cos(\omega t + \phi) = \text{Re}[I_0 e^{j(\omega t+\phi)}] = \text{Re}[\mathbf{I}_s e^{j\omega t}]. \tag{7.116}$$

Instantaneous and phasor forms of field vectors are defined as

$$\mathbf{E}(x, y, z, t) = \mathbf{E}_s(x, y, z)e^{j\omega t}, \tag{7.117}$$

$$\mathbf{D}(x, y, z, t) = \mathbf{D}_s(x, y, z)e^{j\omega t}, \tag{7.118}$$

$$\mathbf{H}(x, y, z, t) = \mathbf{H}_s(x, y, z)e^{j\omega t}, \tag{7.119}$$

$$\mathbf{B}(x, y, z, t) = \mathbf{B}_s(x, y, z)e^{j\omega t}. \tag{7.120}$$

Based on Eq. (7.116), Eq. (7.117) can be written as

$$\mathbf{E}(x, y, z, t) = \text{Re}[\mathbf{E}_s(x, y, z)e^{j\omega t}]. \tag{7.121}$$

Taking partial derivative of Eq. (7.121) yields

$$\frac{\partial \mathbf{E}(x, y, z, t)}{\partial t} = \text{Re}[j\omega \mathbf{E}_s(x, y, z)e^{j\omega}] = j\omega \mathbf{E}(x, y, z, t). \tag{7.122}$$

Equation (7.122) can also be expressed as

$$\frac{\partial \mathbf{E}(x, y, z, t)}{\partial t} = \text{Re}[\omega \mathbf{E}_s(x, y, z)e^{j\frac{\pi}{2}}e^{j\omega}]. \tag{7.123}$$

Then integrating Eq. (7.121) provides

$$\int \mathbf{E}(x, y, z, t)\partial t = \int \text{Re}[\mathbf{E}_s(x, y, z)e^{j\omega t}]\partial t, \tag{7.124}$$

$$\int \mathbf{E}(x, y, z, t)\partial t = \text{Re}\left[\frac{1}{j\omega}\mathbf{E}_s(x, y, z)e^{j\omega t}\right] = \frac{\mathbf{E}(x, y, z, t)}{j\omega}. \tag{7.125}$$

Equation (7.125) can also be expressed as

$$\int \mathbf{E}(x, y, z, t)\partial t = \text{Re}\left[\frac{1}{\omega}\mathbf{E}_s(x, y, z)e^{-j\frac{\pi}{2}}e^{j\omega t}\right]. \tag{7.126}$$

From Eqs. (7.122) and (7.125), it is concluded that the time derivatives are multiplied by $j\omega$ and the integration is divided by $j\omega$.

Maxwell's equations in time-harmonic form are

$$\oint \mathbf{E}.d\mathbf{l} = -j\omega \int \mathbf{B}.d\mathbf{S}, \tag{7.127}$$

$$\oint \mathbf{H}.d\mathbf{l} = \int \mathbf{J}.d\mathbf{S} + j\omega \int \mathbf{D}.d\mathbf{S}, \tag{7.128}$$

$$\oint \mathbf{B}.d\mathbf{S} = 0, \tag{7.129}$$

$$\oint \mathbf{D}.d\mathbf{S} = \int \rho_v dV. \tag{7.130}$$

In point form, Maxwell's equations are

$$\nabla \times \mathbf{E} = -j\omega \mathbf{B}, \tag{7.131}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega \mathbf{D}, \tag{7.132}$$

$$\nabla.\mathbf{B} = 0, \tag{7.133}$$

$$\nabla.\mathbf{D} = \rho_v. \tag{7.134}$$

Equation (7.132) can be expanded as

$$\nabla \times \mathbf{H} = \sigma\mathbf{E} + j\omega\varepsilon\mathbf{E}, \tag{7.135}$$

$$\nabla \times \mathbf{H} = (\sigma + j\omega\varepsilon)\mathbf{E}. \tag{7.136}$$

Consider the complex permittivity ε_c is

$$j\omega\varepsilon_c = \sigma + j\omega\varepsilon, \tag{7.137}$$

$$\varepsilon_c = \varepsilon + \frac{\sigma}{j\omega} = \varepsilon - j\frac{\sigma}{\omega}. \tag{7.138}$$

From Eq. (7.138), it is seen that complex permittivity is related to the permittivity of the media and imaginary ratio of the conductivity and angular frequency.

Example 7.5 A time-harmonic field and a phasor is given by $\mathbf{M} = 2\cos(10^9 t + 5x + 45°)$ and $\mathbf{N}_s = 5j\mathbf{a}_y + 8e^{\frac{j\pi x}{3}}\mathbf{a}_z$. Determine the phasor form of $\mathbf{M}$ and the instantaneous form of $\mathbf{N}_s$.

Solution The phasor form of $\mathbf{M}$ can be determined as

$$\mathbf{M} = \text{Re}[2e^{j(10^9 t + 5x + 45°)}],$$

$$\mathbf{M} = \text{Re}[2e^{j(10^9 t)}e^{j(5x+45°)}],$$

$$\mathbf{M} = \text{Re}[2e^{j\omega t}e^{j(5x+45°)}],$$

where $\omega = 10^9$ and the final expression is

$$\mathbf{M} = \text{Re}[\mathbf{M}_s e^{j\omega t}],$$

where

$$\mathbf{M}_s = 2e^{j(5x+45°)}.$$

The instantaneous form of $\mathbf{N}_s$ can be determined as

$$\mathbf{N}_s = 5j\mathbf{a}_y + 8e^{\frac{j\pi x}{3}}\mathbf{a}_z = 5e^{\frac{j\pi}{2}}\mathbf{a}_y + 8e^{\frac{j\pi x}{3}}\mathbf{a}_z,$$

$$\mathbf{N} = \text{Re}[\mathbf{N}_s e^{j\omega t}],$$

$$\mathbf{N} = \text{Re}[(5e^{\frac{j\pi}{2}}\mathbf{a}_y + 8e^{\frac{j\pi x}{3}}\mathbf{a}_z)e^{j\omega t}],$$

$$\mathbf{N} = \text{Re}[5e^{j\left(\frac{\pi}{2}+\omega t\right)}\mathbf{a}_y + 8e^{j\left(\frac{\pi x}{3}+\omega t\right)}\mathbf{a}_z],$$

$$\mathbf{N} = 5\cos\left(\frac{\pi}{2} + \omega t\right)\mathbf{a}_y + 8\cos\left(\frac{\pi x}{3} + \omega t\right)\mathbf{a}_z,$$

$$\mathbf{N} = -5\sin\omega t\mathbf{a}_y + 8\cos\left(\frac{\pi x}{3} + \omega t\right)\mathbf{a}_z.$$

Practice Problem 7.5 A time-harmonic field and a phasor is given by $Q = 12\cos(10^7 t - 3x - 65°)$ and $\mathbf{R}_s = 9e^{\frac{jx}{2}} j\mathbf{a}_x + 12e^{\frac{-j\pi x}{5}}\mathbf{a}_y$. Calculate the phasor form of Q and the instantaneous form of $\mathbf{R}_s$.

Example 7.6 The electric and magnetic fields in a free space are given by $\mathbf{E} = 5\cos(\omega t + 3y)\mathbf{a}_x$ V/m and $\mathbf{H} = \frac{5}{\eta}\cos(\omega t + 3y)\mathbf{a}_z$ A/m. Determine ω and η using Maxwell's equation.

Solution Faraday's law for electric and magnetic fields is

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}.$$

The curl of the electric field is

$$\nabla \times \mathbf{E} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = (0)\mathbf{a}_x - \left(0 - \frac{\partial E_x}{\partial z}\right)\mathbf{a}_y + \left(-\frac{\partial E_x}{\partial z}\right)\mathbf{a}_z,$$

$$\nabla \times \mathbf{E} = -\frac{\partial E_x}{\partial z}\mathbf{a}_z = -\frac{\partial}{\partial z}[5\cos(\omega t + 3y)]\mathbf{a}_z = 15\sin(\omega t + 3y)\mathbf{a}_z.$$

The time differentiation of the magnetic field is

$$\mu_0 \frac{\partial \mathbf{H}}{\partial t} = -\frac{\mu_0 5\omega}{\eta}\sin(\omega t + 3y)\mathbf{a}_z.$$

Then, the following relation can be written as:

$$15 = \frac{5\mu_0\omega}{\eta},$$

$$\omega = \frac{3\eta}{\mu_0}.$$

The curl of the magnetic field is

$$\nabla \times \mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & H_x \end{vmatrix} = \frac{\partial H_z}{\partial y}\mathbf{a}_x - \frac{\partial H_z}{\partial x}\mathbf{a}_y,$$

$$\nabla \times \mathbf{H} = -\frac{\partial H_z}{\partial y}\mathbf{a}_x = -\frac{\partial}{\partial y}\left[\frac{5}{\eta}\cos(\omega t + 3y)\right]\mathbf{a}_x = -\frac{15}{\eta}\sin(\omega t + 3y)\mathbf{a}_x.$$

The time differentiation of the electric field is

$$\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = -\mu_0 5\omega \sin(\omega t + 3y)\mathbf{a}_x.$$

Then, the following relation can be written as

$$\frac{15}{\eta} = 5\varepsilon_0\omega,$$

$$\frac{15}{\eta} = 5\varepsilon_0\frac{3\eta}{\mu_0},$$

$$\eta = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \Omega.$$

Practice Problem 7.6 In a dielectric medium ($\varepsilon = 6\varepsilon_0$, $\mu = 1.5\mu_0$), the electric and magnetic fields are given by $\mathbf{E} = 15\cos(\omega t - 0.8z)\mathbf{a}_x$ V/m and $\mathbf{H} = \frac{15}{\eta}\cos(\omega t - 0.8z)\mathbf{a}_y$ A/m. Calculate ω and η using Maxwell's equation.

7.11 Exercise Problems

7.1 Figure 7.5 shows a conducting bar which slides over two conducting rails. The bar moves from its initial position to a specific position at a distance of 8 cm. The vertical distance between two conducting rails is 4 cm. Consider the magnetic field is $\mathbf{B} = 1.6\sin 1000t\mathbf{a}_z$ mW/m^2. Determine the induced voltage.

7.2 A square conductor has a cross-sectional area of 0.9 m^2 which rotates at 120 rad/s in a magnetic field. Determine the maximum induced voltage if the conductor has 25 turns and the value of the magnetic field is 0.09 T.

7.3 A circular conductor loop having a radius of 0.4 m^2, lies in the y-plane of the magnetic field. Determine the induced voltage of the conductor if the magnetic field is $-100\sin 314t\mathbf{a}_y$ Wb/m^2.

7.4 Figure 7.6 shows a conducting bar moving over conducting rails. The uniform magnetic field of 0.5 T is directed to the paper. Determine the speed of the conducting bar which generates a current of 0.4 A in the 5 Ω resistor.

7.5 An oil tanker, a long vehicle travels at a speed of 100 km/h in the 401 series highway of Canada. The earth's magnetic field is found to be 0.05 T and the length of the oil tanker is 2 m. The angle between the magnetic field and the normal to the oil tanker is measured to be 45°. Determine the induced voltage.

7.6 The separation distance and area of the plates are 3 mm and 4 cm^2, respectively. The voltage source of $V = 5\sin 314t\ V$ is applied across the parallel plates of a capacitor. Determine the displacement current and density if $\varepsilon = 1.5\varepsilon_0$.

7.7 A sample of seawater is characterized by the properties $\mu = \mu_0, \varepsilon = 80\varepsilon_0$ and 25 S/m. Calculate the conduction and displacement current densities if $\mathbf{E} = 5\cos 314t$ mV/m.

7.8 The expression for magnetic field intensity is given by $\mathbf{H} = 25\cos(10^8 t - \beta x)\mathbf{a}_z$ A/m. Use Maxwell's equation to determine expressions for **B**, **D** and **E**. Consider the permittivity $\varepsilon = 3 \times 10^{-6}\ F/m$.

Fig. 7.5 Conductor bar with slides

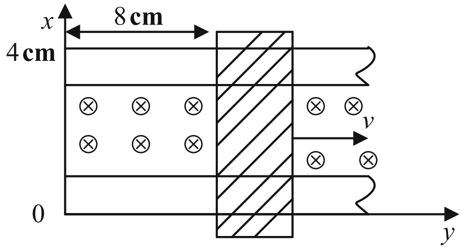
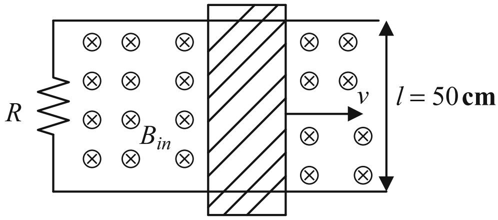


Fig. 7.6 Conductor bar with rails



7.9 A transformer having a uniform cross section of $4\,\text{cm}^2$ is connected to a 110 V, 60 Hz source. Determine the induced voltage in the secondary coil which contains 300 turns and if the magnetic flux density is 0.04 T.

7.10 A cross-sectional area of a 240 V, 50 Hz transformer is $9\,\text{cm}^2$. Calculate the secondary turns if the magnetic flux density is 0.9 T and the induced voltage is 60 V.

7.11 A phasor is given by $\mathbf{A}_s = 15j\mathbf{a}_x + 4e^{-j35^\circ}\mathbf{a}_y$. Determine the instantaneous value of the phasor $\mathbf{A}_s$.

7.12 A time-harmonic field is given by $\mathbf{E} = 5\sin(\omega t + 4x + 10^\circ)\mathbf{a}_x + \cos(\omega t - 4x - 15^\circ)\mathbf{a}_y$. Determine its phasor form.

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Chapter 8
Transmission Lines

8.1 Introduction

The transmission line consists of two or more parallel wires, which is used to transmit electrical energy from the generating station to the distribution system with a low frequency. Coaxial cables, parallel-wire transmission lines, and microstrip lines are used for power transmission. Using high insulation cable reduces electromagnetic interference during power transmission. The parallel-wire transmission line is used for overhead transmission and distribution networks, but the electromagnetic interference increases in this case. The microstrip line is used in the printed circuit board. The transmission line is also used in communication systems with high frequency and is also interconnected between neighboring networks to transfer electrical power under normal and emergency conditions. There are four parameters, namely, resistance, inductance, capacitance, and conductance in the transmission line. In the transmission line, the resistance is given as ohm per meter (Ω/m) or ohm per mile (Ω/mi), inductance is given as henrys per meter (H/m) or henrys per mile (H/mi), capacitance is in farad per meter (F/m), and conductance is in siemens per meter (S/m). The existence of these line parameters are found when the electric and magnetic fields are around the conductor. The transmission line conductors operate in transverse electromagnetic (TEM) mode, which means neither component of **E** nor **H** is in the direction of transmission. In this chapter, transmission line equation, velocity of wave propagation, wavelength, lossless propagation, distortionless transmission line, power, Smith chart, etc. is discussed.

8.2 Transmission Line Equation

Go and return conductors are required to transfer voltage from the sending end terminal to the receiving end terminal. In this case, detailed analysis of the transmission line is required.

The parallel wire and the differential sections of a transmission line are shown in Fig. 8.1. The parallel wire is divided into different sections, each of length Δz. The

Md. A. Salam, *Electromagnetic Field Theories for Engineering*,
DOI 10.1007/978-981-4585-66-8_8,

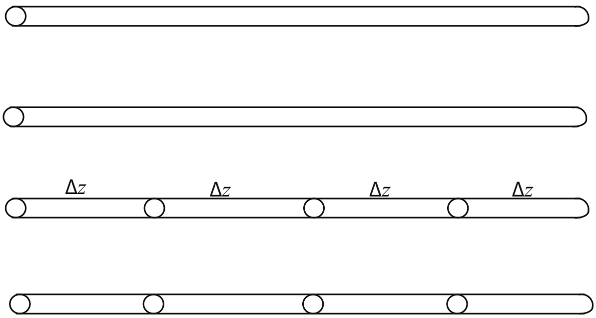


Fig. 8.1 Parallel wire and differential lengths

per unit equivalent circuit of a transmission line is shown in Fig. 8.2. The transmission line is represented by the resistance and inductance, and forms a capacitance along with the return conductor. The current, voltage, and short length of transmission line are shown in Fig. 8.2. In addition, a conductance is considered in parallel with the capacitor to derive the transmission line equation.

Applying Kirchhoff's voltage law (KVL) to the circuit in Fig. 8.2 provides the equation

$$-V(z,t) + RI(z,t)\Delta z + L\Delta z\frac{\partial I(z,t)}{\partial t} + V(z+\Delta z,t) = 0, \tag{8.1}$$

$$V(z+\Delta z,t) - V(z,t) = -\left[RI(z,t) + L\frac{\partial I(z,t)}{\partial t}\right]\Delta z, \tag{8.2}$$

$$\frac{V(z+\Delta z,t) - V(z,t)}{\Delta z} = -\left[RI(z,t) + L\frac{\partial I(z,t)}{\partial t}\right]. \tag{8.3}$$

The differential length Δz is very small compared with the actual length. Then setting $\Delta z \to 0$ into Eq. (8.3) provides

$$\lim_{\Delta z\to 0}\frac{V(z+\Delta z,t) - V(z,t)}{\Delta z} = -\left[RI(z,t) + L\frac{\partial I(z,t)}{\partial t}\right], \tag{8.4}$$

$$\frac{\partial V(z,t)}{\partial z} = -\left[RI(z,t) + L\frac{\partial I(z,t)}{\partial t}\right]. \tag{8.5}$$

Again, applying Kirchhoff's current law (KCL) to the circuit in Fig. 8.2 provides

$$I(z,\ t) = I_g + I_c + I(z+\Delta z,\ t), \tag{8.6}$$

$$I(z+\Delta z,\ t) - I(z,\ t) = -\left[G\Delta z V(z+\Delta z,\ t) + C\Delta z\frac{\partial V(z+\Delta z,\ t)}{\partial t}\right], \tag{8.7}$$

$$\frac{I(z+\Delta z,\ t) - I(z,\ t)}{\Delta z} = -\left[GV(z+\Delta z,\ t) + C\frac{\partial V(z+\Delta z,\ t)}{\partial t}\right]. \tag{8.8}$$

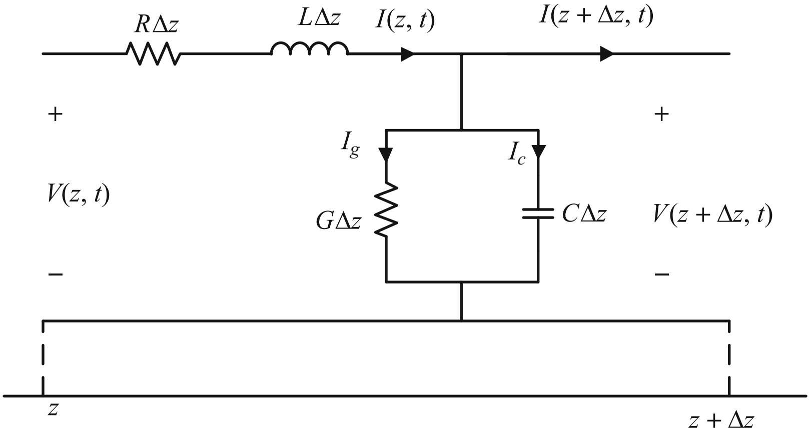


Fig. 8.2 Equivalent circuit of a transmission line with differential length

Again, setting $\Delta z \to 0$ into Eq. (8.8) provides

$$\lim_{\Delta z \to 0} \frac{I(z+\Delta z,\ t) - I(z,\ t)}{\Delta z} = -\lim_{\Delta z \to 0} \left[GV(z+\Delta z,\ t) + C\frac{\partial V(z+\Delta z,\ t)}{\partial t} \right], \tag{8.9}$$

$$\frac{\partial I(z,\ t)}{\partial z} = -\left[GV(z,\ t) + C\frac{\partial V(z,\ t)}{\partial t} \right]. \tag{8.10}$$

Differentiating Eq. (8.5) with respect to z and Eq. (8.10) with respect to t yields

$$\frac{\partial^2 V(z,t)}{\partial z^2} = -\left[R\frac{\partial I(z,t)}{\partial z} + L\frac{\partial^2 I(z,t)}{\partial z \partial t} \right], \tag{8.11}$$

$$\frac{\partial^2 I(z,\ t)}{\partial z \partial t} = -\left[G\frac{\partial V(z,\ t)}{\partial t} + C\frac{\partial^2 V(z,\ t)}{\partial t^2} \right]. \tag{8.12}$$

Substituting Eq. (8.12) into Eq. (8.11) provides

$$\frac{\partial^2 V(z,t)}{\partial z^2} = -\left[R\frac{\partial I(z,t)}{\partial z} - LG\frac{\partial V(z,\ t)}{\partial t} - LC\frac{\partial^2 V(z,\ t)}{\partial t^2} \right]. \tag{8.13}$$

Again, substituting Eq. (8.10) into Eq. (8.13) yields

$$\frac{\partial^2 V}{\partial z^2} = RGV + (RC + LG)\frac{\partial V}{\partial t} + LC\frac{\partial^2 V}{\partial t^2}. \tag{8.14}$$

Differentiating Eq. (8.10) with respect to z and Eq. (8.5) with respect to t yields

$$\frac{\partial^2 I}{\partial z^2} = -\left[G\frac{\partial V}{\partial z} + C\frac{\partial^2 V}{\partial t \partial z} \right], \tag{8.15}$$

$$\frac{\partial^2 V}{\partial t \partial z} = -\left[R\frac{\partial I}{\partial t} + L\frac{\partial^2 I}{\partial t^2}\right]. \tag{8.16}$$

Substituting Eqs. (8.5) and (8.16) into Eq. (8.15) provides

$$\frac{\partial^2 I}{\partial z^2} = -\left[G\left(-RI - L\frac{\partial I}{\partial t}\right) - C\left(R\frac{\partial I}{\partial t} + L\frac{\partial^2 I}{\partial t^2}\right)\right], \tag{8.17}$$

$$\frac{\partial^2 I}{\partial z^2} = RGI + (LG + RC)\,\frac{\partial I}{\partial t} + LC\frac{\partial^2 I}{\partial t^2}. \tag{8.18}$$

Equations (8.14) and (8.18) are the general wave equations for the transmission lines. These wave equations are normally used for energy propagation in free space or in dielectrics.

8.3 Phasor Form Solution of Transmission Line Equation

The transmission line is connected to a source, whose voltage and current waveforms as a function of z and t can be written as

$$V_s(z,\ t) = V_s(z)\cos\omega t = \mathrm{Re}[V_s(z)e^{j\omega t}], \tag{8.19}$$

$$I_s(z,\ t) = I_s(z)\cos\omega t = \mathrm{Re}[I_s(z)e^{j\omega t}]. \tag{8.20}$$

Substituting Eq. (8.20) into Eq. (8.5) provides

$$\frac{dV_s}{dz} = -I_s(R + j\omega L). \tag{8.21}$$

Again, substituting Eq. (8.19) into Eq. (8.10) yields

$$\frac{dI_s}{dz} = -V_s(G + j\omega C). \tag{8.22}$$

Taking differentiation of Eq. (8.21) with respect to z yields

$$\frac{d^2 V_s}{dz^2} = -(R + j\omega L)\frac{dI_s}{dz}. \tag{8.23}$$

Substituting Eq. (8.22) into Eq. (8.23) provides

$$\frac{d^2 V_s}{dz^2} = (R + j\omega L)(G + j\omega C)V_s, \tag{8.24}$$

$$\frac{d^2 V_s}{dz^2} = \gamma^2 V_s. \tag{8.25}$$

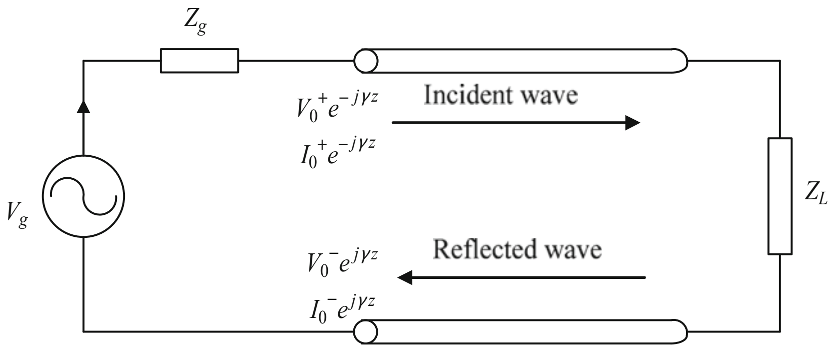


Fig. 8.3 Incident and reflected waves

Differentiating Eq. (8.22) with respect to z provides

$$\frac{d^2 I_s}{dz^2} = -(G + j\omega C)\frac{dV_s}{dz}. \tag{8.26}$$

Substituting Eq. (8.21) into Eq. (8.26) yields

$$\frac{d^2 I_s}{dz^2} = (G + j\omega C)(R + j\omega L)I_s, \tag{8.27}$$

$$\frac{d^2 I_s}{dz^2} - \gamma^2 I_s = 0, \tag{8.28}$$

where the propagation constant of the line is defined as

$$\gamma = \alpha + j\beta = \sqrt{ZY} = \sqrt{(R + j\omega L)(G + j\omega C)}, \tag{8.29}$$

where

α is the attenuation constant in neper per meter (Np/m), (1Np/m=8.68 db/m),
β is the phase constant in radians per meter,
Z is the series impedance in ohm per meter, and
Y is the net shunt admittance in siemens per meter.

The general solutions of Eqs. (8.25) and (8.28) are

$$V_s(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}, \tag{8.30}$$

$$I_s(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}, \tag{8.31}$$

where V_0^+, I_0^+ and V_0^-, I_0^- are the amplitudes of the waveform in the positive and negative z directions, respectively. The terms $V_0^+ e^{-\gamma z}$, $I_0^+ e^{-\gamma z}$ represent incident waves and $V_0^- e^{\gamma z}$, $I_0^- e^{\gamma z}$ represent reflected waves as shown in Fig. 8.3. A standing wave generates in the transmission line when the incident and the reflected waves propagate in opposite directions.

Substituting Eq. (8.30) into Eq. (8.19) provides the instantaneous expression for the voltage

$$V_s(z,\ t) = \text{Re}[(V_0{}^+ e^{-\gamma z} + V_0{}^- e^{\gamma z})e^{j\omega t}], \tag{8.32}$$

$$V_s(z,\ t) = V_0{}^+ e^{-\alpha z}\ \text{Re}[e^{j(\omega t - \beta z)}] + V_0{}^- e^{\alpha z}\text{Re}[e^{j(\omega t + \beta z)}], \tag{8.33}$$

$$V_s(z,\ t) = V_0{}^+ e^{-\alpha z} \cos(\omega t - \beta z) + V_0{}^- e^{\alpha z} \cos(\omega t + \beta z). \tag{8.34}$$

To examine the instantaneous value of the voltage, the following relation can be written as

$$\omega t - \beta z = 2k\pi, \tag{8.35}$$

$$z = \frac{\omega}{\beta}t - \frac{2k\pi}{\beta}. \tag{8.36}$$

Differentiating Eq. (8.36) with respect to t provides

$$\frac{dz}{dt} = \frac{\omega}{\beta}. \tag{8.37}$$

The velocity of the propagation in the z-direction can be written as

$$v = \frac{\omega}{\beta}. \tag{8.38}$$

For a lossless transmission line, i.e., $R = G = 0$ and $\alpha = 0$, Eq. (8.29) can be modified as

$$j\beta = \sqrt{(0 + j\omega L)(0 + j\omega C)}, \tag{8.39}$$

$$\beta = \omega\sqrt{LC}. \tag{8.40}$$

Substituting Eq. (8.40) into Eq. (8.38) yields

$$v = \frac{\omega}{\omega\sqrt{LC}}, \tag{8.41}$$

$$v = \frac{1}{\sqrt{LC}}. \tag{8.42}$$

The wavelength of the wave which results in a phase shift of 2π can be written as

$$\beta\lambda = 2\pi, \tag{8.43}$$

$$\lambda = \frac{2\pi}{\beta}. \tag{8.44}$$

From Eq. (8.44), it is seen that the wavelength is inversely proportional to the phase constant.

8.4 Lossless Propagation

Power utility companies normally want to transfer power to their customer without a loss. To accomplish this task, the value of the resistance and conductance are considered to be zero for transmission lines. Substituting $R = G = 0$ into Eqs. (8.14) and (8.18) yields

$$\frac{\partial^2 V}{\partial z^2} = LC\frac{\partial^2 V}{\partial t^2}, \tag{8.45}$$

$$\frac{\partial^2 I}{\partial z^2} = LC\frac{\partial^2 I}{\partial t^2}. \tag{8.46}$$

Equations (8.45) and (8.46) are the wave equations and the solutions are

$$V(z,t) = V_1(t - \sqrt{LC}z) + V_2(t + \sqrt{LC}z), \tag{8.47}$$

$$I(z,t) = I_1(t - \sqrt{LC}z) + I_2(t + \sqrt{LC}z). \tag{8.48}$$

Differentiating Eq. (8.47) with respect to z provides

$$\frac{\partial V}{\partial z} = -\sqrt{LC}{V_1}'(t - \sqrt{LC}z) + \sqrt{LC}{V_2}'(t + \sqrt{LC}z). \tag{8.49}$$

For a lossless line, Eq. (8.5) can be modified as

$$\frac{\partial V}{\partial z} = -L\frac{\partial I}{\partial t}. \tag{8.50}$$

Substituting Eq. (8.49) into Eq. (8.50) yields

$$\frac{\partial I}{\partial z} = \frac{\sqrt{LC}}{L}V_1'\left(t - \sqrt{LC}z\right) - \frac{\sqrt{LC}}{L}V_2'\left(t + \sqrt{LC}z\right), \tag{8.51}$$

$$\frac{\partial I}{\partial z} = \frac{1}{\sqrt{\frac{L}{C}}}\left[V'_1\left(t - \sqrt{LC}z\right) - V'_2\left(t + \sqrt{LC}z\right)\right]. \tag{8.52}$$

Integrating Eq. (8.52) with respect to t yields

$$I = \frac{1}{\sqrt{\frac{L}{C}}}\left[V_1(t - \sqrt{LC}z) - V_2(t + \sqrt{LC}z)\right], \tag{8.53}$$

$$I = \frac{1}{Z_0}\left[V_1(t - \sqrt{LC}z) - V_2(t + \sqrt{LC}z)\right]. \tag{8.54}$$

The expression of characteristic impedance or resistance is

$$Z_0 = R_0 = \sqrt{\frac{L}{C}}. \tag{8.55}$$

The characteristic impedance can be derived in an alternative way as

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}. \tag{8.56}$$

Setting $R = G = 0$ into Eq. (8.56) provides

$$Z_0 = \sqrt{\frac{L}{C}}.$$

The load voltage and current in terms of the incident and the reflected voltages can be expressed by setting $z = 0$ in Eqs. (8.30) and (8.31) as

$$V_L = {V_0}^+ + {V_0}^-, \tag{8.57}$$

$$I_L = {I_0}^+ + {I_0}^-. \tag{8.58}$$

According to Ohms law, the incident and the reflected currents can be written as

$${I_0}^+ = \frac{{V_0}^+}{R_0}, \tag{8.59}$$

$${I_0}^- = -\frac{{V_0}^-}{R_0}. \tag{8.60}$$

The load resistance can be expressed as

$$R_L = \frac{V_L}{I_L}. \tag{8.61}$$

Substituting Eqs. (8.57) and (8.58) into Eq. (8.61) provides

$$R_L = \frac{{V_0}^+ + {V_0}^-}{{I_0}^+ + {I_0}^-}. \tag{8.62}$$

Substituting Eqs. (8.59) and (8.60) into Eq. (8.62) yields

$$R_L = \frac{{V_0}^+ + {V_0}^-}{\frac{1}{R_0}({V_0}^+ - {V_0}^-)}, \tag{8.63}$$

$$R_L = \frac{1 + \frac{{V_0}^-}{{V_0}^+}}{\frac{1}{R_0}\left(1 - \frac{{V_0}^-}{{V_0}^+}\right)}. \tag{8.64}$$

The voltage reflection coefficient ρ is defined as the ratio of the reflected voltage to the incident voltage; it can be expressed as

$$\rho = \frac{{V_0}^-}{{V_0}^+}. \tag{8.65}$$

Substituting Eq. (8.65) into Eq. (8.64) provides

$$R_L = \frac{R_0(1+\rho)}{(1-\rho)}. \tag{8.66}$$

From Eq. (8.66), solving ρ yields

$$\rho = \frac{R_L - R_0}{R_L + R_0}. \tag{8.67}$$

From Eq. (8.67), it is seen that the voltage reflection coefficient is the ratio of $(R_L - R_0)$ to $(R_L + R_0)$.

Example 8.1 The inductance and the capacitance of a lossless transmission line are 0.7µH/m and 302pF/m, respectively. The source frequency is 1,000Hz and the voltage across the 40Ω load is 120V. Determine (1) the velocity of the wave, (2) characteristic resistance, (3) voltage reflection coefficient, (4) wavelength, (5) phase constant, and (6) incident and reflected voltages.

Solution

1. The velocity of the wave is

$$v = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.7 \times 10^{-6} \times 302 \times 10^{-12}}} = 68.78 \times 10^6 \text{m}.$$

2. The characteristic resistance is

$$R_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.7 \times 10^{-6}}{302 \times 10^{-12}}} = 48.14\Omega.$$

3. The value of the voltage reflection coefficient is

$$\rho = \frac{R_L - R_0}{R_L + R_0} = \frac{40 - 48.14}{40 + 48.14} = -0.09.$$

4. The value of the wavelength is

$$\lambda = \frac{v}{f} = \frac{68.78 \times 10^6}{1000} = 68{,}780\text{m}.$$

5. The value of the phase constant is

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi}{68.78} = 0.091\text{rad/m}.$$

6. The expression of source voltage is

$$V_L = 120 \sin 6283t,$$

$$V_L = {V_0}^+\left(1 + \frac{{V_0}^-}{{V_0}^+}\right),$$

$$120 = {V_0}^+(1 - 0.09),$$

$$\left|{V_0}^+\right| = \frac{120}{1 - 0.09} = 131.87\text{V},$$

$${V_0}^+ = 131.87\sin 6283t\text{V},$$

$$\left|{V_0}^-\right| = 131.87 \times 0.09 = 11.87\text{V},$$

$${V_0}^- = -11.87\sin 6283t\text{V}.$$

Practice problem 8.1 The inductance and the capacitance of a lossless transmission line are 0.97mH/km and 0.012μF/km, respectively. The source frequency is 1,000Hz and the voltage across the 278Ω load is 220V. Calculate (1) the velocity of the wave, (2) characteristic resistance, (3) voltage reflection coefficient, (4) wavelength, (5) phase constant, and (6) incident voltage.

8.5 Low-Loss Transmission Line

The low loss of the transmission line occurs at a higher frequency. In this condition, the relations are defined as $\omega L \geq R$ and $\omega C \geq G$. The expression of the propagation constant can be modified as

$$\gamma = \alpha + j\beta = \sqrt{LC(j\omega)^2\left(1 + \frac{R}{j\omega L}\right)\left(1 + \frac{G}{j\omega C}\right)}, \tag{8.68}$$

$$\gamma = \alpha + j\beta = j\omega\sqrt{LC}\left(1 + \frac{R}{j\omega L}\right)^{\frac{1}{2}}\left(1 + \frac{G}{j\omega C}\right)^{\frac{1}{2}}. \tag{8.69}$$

Expanding Eq. (8.69) and neglecting the higher terms provides

$$\alpha + j\beta = j\omega\sqrt{LC}\left(1 + \frac{R}{j2\omega L}\right)\left(1 + \frac{G}{j2\omega C}\right), \tag{8.70}$$

$$\alpha + j\beta = j\omega\sqrt{LC}\left[1 + \frac{1}{j2\omega}\left(\frac{R}{L} + \frac{G}{C}\right) - \frac{RG}{4\omega^2 LC}\right], \tag{8.71}$$

$$\alpha + j\beta = j\omega\sqrt{LC}\left[1 + \frac{1}{j2\omega}\left(\frac{R}{L} + \frac{G}{C}\right)\right], \tag{8.72}$$

$$\alpha + j\beta = j\omega\sqrt{LC} + \frac{\sqrt{LC}}{2}\left(\frac{R}{L} + \frac{G}{C}\right). \tag{8.73}$$

Now, equating the real and imaginary components of Eq. (8.73) yields

$$\alpha = \frac{\sqrt{LC}}{2}\left(\frac{R}{L} + \frac{G}{C}\right), \tag{8.74}$$

$$\alpha = \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right), \tag{8.75}$$

$$\beta = \omega\sqrt{LC}. \tag{8.76}$$

For low loss (higher frequency), the attenuation and phase constants can be derived in an alternative way. The possible steps are as follows

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}, \tag{8.77}$$

$$\alpha + j\beta = \sqrt{RG + j\omega LG + j\omega RC + (j\omega)^2 LC}, \tag{8.78}$$

$$\alpha + j\beta = j\omega\sqrt{LC}\sqrt{1 + \frac{RG}{(j\omega)^2 LC} + j\omega\frac{LG}{(j\omega)^2 LC} + \frac{j\omega RC}{(j\omega)^2 LC}}, \tag{8.79}$$

$$\alpha + j\beta = j\omega\sqrt{LC}\sqrt{1 - \frac{RG}{\omega^2 LC} + \frac{LG}{j\omega LC} + \frac{RC}{j\omega LC}}. \tag{8.80}$$

If $RG \leq \omega^2 LC$, then Eq. (8.80) can be modified as

$$\alpha + j\beta = j\omega\sqrt{LC}\left(1 + \frac{LG}{j\omega LC} + \frac{RC}{j\omega LC}\right)^{\frac{1}{2}}. \tag{8.81}$$

Expanding Eq. (8.81) in binomial form and neglecting the higher terms provides

$$\alpha + j\beta = j\omega\sqrt{LC}\left[1 + \frac{1}{2j\omega}\left(\frac{LG}{LC} + \frac{RC}{LC}\right)\right], \tag{8.82}$$

$$\alpha + j\beta = \frac{1}{2}\sqrt{LC}\left(\frac{LG}{LC} + \frac{RC}{LC}\right) + j\omega\sqrt{LC}. \tag{8.83}$$

Equating the real and imaginary parts of Eq. (8.83) yields

$$\alpha = \frac{1}{2}\sqrt{LC}\left(\frac{R}{L} + \frac{G}{C}\right), \tag{8.84}$$

$$\alpha = \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right),$$

$$\beta = \omega\sqrt{LC}.$$

From Eq. (8.76), it is concluded that the phase constant is directly proportional to the angular frequency and square root of the product of inductance and capacitance.

Example 8.2 The inductance and the capacitance of a low-loss transmission line are 10mH/m and 56μF/m, respectively. The source frequency is 1,000Hz, while the resistance and the conductance are 40Ω/m and 20S/m, respectively. Determine (1) the attenuation constant and (2) phase constant.

Solution

I. The value of the attenuation constant can be determined as

$$\alpha = \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right) = \frac{1}{2}\left(40\sqrt{\frac{56\times 10^{-6}}{10\times 10^{-3}}} + 20\sqrt{\frac{10\times 10^{-3}}{56\times 10^{-6}}}\right)$$
$$= 135.13\text{db/m}.$$

II. The value of the phase constant is

$$\beta = \omega\sqrt{LC} = 2\pi \times 1000\sqrt{10\times 56\times 10^{-9}} = 4.7\text{rad/m}.$$

Practice problem 8.2 The attenuation and the phase constants of a low-loss transmission line are 140db/m and 5rad/m, respectively. The source frequency is 1,000Hz, while the resistance and the conductance are 30Ω/m and 12S/m, respectively. Determine the value of the inductance and the capacitance.

8.6 Distortionless Line

In distortionless transmission line, the attenuation constant is not frequency dependent and the phase constant is directly proportional to the frequency. In this case, the general expression of the characteristic impedance can be rewritten as

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{L\left(1 - \frac{jR}{\omega L}\right)}{C\left(1 - \frac{jG}{\omega C}\right)}}. \tag{8.85}$$

For low frequency line, the conditions are $\frac{R}{\omega L} \geq 1$ and $\frac{G}{\omega C} \geq 1$. Substituting these conditions into Eq. (8.85) yields

$$Z_0 = \sqrt{\frac{L\left(-\frac{jR}{\omega L}\right)}{C\left(-\frac{jG}{\omega C}\right)}}, \tag{8.86}$$

$$Z_0 = \sqrt{\frac{R}{G}}. \tag{8.87}$$

Comparing Eqs. (8.55) and (8.87) provides

$$\sqrt{\frac{L}{C}} = \sqrt{\frac{R}{G}}, \tag{8.88}$$

$$\frac{L}{C} = \frac{R}{G}. \tag{8.89}$$

Equation (8.89) is known as Heaviside's equation. Equation (8.78) can be modified as

$$\alpha + j\beta = \sqrt{RG\left(1 + \frac{j\omega L}{R} + \frac{j\omega C}{G} + \frac{(j\omega)^2 LC}{RG}\right)}, \tag{8.90}$$

$$\alpha + j\beta = \sqrt{RG}\sqrt{1 + j\omega\left(\frac{L}{R} + \frac{C}{G}\right) - \frac{\omega^2 LC}{RG}}. \tag{8.91}$$

For higher loss (low frequency), $RG \geq \omega^2 LC$, Eq. (8.91) can be modified as

$$\alpha + j\beta = \sqrt{RG}\left[1 + j\omega\left(\frac{L}{R} + \frac{C}{G}\right)\right]^{\frac{1}{2}}. \tag{8.92}$$

Expanding Eq. (8.92) in binomial format and neglecting higher terms yields

$$\alpha + j\beta = \sqrt{RG}\left[1 + \frac{j\omega}{2}\left(\frac{L}{R} + \frac{C}{G}\right)\right], \tag{8.93}$$

$$\alpha + j\beta = \sqrt{RG} + \frac{j\omega}{2}\left(L\sqrt{\frac{G}{R}} + C\sqrt{\frac{R}{G}}\right). \tag{8.94}$$

Separating the real and the imaginary parts of Eq. (8.94) yields

$$\alpha = \sqrt{RG}, \tag{8.95}$$

$$\beta = \frac{\omega}{2}\left(L\sqrt{\frac{G}{R}} + C\sqrt{\frac{R}{G}}\right). \tag{8.96}$$

Substituting Eq. (8.89) into Eq. (8.96) provides

$$\beta = \frac{\omega}{2}\left(L\sqrt{\frac{C}{L}} + C\sqrt{\frac{L}{C}}\right), \tag{8.97}$$

$$\beta = \frac{\omega}{2}(\sqrt{LC} + \sqrt{LC}), \tag{8.98}$$

$$\beta = \omega\sqrt{LC}. \tag{8.99}$$

The expressions of attenuation and phase constants for low-loss and distortionless transmission lines are the same. From Eqs. (8.95) and (8.99), it is concluded that the attenuation constant is not frequency dependent and the phase constant is dependent on the frequency.

Example 8.3 The characteristic impedance and the phase constant of a line are 60Ω and 5rad/m at 120 MHz, respectively. Determine per meter (1) capacitance and (2) inductance.

Solution

1. The characteristic resistance is

$$Z_0 = \sqrt{\frac{L}{C}}. \tag{8.100}$$

Dividing Eq. (8.99) (8.100) provides

$$\frac{\beta}{Z_0} = \frac{\omega\sqrt{LC}}{\sqrt{\frac{L}{C}}}, \tag{8.101}$$

$$\frac{\beta}{Z_0} = \omega C, \tag{8.102}$$

$$C = \frac{\beta}{\omega Z_0} = \frac{5}{2\pi \times 120 \times 10^6 \times 60} = 0.11\,\text{nF/m}.$$

2. The value of the inductance can be determined as

$$L = {Z_0}^2 C = 60^2 \times 0.11 \times 10^{-9} = 0.396\,\mu\text{H/m}.$$

Example 8.4 The inductance and the capacitance of a higher-loss transmission line are 24mH/m and 45μF/m, respectively. The source frequency is 1,000Hz, while the resistance and the conductance are 45Ω/m and 25S/m, respectively. Determine (1) the attenuation constant and (2) phase constant.

Solution

1. The value of the attenuation constant is

$$\alpha = \sqrt{RG} = \sqrt{45 \times 25} = 33.54\,\text{db/m}.$$

2. The value of the phase constant is

$$\beta = \frac{\omega}{2}\left(L\sqrt{\frac{G}{R}} + C\sqrt{\frac{R}{G}}\right) = \frac{2\pi \times 1000}{2}\left(24 \times 10^{-3}\sqrt{\frac{25}{45}} + 45 \times 10^{-6}\sqrt{\frac{45}{25}}\right)$$
$$= 56.39\text{rad/m}.$$

Practice problem 8.3 The characteristic impedance, attenuation constant, and phase constant of a transmission line are 85Ω, $\alpha = 0.07$Np/m, and $\beta = 2.2$rad/m, respectively. Considering the operating frequency is 800 MHz, determine per meter (1) resistance, (2) inductance, and (3) capacitance.

Practice problem 8.4 A high-loss transmission line contains the inductance and the capacitance. The values of these parameters are 14mH/m and 55μF/m, respectively. The source frequency is 1,000Hz, while the resistance and the conductance are 65Ω/m and 35S/m, respectively. Determine (1) the attenuation constant and (2) phase constant.

8.7 Determination of Attenuation Constant

The propagation constant is normally related to the attenuation and the phase constants. The attenuation constant is the real part of the propagation constant of a travelling wave. Mathematically, it can be expressed as

$$\alpha = \mathrm{Re}(\gamma). \tag{8.103}$$

Substituting Eq. (8.77) into Eq. (8.103) yields

$$\alpha = \mathrm{Re}[\sqrt{(R + j\omega L)(G + j\omega C)}]. \tag{8.104}$$

If the waves travel in the $z-$direction, then the reflected waves must be cancelled. Therefore, the terms containing $e^{\gamma z}$ for an infinite line must be cancelled. Then the expressions of the voltage and the current in the $z-$direction by omitting the superscript are

$$V(z) = V_0 e^{-\gamma z} = V_0 e^{-(\alpha + j\beta)z}, \tag{8.105}$$

$$I(z) = I_0 e^{-\gamma z} = \frac{V_0}{Z_0} e^{-(\alpha + j\beta)z}. \tag{8.106}$$

From Eq. (8.106), the conjugate of the current can be expressed as

$$I^*(z) = \frac{V_0}{Z_0} e^{-(\alpha - j\beta)z}. \tag{8.107}$$

The average power propagated along the line in the $z-$direction is

$$P(z) = \frac{1}{2}\mathrm{Re}[V(z)I^*(z)]. \tag{8.108}$$

Substituting Eqs. (8.105) and (8.107) into Eq. (8.108) yields

$$P(z) = \frac{1}{2}\ \mathrm{Re}\left[V_0 e^{-(\alpha + j\beta)z} . \frac{V_0}{Z_0} e^{-(\alpha - j\beta)z}\right], \tag{8.109}$$

$$P(z) = \frac{1}{2}\frac{{V_0}^2}{Z_0} e^{-2\alpha z}. \tag{8.110}$$

The rate of decrease of average power in the $z-$direction is equal to the power loss per unit length and can be written as

$$-\frac{\partial P(z)}{\partial z} = P_L(z). \tag{8.111}$$

Differentiating Eq. (8.110) with respect to z yields

$$\frac{\partial P(z)}{\partial z} = -2\alpha\left(\frac{1}{2}\frac{V_0{}^2}{z_0}e^{-2\alpha z}\right). \tag{8.112}$$

Substituting Eq. (8.110) into Eq. (8.112) yields

$$\frac{\partial P(z)}{\partial z} = -2\alpha P(z). \tag{8.113}$$

Again, substituting Eq. (8.111) into Eq. (8.113) yields

$$P_L(z) = 2\alpha P(z), \tag{8.114}$$

$$\alpha = \frac{P_L(z)}{2P(z)}. \tag{8.115}$$

From Eq. (8.115), it is seen that the attenuation constant is equal to half of the ratio of the power loss per unit length to the average propagated power in the $z-$direction.

Example 8.5 The power loss per unit length and the attenuation constant of a line are found to be 40 W/m and $\alpha = 0.2$ db/m, respectively. Determine the average power propagated along the line in the z–direction.

Solution The value of the average power propagated along the line can be determined as

$$\alpha = \frac{P_L(z)}{2P(z)},$$

$$P(z) = \frac{P_L(z)}{2\alpha} = \frac{40}{2 \times 0.2} = 100\text{W}.$$

Practice problem 8.5 The per unit power loss and the average power propagated along the line are 50 W/m and 120 W, respectively. Calculate the attenuation constant of a line.

8.8 A Finite Transmission Line

A finite transmission line of length, l, propagation constant, γ, and characteristic impedance, Z_0, is connected to a load, Z_L, as shown in Fig. 8.4. The general voltage and current wave Eqs. (8.30) and (8.31), respectively, are used here to analyze the finite transmission line as

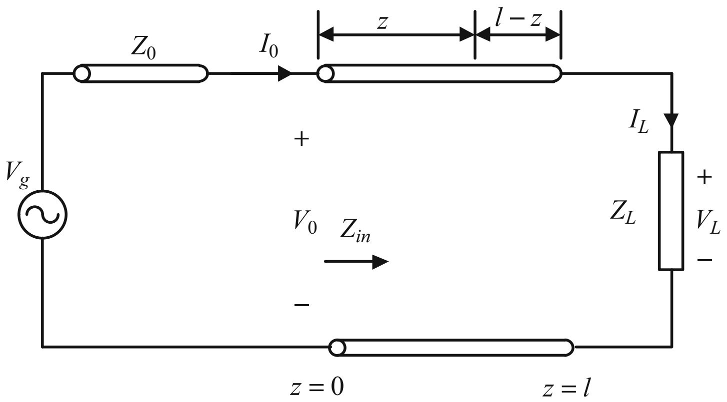
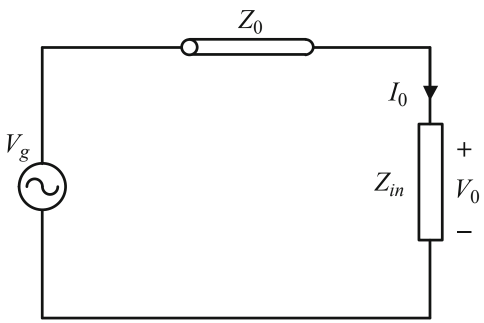


Fig. 8.4 Finite transmission line with load

Fig. 8.5 Equivalent circuit



$$V_s(z) = {V_0}^{+}e^{-\gamma z} + {V_0}^{-}e^{\gamma z}, \tag{8.116}$$

$$I_s(z) = \frac{{V_0}^{+}}{Z_0}e^{-\gamma z} - \frac{{V_0}^{-}}{Z_0}e^{\gamma z}. \tag{8.117}$$

The input voltage, V_0, and the current, I_0, can be determined from Fig. 8.5 as

$$V_0 = \frac{V_g}{Z_g + Z_{in}} Z_{in}, \tag{8.118}$$

$$I_0 = \frac{V_g}{Z_g + Z_{in}}. \tag{8.119}$$

At the beginning of the transmission line, i.e., $z = 0$, the following conditions are written as

$$V_s = V_0, \tag{8.120}$$

$$I_s = I_0. \tag{8.121}$$

Substituting Eqs. (8.120) and (8.121) into Eqs. (8.116) and (8.117) yields

$$V_0 = V_0{}^+ + V_0{}^-, \tag{8.122}$$

$$Z_0 I_0 = V_0{}^+ - V_0{}^-. \tag{8.123}$$

From Eqs. (8.122) and (8.123), the expressions for the incident and the reflected voltages can be derived as

$$V_0{}^+ = \frac{1}{2}(V_0 + I_0 Z_0), \tag{8.124}$$

$$V_0{}^- = \frac{1}{2}(V_0 - I_0 Z_0). \tag{8.125}$$

At the end of the transmission line, i.e., $z = l$, the following conditions are written as

$$V_s = V_L, \tag{8.126}$$

$$I_s = I_L. \tag{8.127}$$

Substituting Eqs. (8.126) and (8.127) into Eqs. (8.116) and (8.117) yields

$$V_L = V_0{}^+ e^{-\gamma l} + V_0{}^- e^{\gamma l}, \tag{8.128}$$

$$Z_0 I_L = V_0{}^+ e^{-\gamma l} - V_0{}^- e^{\gamma l}. \tag{8.129}$$

Again, from Eqs. (8.128) and (8.129), the expressions of the incident and the reflected voltages can be derived as

$$V_0{}^+ = \frac{1}{2}(V_L + I_L Z_0) e^{\gamma l}, \tag{8.130}$$

$$V_0{}^- = \frac{1}{2}(V_L - I_L Z_0) e^{-\gamma l}. \tag{8.131}$$

The following trigonometric formulae are used to finalize the expression

$$\cosh \gamma l = \frac{1}{2}(e^{\gamma l} + e^{-\gamma l}), \tag{8.132}$$

$$\sinh \gamma l = \frac{1}{2}(e^{\gamma l} - e^{-\gamma l}), \tag{8.133}$$

$$\tanh \gamma l = \frac{e^{\gamma l} - e^{-\gamma l}}{e^{\gamma l} + e^{-\gamma l}}. \tag{8.134}$$

Adding Eqs. (8.130) and (8.131) yields

$$V_0{}^+ + V_0{}^- = V_L \frac{e^{\gamma l} + e^{-\gamma l}}{2} + I_L Z_0 \frac{e^{\gamma l} - e^{-\gamma l}}{2}. \tag{8.135}$$

Substituting Eqs. (8.132) and (8.133) into Eq. (8.135) yields

$$V_0{}^+ + V_0{}^- = V_L \cosh \gamma l + I_L Z_0 \sinh \gamma l. \tag{8.136}$$

Subtracting Eq. (8.131) from Eq. (8.130) yields

$$V_0{}^+ - V_0{}^- = V_L \frac{e^{\gamma l} - e^{-\gamma l}}{2} + I_L Z_0 \frac{e^{\gamma l} + e^{-\gamma l}}{2}. \tag{8.137}$$

Substituting Eqs. (8.132) and (8.133) into Eq. (8.137) yields

$$V_0{}^+ - V_0{}^- = V_L \sinh \gamma l + I_L Z_0 \cosh \gamma l. \tag{8.138}$$

The input impedance at any point of the line is

$$Z_{in} = \frac{V_s(z)}{I_s(z)}. \tag{8.139}$$

The input impedance at $z = 0$ is

$$Z_{in} = \frac{V_s(z=0)}{I_s(z=0)} = Z_0 \frac{V_0{}^+ + V_0{}^-}{V_0{}^+ - V_0{}^-}. \tag{8.140}$$

Substituting Eqs. (8.136) and (8.138) into Eq. (8.140) yields

$$Z_{in} = Z_0 \frac{V_L \cosh \gamma l + I_L Z_0 \sinh \gamma l}{V_L \sinh \gamma l + I_L Z_0 \cosh \gamma l}, \tag{8.141}$$

$$Z_{in} = Z_0 \frac{I_L \cosh \gamma l \left(\frac{V_L}{I_L} + Z_0 \frac{\sinh \gamma l}{\cosh \gamma l} \right)}{I_L \cosh \gamma l \left(\frac{V_L}{I_L} \frac{\sinh \gamma l}{\cosh \gamma l} + Z_0 \right)}. \tag{8.142}$$

At the end of the transmission line, the load impedance is expressed as

$$Z_L = \frac{V_L}{I_L}. \tag{8.143}$$

Substituting Eq. (8.143) into Eq. (8.142) yields

$$Z_{in} = Z_0 \frac{Z_L + Z_0 \tanh \gamma l}{Z_0 + Z_L \tanh \gamma l}. \tag{8.144}$$

Equation (8.144) represents the general expression for the input impedance at a distance l from the load.

Again, the voltage reflection coefficient at the load is defined as the ratio of reflected wave to the incident wave and it is represented by ρ. Mathematically, the reflection coefficient can be written as

$$\rho = \frac{V_0{}^{-}e^{\gamma l}}{V_0{}^{+}e^{-\gamma l}}. \tag{8.145}$$

Substituting Eqs. (8.130) and (8.131) into Eq. (8.145) yields

$$\rho = \frac{\frac{1}{2}(V_L - I_L Z_0)e^{-\gamma l}e^{\gamma l}}{\frac{1}{2}(V_L + I_L Z_0)e^{\gamma l}e^{-\gamma l}}, \tag{8.146}$$

$$\rho = \frac{\frac{V_L}{I_L} - Z_0}{\frac{V_L}{I_L} + Z_0}. \tag{8.147}$$

Substituting Eq. (8.143) into Eq. (8.147) yields

$$\rho = \frac{Z_L - Z_0}{Z_L + Z_0}. \tag{8.148}$$

From Eq. (8.148), it is seen that the reflection coefficient is a dimensionless parameter and it is also dependent on the value of the load and characteristic impedances.

Example 8.6 The parameters of a parallel wire telephone line are given as $R = 3$ Ω/km, $G = 0.21\,\mu$S/km, $L = 0.004$ H/km, and $C = 0.006\,\mu$F/km. The load of $Z_L = 30 + j30\,\Omega$ is connected at the end of a 35 km transmission line and the operating frequency is 1,000 Hz. Determine (1) the characteristic impedance, Z_0, (2) propagation constant, γ, and (3) input impedance, Z_{in}.

Solution

1. The value of the characteristic impedance is

$$Z_0 = \sqrt{\frac{(R + j\omega L)}{(G + j\omega C)}} = \sqrt{\frac{3 + j2\pi(1000)(0.004)}{0.21 + j2\pi(1000)(0.006)}} \times 1000,$$

$$Z_0 = \sqrt{0.67\,\underline{|-6.49^\circ}} \times 1000 = 818.54\,\underline{|-3.25^\circ}\ \Omega.$$

2. The value of the propagation constant is

$$\begin{aligned}\gamma &= \sqrt{(R + j\omega L)(G + j\omega C)}\\ &= \sqrt{(3 + j2\pi(1000)(0.004))(0.21 + j2\pi(1000)(0.006))} \times 10^{-3},\\ \gamma &= \sqrt{(3 + j25.13)(0.21 + j37.7)} \times 10^{-3}\\ &= \sqrt{954.14\,\underline{|172.87^\circ}} \times 10^{-3} = 30.89\,\underline{|86.44^\circ} \times 10^{-3},\\ \gamma &= \alpha + j\beta = (1.92 + j30.83) \times 10^{-3} = 0.00192 + j0.0308.\end{aligned}$$

3. The values of the hyperbolic functions are

$$\cosh\gamma l = \cosh[35(0.00192 + j0.0308)] = \cosh(0.0672 + j1.078),$$

$$\cosh\gamma l = \cosh(0.0672)\cos(1.078) + j\sinh(0.0672)\sin(1.078),$$

$$\cosh\gamma l = 0.474 + j0.0592 = 0.477\,\underline{|7.12^\circ},$$

$$\sinh\gamma l = \sinh[35(0.00192 + j0.0308)] = \sinh(0.0672 + j1.078),$$

$$\sinh\gamma l = \sinh(0.0672)\cos(1.078) + j\cosh(0.0672)\sin(1.078),$$

$$\sinh\gamma l = 0.0318 + j0.883 = 0.883\,\underline{|87.94^\circ},$$

$$\tanh\gamma l = \frac{\sinh\gamma l}{\cosh\gamma l} = \frac{0.883\,\underline{|87.94^\circ}}{0.447\,\underline{|7.12^\circ}} = 1.98\,\underline{|80.82^\circ}.$$

The value of the input impedance can be determined as

$$Z_{in} = Z_0\frac{Z_L + Z_0\tanh\gamma l}{Z_0 + Z_L\tanh\gamma l}$$

$$= 818.54\,\underline{|-3.25^\circ}\,\frac{30 + j30 + 818.54\,\underline{|-3.25^\circ} \times 1.98\,\underline{|80.82^\circ}}{818.54\,\underline{|-3.25^\circ} + (30 + j30) \times 1.98\,\underline{|80.82^\circ}},$$

$$Z_{in} = 1764.79\,\underline{|71.93^\circ}\,\Omega.$$

Practice problem 8.6 The parameters of a parallel wire telephone line are given as $R = 5\Omega/\text{km}$, $G = 0.26\mu\text{S}/\text{km}$, $L = 0.0032\text{H}/\text{km}$, and $C = 0.0065\mu\text{F}/\text{km}$. The load impedance $Z_L = 40 + j40\Omega$ is connected at the end of a 25 km transmission line and the operating frequency is 1,000 Hz. Determine (1) the characteristic impedance, Z_0, (2) propagation constant, γ, and (3) input impedance, Z_{in}.

8.9 Input Impedance for Lossless Transmission Line

For a lossless transmission line, the attenuation constant α is zero. Then the propagation constant can be modified as

$$\gamma = \alpha + j\beta = j\beta. \tag{8.149}$$

The following trigonometry formula can be written as

$$\tanh\gamma l = \tanh(j\beta)l = j\tan\beta l. \tag{8.150}$$

From Eq. (8.144), the expression of the input impedance can be modified as

$$Z_{in} = Z_0\frac{Z_L + jZ_0\tan\beta l}{Z_0 + jZ_L\tan\beta l}. \tag{8.151}$$

From Eq. (8.151), it is concluded that the input impedance will be different for different values of β, Z_0, and l. The following conditions are considered to analyze the input impedance:

For a short circuit termination, $Z_L = 0$ and $Z_{in} = Z_{sc}$. Equation (8.151) can be modified as

$$Z_{sc} = Z_0 \frac{0 + jZ_0 \tan \beta l}{Z_0 + 0} = jZ_0 \tan \beta l. \tag{8.152}$$

For an open circuit termination, $Z_L = \infty$ and $Z_{in} = Z_{oc}$. Again, Eq. (8.151) can be modified as

$$Z_{oc} = Z_0 \frac{\infty + jZ_0 \tan \beta l}{Z_0 + j\infty \tan \beta l}, \tag{8.153}$$

$$Z_{oc} = Z_0 \frac{\infty \left(1 + \frac{jZ_0 \tan \beta l}{\infty}\right)}{\infty \left(\frac{Z_0}{\infty} + j \tan \beta l\right)}, \tag{8.154}$$

$$Z_{oc} = Z_0 \frac{1}{j \tan \beta l}. \tag{8.155}$$

Multiplying Eqs. (8.152) and (8.155) yields

$${Z_0}^2 = Z_{sc} Z_{oc}, \tag{8.156}$$

$$Z_0 = \sqrt{Z_{sc} Z_{oc}}. \tag{8.157}$$

Dividing Eq. (8.152) by Eq. (8.155) provides

$$\frac{Z_{sc}}{Z_{oc}} = \frac{jZ_0 \tan(\beta l)}{\dfrac{1}{jZ_0 \tan(\beta l)}}, \tag{8.158}$$

$$\frac{Z_{sc}}{Z_{oc}} = -\tan^2(\beta l). \tag{8.159}$$

From Eqs. (8.157) and (8.159), it is concluded that the characteristic impedance Z_0 and the phase constant β for a given length can be determined if the short circuit and open circuit impedances are given.

For a lossless finite transmission line, consider $z = -l$, $\alpha = 0$, $\gamma = j\beta = j\omega\sqrt{LC}$. Then Eqs. (8.116) and (8.117) become

$$V_s(z = -l) = {V_0}^+ e^{j\omega\sqrt{LC}l} + {V_0}^- e^{-j\omega\sqrt{LC}l}, \tag{8.160}$$

$$I_s(z = -l) = \frac{1}{Z_0}[{V_0}^+ e^{j\omega\sqrt{LC}l} - {V_0}^- e^{-j\omega\sqrt{LC}l}]. \tag{8.161}$$

From Eq. (8.65), substituting the expression ${V_0}^- = {V_0}^+\rho$ into Eqs. (8.160) and (8.161) yields

$$V_s = {V_0}^+ (e^{j\omega\sqrt{LC}l} + \rho e^{-j\omega\sqrt{LC}l}), \tag{8.162}$$

$$I_s = \frac{V_0{}^+}{Z_0}[e^{j\omega\sqrt{LC}l} - \rho e^{-j\omega\sqrt{LC}l}]. \tag{8.163}$$

The maximum and the minimum input impedances can be obtained by the following relations

$$(Z_{in})_{\max} = \frac{V_{\max}}{\mathrm{I}_{\min}}, \tag{8.164}$$

$$(Z_{in})_{\min} = \frac{V_{\min}}{\mathrm{I}_{\max}}. \tag{8.165}$$

The maximum value of the voltage and the minimum value of the current happen when $e^{j\beta l} = e^{-j\beta l} = 1$. Then Eqs. (8.162) and (8.163) can be modified as

$$V_{\max} = \left|V_0{}^+\right|(1 + |\rho|), \tag{8.166}$$

$$I_{\min} = \frac{\left|V_0{}^+\right|}{Z_0}(1 - |\rho|). \tag{8.167}$$

Substituting Eqs. (8.166) and (8.167) into Eq. (8.164) yields

$$Z_{\max} = \frac{Z_0(1 + |\rho|)}{(1 - |\rho|)}. \tag{8.168}$$

For the minimum voltage and the maximum current setting $e^{j\beta l} = 1$ and $e^{-j\beta l} = -1$, Eqs. (8.162) and (8.163) provide

$$V_{\min} = \left|V_0{}^+\right|(1 - |\rho|), \tag{8.169}$$

$$I_{\max} = \frac{\left|V_0{}^+\right|}{Z_0}(1 + |\rho|). \tag{8.170}$$

The standing wave ratio is defined as the ratio of the magnitudes of the maximum voltage to the minimum voltage and it is represented as

$$s = \frac{V_{\max}}{V_{\min}}. \tag{8.171}$$

Substituting Eqs. (8.166) and (8.169) into Eq. (8.171) yields

$$s = \frac{1 + |\rho|}{1 - |\rho|}. \tag{8.172}$$

Equation (8.172) can be modified as

$$|\rho| = \frac{s - 1}{s + 1}. \tag{8.173}$$

If the transmission line is terminated by a load having a length of $l = \frac{\lambda}{4}$, then the value of the phase constant is

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi}{4l}, \tag{8.174}$$

$$\beta l = \frac{2\pi}{4l} l = \frac{\pi}{2}. \tag{8.175}$$

The expression of the input impedance can be modified as

$$Z_{in} = Z_0 \frac{Z_L \cos\left(\frac{\pi}{2}\right) + jZ_0 \sin\left(\frac{\pi}{2}\right)}{Z_0 \cos\left(\frac{\pi}{2}\right) + jZ_L \sin\left(\frac{\pi}{2}\right)}, \tag{8.176}$$

$$Z_{in} = Z_0 \frac{jZ_0}{jZ_L}, \tag{8.177}$$

$$Z_{in} = \frac{{Z_0}^2}{Z_L}. \tag{8.178}$$

From Eq. (8.178), it is concluded that the input impedance is directly proportional to the square of the characteristic impedance and inversely proportional to the load impedance if the transmission line is terminated by a one-quarter long wave length.

Again, consider the transmission line is terminated exactly by one-half of the wave length, i.e., $l = \frac{\lambda}{2}$. Again, the value of the phase constant is

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi}{2l}, \tag{8.179}$$

$$\beta l = \frac{2\pi}{2l} l = \pi. \tag{8.180}$$

The expression of the input impedance can be modified as

$$Z_{in} = Z_0 \frac{Z_L \cos\pi + jZ_0 \sin\pi}{Z_0 \cos\pi + jZ_L \sin\pi}, \tag{8.181}$$

$$Z_{in} = Z_0 \frac{Z_L(-1)}{Z_0(-1)}, \tag{8.182}$$

$$Z_{in} = Z_L. \tag{8.183}$$

From Eq. (8.183), it is seen that the input impedance is equal to the load impedance, regardless of the characteristic impedance or phase constant.

If the load impedance is equal to the characteristic impedance, i.e., $Z_L = Z_0$, then the expression of the input impedance can be modified as

$$Z_{in} = Z_0 \frac{Z_0 \cos\pi + jZ_0 \sin\pi}{Z_0 \cos\pi + jZ_0 \sin\pi}, \tag{8.184}$$

$$Z_{in} = Z_0. \tag{8.185}$$

From Eq. (8.185), it is concluded that the input impedance is equal to the characteristic impedance regardless of the length of the line.

Example 8.7 The open circuit and the short circuit impedances of a 2 m length lossless transmission lines are $-j35\Omega$ and $j80\Omega$, respectively. Determine (1) the characteristic impedance, Z_0, (2) phase constant, β, and (3) short circuit impedance if the length is double of the given length considering the same operating frequency.

Solution

I. The value of the characteristic impedance is determined as

$$Z_0 = \sqrt{Z_{sc}Z_{oc}} = \sqrt{(-j35)(j80)} = 52.92\Omega.$$

II. The value of the phase constant can be determined as

$$\tan \beta l = \sqrt{\frac{-Z_{sc}}{Z_{oc}}},$$

$$\beta = \frac{1}{l}\tan^{-1}\left(\sqrt{\frac{-Z_{sc}}{Z_{oc}}}\right) = \frac{1}{2}\tan^{-1}\left(\sqrt{\frac{j80}{j35}}\right) = 0.493\text{rad/m}.$$

III. The value of the short circuit impedance is determined as

$$\beta l = 0.493(2 \times 2) = 1.972\,\text{rad},$$

$$Z_{sc} = jZ_0 \tan \beta l = j52.92 \tan(1.972) = j52.92(-2.36) = -j124.88\Omega.$$

Practice problem 8.7 A lossless transmission line is 3 m long. The open circuit and the short circuit impedances of this line are $-j55\Omega$ and $j110\Omega$, respectively. Calculate (1) the characteristic impedance, Z_0, (2) phase constant, β, and (3) short circuit impedance if the length is double of the given length considering the same operating frequency.

Example 8.8 A 10 m length lossless transmission line having characteristic impedance of $Z_0 = 30\Omega$ operates at 3 MHz. A load impedance of $Z_L = 40 + j50\Omega$ is used to terminate the transmission line. Assume that the velocity of the light in the vacuum is $v = 1.5 \times 10^8$ m/s. Determine the reflection coefficient, standing wave ratio, and input impedance.

Solution The value of the reflection coefficient can be determined as

$$\rho = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{40 + j50 - 30}{40 + j50 + 30} = 0.59 \angle 43.15^\circ.$$

The value of the standing wave ratio is

$$s = \frac{1 + |\rho|}{1 - |\rho|} = \frac{1 + 0.59}{1 - 0.59} = 3.88.$$

The following parameter can be determined as

$$\beta l = \frac{\omega l}{v} = \frac{2\pi \times 3 \times 10^6 \times 10}{1.5 \times 10^8} = 0.4\pi = 72^\circ.$$

The value of the input impedance can be determined as

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l} = 30 \frac{(40 + j50) + j30 \tan 72^\circ}{30 + j(40 + j50) \tan 72^\circ} = 20.43 \underline{|29.16^\circ} \ \Omega.$$

Practice problem 8.8 The characteristic impedance of a 5 m length lossless transmission line is 20Ω which operates at 2 MHz. A load impedance of $Z_L = 35 + j45\Omega$ is used to terminate the transmission line and the velocity of the light in the vacuum is assumed to be $v = 1.2 \times 10^8$ m/s. Determine the reflection coefficient, standing wave ratio, and input impedance.

8.10 Power of Lossless Transmission Line

The power in a transmission line is the combination of the incident power and the reflected power. Some portions of the power are absorbed by the load and the rest of the power is returned back to the source. At a distance l, from Eqs. (8.116) and (8.17), the expressions for voltage and current can be expressed as

$$V_s(l) = V_0{}^+(e^{j\beta l} + \rho e^{-j\beta l}), \tag{8.186}$$

$$I_s(l) = \frac{V_0{}^+}{Z_0}(e^{j\beta l} - \rho e^{-j\beta l}). \tag{8.187}$$

The conjugate of the current is

$$I_s{}^*(l) = \frac{V_0{}^+}{Z_0}(e^{-j\beta l} - \rho e^{j\beta l}). \tag{8.188}$$

The average power at the input terminals of the transmission line is

$$P_{av} = \frac{1}{2}\mathrm{Re}[V_s(l) I_s{}^*(l)]. \tag{8.189}$$

Substituting Eqs. (8.186) and (8.188) into Eq. (8.189) yields

$$P_{av} = \frac{1}{2}\mathrm{Re}[V_0{}^+(e^{j\beta l} + \rho e^{-j\beta l})\frac{V_0{}^+}{Z_0}(e^{-j\beta l} - \rho e^{j\beta l})], \tag{8.190}$$

$$P_{av} = \frac{\left|V_0{}^+\right|^2}{2Z_0}[1 - |\rho|^2] + \frac{\left|V_0{}^+\right|^2}{2Z_0}\mathrm{Re}[\rho \cos 2\beta l - j\rho \sin 2\beta l - \rho \cos 2\beta l - j\rho \sin 2\beta l], \tag{8.191}$$

$$P_{av} = \frac{|V_0{}^+|^2}{2Z_0}[1 - |\rho|^2], \tag{8.192}$$

$$P_{av} = \frac{|V_0{}^+|^2}{2Z_0} - \frac{|V_0{}^+|^2}{2Z_0}|\rho|^2, \tag{8.193}$$

$$P_{av} = P_{av}{}^i - P_{av}{}^r, \tag{8.194}$$

where the expression of the incident power is

$$P_{av}{}^i = \frac{|V_0{}^+|^2}{2Z_0}. \tag{8.195}$$

And the expression of the reflected power is

$$P_{av}{}^r = |\rho|^2 P_{av}{}^i. \tag{8.196}$$

From Eq. (8.196), it is concluded that the average reflected power is equal to $|\rho|^2$ times the average incident power.

Example 8.9 The load of $Z_L = 80 + j40\Omega$ is connected to a 50Ω transmission line. If $V_0{}^+ = 30$V, determine (1) the reflection coefficient, ρ, (2) standing wave ratio, s, (3) incident power, (4) reflected power, and (5) net power.

Solution

I. The value of the reflection coefficient is calculated as

$$\rho = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{80 + j40 - 50}{80 + j40 + 50} = 0.37\,\underline{|36.03}\,.$$

II. The value of the standing wave ratio can be determined as

$$s = \frac{1 + |\rho|}{1 - |\rho|} = \frac{1 + 0.37}{1 - 0.37} = 2.17.$$

III. The value of the incident power can be calculated as

$$P_{av}{}^i = \frac{|V_0{}^+|^2}{2Z_0} = \frac{40^2}{2 \times 50} = 16\text{W}.$$

IV. The value of the reflected power is determined as

$$P_{av}{}^r = |\rho|^2 P_{av}{}^i = (0.37)^2 \times 16 = 2.19\text{W}.$$

V. The value of the net power is calculated as

$$P_{net} = P_{av}{}^i - P_{av}{}^r = 16 - 2.19 = 13.81\text{W}.$$

Practice problem 8.9 The load of $Z_L = 100 + j60\Omega$ is connected to a 55Ω transmission line. If $V_0{}^+ = 40\text{V}$, determine (1) the reflection coefficient, ρ, (2) standing wave ratio, s, (3) incident power, (4) reflected power, and (5) net power.

8.11 Basics of Smith Chart

The cumbersome manipulation of complex phasor algebra is usually required to determine the impedance, current, voltage, and power of the transmission line. The input impedance of a transmission line depends on three parameters, namely, load impedance, characteristic impedance, and the observation point. The Smith chart is a graphical method that can determine transmission line parameters easily. The Smith chart is a set of coordinates of normalized impedances or admittances within a unit circle whose radius is $\rho = 1$. The reflection coefficient and standing wave ratio do not change in the Smith chart. The normalized impedance or admittance can be obtained by dividing Z_0 or Y_0 of the respective line parameters, e.g., Z_L or Y_L. After normalized, the properties of the parameters remain unchanged. The normalized impedance is

$$z_l = \frac{Z_L}{Z_0} = \frac{R_L}{Z_0} + j\frac{X_L}{Z_0} = r + jx. \tag{8.197}$$

The reflection coefficient is modified as

$$\rho = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{\frac{Z_L}{Z_0} - 1}{\frac{Z_L}{Z_0} + 1}. \tag{8.198}$$

Substituting Eq. (8.197) into Eq. (8.198) yields

$$\rho = \frac{z_l - 1}{z_l + 1}, \tag{8.199}$$

$$\rho = \frac{r + jx - 1}{r + jx + 1}. \tag{8.200}$$

Equation (8.199) can be modified as

$$\rho z_l + \rho = z_l - 1, \tag{8.201}$$

$$(1 - \rho)z_l = 1 + \rho, \tag{8.202}$$

$$z_l = \frac{1 + \rho}{1 - \rho}. \tag{8.203}$$

The rectangular and polar forms of the reflection coefficient are

$$\rho = a + jb = Re^{j\theta}, \tag{8.204}$$

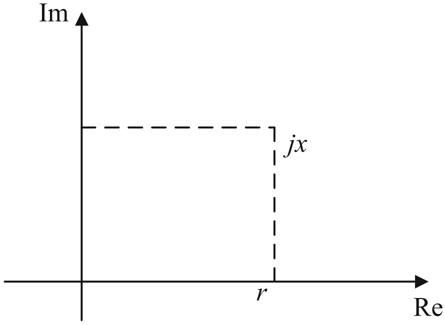


Fig. 8.6 Normalized impedance

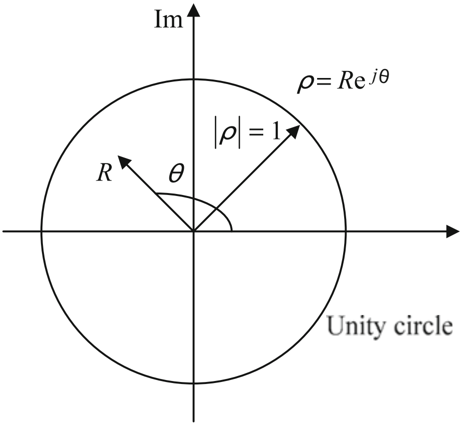


Fig. 8.7 Complex reflection plane with unit circle

where R is the magnitude of the reflection coefficient. The normalized load impedance and the complex reflection coefficient are shown in Figs. 8.6 and 8.7, respectively. Substituting Eqs. (8.197) and (8.204) into Eq. (8.203) yields

$$r + jx = \frac{1 + a + jb}{1 - (a + jb)}, \tag{8.205}$$

$$r + jx = \frac{(1 + a + jb)(1 - a + jb))}{(1 - a)^2 + b^2}, \tag{8.206}$$

$$r[(1 - a)^2 + b^2] + jx[(1 - a)^2 + b^2] = 1 - a + jb + a - a^2 + jab + jb - jab - b^2, \tag{8.207}$$

$$r[(1 - a)^2 + b^2] + jx[(1 - a)^2 + b^2] = 1 - a^2 - b^2 + j2b. \tag{8.208}$$

Separating the real and the imaginary parts of Eq. (8.208) yields

$$r(1 - 2a + a^2 + b^2) = 1 - a^2 - b^2, \tag{8.209}$$

$$a^2(r + 1) + b^2(r + 1) - 2ar + r - 1 = 0, \tag{8.210}$$

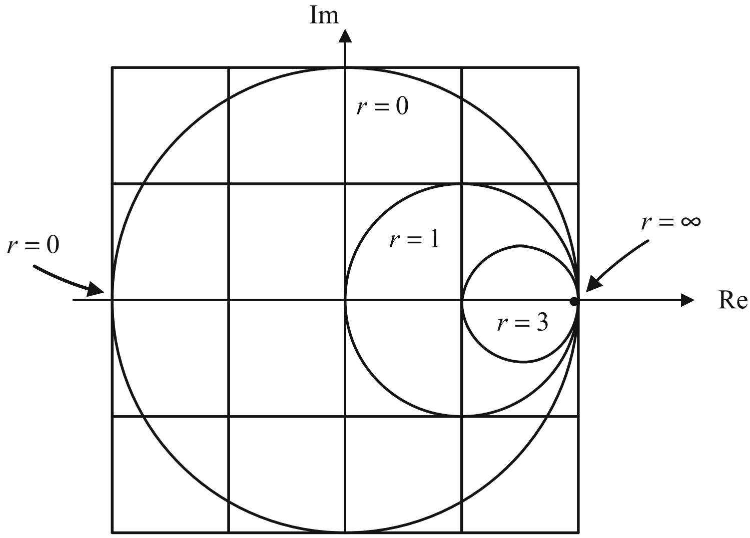


Fig. 8.8 Constant resistance circles with different radius

$$a^2 + b^2 - 2\frac{r}{(r+1)}a + \frac{r-1}{(r+1)} = 0, \quad (8.211)$$

$$\left(a - \frac{r}{r+1}\right)^2 + (b-0)^2 = \frac{r^2}{(r+1)^2} - \frac{r-1}{r+1}, \quad (8.212)$$

$$\left(a - \frac{r}{r+1}\right)^2 + (b-0)^2 = \left(\frac{1}{r+1}\right)^2. \quad (8.213)$$

Equation (8.213) is known as constant resistance circle, whose center is at $\left(\frac{r}{r+1}, 0\right)$ and the radius is $\left(\frac{1}{r+1}\right)$. For different values of r, many circles can be drawn in the constant resistance circle. At $r = 0$, the center and the radius of the circle are found to be (0,0) and 1, respectively. This circle is drawn at the real and imaginary plane as shown in Fig. 8.8. At $r = \infty$, the center of the circle is at (1,0) and the radius is zero as shown in Fig. 8.8. It is found that the constant resistance circle lies on the real line. The constant resistance circles can be drawn by the following conditions:

I. At $r = 1$, center = (0.5,0), radius = 0.5.
II. At $r = 3$, center = (0.25,0), radius = 0.25.
III. At $r = 4$, center = (0.2,0), radius = 0.2.

The imaginary part of Eq. (8.208) is

$$x[(1-a)^2 + b^2] = 2b, \quad (8.214)$$

$$(1-a)^2 + b^2 - \frac{2}{x}b = 0. \quad (8.215)$$

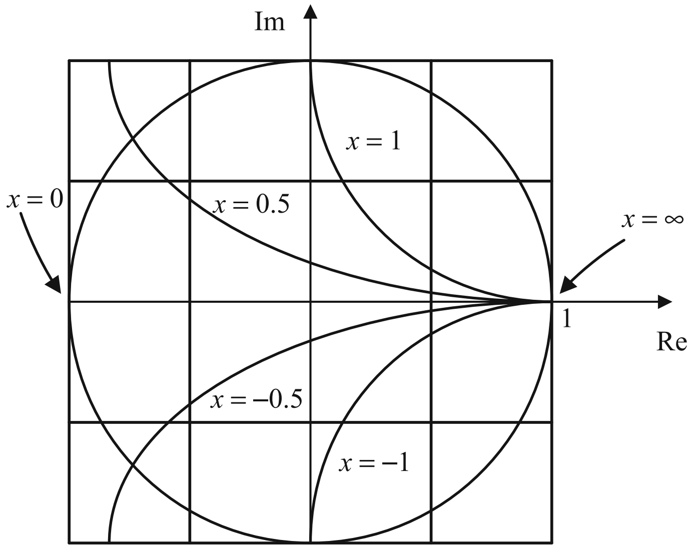


Fig. 8.9 Constant reactance circles with different radius

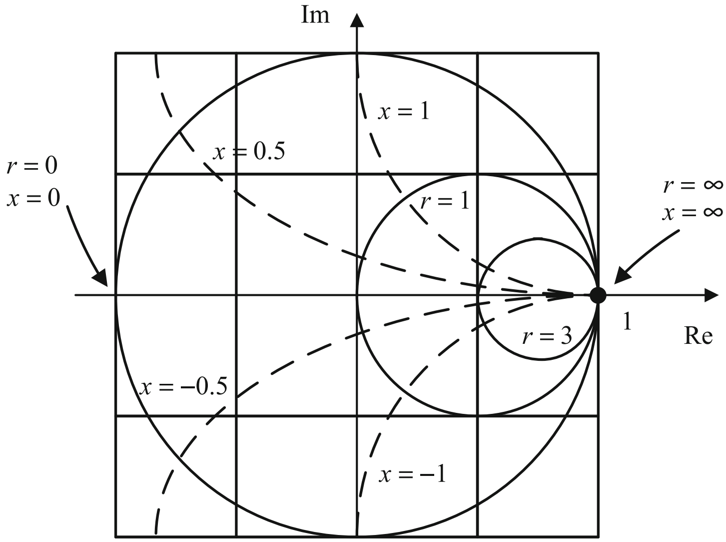


Fig. 8.10 Constant resistance and constant reactance circles

$$(a-1)^2 + \left(b - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2. \tag{8.216}$$

Equation (8.216) is known as constant reactance circle. The center and the radius of this circuit are $\left(1, \frac{1}{x}\right)$ and $\frac{1}{x}$, respectively. At $x = 0$, the center and radius of the constant reactance circle are $(1, \infty)$ and ∞, respectively. In this case, the circle lies on the real line. At $x = \infty$, the center is at (1,0) and the radius is 1. Families of constant reactance circles are drawn by the conditions; at $x = 0.5$, center = (1, 2), radius = 2, at $x = 1$, center = (1, 1), radius = 1, at $x = 2$, center = (1, 0.5), radius = 0.5. The quarter parts of the constant reactance circle lies above the real axis and negative parts below the real axis within the unity circle as shown in Fig. 8.9. Again, the constant resistance and the constant reactance circles are drawn together as shown in Fig. 8.10. From Fig. 8.10, it is seen that the constant resistance and the constant reactance circles are orthogonal to each other. The reflection coefficient and other related transmission line parameters can be determined by the point of intersection between the two circles.

8.12 Exercise Problems

8.1 A lossless transmission line has an inductance of 0.4μH/m and capacitance of 250pF/m. The voltage across the 30Ω load is 100V and the source frequency is 1,000Hz. Determine (1) the velocity of the wave, (2) characteristic resistance, (3) reflection coefficient, (4) wavelength, (5) phase constant, and (6) incident and reflected voltages.

8.2 The inductance and the capacitance of a lossless transmission line are 0.35mH/km and 0.025μF/km, respectively. The source frequency is 1,000Hz and the voltage across the 128Ω load is 210V. Find (1) the velocity of the wave, (2) characteristic resistance, (3) reflection coefficient, (4) wavelength, (5) phase constant, and (6) incident voltage.

8.3 A low-loss transmission line has an inductance of 15mH/m and capacitance of 46μF/m. The source frequency is 1,000Hz and the resistance and the conductance are 30Ω/m and 10S/m, respectively. Find (1) the attenuation constant and (2) phase constant.

8.4 The characteristic impedance, attenuation constant, and phase constant of a transmission line are 75Ω, $\alpha = 0.75$ db/m, and $\beta = 1.2$ rad/m, respectively. Assume the operating frequency is 1,000 MHz. Determine per meter (1) resistance, (2) inductance, and (3) capacitance.

8.5 A lossy transmission line operates at 120 MHz and the related parameters are found to be $Z_0 = 40\Omega$, $\alpha = 0.05$ db/m, and $\beta = 0.6\pi$ rad/m. Determine the resistance, inductance, capacitance, and conductance of the line.

8.6 A high-loss transmission line has an inductance of 20mH/m and capacitance of 65μF/m. The source frequency is 900Hz and the resistance and the conductance are 45Ω/m and 45S/m, respectively. Calculate (1) the attenuation constant and (2) phase constant.

8.7 A line has a power loss of 50 W per meter and the attenuation constant of $\alpha = 0.5$ db/m. Calculate the average power propagated along the line in the $z-$direction.

8.8 The power loss and the average power propagated along the line are 60 W/m and 150 W, respectively. Determine the attenuation constant of the line.

8.9 A parallel wire telephone line has the parameters $R = 2\Omega/\text{km}$, $G = 0.35\mu\text{S}/\text{km}$, $L = 0.009\text{H}/\text{km}$, and $C = 0.003\mu\text{F}/\text{km}$. The 35 km transmission line is terminated by the load of $Z_L = 20 + j35\Omega$ and the operating frequency is assumed to be 1,000 Hz. Calculate (1) the characteristic impedance, Z_0, (2) propagation constant, γ, and (3) input impedance, Z_{in}.

8.10 The required parameters of a parallel wire telephone line are given by $R = 6\Omega/\text{km}$, $G = 0.16\mu\text{S}/\text{km}$, $L = 0.0045\text{H}/\text{km}$, and $C = 0.0035\mu\text{F}/\text{km}$. A load impedance of $Z_L = 35 + j45\Omega$ is connected at the end of a 55 km transmission line and the operating frequency is considered as 1,000 Hz. Find (1) the characteristic impedance, Z_0, (2) propagation constant, γ, and (3) input impedance, Z_{in}.

8.11 Lossless transmission lines of 4 m length have the open circuit and the short circuit impedances of $-j40\Omega$ and $j95\Omega$, respectively. Calculate (1) the characteristic impedance, Z_0, (2) phase constant, β, and (3) short circuit impedance if the length is half of the given length considering the same operating frequency.

8.12 A lossless transmission line is 5 m long and its open circuit and short circuit impedances are measured as $-j50\Omega$ and $j100\Omega$, respectively. Find (1) the characteristic impedance, Z_0, (2) phase constant, β, and (3) short circuit impedance if the length is three times of the given length considering the same operating frequency.

8.13 The open circuit and short circuit impedances of a 5-m-long transmission line are found to be $200\,\underline{|-40^\circ}\ \Omega$ and $300\,\underline{|30^\circ}\ \Omega$, respectively. Find (1) Z_0, α, β and (2) R, L, C, G.

8.14 A 15 m length lossless transmission line has a characteristic impedance of $Z_0 = 45\Omega$, and it operates at 5 MHz. A load impedance of $Z_L = 60 + j55\Omega$ is used to terminate the transmission line. Assume the velocity of the light in the vacuum to be $v = 2.5 \times 10^8$ m/s. Find the reflection coefficient, standing wave ratio, and input impedance.

8.15 A 55Ω transmission line is terminated by the load of $Z_L = 65 + j45\Omega$. If ${V_0}^+ = 45\text{V}$, calculate (1) the reflection coefficient, ρ, (2) standing wave ratio, s, (3) incident power, (4) reflected power, and (5) net power.

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Chapter 9
Uniform Plane Waves

9.1 Introduction

The uniform plane wave is defined as the magnitude of the electric and magnetic fields. They are the same at all points in the direction of propagation. The electric and magnetic fields are orthogonal to the direction of propagation. In terms of energy, the wave phenomenon is defined as the exchange of two different forms of energy. It is meant that the time rate of change of one form triggers the other to a spatial change. Waves do not have mass but contain energy, momentum, and velocity. Examples of waves are the voltage and current waves on the transmission line, seismic waves, sound waves, and water waves. In this chapter, time-domain Maxwell's equation, the solution of Maxwell's equation, different parameters of electromagnetic wave propagation in different media, incident wave, reflected wave, transmitted wave, and wave polarization will be discussed.

9.2 Time-Domain Maxwell's Equations

Consider a homogeneous, linear, and isotropic unbound medium to express time-domain Maxwell's equation. The net free charge in this region is considered to be zero ($\rho = 0$) and the current density is $\mathbf{J} = \sigma\mathbf{E}$. In the lossless region, the conductivity is zero ($\sigma = 0$). The four differential forms of Maxwell's equations are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{9.1}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}, \tag{9.2}$$

$$\nabla \cdot \mathbf{B} = \mathbf{0}, \tag{9.3}$$

$$\nabla \cdot \mathbf{D} = \mathbf{0}. \tag{9.4}$$

Md. A. Salam, *Electromagnetic Field Theories for Engineering*,
DOI 10.1007/978-981-4585-66-8_9,

The following general relations are

$$\mathbf{B} = \mu \mathbf{H}, \tag{9.5}$$

$$\mathbf{D} = \varepsilon \mathbf{E}. \tag{9.6}$$

Substituting Eq. (9.5) into Eq. (9.3) yields

$$\nabla \cdot \mu \mathbf{H} = \mathbf{0}, \tag{9.7}$$

$$\nabla \cdot \mathbf{H} = \mathbf{0}. \tag{9.8}$$

Equation (9.4) can be modified by substituting Eq. (9.6) as

$$\nabla \cdot \varepsilon \mathbf{E} = \mathbf{0}, \tag{9.9}$$

$$\nabla \cdot \mathbf{E} = \mathbf{0}. \tag{9.10}$$

Substituting Eq. (9.5) into Eq. (9.1) provides

$$\nabla \times \mathbf{E} = -\frac{\partial(\mu \mathbf{H})}{\partial t}, \tag{9.11}$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}. \tag{9.12}$$

Again, substituting Eq. (9.6) into Eq. (9.2) yields

$$\nabla \times \mathbf{H} = \frac{\partial(\varepsilon \mathbf{E})}{\partial t}, \tag{9.13}$$

$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}. \tag{9.14}$$

From Eq. (9.12) and (9.14), it is concluded that time-varying electric and magnetic fields have coexistence.

Taking the curl of Eq. (9.12) yields

$$\nabla \times \nabla \times \mathbf{E} = -\mu \frac{\partial(\nabla \times \mathbf{H})}{\partial t}. \tag{9.15}$$

Substituting Eq. (9.14) into Eq. (9.15) provides

$$\nabla \times \nabla \times \mathbf{E} = -\mu \frac{\partial}{\partial t}\left(\frac{\varepsilon \partial \mathbf{E}}{\partial t}\right), \tag{9.16}$$

$$\nabla \times \nabla \times \mathbf{E} = -\mu \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}. \tag{9.17}$$

For any vector **A**, the following vector identity can be written as

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}. \tag{9.18}$$

Based on Eq. (9.18), Eq. (9.17) can be expressed as

$$(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} = -\mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}. \tag{9.19}$$

Substituting Eq. (9.10) into Eq. (9.19) yields

$$-\nabla^2\mathbf{E} = -\mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}, \tag{9.20}$$

$$\nabla^2\mathbf{E} = \mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}. \tag{9.21}$$

Again taking the curl of Eq. (9.14) provides

$$\nabla \times \nabla \times \mathbf{H} = \varepsilon\frac{\partial(\nabla \times \mathbf{E})}{\partial t}. \tag{9.22}$$

Substituting Eq. (9.12) into Eq. (9.22) provides

$$\nabla \times \nabla \times \mathbf{H} = \varepsilon\frac{\partial}{\partial t}\left(-\mu\frac{\partial\mathbf{H}}{\partial t}\right), \tag{9.23}$$

$$\nabla \times \nabla \times \mathbf{H} = -\mu\varepsilon\frac{\partial^2\mathbf{H}}{\partial t^2}. \tag{9.24}$$

Based on Eq. (9.18), Eq. (9.24) can be expressed as

$$(\nabla \cdot \mathbf{H}) - \nabla^2\mathbf{H} = -\mu\varepsilon\frac{\partial^2\mathbf{H}}{\partial t^2}. \tag{9.25}$$

Substituting Eq. (9.8) into Eq. (9.25) yields

$$-\nabla^2\mathbf{H} = -\mu\varepsilon\frac{\partial^2\mathbf{H}}{\partial t^2}, \tag{9.26}$$

$$\nabla^2\mathbf{H} = \mu\varepsilon\frac{\partial^2\mathbf{H}}{\partial t^2}. \tag{9.27}$$

Equations (9.21) and (9.27) are the wave equations in three-dimensional spaces.

9.3 Wave Equation in Time-Harmonic Fields

Consider a field in space coordinates and time function to derive the wave equation in time-harmonic fields. This field can be written as

$$\mathbf{E}(x, y, z, t) = \mathbf{E}(x, y, z)e^{j\omega t}, \tag{9.28}$$

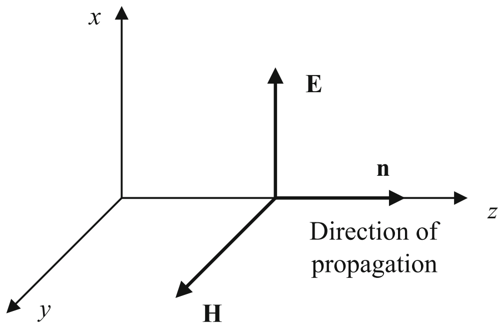


Fig. 9.1 Schematic of uniform plane waves

where the vector **E** is a function of space coordinates (x, y, z).

Taking derivatives of Eq. (9.28) provides

$$\frac{\partial \mathbf{E}(x, y, z, t)}{\partial t} = j\omega \mathbf{E}(x, y, z)e^{j\omega t}. \tag{9.29}$$

Again taking derivatives of Eq. (9.29) yields

$$\frac{\partial^2 \mathbf{E}(x, y, z, t)}{\partial t^2} = j^2\omega^2 \mathbf{E}(x, y, z)e^{j\omega t}, \tag{9.30}$$

$$\frac{\partial^2 \mathbf{E}(x, y, z, t)}{\partial t^2} = j^2\omega^2 \mathbf{E}(x, y, z, t). \tag{9.31}$$

Omitting the term $\mathbf{E}(x, y, z, t)$ from both sides of Eq. (9.31) yields

$$\frac{\partial^2}{\partial t^2} = -\omega^2. \tag{9.32}$$

Substituting Eq. (9.32) into Eqs. (9.21) and (9.27) provides

$$\nabla^2 \mathbf{E} = -\omega^2 \mu\varepsilon \mathbf{E}, \tag{9.33}$$

$$\nabla^2 \mathbf{H} = -\omega^2 \mu\varepsilon \mathbf{H}. \tag{9.34}$$

9.4 Solution of a Wave Equation in the Frequency Domain

Consider if the x-component of the electric field does not vary in the xy plane, i.e., constant and perpendicular to the direction of propagation as shown in Fig. 9.1. Assuming the solution of the uniform plane wave is

$$\mathbf{E}(x, y, z, t) = E_x(z)\mathbf{a}_x. \tag{9.35}$$

The E_x is constant in the x and y coordinates, the following condition can be written as

$$\frac{\partial E_x}{\partial x} = \frac{\partial E_x}{\partial y} = 0. \tag{9.36}$$

Based on Eq. (9.33), the following equation with full derivatives can be written as

$$\frac{d^2 E_x(z)}{dz^2} = -\omega^2 \mu \varepsilon E_x(z). \tag{9.37}$$

Consider the following relation

$$\gamma^2 = -\omega^2 \varepsilon \mu. \tag{9.38}$$

The complex propagation constant is

$$\gamma = \sqrt{-\omega^2 \varepsilon \mu}, \tag{9.39}$$

$$\gamma = \sqrt{j^2 \omega^2 \varepsilon \mu}, \tag{9.40}$$

$$\gamma = \alpha + j\beta = j\omega\sqrt{\varepsilon\mu}, \tag{9.41}$$

$$\alpha = 0, \tag{9.42}$$

$$\beta = \omega\sqrt{\varepsilon\mu} = \frac{\omega}{v}, \tag{9.43}$$

where the expression of the propagation velocity of the wave is

$$v = \frac{1}{\sqrt{\varepsilon\mu}}. \tag{9.44}$$

Substituting Eq. (9.38) into Eq. (9.37) provides

$$\frac{d^2 E_x(z)}{dz^2} - \gamma^2 E_x(z) = 0. \tag{9.45}$$

The general solution of Eq. (9.45) is

$$E_x(z) = C_1 e^{-\gamma z} + C_2 e^{\gamma z}. \tag{9.46}$$

Equation (9.46) with attenuation and phase constants can be modified as

$$E_x(z) = C_1 e^{-\alpha z} e^{-j\beta z} + C_2 e^{\alpha z} e^{j\beta z}, \tag{9.47}$$

$$E_x(z) = E_x{}^{+} e^{-j\beta z} + E_x{}^{-} e^{j\beta z}, \tag{9.48}$$

where

$E_x{}^{+} = C_1$ is the wave travelling in the positive z-direction,
$E_x{}^{-} = C_2$ is the wave travelling in the negative z-direction.

Equation (9.12) can be expressed as

$$\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -j\omega\mu\mathbf{H}. \tag{9.49}$$

Substituting Eq. (9.35) into Eq. (9.49) yields

$$\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = -j\omega\mu\mathbf{H}, \tag{9.50}$$

$$\frac{\partial E_x}{\partial z}\mathbf{a}_y = -j\omega\mu\mathbf{H}. \tag{9.51}$$

Substituting **H** in Cartesian coordinates in Eq. (9.51) provides

$$\frac{\partial E_x}{\partial z}\mathbf{a}_y = -j\omega\mu(H_x\mathbf{a}_x + H_y\mathbf{a}_y + H_z\mathbf{a}_z). \tag{9.52}$$

Equating the coefficient of $\mathbf{a}_y$ in Eq. (9.52) yields

$$\frac{\partial E_x}{\partial z} = -j\omega\mu H_y. \tag{9.53}$$

Substituting Eq. (9.48) into Eq. (9.53) yields

$$\frac{\partial}{\partial z}({E_x}^+e^{-j\beta z} + {E_x}^-e^{j\beta z}) = -j\omega\mu H_y, \tag{9.54}$$

$$-j\beta {E_x}^+e^{-j\beta z} + j\beta {E_x}^-e^{j\beta z} = -j\omega\mu H_y, \tag{9.55}$$

$$H_y = \frac{\beta}{\omega\mu}{E_x}^+e^{-j\beta z} - \frac{\beta}{\omega\mu}{E_x}^-e^{j\beta z}. \tag{9.56}$$

The ratio of the electric and the magnetic fields for a wave travelling in the positive z-direction, and the negative z-direction, can be determined as

$$\frac{E_x}{H_y} = \frac{{E_x}^+e^{-j\beta z}}{\frac{\beta}{\omega\mu}{E_x}^+e^{-j\beta z}} = \frac{\omega\mu}{\beta} = \eta, \tag{9.57}$$

$$\frac{E_x}{H_y} = \frac{{E_x}^-e^{j\beta z}}{-\frac{\beta}{\omega\mu}{E_x}^-e^{j\beta z}} = -\frac{\omega\mu}{\beta} = -\eta. \tag{9.58}$$

In general, the electric and the magnetic fields can be related as

$$\mathbf{H} = \frac{1}{\eta}\mathbf{n} \times \mathbf{E} \text{ and } \mathbf{E} = \eta\mathbf{H} \times \mathbf{n} \tag{9.59}$$

where **n** is the unit vector which represents the direction of wave propagation.

The unit of the electric field is V/m and the magnetic field is A/m. Therefore, the unit of the parameter η is V/A or Ω. This parameter is called the intrinsic impedance of the medium and it is denoted by the symbol η (eta). Mathematically, it can be expressed as

$$\eta = \frac{\omega\mu}{\beta}. \tag{9.60}$$

Substituting Eq. (9.43) into Eq. (9.60) yields

$$\eta = \frac{\omega\mu}{\omega\sqrt{\varepsilon\mu}}, \tag{9.61}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}. \tag{9.62}$$

For a free space, Eq. (9.62) can be written as

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}. \tag{9.63}$$

Substituting $\mu_0 = 4\pi \times 10^{-7}$ H/m and $\varepsilon_0 = \frac{1}{36\pi} \times 10^{-9}$ F/m into Eq. (9.63) provides the value of intrinsic impedance as

$$\eta_0 = \sqrt{\frac{4\pi \times 10^{-7}}{\frac{1}{36\pi} \times 10^{-9}}} \approx 120\pi = 377\ \Omega. \tag{9.64}$$

Substituting Eq. (9.63) into Eq. (9.62) yields

$$\eta = \eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}}. \tag{9.65}$$

The expression of phase constant for a free space is

$$\beta_0 = \omega\sqrt{\varepsilon_0\mu_0}. \tag{9.66}$$

Substituting Eq. (9.44) into Eq. (9.66) yields

$$\beta_0 = \frac{2\pi f}{v_0}, \tag{9.67}$$

$$\beta_0 = \frac{2\pi}{\lambda_0}, \tag{9.68}$$

$$\lambda_0 = \frac{2\pi}{\beta_0}, \tag{9.69}$$

where the expression of wavelength for a free space is

$$\lambda_0 = \frac{v_0}{f}. \tag{9.70}$$

Substituting Eq. (9.66) into Eq. (9.43) yields

$$\beta = \beta_0\sqrt{\varepsilon_r \mu_r}. \tag{9.71}$$

The expression of wavelength in any medium is

$$\lambda = \frac{2\pi}{\beta}. \tag{9.72}$$

Substituting Eq. (9.43) into Eq. (9.72) yields

$$\lambda = \frac{2\pi}{2\pi f\sqrt{\mu_0\mu_r\varepsilon_0\varepsilon_r}}. \tag{9.73}$$

Substituting Eq. (9.44) for a free space into Eq. (9.73) yields

$$\lambda = \frac{v_0}{f\sqrt{\mu_r\varepsilon_r}}. \tag{9.74}$$

Again, substituting Eq. (9.70) into Eq. (9.74) yields

$$\lambda = \frac{\lambda_0}{\sqrt{\mu_r\varepsilon_r}}. \tag{9.75}$$

Example 9.1 In free space, the electric field of a uniform plane wave is given by $\mathbf{E} = 300\cos(10^8 t - \beta_0 x)\mathbf{a}_z$ V/m. Determine (a) the phase constant, (b) wavelength, and (c) magnetic field at the point (0.2,1.3,0.4) m and $t = 5$ ns.

Solution

(a) The value of the phase constant is determined as,
$\beta_0 = \omega\sqrt{\mu_0\varepsilon_0} = 10^8\sqrt{\frac{1}{36\pi} \times 10^{-9} \times 4\pi \times 10^{-7}} = 0.33$ rad/m.

(b) The value of the wavelength is
$\lambda_0 = \frac{2\pi}{\beta_0} = \frac{2\pi}{0.33} = 19.04$ m.

(c) Here, the wave is propagated in the x-direction and the electric field is directed in the z-direction. Hence, the magnetic field is directed in the y-direction (Fig. 9.2).

The magnetic field can be determined as

$$\mathbf{H} = \frac{\mathbf{E}}{\eta_0} = \frac{300}{377}e^{-j\beta_0 x}\mathbf{a}_y = 0.8e^{-j\beta_0 x}\mathbf{a}_y \text{A/m}.$$

The instantaneous magnetic field is

$$\mathbf{H}(x, y, z, t) = 0.8\cos(10^8 t - \beta_0 x)\mathbf{a}_y \text{A/m}.$$

The magnetic field at the point (0.2,1.3,0.4) m is

$$\mathbf{H}(0.2,1.3,0.4,5) = 0.8\cos(10^8 \times 5 \times 10^{-9} - 0.33 \times 0.2)\mathbf{a}_y = 0.8\mathbf{a}_y \text{A/m}.$$

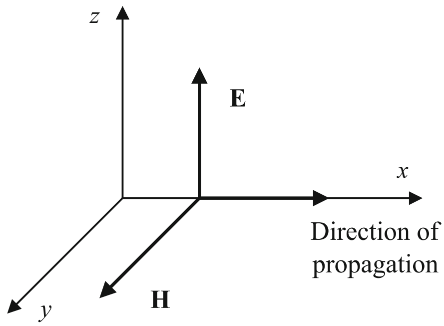


Fig. 9.2 Direction of electromagnetic fields

Practice Problem 9.1 In a isotropic, lossless, and uniform dielectric medium, the electric and the magnetic fields are given by $\mathbf{E} = 600\cos(10^8 t - \beta y)\mathbf{a}_x$V/m and $\mathbf{H} = 2.5\cos(10^8 t - \beta y)\mathbf{a}_z$ A/m. Determine (a) the intrinsic impedance, (b) relative permeability μ_r and relative permittivity ε_r, (c) phase constant, and (d) wavelength. Assume $v = 0.6v_0$ m/s.

Example 9.2 In a lossless, isotropic, and uniform dielectric medium, the electric field of a uniform plane wave is given by $\mathbf{E} = (2\mathbf{a}_x + j5\mathbf{a}_y)e^{j3z}$ V/m. Determine (a) the phase constant, (b) angular frequency, (c) velocity, and (d) intrinsic impedance. The frequency of the uniform plane wave is 40 MHz.

Solution

(a) The value of the phase constant is
$\beta = 3$ rad/m, propagation direction is $-\mathbf{a}_z$.

(b) The value of the angular frequency is
$\omega = 2\pi f = 2\pi \times 40 \times 10^6 = 25.13 \times 10^7$ rad/s.

(c) The velocity of the wave is
$v = \frac{\omega}{\beta} = \frac{25.13\times10^7}{3} = 8.38 \times 10^7$ m/s.

(d) The value of the permittivity is determined as
$v = \frac{v_0}{\sqrt{\mu_r\varepsilon_r}}, \mu_r\varepsilon_r = \left(\frac{v_0}{v}\right)^2, \mu_r\varepsilon_r = \left(\frac{3\times10^8}{8.37\times10^7}\right)^2, \varepsilon_r = 12.82.$
The value of the intrinsic impedance can be determined as

$$\eta = \eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}} = 377\sqrt{\frac{1}{12.82}} = 105.29\ \Omega.$$

Practice Problem 9.2 The electric field of a uniform plane wave in a lossless, isotropic, and uniform dielectric medium is given by $\mathbf{E} = (4\mathbf{a}_x + j9\mathbf{a}_y)e^{-j2z}$ V/m. Calculate (a) the phase constant, (b) angular frequency, (c) velocity, and (d) intrinsic impedance. Consider that the frequency of the uniform plane wave is 30 MHz.

9.5 Solution of a Wave Equation in the Time Domain

Consider that the x-component of the electric field is a function of z and t. The time-domain solution of the wave equation is

$$\mathbf{E}(z,t) = E_x(z,t) = \text{Re}[E_x(z)e^{j\omega t}], \tag{9.76}$$

where $E_x(z)$ is a phasor form. The wave equation (9.21) can be expressed as

$$\frac{d^2E_x(z,t)}{dz^2} - \mu\varepsilon\frac{\partial E_x(z,t)}{\partial t^2} = 0. \tag{9.77}$$

Substituting Eq. (9.76) into Eq. (9.77) yields

$$\left[\frac{d^2E_x(z)}{dz^2} - (j\omega)^2\mu\varepsilon E_x(z)\right]e^{j\omega t} = 0, \tag{9.78}$$

$$\frac{d^2E_x(z)}{dz^2} + \left(\frac{\omega}{v}\right)^2 E_x(z) = 0, \tag{9.79}$$

where the wave number of the wave is defined as

$$k = \beta = \frac{\omega}{v}. \tag{9.80}$$

Substituting Eq. (9.80) into Eq. (9.79) yields

$$(D^2 + k^2)E_x(z) = 0, \tag{9.81}$$

where $\frac{d^2}{dz^2} = D^2$.

Equation (9.81) can be written as

$$D^2 + k^2 = 0, \tag{9.82}$$

$$D^2 = -k^2 = j^2k^2, \tag{9.83}$$

$$D = \pm jk. \tag{9.84}$$

The solution is

$$E_x(z) = Ae^{-jkz} + Be^{jkz}. \tag{9.85}$$

Substituting Eq. (9.85) into Eq. (9.76) yields

$$E_x(z,t) = \mathrm{Re}[(Ae^{-jkz} + Be^{jkz})e^{j\omega t}], \tag{9.86}$$

$$E_x(z,t) = \mathrm{Re}[Ae^{j(\omega t - kz)} + Be^{j(\omega t + kz)}]. \tag{9.87}$$

The real and imaginary parts of Eq. (9.87) are

$$E_x(z,t) = A\cos(\omega t - kz) + B\cos(\omega t + kz). \tag{9.88}$$

The instantaneous phase of the wave is

$$\phi = \omega t - kz = \omega t - \beta z, \tag{9.89}$$

$$z = \frac{\omega}{\beta}t - \frac{\phi}{\beta}. \tag{9.90}$$

Differentiating Eq. (9.90) with respect to t yields

$$\frac{dz}{dt} = \frac{\omega}{\beta}, \tag{9.91}$$

$$v = \frac{\omega}{\beta}. \tag{9.92}$$

The frequency-domain solution of the wave equation in a time-domain form can be written as

$$E_x(z,t) = \text{Re}(E_x(z)e^{j\omega t}). \tag{9.93}$$

Substituting Eq. (9.46) into Eq. (9.93) yields

$$E_x(z,t) = \text{Re}[C_1 e^{-\alpha z} e^{j(\omega t - \beta z)}] + \text{Re}[C_2 e^{\alpha z} e^{j(\omega t + \beta z)}], \tag{9.94}$$

$$E_x(z,t) = E_x{}^{+} e^{-\alpha z} \cos(\omega t - \beta z) + E_x{}^{-} e^{\alpha z} \cos(\omega t + \beta z). \tag{9.95}$$

Example 9.3 An electric field of a uniform plane is given by $\mathbf{E} = 150\mathbf{a}_x$V/m. This wave propagates in the z-direction to a medium whose properties are $\sigma = 0$, $\mu_r = 1$, and $\varepsilon_r = 6$. Assume that the frequency of the wave is 10 MHz. Determine the instantaneous electric and magnetic fields.

Solution The electric field is

$$\mathbf{E}(z) = 150e^{-j\beta z}\mathbf{a}_x.$$

The instantaneous expression of electric field is

$$\mathbf{E}(z,t) = \text{Re}[150e^{-j\beta z}e^{j\omega t}]\mathbf{a}_x,$$

$$\mathbf{E}(z,t) = \text{Re}[150e^{j(\omega t - \beta z)}]\mathbf{a}_x,$$

$$\mathbf{E}(z,t) = 150\cos(\omega t - \beta z)\mathbf{a}_x.$$

The value of the phase constant is

$$\beta = \frac{\omega}{v_0} = \frac{2\pi \times 10 \times 10^6}{3 \times 10^8} = 0.21 rad/m,$$

$$\mathbf{E}(z,t) = 150\cos(\omega t - 0.21z)\mathbf{a}_x \text{V/m}.$$

The intrinsic impedance can be determined as

$$\eta = \eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}} = 377\sqrt{\frac{1}{6}} = 153.91\Omega.$$

The instantaneous value of the magnetic field can be calculated as

$$\mathbf{H}(z,t) = \frac{\mathbf{n} \times \mathbf{E}(z,t)}{\eta},$$

$$\mathbf{H}(z,t) = \frac{150\cos(\omega t - 0.21z)}{153.91}\mathbf{a}_z \times \mathbf{a}_x = 0.97150\cos(\omega t - 0.21z)\mathbf{a}_y \text{A/m}.$$

Practice Problem 9.3 A uniform plane wave having an electric field $\mathbf{E} = 200\mathbf{a}_y$ V/m propagates in the z-direction to a medium whose properties are $\sigma = 0$, $\mu_r = 1$, and $\varepsilon_r = 9$. Assume that the frequency of the wave is 30 MHz. Determine the instantaneous electric and magnetic fields.

9.6 Wave Propagation in Lossy Medium

In a lossy medium, the conductivity is not zero, i.e., $\sigma \neq 0$ and the conductivity is zero ($\sigma = 0$) in a lossless medium. The conduction current in a lossy medium is

$$\mathbf{J}_c = \sigma \mathbf{E}. \tag{9.96}$$

The displacement current density is

$$\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}. \tag{9.97}$$

The expression of the total current density is

$$\mathbf{J} = \mathbf{J}_c + \mathbf{J}_d. \tag{9.98}$$

Substituting Eqs. (9.96) and (9.97) into Eq. (9.98) yields

$$\mathbf{J} = \sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} = \sigma \mathbf{E} + j\varepsilon\omega \mathbf{E}. \tag{9.99}$$

The Ampere's law is

$$\nabla \times \mathbf{H} = \mathbf{J}. \tag{9.100}$$

Substituting Eq. (9.99) into Eq. (9.100) yields

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + j\varepsilon\omega \mathbf{E}, \tag{9.101}$$

$$\nabla \times \mathbf{H} = j\omega \left(\varepsilon + \frac{\sigma}{j\omega}\right) \mathbf{E}, \tag{9.102}$$

$$\nabla \times \mathbf{H} = j\omega \left(\varepsilon - j\frac{\sigma}{\omega}\right) \mathbf{E}, \tag{9.103}$$

$$\nabla \times \mathbf{H} = j\omega\varepsilon_{cp} \mathbf{E}, \tag{9.104}$$

where the expression of the complex permittivity of this medium is

$$\varepsilon_{cp} = \varepsilon - j\frac{\sigma}{\omega}. \tag{9.105}$$

In a lossy medium, the ratio of the conduction current density to the displacement current density is known as loss tangent. The loss tangent is the tangent of the phase

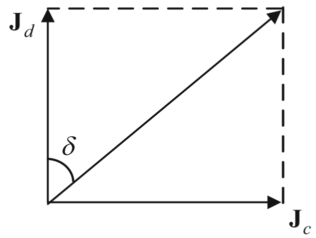


Fig. 9.3 Representation of conduction and displacement current densities

of the complex dielectric constant of the medium. Mathematically, the loss tangent can be expressed as

$$\tan\delta = \frac{|\mathbf{J}_c|}{|\mathbf{J}_d|}. \tag{9.106}$$

Substituting Eqs. (9.96) and (9.97) into Eq. (9.106) yields

$$\tan\delta = \frac{|\sigma\mathbf{E}|}{|j\omega\varepsilon\mathbf{E}|}, \tag{9.107}$$

$$\tan\delta = \frac{\sigma}{\omega\varepsilon}. \tag{9.108}$$

The representation of the conduction current and the displacement current densities is shown in Fig. 9.3. For a good conductor, the value of $\tan\delta$ is very large, i.e., $\sigma \gg \omega\varepsilon$ or $\frac{\sigma}{\omega\varepsilon} \gg 1$. In case of a good dielectric, the value of $\tan\delta$ is very small, i.e., $\sigma \ll \omega\varepsilon$ or $\frac{\sigma}{\omega\varepsilon} \ll 1$. The value of the loss tangent is zero for a lossless dielectric and for a good dielectric it is in the range $10^{-4} - 10^{-3}$.

Wave equations for a lossy medium $\left(\frac{\sigma}{\omega\varepsilon} \ll 1\right)$ can be derived as

$$\nabla \times \mathbf{H} = \sigma\mathbf{E} + \varepsilon\frac{\partial\mathbf{E}}{\partial t}. \tag{9.109}$$

Taking the curl of Eq. (9.109) yields

$$\nabla \times \nabla \times \mathbf{H} = \sigma(\nabla \times \mathbf{E}) + \varepsilon\frac{\partial(\nabla \times \mathbf{E})}{\partial t}, \tag{9.110}$$

$$\nabla(\nabla \cdot \mathbf{H}) - \nabla^2\mathbf{H} = \sigma(\nabla \times \mathbf{E}) + \varepsilon\frac{\partial(\nabla \times \mathbf{E})}{\partial t}. \tag{9.111}$$

Substituting Eqs. (9.8) and (9.12) into Eq. (9.111) yields

$$0 - \nabla^2\mathbf{H} = -\sigma\mu\frac{\partial\mathbf{H}}{\partial t} - \varepsilon\mu\frac{\partial^2\mathbf{H}}{\partial t^2}, \tag{9.112}$$

$$\nabla^2\mathbf{H} = \sigma\mu\frac{\partial\mathbf{H}}{\partial t} + \varepsilon\mu\frac{\partial^2\mathbf{H}}{\partial t^2}. \tag{9.113}$$

In phasor form, Eq. (9.113) can be modified as

$$\nabla^2\mathbf{H} = \sigma\mu(j\omega\mathbf{H}) + \varepsilon\mu(j\omega)^2\mathbf{H}, \tag{9.114}$$

$$\nabla^2\mathbf{H} = j\omega\mu(\sigma + j\omega\varepsilon)\mathbf{H}. \tag{9.115}$$

Similarly, for electric fields, Eq. (9.115) can be written as

$$\nabla^2\mathbf{E} = j\omega\mu(\sigma + j\omega\varepsilon)\mathbf{E}. \tag{9.116}$$

Then the square of the propagation constant can be written as

$$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon), \tag{9.117}$$

$$\gamma = \sqrt{j\omega\mu \times j\omega\varepsilon\left(\frac{\sigma}{j\omega\varepsilon} + 1\right)}, \tag{9.118}$$

$$\gamma = j\omega\sqrt{\mu\varepsilon}\left(\frac{\sigma}{j\omega\varepsilon} + 1\right)^{\frac{1}{2}}, \tag{9.119}$$

$$\gamma = j\omega\sqrt{\mu\varepsilon}\left[1 + \frac{\frac{1}{2}}{1!}\left(\frac{\alpha}{j\omega\varepsilon}\right) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{\alpha}{j\omega\varepsilon}\right)^2 + \ldots\right], \tag{9.120}$$

$$\gamma = j\omega\sqrt{\mu\varepsilon}\left[1 + \frac{1}{2}\left(\frac{\alpha}{j\omega\varepsilon}\right) + \frac{1}{8}\frac{\alpha^2}{\omega^2\varepsilon^2} + \ldots\right]. \tag{9.121}$$

The expression of the attenuation constant can be determined as

$$\alpha = \mathrm{Re}(\gamma) = \frac{1}{2}\frac{\sigma}{\varepsilon}\sqrt{\mu\varepsilon} = \frac{1}{2}\sigma\sqrt{\frac{\mu}{\varepsilon}}. \tag{9.122}$$

The expression of the phase constant is

$$\beta = \mathrm{Im}(\gamma) = \omega\sqrt{\mu\varepsilon}\left[1 + \frac{\sigma^2}{8\omega^2\varepsilon^2}\right], \tag{9.123}$$

$$\beta = \omega\sqrt{\mu\varepsilon}\left[1 + \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]. \tag{9.124}$$

Substituting Eq. (9.124) into Eq. (9.43) yields the expression of velocity,

$$\beta = \frac{\omega}{\omega\sqrt{\mu\varepsilon}\left[1 + \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]} = \frac{1}{\sqrt{\mu\varepsilon}\left[1 + \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]}. \tag{9.125}$$

Considering the condition $\frac{\sigma}{\omega\varepsilon} \ll 1$ for good dielectrics (lossless), Eq. (9.124) can be modified as

$$\beta = \omega\sqrt{\mu\varepsilon}. \tag{9.126}$$

Equations (9.124) and (9.126) are slightly different, but in both cases, the phase constant is directly proportional to the angular frequency.

For good dielectrics, Eq. (9.122) can be modified again as

$$\alpha = \frac{1}{2}\frac{\sigma}{\omega\varepsilon}\omega\sqrt{\mu\varepsilon}. \tag{9.127}$$

Substituting Eqs. (9.108) and (9.126) into Eq. (9.127) yields

$$\alpha = \frac{1}{2}\beta \tan\delta. \tag{9.128}$$

From Eq. (9.128), it is concluded that the attenuation constant is equal to half of the product of phase constant and loss tangent.

Example 9.4 A uniform plane wave having a frequency of 4 GHz travels through a good dielectric medium whose relative permittivity and loss tangent are 6 and 4×10^{-5}, respectively. Determine (a) the phase constant and (b) attenuation constant.

Solution

(a) The value of the phase constant is
$\beta = \omega\sqrt{\mu\varepsilon} = 2\pi f\sqrt{\mu_0\varepsilon_r\varepsilon_0} = 2\pi f\frac{\sqrt{\varepsilon_r}}{v_0} = \frac{2\pi\times 4\times 10^9\sqrt{6}}{3\times 10^8} = 205.21$ rad/m.

(b) The value of the attenuation constant can be determined as
$\tan\delta = 4 \times 10^{-5}$,
$\alpha = \frac{1}{2}\beta\tan\delta = \frac{205.21\times 4\times 10^{-4}}{2} = 0.041$ Np/m.

Practice Problem 9.4 A 6-MHz uniform plane wave travels through a good dielectric medium whose relative permittivity and attenuation constant are 5 and 0.032 Np/m, respectively. Determine (a) the phase constant and (b) loss tangent.

9.7 Wave Propagation in Good Conductors

The medium or material is said to be a good conductor if it satisfies the condition $\sigma >> \omega\varepsilon$ or $\frac{\sigma}{\omega\varepsilon} >> 1$. The expression of the propagation constant can be written as

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}, \tag{9.129}$$

$$\gamma = \sqrt{j\omega\mu \times j\omega\varepsilon\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)}. \tag{9.130}$$

Applying the condition of a good conductor to Eq. (9.130) yields

$$\gamma = \sqrt{j\omega\mu \times j\omega\varepsilon\left(\frac{\sigma}{j\omega\varepsilon}\right)}, \tag{9.131}$$

$$\gamma = \sqrt{j\omega\mu\sigma}. \tag{9.132}$$

But the following relations can be written as:

$$j = e^{j\frac{\pi}{2}} = \cos\frac{\pi}{2} + j\sin\frac{\pi}{2}, \tag{9.133}$$

$$\sqrt{j} = \sqrt{e^{j\frac{\pi}{2}}} = e^{j\frac{\pi}{4}} = \cos\frac{\pi}{4} + j\sin\frac{\pi}{4} = \frac{1+j}{\sqrt{2}}. \tag{9.134}$$

Substituting Eq. (9.134) into Eq. (9.132) yields

$$\gamma = \frac{1+j}{\sqrt{2}}\sqrt{\omega\mu\sigma}. \tag{9.135}$$

The expressions of attenuation and phase constants are

$$\alpha = \mathrm{Re}(\gamma) = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma}, \tag{9.136}$$

$$\beta = \mathrm{Im}(\gamma) = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma}. \tag{9.137}$$

From Eqs. (9.135) and (9.136), it is concluded that the attenuation and phase constants are the same in a good conductor.

The expression of the velocity of propagation can be obtained by substituting Eq. (9.137) into Eq. (9.43) as

$$v = \frac{\omega}{\sqrt{\frac{\omega\mu\sigma}{2}}}, \tag{9.138}$$

$$v = \sqrt{\frac{2\omega}{\mu\sigma}}. \tag{9.139}$$

The intrinsic impedance can be derived as

$$\eta = \sqrt{\frac{\mu}{\varepsilon_{cp}}}. \tag{9.140}$$

Substituting Eq. (9.105) into Eq. (9.140) yields

$$\eta = \sqrt{\frac{\mu}{\varepsilon - j\frac{\sigma}{\omega}}}. \tag{9.141}$$

Applying the condition $\frac{\sigma}{\omega\varepsilon} \gg 1$ or $\frac{\sigma}{\omega} \gg \varepsilon$ into Eq. (9.141) yields

$$\eta = \sqrt{\frac{\mu}{-j\frac{\sigma}{\omega}}}, \tag{9.142}$$

$$\eta = \sqrt{j\frac{\omega\mu}{\sigma}}. \tag{9.143}$$

Substituting Eq. (9.134) into Eq. (9.142) yields

$$\eta = \sqrt{\frac{\omega\mu}{2\sigma}}(1 + j), \tag{9.144}$$

$$\eta = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ . \tag{9.145}$$

From Eq. (9.95), consider the positive direction of the travelling wave for a lossy material. This equation is

$$E_x(z,t) = E_x{}^{+} e^{-\alpha z} \cos(\omega t - \beta z)\mathbf{a}_x. \tag{9.146}$$

From Eq. (9.146), it is concluded that the amplitude of the wave will be decreased by the factor $e^{-\alpha z}$ over a distance of $\frac{1}{\alpha}$. This distance is known as skin depth of the material and it is represented by the Greek letter δ. The skin depth is the depth of the material by which the uniform plane wave can penetrate the material before it is decreased by the factor $\frac{1}{e}$ or 37 %. Mathematically, the expression of skin depth is

$$\delta = \frac{1}{\alpha}. \tag{9.147}$$

Substituting Eq. (9.136) into Eq. (9.147) yields the expression of skin depth as

$$\delta = \frac{1}{\sqrt{\pi f \mu\sigma}}. \tag{9.148}$$

From Eq. (9.148), it is concluded that the skin depth increases as both the frequency and conductivity decrease.

Example 9.5 A 4-GHz uniform plane wave travels through a large copper conductor. Consider $\sigma = 5.8 \times 10^7$ S/m, $\varepsilon_r = \mu_r = 1$ for a copper conductor. Calculate (a) the attenuation constant, (b) phase constant, (c) intrinsic impedance, (d) wavelength, and (e) velocity of propagation.

Solution

(a) The value of the attenuation constant is
$\alpha = \sqrt{f\pi\mu\sigma} = \sqrt{4 \times 10^9 \times \pi \times 4\pi \times 10^{-7} \times 5.8 \times 10^7} = 9.57 \times 10^5$ Np/m.

(b) The value of the phase constant is
$\beta = \sqrt{f\pi\mu\sigma} = \sqrt{4 \times 10^9 \times \pi \times 4\pi \times 10^{-7} \times 5.8 \times 10^7} = 9.57 \times 10^5$ rad/m.

(c) The value of the intrinsic impedance can be determined as
$\eta = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ = \sqrt{\frac{2\pi \times 4 \times 10^9 \times 4\pi \times 10^{-7}}{5.8 \times 10^7}} \angle 45^\circ = 0.023 \angle 45^\circ\ \Omega.$

(d) The value of the wavelength is determined as
$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{9.57 \times 10^5} = 0.66 \times 10^{-5}$ m.

(e) The velocity of propagation can be determined as
$v = \sqrt{\frac{2\omega}{\mu\sigma}} = \sqrt{\frac{2 \times 2 \times \pi \times 4 \times 10^9}{4\pi \times 10^{-7} \times 5.8 \times 10^7}} = 26261.29$ m/s.

Example 9.6 A uniform plane wave of $\mathbf{E} = 150\cos(10^8 t)\mathbf{a}_x$ is travelling through seawater in the positive z-direction at $z = 0$. Consider $\sigma = 4$ S/m, $\varepsilon_r = 72$ and $\mu_r = 1$ for seawater. Calculate (a) the attenuation constant, (b) phase constant, (c) intrinsic impedance, (d) wavelength, (e) velocity of propagation, and (f) skin depth.

Solution

(a) The value of the attenuation constant is
$\alpha = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\frac{10^8\times 4\pi\times 10^{-7}\times 4}{2}} = 15.85$ Np/m.
(b) The value of the phase constant is
$\beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\frac{10^8\times 4\pi\times 10^{-7}\times 4}{2}} = 15.85$ rad/m.
(c) The value of the intrinsic impedance can be determined as
$\eta = \sqrt{\frac{\omega\mu}{\sigma}}\,\underline{|45^\circ} = \sqrt{\frac{10^8\times 4\pi\times 10^{-7}}{4}}\,\underline{|45^\circ} = 5.6\,\underline{|45^\circ}\ \Omega.$
(d) The value of the wavelength is determined as
$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{15.85} = 0.40$ m.
(e) The velocity of propagation can be determined as
$v = \sqrt{\frac{2\omega}{\mu\sigma}} = \sqrt{\frac{2\times 10^8}{4\pi\times 10^{-7}\times 4}} = 0.20\times 10^8$ m/s.
(f) The value of the skin depth is calculated as
$\delta = \frac{1}{\sqrt{\pi f\mu\sigma}} = \frac{\sqrt{2}}{\sqrt{\omega\mu\sigma}} = \frac{\sqrt{2}}{\sqrt{10^8\times 4\pi\times 10^{-7}\times 4}} = 0.06$ m.

Practice problem 9.5 A uniform plane wave having the frequency of 6 GHz travels through a large aluminum conductor. Consider $\sigma = 3.54\times 10^7$ S/m, $\varepsilon_r = \mu_r = 1$ for a copper conductor. Calculate (a) the attenuation constant, (b) phase constant, (c) intrinsic impedance, (d) wavelength, and (e) velocity of propagation.

Practice problem 9.6 A uniform plane wave of $\mathbf{E} = 100\cos(10^8 t)\mathbf{a}_x$ is travelling through air in the positive z-direction at $z = 0$. Consider $\sigma = 3\times 10^{-6}$ S/m, $\varepsilon_r = 1$ and $\mu_r = 1$ for air. Find (a) the attenuation constant, (b) phase constant, (c) intrinsic impedance, (d) wavelength, (e) velocity of propagation, and (f) skin depth.

9.8 Power Flow and Poynting Vector

It is important to know how much power flows in the direction of a uniform plane wave. In this case, it is necessary to develop a theorem to calculate the power of a uniform plane wave. This theorem is known as Poynting theorem. In 1884, English physicist John H. Poynting developed this theory. Consider the following Maxwell's equation to derive the Poynting theorem:

$$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}. \tag{9.149}$$

Taking $\mathbf{E}$ dot of Eq. (9.149) yields

$$\mathbf{E}.(\nabla\times\mathbf{H}) = \mathbf{E}.\mathbf{J} + \mathbf{E}.\frac{\partial\mathbf{D}}{\partial t}. \tag{9.150}$$

Again taking **H** dot of Eq. (9.1) yields

$$\mathbf{H}.(\nabla \times \mathbf{E}) = -\mathbf{H}.\frac{\partial \mathbf{B}}{\partial t}. \tag{9.151}$$

Subtracting Eq. (9.150) from Eq. (9.151) and using the vector identity yields

$$\mathbf{H}.(\nabla \times \mathbf{E}) - \mathbf{E}.(\nabla \times \mathbf{H}) = \nabla.(\mathbf{E} \times \mathbf{H}). \tag{9.152}$$

Substituting Eqs. (9.151) and (9.149) into Eq. (9.152) yields

$$\mathbf{H}.\left(-\frac{\partial \mathbf{B}}{\partial t}\right) - \mathbf{E}.\mathbf{J} - \mathbf{E}.\frac{\partial \mathbf{D}}{\partial t} = \nabla.(\mathbf{E} \times \mathbf{H}), \tag{9.153}$$

$$-\mu\mathbf{H}.\frac{\partial \mathbf{H}}{\partial t} - \mathbf{E}.\mathbf{J} - \varepsilon\mathbf{E}.\frac{\partial \mathbf{E}}{\partial t} = \nabla.(\mathbf{E} \times \mathbf{H}). \tag{9.154}$$

The following differential equations can be written as:

$$\mu\mathbf{H}.\frac{\partial \mathbf{H}}{\partial t} = \mu\frac{\partial(\mathbf{H}.\mathbf{H})}{\partial t} = \mu\frac{\partial H^2}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\mu H^2) = \frac{\partial}{\partial t}\left(\frac{\mu H^2}{2}\right), \tag{9.155}$$

$$\mathbf{E}.\mathbf{J} = \sigma\mathbf{E}.\mathbf{E} = \sigma E^2, \tag{9.156}$$

$$\varepsilon\mathbf{E}.\frac{\partial \mathbf{E}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\varepsilon E^2}{2}\right). \tag{9.157}$$

Substituting Eqs. (9.155), (9.156), and (9.157) into Eq. (9.154) yields

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\sigma E^2 - \frac{\partial}{\partial t}\left(\frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2}\right). \tag{9.158}$$

Taking volume integral of Eq. (9.158) yields

$$\int_v \nabla \cdot (\mathbf{E} \times \mathbf{H}) dv = -\int_v \sigma E^2 dv - \int_v \frac{\partial}{\partial t}\left(\frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2}\right) dv. \tag{9.159}$$

Applying the divergence theorem to the left-hand side of Eq. (9.159) yields

$$\oint_s (\mathbf{E} \times \mathbf{H}).d\mathbf{S} = -\int_v \sigma E^2 dv - \int_v \frac{\partial}{\partial t}\left(\frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2}\right) dv. \tag{9.160}$$

The left side of Eq. (9.160) represents the total power flowing out of the volume. In the right side of Eq. (9.160), the first and second terms represent ohmic losses and the time rate of stored energy within the volume. The Poynting theorem states that the power flowing out of a given volume is equal to the rate of decreased energy minus the ohmic losses. The quantity $\mathbf{E} \times \mathbf{H}$ is known as Poynting vector **P**

$$\mathbf{P} = \mathbf{E} \times \mathbf{H} \ \mathrm{W/m^2}. \tag{9.161}$$

The time-average Poynting vector over one cycle is

$$\mathbf{P}_{av} = \frac{1}{T}\int_0^T P(z,t)dt. \qquad (9.162)$$

In general, the average Poynting vector is

$$\mathbf{P}_{av} = \frac{1}{2}\mathrm{Re}(\mathbf{E} \times \mathbf{H}^*). \qquad (9.163)$$

The time-average power crossing any surface is given as

$$P_s = \int_s \mathbf{P}.d\mathbf{S}. \qquad (9.164)$$

Consider a uniform plane wave travels in the z-direction. The electric field is directed in the x-direction and the magnetic field is directed in the y-direction. These electromagnetic fields are

$$\mathbf{E} = E_0 e^{j\omega t}\mathbf{a}_x. \qquad (9.165)$$

Using Eq. (9.59), the expression of magnetic field is

$$\mathbf{H} = \frac{\mathbf{n} \times \mathbf{E}}{\eta} = \frac{E_0}{\eta}e^{j\omega t}\mathbf{a}_z \times \mathbf{a}_x = \frac{E_0}{\eta}e^{j\omega t}\mathbf{a}_y. \qquad (9.166)$$

Substituting Eqs. (9.165) and (9.166) into Eq. (9.163) yields

$$\mathbf{P}_{av} = \frac{1}{2}\mathrm{Re}\left[E_0 e^{j\omega t}\left(\frac{E_0}{\eta}e^{j\omega t}\right)^*\right](\mathbf{a}_x \times \mathbf{a}_y), \qquad (9.167)$$

$$\mathbf{P}_{av} = \frac{1}{2}\mathrm{Re}\left[\frac{|E_0|^2}{\eta^*}\right]\mathbf{a}_z, \qquad (9.168)$$

$$\mathbf{P}_{av} = \frac{|E_0|^2}{2}\mathrm{Re}\left[\frac{1}{\eta^*}\right]\mathbf{a}_z = \frac{|H_0|^2}{2}\mathrm{Re}[\eta]\mathbf{a}_z. \qquad (9.169)$$

Substituting Eq. (9.144) into Eq. (9.169) yields

$$\mathbf{P}_{av} = \frac{|H_0|^2}{2}\mathrm{Re}\left[\sqrt{\frac{\omega\mu}{2\sigma}}(1+j)\right], \qquad (9.170)$$

$$\mathbf{P}_{av} = \frac{|H_0|^2}{2}\sqrt{\frac{\omega\mu}{2\sigma}} = \frac{|H_0|^2}{2}\sqrt{\frac{\pi f\mu}{\sigma}}, \qquad (9.171)$$

$$\mathbf{P}_{av} = \frac{|H_0|^2}{2}\sqrt{\frac{\pi f\sigma\mu}{\sigma^2}}, \qquad (9.172)$$

$$\mathbf{P}_{av} = \frac{|H_0|^2}{2}\frac{1}{\sigma}\sqrt{\frac{\pi f\sigma\mu}{1}}, \qquad (9.173)$$

$$\mathbf{P}_{av} = \frac{|H_0|^2}{2}\frac{1}{\sigma}\frac{1}{\frac{1}{\sqrt{\pi f \sigma \mu}}}. \tag{9.174}$$

Substituting Eq. (9.148) into Eq. (9.172) yields

$$\mathbf{P}_{av} = \frac{|H_0|^2}{2}\frac{1}{\sigma\delta}. \tag{9.175}$$

Substituting $\eta = \sqrt{\frac{\mu}{\varepsilon}}$ into Eq. (9.169) provides the expression of average power density as

$$\mathbf{P}_{av} = \frac{|H_0|^2}{2}\sqrt{\frac{\mu}{\varepsilon}}. \tag{9.176}$$

Example 9.7 The electric field directed in the nonmagnetic medium in the z- direction is given by $\mathbf{E} = 5\cos(10^8 t - 0.5x)\mathbf{a}_z$ V/m. Determine (a) the relative permittivity, (b) intrinsic impedance, and (c) time-average power.

Solution

(a) The value of the phase constant is
$\beta = 0.5$ rad/m.
The relative permittivity can be determined as
$\beta = \omega\sqrt{\varepsilon\mu}$,
$\sqrt{\varepsilon_r} = \frac{\beta}{\omega\sqrt{\varepsilon_0\mu_0}} = \frac{0.5}{10^8\sqrt{4\pi\times10^{-7}\times\frac{1}{36\pi}\times10^{-9}}} = 1.5$,
$\varepsilon_r = 2.25$.

(b) The value of the intrinsic impedance can be determined as
$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0}{\varepsilon_0\varepsilon_r}} = \sqrt{\frac{4\pi\times10^{-7}}{\frac{1}{36\pi}\times10^{-9}}} \times \frac{1}{1.5} = 120\pi \times \frac{1}{1.5} = 251.33\Omega$.

(c) The value of the time-average power is
$P_{av} = \frac{E_0{}^2}{2\eta}\mathbf{a}_x = \frac{25}{2\times251.33}\mathbf{a}_x = 0.05\ \text{W/m}^2$.

Practice problem 9.7 An electric field directed in the lossless dielectric in the positive z-direction is given by $\mathbf{E} = 150\sin(\omega t - 0.4z)\mathbf{a}_x$ V/m and the time-average power is found to be 140 W/m^2. Find (a) the relative permittivity, (b) frequency, and (c) equation of the magnetic field.

9.9 Incident and Reflected Waves

Consider a region which consists of two media and these two media are separated by a boundary as shown in Fig. 9.4. Two media have material properties ε_1, μ_1, η_1 for medium 1 and ε_2, μ_2, η_2 for medium 2. The term reflection occurs when a uniform plane wave is incident onto the boundary of two media. The wave from medium 1 incident wave ($E_1{}^+$, $H_1{}^+$) onto the boundary which creates the reflection wave ($E_1{}^-$, $H_1{}^-$). Expressions of electric and magnetic fields are

$$E_1(z) = (E_1{}^+ e^{-\gamma_1 z} + E_1{}^- e^{\gamma_1 z})\mathbf{a}_x, \tag{9.177}$$

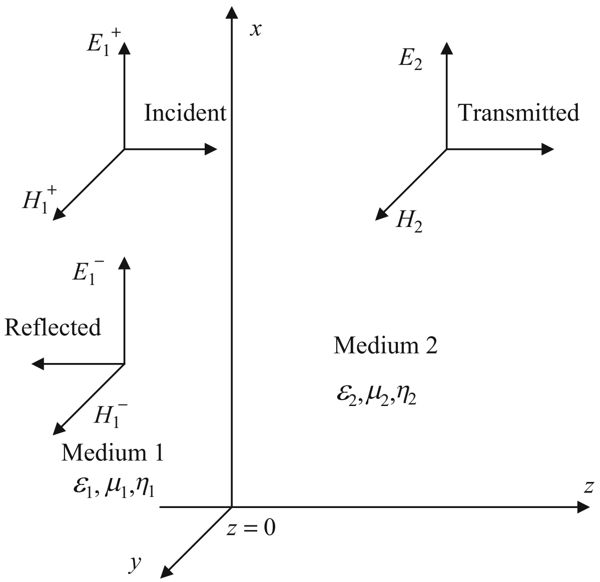


Fig. 9.4 Two media with incident, reflected, and transmitted waves

$$H_1(z) = (H_1{}^+ e^{-\gamma_1 z} + H_1{}^- e^{\gamma_1 z})\mathbf{a}_y, \tag{9.178}$$

$$H_1(z) = \frac{E_1{}^+}{\eta_1} e^{-\gamma_1 z}\mathbf{a}_y - \frac{E_1{}^-}{\eta_1} e^{\gamma_1 z}\mathbf{a}_y, \tag{9.179}$$

where the propagation constant and intrinsic impedance are

$$\gamma_1 = \alpha_1 + j\beta_1 = \sqrt{j\omega\mu_1(\sigma_1 + j\omega\varepsilon_1)}, \tag{9.180}$$

$$\eta_1 = \eta_1 \underline{|\theta_{\eta 1}} = \sqrt{\frac{j\omega\mu_1}{\sigma_1 + j\omega\varepsilon_1}}. \tag{9.181}$$

At the boundary, set $z = 0$ to Eqs. (9.177) and (9.179) and omitting the unit vector yields

$$E_1 = E_1{}^+ + E_1{}^-, \tag{9.182}$$

$$H_1 = \frac{E_1{}^+}{\eta_1} - \frac{E_1{}^-}{\eta_1}. \tag{9.183}$$

Some portions of the incident will be transmitted to medium 2. The electric and magnetic fields of this medium are

$$E_2(z) = E_2{}^+ e^{-\gamma_2 z}\mathbf{a}_x, \tag{9.184}$$

$$H_2(z) = H_2{}^+ e^{-\gamma_2 z}\mathbf{a}_y = \frac{E_2{}^+}{\eta_2} e^{-\gamma_2 z}\mathbf{a}_y, \tag{9.185}$$

where the propagation constant and intrinsic impedance are

$$\gamma_2 = \alpha_2 + j\beta_2 = \sqrt{j\omega\mu_2(\sigma_2 + j\omega\varepsilon_2)}, \tag{9.186}$$

$$\eta_2 = \eta_2 \left|\underline{\theta_{\eta 2}}\right. = \sqrt{\frac{j\omega\mu_2}{\sigma_2 + j\omega\varepsilon_2}}. \tag{9.187}$$

At the boundary, $z = 0$, then Eqs. (9.184) and (9.185) and omitting the unit vector becomes

$$E_2 = E_2{}^+, \tag{9.188}$$

$$H_2 = H_2{}^+ = \frac{E_2{}^+}{\eta_2}. \tag{9.189}$$

Total electric and magnetic fields are the same at the boundary, and then the following equations can be written as

$$E_1 = E_2, \tag{9.190}$$

$$H_1 = H_2. \tag{9.191}$$

Substituting Eqs. (9.182) and (9.188) into Eq. (9.190) yields

$$E_2{}^+ = E_1{}^+ + E_1{}^-. \tag{9.192}$$

Again, substituting Eqs. (9.183) and (9.189) into Eq. (9.191) yields

$$\frac{E_2{}^+}{\eta_2} = \frac{E_1{}^+}{\eta_1} - \frac{E_1{}^-}{\eta_1}, \tag{9.193}$$

$$\frac{E_2{}^+}{\eta_2} = \frac{E_1{}^+ - E_1{}^-}{\eta_1}, \tag{9.194}$$

$$\frac{E_2{}^+}{E_1{}^+ - E_1{}^-} = \frac{\eta_2}{\eta_1}. \tag{9.195}$$

Substituting Eq. (9.192) into Eq. (9.195) yields

$$\frac{E_1{}^+ + E_1{}^-}{E_1{}^+ - E_1{}^-} = \frac{\eta_2}{\eta_1}. \tag{9.196}$$

Applying addition–subtraction rules in Eq. (9.196) yields

$$\frac{E_1{}^+ + E_1{}^- - (E_1{}^+ - E_1{}^-)}{(E_1{}^+ + E_1{}^-) + (E_1{}^+ - E_1{}^-)} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}, \tag{9.197}$$

$$\frac{2E_1{}^-}{2E_1{}^+} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}, \tag{9.198}$$

$$\frac{E_1{}^-}{E_1{}^+} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}. \tag{9.199}$$

The ratio of the reflected wave's electric field to the incident wave's electric field at the boundary is known as reflection coefficient ρ and its expression is given as

$$\rho = \frac{E_1{}^-}{E_1{}^+} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}. \tag{9.200}$$

Equation (9.192) is modified as

$$E_2{}^+ = E_1{}^+ \left(1 + \frac{E_1{}^-}{E_1{}^+}\right). \tag{9.201}$$

Substituting Eq. (9.200) into Eq. (9.201) yields

$$\frac{E_2{}^+}{E_1{}^+} = 1 + \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}, \tag{9.202}$$

$$\frac{E_2{}^+}{E_1{}^+} = \frac{2\eta_2}{\eta_2 + \eta_1}. \tag{9.203}$$

The ratio of the transmitted wave's electric field to the incident wave's electric field is known as the transmission coefficient. This coefficient is represented by τ and its expression becomes

$$\tau = \frac{E_2{}^+}{E_1{}^+} = \frac{2\eta_2}{\eta_2 + \eta_1}. \tag{9.204}$$

If medium 1 and medium 2 are perfect dielectrics, i.e., lossless dielectric, then η_1 and η_2 are real numbers and $\mu_1 = \mu_2 = \mu_0$. The expression of reflection coefficient is modified as

$$\rho = \frac{\sqrt{\frac{\mu_0}{\varepsilon_2}} - \sqrt{\frac{\mu_0}{\varepsilon_1}}}{\sqrt{\frac{\mu_0}{\varepsilon_2}} + \sqrt{\frac{\mu_0}{\varepsilon_1}}}, \tag{9.205}$$

$$\rho = \frac{\sqrt{\frac{\mu_0}{\varepsilon_2}}\left(1 - \sqrt{\frac{\mu_0}{\varepsilon_1}} \times \sqrt{\frac{\varepsilon_2}{\mu_0}}\right)}{\sqrt{\frac{\mu_0}{\varepsilon_2}}\left(1 + \sqrt{\frac{\mu_0}{\varepsilon_1}} \times \sqrt{\frac{\varepsilon_2}{\mu_0}}\right)}, \tag{9.206}$$

$$\rho = \frac{1 - \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}}{1 + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}}. \tag{9.207}$$

The expression of the transmission coefficient can be modified as

$$\tau = \frac{2\sqrt{\frac{\mu_0}{\varepsilon_2}}}{\sqrt{\frac{\mu_0}{\varepsilon_2}} + \sqrt{\frac{\mu_0}{\varepsilon_1}}}, \tag{9.208}$$

$$\tau = \frac{2\sqrt{\frac{\mu_0}{\varepsilon_2}}}{\sqrt{\frac{\mu_0}{\varepsilon_2}}\left(1 + \sqrt{\frac{\mu_0}{\varepsilon_1}} \times \sqrt{\frac{\varepsilon_2}{\mu_0}}\right)}, \tag{9.209}$$

$$\tau = \frac{2}{1 + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}}. \tag{9.210}$$

From Eqs. (9.202) and (9.204), the following relation can be written as

$$\tau = 1 + \rho. \tag{9.211}$$

Substituting Eq. (9.200) into Eq. (9.177) yields

$$E_1(z) = (E_1{}^+ e^{-\gamma_1 z} + \rho E_1{}^+ e^{\gamma_1 z})\mathbf{a}_x. \tag{9.212}$$

The incident wave of the electric field is

$$E_i = E_1{}^+ e^{-\gamma_1 z}\mathbf{a}_x. \tag{9.213}$$

The incident wave of the magnetic field is

$$H_i = \frac{E_1{}^+}{\eta_1} e^{-\gamma_1 z}\mathbf{a}_y. \tag{9.214}$$

The reflected wave of the electric field is

$$E_r = \rho E_1{}^+ e^{\gamma_1 z}\mathbf{a}_x. \tag{9.215}$$

The reflected wave of the magnetic field is

$$H_r = -\frac{\rho E_1{}^+}{\eta_1} e^{\gamma_1 z}\mathbf{a}_y. \tag{9.216}$$

Substituting Eq. (9.204) into Eq. (9.184) yields the transmitted wave electric field

$$E_2 = \tau E_1{}^+ e^{-\gamma_2 z}\mathbf{a}_x. \tag{9.217}$$

The transmitted wave magnetic field is

$$H_2 = \frac{\tau E_1{}^+}{\eta_2} e^{-\gamma_2 z}\mathbf{a}_y. \tag{9.218}$$

Time-domain forms of the field vectors can be written as

$$E_i = E_1{}^+ e^{-\alpha_1 z}\mathrm{Re}[e^{j(\omega t - \beta_1 z)}]\mathbf{a}_x, \tag{9.219}$$

$$H_i = \frac{E_1{}^+}{\eta_1} e^{-\alpha_1 z}\mathrm{Re}[e^{j(\omega t - \beta_1 z)}]\mathbf{a}_y, \tag{9.220}$$

$$E_r = \rho E_1{}^+ e^{\alpha_1 z} \operatorname{Re}[e^{j(\omega t+\beta_1 z)}]\mathbf{a}_x, \tag{9.221}$$

$$H_r = -\frac{\rho E_1{}^+}{\eta_1} e^{\alpha_1 z} \operatorname{Re}[e^{j(\omega t+\beta_1 z)}]\mathbf{a}_y, \tag{9.222}$$

$$E_2 = \tau E_1{}^+ e^{-\alpha_2 z} \operatorname{Re}[e^{j(\omega t-\beta_2 z)}]\mathbf{a}_x, \tag{9.223}$$

$$H_2 = \frac{\tau E_1{}^+}{\eta_2} e^{-\alpha_2 z} \operatorname{Re}[e^{j(\omega t-\beta_2 z)}]\mathbf{a}_y. \tag{9.224}$$

For lossless media ($\sigma_1 = \sigma_2 = 0$, $\alpha_1 = \alpha_2 = 0$), in sinusoidal forms, Eqs. (9.219)–(9.224) can be written as

$$E_i = E_1{}^+ \cos(\omega t - \beta_1 z)\mathbf{a}_x, \tag{9.225}$$

$$H_i = \frac{E_1{}^+}{\eta_1} \cos(\omega t - \beta_1 z)\mathbf{a}_y, \tag{9.226}$$

$$E_r = \rho E_1{}^+ \cos(\omega t + \beta_1 z)\mathbf{a}_x, \tag{9.227}$$

$$H_r = -\frac{\rho E_1{}^+}{\eta_1} \cos(\omega t + \beta_1 z)\mathbf{a}_y, \tag{9.228}$$

$$E_2 = \tau E_1{}^+ \cos(\omega t - \beta_2 z)\mathbf{a}_x, \tag{9.229}$$

$$H_2 = \frac{\tau E_1{}^+}{\eta_2} \cos(\omega t - \beta_2 z)\mathbf{a}_y. \tag{9.230}$$

Like the transmission line, the input impedance for a uniform plane wave can be derived as

$$\eta_{in} = \frac{E_1}{H_1}. \tag{9.231}$$

Substituting Eqs. (9.182) and (9.183) into Eq. (9.231) yields

$$\eta_{in} = \frac{E_1{}^+ + E_1{}^-}{\frac{E_1{}^+}{\eta_1} - \frac{E_1{}^-}{\eta_1}}. \tag{9.232}$$

For lossless medium, modify Eqs. (9.212), (9.214), and (9.216) by putting $\alpha_1 = \alpha_2 = 0$ and substituting in Eq. (9.232) as

$$\eta_{in} = \frac{E_1{}^+(e^{-j\beta_1 z} + \rho e^{j\beta_1 z})}{\frac{E_1{}^+}{\eta_1}(e^{-j\beta_1 z} - \rho e^{j\beta_1 z})}, \tag{9.233}$$

$$\eta_{in} = \eta_1 \frac{(e^{-j\beta_1 z} + \rho e^{j\beta_1 z})}{(e^{-j\beta_1 z} - \rho e^{j\beta_1 z})}. \tag{9.234}$$

Substituting Eq. (9.200) into Eq. (9.234) yields

$$\eta_{in} = \eta_1 \frac{e^{-j\beta_1 z} + \frac{\eta_2-\eta_1}{\eta_2+\eta_1} e^{j\beta_1 z}}{e^{-j\beta_1 z} - \frac{\eta_2-\eta_1}{\eta_2+\eta_1} e^{j\beta_1 z}}, \tag{9.235}$$

$$\eta_{in} = \eta_1 \left[\frac{\eta_2(e^{-j\beta_1 z} + e^{j\beta_1 z}) + \eta_1(e^{-j\beta_1 z} - e^{j\beta_1 z})}{\eta_1(e^{-j\beta_1 z} + e^{j\beta_1 z}) + \eta_2(e^{-j\beta_1 z} - e^{j\beta_1 z})} \right]. \tag{9.236}$$

Dividing numerator and denominator of Eq. (9.236) by $(e^{-j\beta_1 z} + e^{j\beta_1 z})$ yields

$$\eta_{in} = \eta_1 \left[\frac{\eta_2 + \eta_1 \frac{(e^{-j\beta_1 z} - e^{j\beta_1 z})}{(e^{-j\beta_1 z} + e^{j\beta_1 z})}}{\eta_1 + \eta_2 \frac{(e^{-j\beta_1 z} - e^{j\beta_1 z})}{(e^{-j\beta_1 z} + e^{j\beta_1 z})}} \right], \tag{9.237}$$

$$\eta_{in} = \eta_1 \left[\frac{\eta_2 - \eta_1 \frac{(e^{j\beta_1 z} - e^{-j\beta_1 z})}{(e^{-j\beta_1 z} + e^{j\beta_1 z})}}{\eta_1 - \eta_2 \frac{(e^{j\beta_1 z} - e^{-j\beta_1 z})}{(e^{-j\beta_1 z} + e^{j\beta_1 z})}} \right]. \tag{9.238}$$

The following trigonometry formulae can be written as

$$\tanh(j\beta_1 z) = \frac{e^{j\beta_1 z} - e^{-j\beta_1 z}}{e^{-j\beta_1 z} + e^{j\beta_1 z}}, \tag{9.239}$$

$$\tanh(j\beta_1 z) = j \tan \beta_1 z. \tag{9.240}$$

Substituting Eq. (9.240) into Eq. (9.239) yields

$$j \tan \beta_1 z = \frac{e^{j\beta_1 z} - e^{-j\beta_1 z}}{e^{-j\beta_1 z} + e^{j\beta_1 z}}. \tag{9.241}$$

Substituting Eq. (9.241) into Eq. (9.238) yields

$$\eta_{in} = \eta_1 \left[\frac{\eta_2 - j\eta_1 \tan \beta_1 z}{\eta_1 - j\eta_2 \tan \beta_1 z} \right]. \tag{9.242}$$

The expression of average power density for the incident wave can be derived as

$$P_i = \frac{1}{2} \mathrm{Re}[E_i \times {H_i}^*]. \tag{9.243}$$

Substituting Eqs. (9.212) and (9.213) into Eq. (9.243) yields

$$P_i = \frac{1}{2} \mathrm{Re}[{E_1}^+ e^{-\gamma_1 z} \times \frac{{E_1}^+ e^{\gamma_1 z}}{\eta_1}](\mathbf{a}_x \times \mathbf{a}_y), \tag{9.244}$$

$$P_i = \frac{1}{2} \frac{({E_1}^+)^2}{\eta_1} \mathbf{a}_z. \tag{9.245}$$

The expression of average power density for the reflected wave can be derived as

$$P_r = \frac{1}{2}\mathrm{Re}[E_r \times H_r{}^*]. \tag{9.246}$$

Substituting Eqs. (9.214) and (9.215) into Eq. (9.246) yields

$$P_r = \frac{1}{2}\mathrm{Re}[\rho E_1{}^+ e^{\gamma_1 z} \times \frac{-\rho E_1{}^+ e^{-\gamma_1 z}}{\eta_1}](\mathbf{a}_x \times \mathbf{a}_y), \tag{9.247}$$

$$P_r = -\frac{1}{2}\frac{(\rho E_1{}^+)^2}{\eta_1}\mathbf{a}_z. \tag{9.248}$$

The expression of average power density for the transmitted wave can be derived as

$$P_2 = \frac{1}{2}\mathrm{Re}[E_2 \times H_2{}^*]. \tag{9.249}$$

Substituting Eqs. (9.216) and (9.217) into Eq. (9.249) yields

$$P_2 = \frac{1}{2}\mathrm{Re}[\tau E_1{}^+ e^{-\gamma_1 z} \times \frac{\tau E_1{}^+ e^{\gamma_1 z}}{\eta_2}](\mathbf{a}_x \times \mathbf{a}_y), \tag{9.250}$$

$$P_2 = \frac{1}{2}\frac{(\tau E_1{}^+)^2}{\eta_2}\mathbf{a}_z. \tag{9.251}$$

Example 9.8 The magnitude and the frequency of a uniform plane wave are 150 V/m and 100 MHz, respectively. The wave propagates in free space and is incident onto the boundary of two media. The material of medium 2 is characterized by $\mu_r = 1$ and $\varepsilon_r = 8$. Calculate the (a) β_1, (b) β_2, (c) η_1, (d) η_2, (e) time-domain incident, reflected and transmitted fields.

Solution

(a) The value of the phase constant for medium 1 is
$\beta_1 = \omega\sqrt{\mu_0\varepsilon_0} = 2\pi \times 10^8\sqrt{4\pi \times 10^{-7} \times \frac{1}{36\pi} \times 10^{-9}} = 2.09$ rad/m.

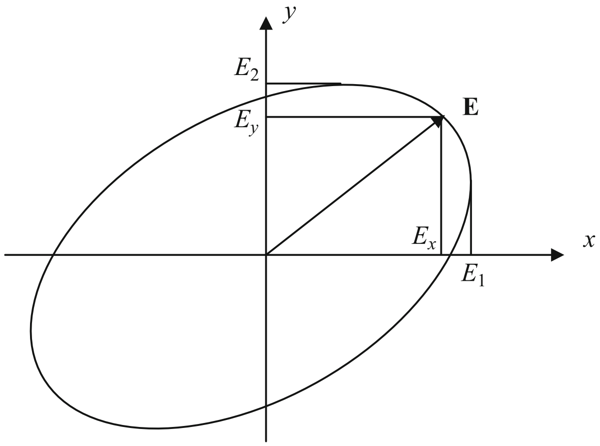
(b) The value of the phase constant for medium 1 is
$\beta_2 = \omega\sqrt{\mu_0\varepsilon_0\mu_{r2}\varepsilon_{r2}} = 2\pi \times 10^8\sqrt{4\pi \times 10^{-7} \times \frac{1}{36\pi} \times 8 \times 10^{-9}} = 5.92$ rad/m.

(c) The value of the intrinsic impedance for medium 1 is
$\eta_1 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\frac{4\pi \times 10^{-7}}{\frac{1}{36\pi} \times 10^{-9}}} = 377\Omega.$

(d) The value of the intrinsic impedance for medium 2 is
$\eta_2 = \sqrt{\frac{\mu_0\mu_{r2}}{\varepsilon_0\varepsilon_{r2}}} = \sqrt{\frac{4\pi \times 10^{-7}}{\frac{1}{36\pi} \times 10^{-9} \times 8}} = 133.29\Omega.$

(e) The value of the reflection coefficient is
$\rho = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{133.29 - 377}{133.29 + 377} = -0.48.$
The value of the transmission coefficient

Fig. 9.5 Elliptical polarization of fields



$\tau = \frac{2\eta_2}{\eta_2+\eta_1} = \frac{2\times 133.29}{133.29+377} = 0.52.$

Time-domain fields are

$E_i = 150\cos(2\pi \times 10^8 t - 2.09z)\mathbf{a}_x$ V/m,

$H_i = \frac{E_i}{\eta 1} = 0.4\cos(2\pi \times 10^8 t - 2.09z)\mathbf{a}_y$ A/m,

$E_r = \rho E_1{}^+ \cos(\omega t + \beta_1 z)\mathbf{a}_x = -72\cos(2\pi \times 10^8 t + 2.09z)\mathbf{a}_x$ V/m,

$H_r = -\frac{\rho E_1{}^+}{\eta_1} \cos(\omega t + \beta_1 z)\mathbf{a}_y = 0.19\cos(2\pi \times 10^8 t + 2.09z)\mathbf{a}_y$ A/m,

$E_2 = \tau E_1{}^+ \cos(\omega t - \beta_2 z)\mathbf{a}_x = 37.44\cos(2\pi \times 10^8 t - 5.92z)\mathbf{a}_x$ V/m,

$H_2 = \frac{\tau E_1{}^+}{\eta_2} \cos(\omega t - \beta_2 z)\mathbf{a}_y = 0.28\cos(2\pi \times 10^8 t - 5.92z)\mathbf{a}_y$ A/m.

Practice Problem 9.8 A uniform plane wave travels through air and is incident onto the boundary of two media. The expression of the incident wave is $E_i = 50e^{-j\beta_1 z}$ V/m. The material of medium 2 is characterized by $\mu_r = 1$, $\varepsilon_r = 4.5$. Calculate (a) η_1, (b) η_2, (c) P_i, (d) P_r, and (e) P_2.

9.10 Uniform Wave Polarization

Polarization is normally used to describe the fundamental characteristic of a wave. The polarization of a uniform plane wave is defined as the behavior of an electric field in time at a given point in space. The general elliptical polarization is shown in Fig. 9.5. Two electric fields E_x and E_y are directed in the $x-$ and $y-$ directions, respectively. In addition, E_y leads E_x by a time-phase angle ϕ. Then, equations of electric fields in the time domain are

$$E_x = \text{Re}[E_1 e^{j(\omega t - \beta z)}] = E_1 \cos(\omega t - \beta z), \tag{9.252}$$

$$E_y = \text{Re}[E_2 e^{j(\omega t + \phi - \beta z)}] = E_2 \cos(\omega t - \beta z + \phi). \tag{9.253}$$

Consider $z = 0$ without losing the fundamental property of a wave. Then Eqs. (9.252) and (9.253) can be modified as

$$E_x = E_1 \cos \omega t, \tag{9.254}$$

$$\cos \omega t = \frac{E_x}{E_1}, \tag{9.255}$$

$$\sin \omega t = \sqrt{1 - \cos^2 \omega t} = \sqrt{1 - \left(\frac{E_x}{E_1}\right)^2}, \tag{9.256}$$

$$E_y = E_2 \cos(\omega t + \phi) = E_2(\cos \omega t \cos \phi - \sin \omega t \sin \phi). \tag{9.257}$$

Substituting Eqs. (9.255) and (9.256) into Eq. (9.257) yields

$$\frac{E_y}{E_2} = \frac{E_x}{E_1} \cos \phi - \sqrt{1 - \frac{{E_x}^2}{{E_1}^2}} \sin \phi, \tag{9.258}$$

$$\frac{E_y}{E_2} - \frac{E_x}{E_1} \cos \phi = -\sqrt{1 - \frac{{E_x}^2}{{E_1}^2}} \sin \phi. \tag{9.259}$$

Squaring both sides of Eq. (9.259) yields

$$\frac{{E_y}^2}{{E_2}^2} - \frac{2E_x E_y}{E_1 E_2} \cos \phi + \frac{{E_x}^2}{{E_1}^2} \cos^2 \phi = \sin^2 \phi - \frac{{E_x}^2}{{E_1}^2} \sin^2 \phi, \tag{9.260}$$

$$\frac{{E_y}^2}{{E_2}^2} - \frac{2E_x E_y}{E_1 E_2} \cos \phi + \frac{{E_x}^2}{{E_1}^2} = \sin^2 \phi. \tag{9.261}$$

Equation (9.261) is known as the equation of an ellipse. Therefore, the wave is elliptically polarized.

9.11 Exercise Problems

9.1 The electric field of a uniform plane wave in free space is given by $\mathbf{E} = 150 \cos(2\pi 10^8 t - \beta_0 x)\mathbf{a}_z$ V/m. Determine (a) the phase constant, (b) wavelength, and (c) magnetic field at the point (0.1,1.03,0.2) m and $t = 2$ ns.

9.2 The electric field of a lossless, isotropic, and uniform dielectric medium is given by $\mathbf{E} = (3\mathbf{a}_x + j8\mathbf{a}_y)e^{-j1.5z}$ V/m. Calculate (a) the phase constant, (b) angular frequency, (c) velocity, and (d) intrinsic impedance. Consider that the frequency of the uniform plane wave is 40 MHz.

9.3 Convert the expression $3\cos\omega t + 4\sin\omega t$ into the format $A\cos(\omega t - \theta)$. Find the values of A and θ.

9.4 In free space, the magnetic field is given by $\mathbf{H}(z,t) = 250\cos(10^8 t - \beta z)\mathbf{a}_y$ A/m. Determine (a) the phase constant, (b) wavelength, and (c) $\mathbf{E}(z,t)$.

9.5 A uniform plane wave travels through a good dielectric medium with a frequency of 8 GHz. The relative permittivity and loss tangent of this medium are 9 and 5×10^{-6}, respectively. Determine (a) the phase constant and (b) attenuation constant.

9.6 The permittivity and attenuation constant of a good dielectric are 6 and 0.045 Np/m, respectively. The frequency of a uniform plane wave is 8 MHz. Calculate (a) the phase constant and (b) loss tangent.

9.7 The electric field of a uniform plane wave is given by $\mathbf{E} = 25e^{-0.08y}\cos(10^8 t - 2.5y)\mathbf{a}_z$ V/m. Determine (a) the propagation constant, (b) phase constant, (c) wavelength, (d) speed, and (e) skin depth.

9.8 The electric and magnetic fields in a isotropic, lossless, and uniform dielectric medium are given by $\mathbf{E} = 750\cos(2\pi 10^8 t - \beta z)\mathbf{a}_x$ V/m and $\mathbf{H} = 5\cos(10^8 t - \beta z)\mathbf{a}_y$A/m. Determine (a) the intrinsic impedance, (b) relative permeability μ_r and relative permittivity ε_r, (c) phase constant, and (d) wavelength. Assume the propagating velocity of a uniform plane wave is 1.5×10^8 m/s.

9.9 The electric field of a uniform plane wave in a lossless, isotropic, and uniform dielectric medium is given by $\mathbf{E} = (2\mathbf{a}_x + j5\mathbf{a}_y)e^{-j3z}$ V/m. Calculate (a) the phase constant, (b) angular frequency, (c) velocity, (d) intrinsic impedance, and (e) $\mathbf{H}$. Consider that the frequency of the uniform plane wave is 40 MHz.

9.10 The electric field of a uniform plane wave directed in the nonmagnetic medium in the z-direction is given by $\mathbf{E} = 25\cos(10^8 t - 0.8x)\mathbf{a}_z$ V/m. Find (a) the relative permittivity, (b) intrinsic impedance, and (c) time-average power.

9.11 The magnitude and the frequency of a uniform plane wave are 200 V/m and 5 GHz, respectively. The wave propagates in free space and is incident onto the boundary of two media. The material of medium 2 is characterized by $\mu_r = 1$ and $\varepsilon_r = 6$. Determine the (a) β_1, (b) β_2, (c) η_1, (d) η_2, (e) time domain incident, reflected and transmitted fields.

9.12 A uniform plane wave travels through air and is incident onto the boundary of two media. The expression of the incident wave is $E_i = 30e^{-j\beta_1 z}$ V/m. The material of medium 2 is characterized by $\mu_r = 1$, $\varepsilon_r = 3$. Calculate (a) η_1, (b) η_2, (c) P_i, (d) P_r, and (e) P_2.

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Chapter 10
Basics of Antennas

10.1 Introduction

Electromagnetic fields are generated from charge. These time-varying electromagnetic fields propagate in different directions of free space and radiate energy. This radiation of energy into free space is accomplished efficiently with the help of a special electromagnetic device. This electromagnetic device is known as antenna. An antenna is an electromagnetic device that can transmit and receive radio waves efficiently. The antenna provides a transition between a guided electromagnetic wave and free space. The German physicist Heinrich Rudolf Hertz (1857–1894) first introduced the existence of electromagnetic waves, radiation of electromagnetic waves and antenna. Practical applications are ship navigation, aircrafts, air traffic control, satellites, radio and TV broadcasting.

In this chapter, principles of antennas, potential functions for antennas, Hertzian dipole, Faraday's law, conduction current, displacement current, motional voltage, Maxwell's equation, transformers, time-varying potentials, fields of a series circuit, and time-harmonic fields will be discussed.

10.2 Working Principles of Antennas

A radio frequency signal is generated in the transmitter. Then this signal, in the form of electromagnetic waves, is radiated into free space at the speed of light through a device known as the transmitting antenna. These electromagnetic waves arrive at the receiving antenna and are captured by its surrounding space. Then voltage is induced in the receiving antenna and passed to the receiver. This voltage is then converted back to its original radio frequency signal. Antennas are classified into different ways. The frequency band of operation and physical structure are two of them. Different types of antennas include wire, yagi, loop, vertical, folded dipole and horn.

Md. A. Salam, *Electromagnetic Field Theories for Engineering*,
DOI 10.1007/978-981-4585-66-8_10,

10.3 Potential Functions for Antennas

Four Maxwell's equations with sources are considered for deriving electromagnetic fields. These four equations, which have already been discussed in some previous chapters, are

$$\nabla.\mathbf{E} = \frac{\rho}{\varepsilon}, \tag{10.1}$$

$$\nabla.\mathbf{B} = 0, \tag{10.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu \frac{\partial \mathbf{H}}{\partial t}, \tag{10.3}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t}. \tag{10.4}$$

The following vector identity can be written as

$$\nabla \cdot \nabla \times \mathbf{A} = 0. \tag{10.5}$$

From Eqs. (10.2) and (10.5), the following equations can be written as

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{10.6}$$

and

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A}. \tag{10.7}$$

Substituting Eq. (10.7) into Eq. (10.3) yields

$$\nabla \times \mathbf{E} = -\mu \frac{\partial}{\partial t} \left(\frac{1}{\mu} \nabla \times \mathbf{A} \right), \tag{10.8}$$

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0. \tag{10.9}$$

Again consider the following vector identity:

$$\nabla \times \nabla V = 0. \tag{10.10}$$

From Eqs. (10.9) and (10.10), the following equation can be written as

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V, \tag{10.11}$$

where $\mathbf{A}$ is the vector magnetic potential and V is the scalar potential. The negative sign of Eq. (10.11) is considered for suitable mathematical manipulation. Equation (10.11) can be written as

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}. \tag{10.12}$$

Substituting Eqs. (10.7) and (10.12) into Eq. (10.4) yields

$$\frac{1}{\mu}\nabla \times \nabla \times \mathbf{A} = \mathbf{J} + \varepsilon\frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \mathbf{A}}{\partial t}\right). \tag{10.13}$$

The following vector identity can be written as

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla.\mathbf{A}) - \nabla^2\mathbf{A}. \tag{10.14}$$

Substituting Eq. (10.14) into Eq. (10.13) yields

$$\nabla(\nabla.\mathbf{A}) - \nabla^2\mathbf{A} = \mu\mathbf{J} + \varepsilon\mu\frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \mathbf{A}}{\partial t}\right). \tag{10.15}$$

Applying Lorentz gauge condition yields

$$\nabla.\mathbf{A} = -\varepsilon\mu\frac{\partial V}{\partial t}, \tag{10.16}$$

$$-\nabla^2\mathbf{A} = \mu\mathbf{J} - \varepsilon\mu\frac{\partial^2\mathbf{A}}{\partial t^2} \tag{10.17}$$

$$\nabla^2\mathbf{A} - \varepsilon\mu\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu\mathbf{J}. \tag{10.18}$$

In case of sinusoidal, Eq. (10.18) can be written as

$$\nabla^2\mathbf{A} + \varepsilon\mu\omega^2\mathsf{A} = -\mu\mathbf{J}. \tag{10.19}$$

The source point with vector magnetic potential, electric and magnetic fields is shown in Fig. 10.1. The general solution of Eq. (10.19) is

$$\mathbf{A} = \frac{\mu}{4\pi}\int_v \frac{\mathbf{J}e^{-j\beta r}}{r}dv. \tag{10.20}$$

10.4 Hertzian Dipole

The Hertzian dipole is a simple form of a radiating antenna that can be used to calculate all fields. It consists of two equal and opposite charges. These charges are separated by an infinitesimal distance *dl* as shown in Fig. 10.2. From Fig. 10.2, the following equations can be written as

$$\mathbf{A}_r = \mathbf{A}_z \cos\theta, \tag{10.21}$$

$$\mathbf{A}_\theta = -\mathbf{A}_z \sin\theta \tag{10.22}$$

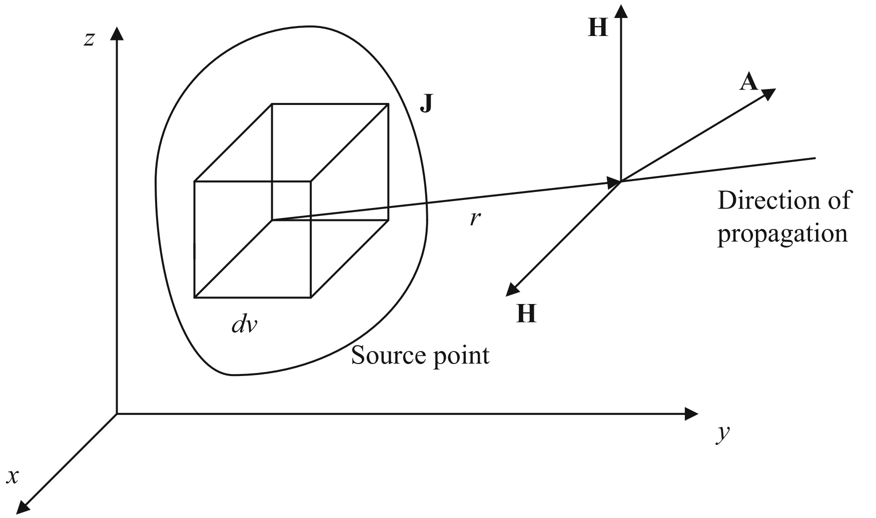

Fig. 10.1 Source with electric and magnetic fields

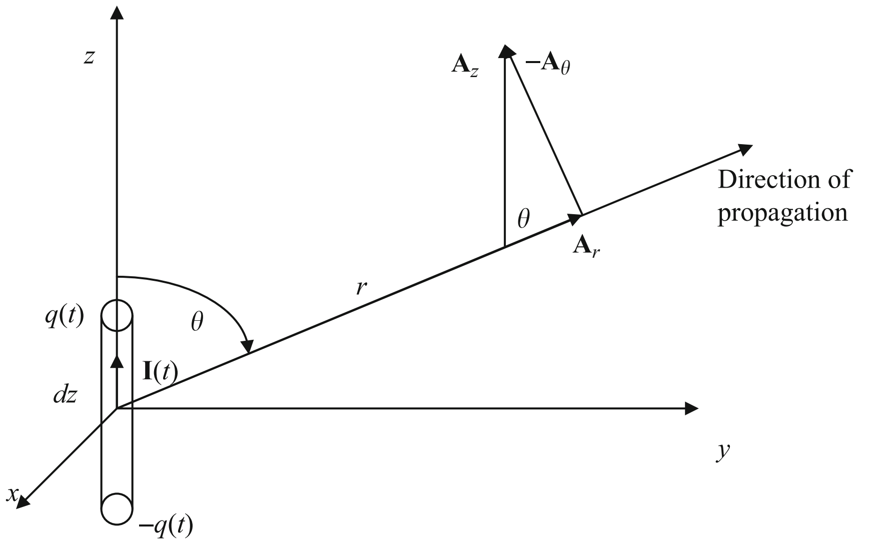

Fig. 10.2 Current through an infinitesimal length of wire

and

$$\mathbf{A}_\phi = 0. \tag{10.23}$$

The vector magnetic potential in the z-direction is

$$\mathbf{A}_z = \frac{\mu \mathbf{I}(z)}{4\pi r} e^{-j\beta r} dz. \tag{10.24}$$

Substituting Eq. (10.24) into Eqs. (10.21) and (10.22) yields

$$\mathbf{A}_r = \frac{\mu \mathbf{I}(z)}{4\pi r} e^{-j\beta r} \cos\theta dz \tag{10.25}$$

and

$$\mathbf{A}_\theta = -\frac{\mu \mathbf{I}(z)}{4\pi r} e^{-j\beta r} \sin\theta dz. \tag{10.26}$$

From the spherical coordinate system, the curl of vector magnetic potential is

$$\nabla \times \mathbf{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \mathbf{a}_r & r\mathbf{a}_\theta & r\sin\theta \mathbf{a}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}, \tag{10.27}$$

$$\nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\left[\frac{\partial(A_\phi \sin\theta)}{\partial\theta} - \frac{\partial A_\theta)}{\partial\phi}\right]\mathbf{a}_r + \frac{1}{r}\left[\frac{\partial A_r}{\partial\phi} - \frac{\partial(rA_\phi)}{\partial r}\right]\mathbf{a}_\theta + \frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]\mathbf{a}_\phi, \tag{10.28}$$

$$\mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{A}. \tag{10.29}$$

Since $A_\phi = 0$ and $\partial/\partial\phi = 0$, the expression of magnetic field from Eqs. (10.28) and (10.29) in a_ϕ direction is

$$\mathbf{H}_\phi = \frac{1}{\mu r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]. \tag{10.30}$$

Substituting Eqs. (10.25) and (10.26) into Eq. (10.30) yields

$$\mathbf{H}_\phi = \frac{\mathbf{I}(z)dz}{4\pi r}\left[\frac{\partial}{\partial r}\left(-e^{-j\beta r}\sin\theta\right) - \frac{\partial}{\partial\theta}\left(\frac{e^{-j\beta r}}{r}\cos\theta\right)\right], \tag{10.31}$$

$$\mathbf{H}_\phi = \frac{\mathbf{I}(z)dz}{4\pi r}\left[-\sin\theta e^{-j\beta r}(-j\beta) + \frac{e^{-j\beta r}}{r}\sin\theta\right], \tag{10.32}$$

$$\mathbf{H}_\phi = \frac{\sin\theta \mathbf{I}(z)dz}{4\pi}\left[\frac{j\beta}{r} + \frac{1}{r^2}\right]e^{-j\beta r}, \tag{10.33}$$

$$\mathbf{H}_r = 0 \tag{10.34}$$

and

$$\mathbf{H}_\theta = 0. \tag{10.35}$$

Substituting $\mathbf{A}_r = \mathbf{H}_r = 0$ and $\mathbf{A}_\theta = \mathbf{H}_\theta = 0$ into Eq. (10.28) yields

$$\nabla \times \mathbf{H} = \frac{1}{r \sin\theta} \frac{\partial(H_\phi \sin\theta)}{\partial\theta} \mathbf{a}_r - \frac{1}{r} \frac{\partial(r A_\phi)}{\partial r} \mathbf{a}_\theta. \tag{10.36}$$

Consider the following Maxwell's equation to find the electric field:

$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E}. \tag{10.37}$$

Substituting Eq. (10.36) into Eq. (10.37) yields

$$\mathbf{E} = \frac{1}{j\omega\varepsilon} \left[\frac{1}{r \sin\theta} \frac{\partial(H_\phi \sin\theta)}{\partial\theta} \mathbf{a}_r - \frac{1}{r} \frac{\partial(r A_\phi)}{\partial r} \mathbf{a}_\theta \right]. \tag{10.38}$$

The r-component of the electric field is

$$\mathbf{E}_r = \frac{1}{j\omega\varepsilon} \frac{1}{r \sin\theta} \frac{\partial(H_\phi \sin\theta)}{\partial\theta}. \tag{10.39}$$

Substituting Eq. (10.33) into Eq. (10.39) yields

$$\mathbf{E}_r = \frac{1}{j\omega\varepsilon} \frac{1}{r \sin\theta} \frac{\partial(\sin^2\theta)}{\partial\theta} \frac{\mathbf{I}(z)dz}{4\pi} \left[\frac{j\beta}{r} + \frac{1}{r^2} \right] e^{-j\beta r}, \tag{10.40}$$

$$\mathbf{E}_r = \frac{1}{j\omega\varepsilon} \frac{2 \sin\theta \cos\theta}{r \sin\theta} \frac{\mathbf{I}(z)dz}{4\pi} \left[\frac{j\beta}{r} + \frac{1}{r^2} \right] e^{-j\beta r}. \tag{10.41}$$

The expression of the phase constant is

$$\beta = \omega\sqrt{\mu\varepsilon}, \tag{10.42}$$

$$\frac{\beta}{\omega} = \sqrt{\mu\varepsilon}. \tag{10.43}$$

The expression of the intrinsic impedance is

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\varepsilon\mu}{\varepsilon^2}} = \frac{\sqrt{\mu\varepsilon}}{\varepsilon}, \tag{10.44}$$

$$\eta\varepsilon = \sqrt{\mu\varepsilon}. \tag{10.45}$$

Substituting Eq. (10.43) into Eq. (10.45) yields

$$\frac{\beta}{\omega} = \varepsilon\eta, \tag{10.46}$$

$$\omega\varepsilon = \frac{\beta}{\eta}. \tag{10.47}$$

Substituting Eq. (10.47) into Eq. (10.41) yields

$$\mathbf{E}_r = \frac{\cos\theta}{j\frac{\beta}{\eta}} \frac{\mathbf{I}(z)dz}{2\pi} \left[\frac{j\beta}{r^2} + \frac{1}{r^3} \right] e^{-j\beta r}, \tag{10.48}$$

$$\mathbf{E}_r = \eta \cos\theta \frac{\mathbf{I}(z)dz}{2\pi}\left[\frac{1}{r^2} + \frac{1}{j\beta r^3}\right] e^{-j\beta r}. \tag{10.49}$$

The θ-component of the electric field is

$$\mathbf{E}_\theta = -\frac{1}{r}\frac{1}{j\omega\varepsilon}\frac{\partial(rH_\phi)}{\partial r}. \tag{10.50}$$

Substituting Eq. (10.33) into Eq. (10.50) yields

$$\mathbf{E}_\theta = -\frac{1}{r}\frac{1}{j\omega\varepsilon}\frac{\sin\theta\mathbf{I}(z)dz}{4\pi}\frac{\partial}{\partial r}\left[r\left(\frac{j\beta}{r} + \frac{1}{r^2}\right)e^{-j\beta r}\right], \tag{10.51}$$

$$\mathbf{E}_\theta = -\frac{1}{r}\frac{1}{j\omega\varepsilon}\frac{\sin\theta\mathbf{I}(z)dz}{4\pi}\frac{\partial}{\partial r}\left[\left(j\beta + \frac{1}{r}\right)e^{-j\beta r}\right], \tag{10.52}$$

$$\mathbf{E}_\theta = -\frac{1}{r}\frac{1}{j\omega\varepsilon}\frac{\sin\theta\mathbf{I}(z)dz}{4\pi}\left[-\frac{1}{r^2}e^{-j\beta r} + (-j\beta)\left(j\beta + \frac{1}{r}\right)e^{-j\beta r}\right], \tag{10.53}$$

$$\mathbf{E}_\theta = \frac{1}{j\omega\varepsilon}\frac{\sin\theta\mathbf{I}(z)dz}{4\pi}\left[\frac{1}{r^3} - \frac{\beta^2}{r} + j\frac{\beta}{r^2}\right]e^{-j\beta r}. \tag{10.54}$$

Substituting Eq. (10.47) into Eq. (10.54) yields

$$\mathbf{E}_\theta = \frac{1}{j\frac{\beta}{\eta}}\frac{\sin\theta\mathbf{I}(z)dz}{4\pi}\left[\frac{1}{r^3} - \frac{\beta^2}{r} + j\frac{\beta}{r^2}\right]e^{-j\beta r}, \tag{10.55}$$

$$\mathbf{E}_\theta = \frac{\sin\theta\mathbf{I}(z)dz}{4\pi}\eta\left[\frac{1}{j\beta r^3} - \frac{\beta}{jr} + \frac{1}{r^2}\right]e^{-j\beta r}, \tag{10.56}$$

$$\mathbf{E}_\theta = \frac{\sin\theta\mathbf{I}(z)dz}{4\pi}\eta\left[\frac{1}{j\beta r^3} + \frac{j\beta}{r} + \frac{1}{r^2}\right]e^{-j\beta r}. \tag{10.57}$$

For zero frequency, $\beta = 0$ and Eq. (10.33) can be modified as

$$\mathbf{H}_\phi = \frac{\sin\theta\mathbf{I}(z)dz}{4\pi}\left[\frac{0}{r} + \frac{1}{r^2}\right]e^0, \tag{10.58}$$

$$\mathbf{H}_\phi = \frac{\sin\theta\mathbf{I}(z)dz}{4\pi r^2}. \tag{10.59}$$

Equation (10.59) represents the expression of the Biot–Savart law for an infinitesimal current element.

On the basis of the variation of distance, the fields can be divided into three categories. These are (1) electrostatic fields or near fields, which vary as $1/r^3$; (2) induction fields, which vary as $1/r^2$ and (3) radiation fields or far fields, which vary as $1/r$. The components of near fields are

$$\mathbf{E}_r\left(\frac{1}{r^3}\right) = \eta\cos\theta\frac{\mathbf{I}(z)dz}{j\beta r^3 2\pi}e^{-j\beta r} = -j2\eta\cos\theta\frac{\mathbf{I}(z)dz}{4\pi\beta r^3}e^{-j\beta r} \tag{10.60}$$

and

$$\mathbf{E}_\theta\left(\frac{1}{r^3}\right) = \frac{-j\,\sin\theta\,\mathbf{I}(z)dz}{4\pi\beta r^3}\eta e^{-j\beta r}. \tag{10.61}$$

The resultant near field can be determined as

$$\left|\mathbf{E}_{nf}\right| = \sqrt{\left|\mathbf{E}_r\left(\frac{1}{r^3}\right)\right|^2 + \left|\mathbf{E}_\theta\left(\frac{1}{r^3}\right)\right|^2}. \tag{10.62}$$

Substituting Eqs. (10.60) and (10.61) into Eq. (10.62) yields

$$\left|\mathbf{E}_{nf}\right| = \frac{\mathbf{I}(z)dz}{4\pi\beta r^3}\eta\sqrt{4\cos^2\theta + \sin^2\theta}, \tag{10.63}$$

$$\mathbf{E}_{nf} = \frac{\mathbf{I}(z)dz}{4\pi\beta r^3}\eta\sqrt{1 + 3\cos^2\theta}, \tag{10.64}$$

$$\mathbf{E}_{nf} = \frac{\mathbf{I}(z)dz}{4\pi\beta r^3}\eta\sqrt{1 + \frac{3}{2}(1 + \cos 2\theta)}. \tag{10.65}$$

At $\theta = \pi/2$, the near field is minimum and its value is

$$(\mathbf{E}_{nf})_{\text{min}} = \frac{\mathbf{I}(z)dz}{4\pi\beta r^3}\eta. \tag{10.66}$$

At $\theta = 0°, \pi$, the near fields are maximum and the values are

$$(\mathbf{E}_{nf})_{\text{max}} = 1.59\frac{\mathbf{I}(z)dz}{4\pi\beta r^3}\eta, \tag{10.67}$$

$$(\mathbf{E}_{nf})_{\text{max}} = 2\frac{\mathbf{I}(z)dz}{4\pi\beta r^3}\eta. \tag{10.68}$$

The far-field components of the electric and the magnetic fields are

$$\mathbf{E}_\theta\left(\frac{1}{r}\right) = j\beta\frac{\sin\theta\,\mathbf{I}(z)dz}{4\pi r}\eta e^{-j\beta r} \tag{10.69}$$

and

$$\mathbf{H}_\phi\left(\frac{1}{r}\right) = \frac{j\beta\sin\theta\,\mathbf{I}(z)dz}{4\pi r}e^{-j\beta r}. \tag{10.70}$$

The ratio of far electric and magnetic fields is

$$\frac{\mathbf{E}_\theta\left(\frac{1}{r}\right)}{\mathbf{H}_\theta\left(\frac{1}{r}\right)} = \frac{j\beta\frac{\sin\theta\,\mathbf{I}(z)dz}{4\pi r}\eta e^{-j\beta r}}{\frac{j\beta\sin\theta\,\mathbf{I}(z)dz}{4\pi r}e^{-j\beta r}} = \eta. \tag{10.71}$$

The magnitudes of the far-field components (radiated fields) are

$$|\mathbf{E}_{rad}| = \left|\mathbf{E}_{\theta}\left(\frac{1}{r}\right)\right| = \beta\frac{\sin\theta\,\mathbf{I}(z)dz}{4\pi r}\eta \tag{10.72}$$

and

$$|\mathbf{H}_{rad}| = \left|\mathbf{H}_{\phi}\left(\frac{1}{r}\right)\right| = \frac{\beta\sin\theta\,\mathbf{I}(z)dz}{4\pi r}. \tag{10.73}$$

The average radiated power density can be determined as

$$P_{av} = \frac{1}{2}\,|\mathbf{E}_{rad}|\,|\mathbf{H}_{rad}|\,. \tag{10.74}$$

Substituting Eqs. (10.72) and (10.73) into Eq. (10.74) yields

$$P_{av} = \frac{1}{2}\frac{\eta\beta^2\mathbf{I}^2(z)(dz)^2}{(4\pi r)^2}\sin^2\theta. \tag{10.75}$$

The total power radiated by the antenna can be determined by integrating the average power density over a sphere of any radius as

$$P_{rad} = \oint_s P_{av}.d\mathrm{S}. \tag{10.76}$$

Substituting Eq. (10.75) into Eq. (10.76) yields

$$P_{rad} = \frac{1}{2}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{\eta\beta^2\mathbf{I}^2(z)(dz)^2}{(4\pi r)^2}\sin^2\theta(r^2\sin\theta\,d\theta d\phi). \tag{10.77}$$

$$P_{rad} = \frac{\mathbf{I}^2(z)(dz)^2\beta^2\eta}{32\pi^2}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\sin^3\theta d\theta d\phi, \tag{10.78}$$

$$P_{rad} = \frac{\mathbf{I}^2(z)(dz)^2\beta^2\eta}{32\pi^2}(2\pi)\int_{\theta=0}^{\pi}\left(\frac{4}{3}\sin\theta - \frac{1}{4}\sin 3\theta\right)d\theta\,, \tag{10.79}$$

$$P_{rad} = \frac{\mathbf{I}^2(z)(dz)^2\beta^2\eta}{32\pi^2}(2\pi)\left[\frac{3}{4}\left(-\cos\theta\right)_0^{\pi} + \frac{1}{12}\left(\cos 3\theta\right)_0^{\pi}\right], \tag{10.80}$$

$$P_{rad} = \frac{\mathbf{I}^2(z)(dz)^2\beta^2\eta}{32\pi^2}(2\pi)\left[-\frac{3}{4}\left(-1-1\right) + \frac{1}{12}(-1-1)\right], \tag{10.81}$$

$$P_{rad} = \frac{\mathbf{I}^2(z)(dz)^2\beta^2\eta}{32\pi^2}(2\pi)\frac{4}{3}\,. \tag{10.82}$$

Substituting $\eta = 120\pi$ and $\beta = 2\pi/\lambda$ into Eq. (10.82) yields

$$P_{rad} = \frac{\mathbf{I}^2(z)(dz)^2(120\pi)}{32\pi^2}(2\pi)\frac{4}{3}\left(\frac{2\pi}{\lambda}\right)^2, \tag{10.83}$$

$$P_{rad} = 40\pi^2 \mathbf{I}^2(z) \left(\frac{dz}{\lambda}\right)^2. \tag{10.84}$$

The rms value of the current is

$$\mathbf{I}_{rms} = \frac{\mathbf{I}(z)}{\sqrt{2}}. \tag{10.85}$$

Substituting Eq. (10.85) into Eq. (10.85) yields

$$P_{rad} = 80\pi^2 \mathrm{I}_{rms}{}^2 \left(\frac{dz}{\lambda}\right)^2, \tag{10.86}$$

$$\frac{P_{rad}}{\mathbf{I}_{rms}{}^2} = 80\pi^2 \left(\frac{dz}{\lambda}\right)^2, \tag{10.87}$$

$$R_{rad} = 80\pi^2 \left(\frac{dz}{\lambda}\right)^2. \tag{10.88}$$

The radiation resistance represents hypothetical resistance that will absorb the same amount of power as radiated by the Hertzian dipole when both carry the same current.

Example 10.1 The ratio of an infinitesimal distance to the wavelength of the Hertzian dipole is found to be 0.12. Determine the radiation resistance.

Solution The value of the radiation resistance can be determined as

$$R_{rad} = 80\pi^2 \left(\frac{dz}{\lambda}\right)^2 = 80\pi^2 (0.12)^2 = 11.37\ \Omega.$$

Example 10.2 A 0.5 m infinitesimal dipole radiates at the frequency of 20 MHz. Determine the radiation resistance.

Solution The value of the wavelength is calculated as

$$R_{rad} = 80\pi^2 \left(\frac{dz}{\lambda}\right)^2 = 80\pi^2 (0.12)^2 = 11.37\ \Omega.$$

The value of the radiation resistance can be determined as

$$R_{rad} = 80\pi^2 \left(\frac{dz}{\lambda}\right)^2 = 80\pi^2 \left(\frac{0.5}{15}\right)^2 = 0.88\ \Omega.$$

Practice Problem 10.1 The radiation resistance of the Hertzian dipole is found to be 5 Ω. Calculate the ratio of an infinitesimal distance to the wavelength.

Practice Problem 10.2 An infinitesimal dipole radiates at the frequency of 10 MHz. Determine the elemental length of the dipole if the radiation resistance is found to be 0.65 Ω.

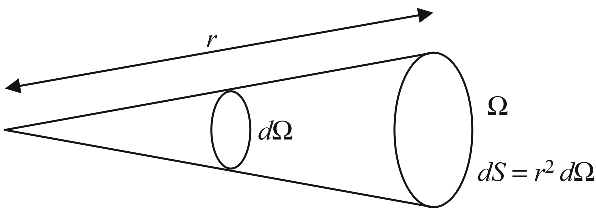

Fig. 10.3 Representation of a solid angle

10.5 Antenna Gain and Directivity

The term gain is used to determine the performance of any device. In antennas, the gain is defined as the amount of power density radiated in all directions. The radiation intensity of an antenna in the direction of (θ, ϕ) is equal to square of the radius time average power and it can be expressed as

$$U(\theta, \phi) = r^2 P_{av}. \tag{10.89}$$

From Eq. (10.76), the total power radiated by the antenna is

$$P_{rad} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} P_{av} r^2 \sin\theta d\theta d\phi. \tag{10.90}$$

Substituting Eq. (10.89) into Eq. (10.90) yields

$$P_{rad} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} U(\theta, \phi) \sin\theta d\theta d\phi. \tag{10.91}$$

The solid angle and the differential solid angle are shown in Fig. 10.3.

The expression of the differential solid angle is

$$d\Omega = \sin\theta d\theta d\phi. \tag{10.92}$$

The unit of the differential solid angle is steradian (sr). For $U(\theta, \phi) = 1$, the value of the radiation power from Eq. (10.91) can be determined as

$$P_{rad} = 2\pi[-\cos\theta]_0^{\pi} = 4\pi. \tag{10.93}$$

Substituting Eq. (10.92) into Eq. (10.91) yields

$$P_{rad} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} U(\theta, \phi) d\Omega. \tag{10.94}$$

The unit of $U(\theta, \phi)$ is watts per steradian (W/sr). The average radiation intensity is defined as the total radiated power divided by 4π sr. Mathematically, it can be expressed as

$$U_{av} = \frac{P_{rad}}{4\pi}. \tag{10.95}$$

The ratio of the radiation intensity to the average radiation intensity in the (θ, ϕ) direction is known as the directive gain of an antenna. Mathematically, it can be written as

$$G_{\mathrm{D}}(\theta, \phi) = \frac{U(\theta, \phi)}{U_{av}}. \tag{10.96}$$

Substituting Eq. (10.95) into Eq. (10.96) yields

$$G_{\mathrm{D}}(\theta, \phi) = \frac{4\pi U(\theta, \phi)}{P_{rad}}, \tag{10.97}$$

$$U(\theta, \phi) = \frac{G_D(\theta, \phi) P_{rad}}{4\pi}. \tag{10.98}$$

Substituting Eq. (10.98) into Eq. (10.89) yields

$$\frac{G_{\mathrm{D}}(\theta, \phi) P_{rad}}{4\pi} = r^2 P_{av}, \tag{10.99}$$

$$P_{av} = \frac{G_{\mathrm{D}}(\theta, \phi) P_{rad}}{4\pi r^2}. \tag{10.100}$$

The ratio of maximum radiation intensity to the average radiation intensity is known as the directivity of the antenna. It is represented by the letter D and it can be expressed as

$$D = \frac{U_{max}}{U_{av}}. \tag{10.101}$$

Substituting Eq. (10.95) into Eq. (10.101) yields

$$D = \frac{4\pi U_{max}}{P_{rad}}. \tag{10.102}$$

Substituting Eq. (10.94) into Eq. (10.102) yields

$$D = \frac{4\pi U_{max}}{\int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} U(\theta, \phi) d\Omega}, \tag{10.103}$$

$$D = \frac{4\pi}{\int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{U(\theta,\phi)}{U_{max}} d\Omega}, \tag{10.104}$$

$$D = \frac{4\pi}{\int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} U_n(\theta, \phi) d\Omega}, \tag{10.105}$$

where normalized radiation intensity is

$$U_n(\theta, \phi) = \frac{U(\theta, \phi)}{U_{max}} = \sin^2\theta. \tag{10.106}$$

Substituting Eqs. (10.92) and (10.106) into Eq. (10.105) yields

$$D = \frac{4\pi}{\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\sin^3\theta d\theta d\Omega} = \frac{4\pi}{2\pi \times \frac{4}{3}} = \frac{3}{2}. \tag{10.107}$$

The directive gain is basically equal to antenna efficiency time directivity and it is related as

$$G_D(\theta, \phi) = \eta D. \tag{10.108}$$

For $\eta = 1$, the directive gain and directivity of an antenna are the same. Practically, it is not possible due to different losses. Considering A_e is the effective area of a receiving antenna, then the expression of power received by the antenna is

$$P_R = A_e P_{av}. \tag{10.109}$$

The electric field for a differential length dz is

$$E_{rms} = \frac{V_{rms}}{dz}. \tag{10.110}$$

According to the circuit theory, the maximum power transfer to the receiving antenna is

$$P_R = \frac{{V_{rms}}^2}{4R_{rad}}. \tag{10.111}$$

Substituting Eqs. (10.88) and (10.110) into Eq. (10.111) yields

$$P_R = \frac{{E_{rms}}^2(dz)^2}{4 \times 80\pi^2\left(\frac{dz}{\lambda}\right)^2}, \tag{10.112}$$

$$P_R = \frac{{E_{rms}}^2\lambda^2}{8 \times 40\pi^2}, \tag{10.113}$$

$$P_R = \frac{3\lambda^2}{8\pi}\frac{{E_{rms}}^2}{120\pi}, \tag{10.114}$$

$$P_R = \frac{3\lambda^2}{8\pi}\frac{{E_{rms}}^2}{\sqrt{\frac{\mu}{\varepsilon}}}. \tag{10.115}$$

Substituting Eq. (10.109) into Eq. (10.115) yields

$$A_e P_{av} = \frac{3\lambda^2}{8\pi}\frac{{E_{rms}}^2}{\sqrt{\frac{\mu}{\varepsilon}}}, \tag{10.116}$$

$$A_e \frac{P_{av}}{{E_{rms}}^2} = \frac{3\lambda^2}{8\pi}\frac{1}{\sqrt{\frac{\mu}{\varepsilon}}}, \tag{10.117}$$

$$A_e \frac{1}{Z} = \frac{3\lambda^2}{8\pi} \frac{1}{\sqrt{\frac{\mu}{\varepsilon}}}, \tag{10.118}$$

$$A_e \frac{1}{\sqrt{\frac{\mu}{\varepsilon}}} = \frac{3\lambda^2}{8\pi} \frac{1}{\sqrt{\frac{\mu}{\varepsilon}}}, \tag{10.119}$$

$$A_e = \frac{3\lambda^2}{8\pi}. \tag{10.120}$$

From Eq. (10.75), the expression of the maximum value of the average power can be determined as

$$(P_{av})_{\max} = \frac{1}{2} \frac{\eta\beta^2 \mathbf{I}^2(z)(dz)^2}{(4\pi r)^2}. \tag{10.121}$$

For maximum directive gain of the antenna, Eq. (10.100) can be rearranged as

$$G_{\mathrm{D}}(\theta, \phi) = \frac{(P_{av})_{\max}}{4\pi r^2 P_{rad}}. \tag{10.122}$$

The directive gain of an antenna can be obtained by substituting Eqs. (10.82) and (10.121) into Eq. (10.122) as

$$G_{\mathrm{D}}(\theta, \phi) = \frac{\frac{1}{2} \frac{\eta\beta^2 \mathbf{I}^2(z)(dz)^2}{(4\pi r)^2}}{4\pi r^2 \times \frac{\mathbf{I}^2(z)(dz)^2\beta^2\eta}{32\pi^2}(2\pi)\frac{4}{3}}, \tag{10.123}$$

$$G_{\mathrm{D}}(\theta, \phi) = \frac{4\pi r^2 \times \eta\beta^2 \mathbf{I}^2(z)(dz)^2}{2(4\pi r)^2} \times \frac{(32\pi^2)}{\mathbf{I}^2(z)(dz)^2\beta^2\eta} \times \frac{3}{4(2\pi)}, \tag{10.124}$$

$$G_{\mathrm{D}}(\theta, \phi) = \frac{3}{2}. \tag{10.125}$$

From Eqs. (10.120) and (10.125), the ratio of the directive gain to the aperture area of the antenna can be determined as

$$\frac{G_{\mathrm{D}}(\theta, \phi)}{A_e} = \frac{\frac{3}{2}}{\frac{3\lambda^2}{8\pi}}, \tag{10.126}$$

$$G_{\mathrm{D}}(\theta, \phi) = \frac{4\pi A_e}{\lambda^2}. \tag{10.127}$$

The gain and directivity in decibels (db) can be expressed as

$$G_{\mathrm{D}}(\mathrm{db}) = 10 \log G_{\mathrm{D}}, \tag{10.128}$$

$$D(\mathrm{db}) = 10 \log D. \tag{10.129}$$

From Eq. (10.127), the value of the directive gain can be determined if other relative parameters are given.

Example 10.3 A 1.5 m diameter parabolic reflector antenna is used in the receiving side of a house. Determine the directive gain at 4 GHz, if the effective aperture is 60 % of the physical aperture area.

Solution The value of the effective aperture is

$$A_e = 0.60 \times \pi \frac{1.5^2}{4} = 1.06 \text{ m}.$$

The value of the wavelength can be determined as

$$\lambda f = v,$$

$$\lambda = \frac{v}{f} = \frac{3 \times 10^8}{4 \times 10^9} = 0.075 \text{ m}.$$

The directive gain or gain can be determined as

$$G_{\mathrm{D}}(\theta, \phi) = \frac{4\pi \times 1.06}{0.075^2} = 2368.06.$$

The value of the gain in decibel (db) is

$$G_{\mathrm{db}} = 10 \log G_{\mathrm{D}} = 10 \log 2368.06 = 33.74.$$

Practice Problem 10.3 The directive gain of an antenna is found to be 20 db. Calculate the aperture area of the antenna if the wavelength is 0.85 m.

10.6 Long Dipole Antennas

It is seen that the Hertzian dipole or a short dipole antenna has small radiation resistance and low radiation efficiency. Therefore, there is a need to have a high radiation efficiency antenna to overcome those drawbacks. In this case, a long dipole antenna is an alternative option to transmit or radiate signal efficiently. The current distribution in the linear dipole antenna is sinusoidal. The current in the antenna is equal in magnitude but flows in opposite direction. Consider a centre-fed long dipole antenna whose length ($2h$) is comparable to the wavelength as shown in Fig. 10.4. The current distribution in the z-direction is given as

$$I(z) = I_m \sin \beta(h - z), \; z > 0 \tag{10.130}$$

$$I(z) = I_m \sin \beta(h + z), \; z < 0. \tag{10.131}$$

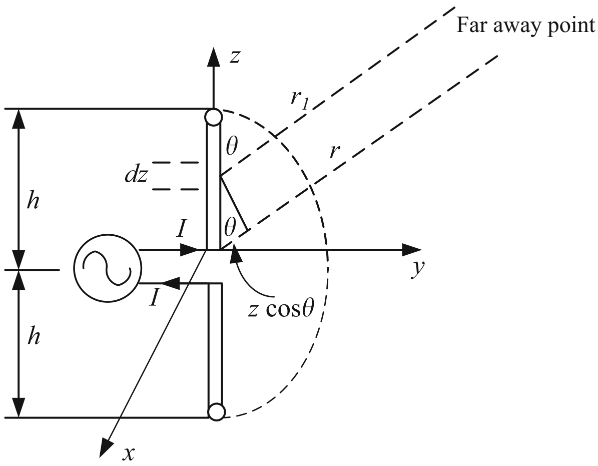

Fig. 10.4 Representation of a linear dipole antenna

Consider the long dipole antenna has only a far electric field which varies only θ in the $r-$direction. For differential current element $I(z)dz$, the differential electric field, dE_θ from Eq. (10.57) is given as

$$dE_\theta = j\frac{\beta\eta \sin\theta I(z)dz}{4\pi r_1}e^{-j\beta r_1}. \tag{10.132}$$

From Fig. 10.4, the following equations can be written as

$$r - r_1 = z\cos\theta, \tag{10.133}$$

$$r_1 = r - z\cos\theta. \tag{10.134}$$

Physically and practically, the parameters r and r_1 are the same. For mathematical simplification, consider $r = r_1$ in the denominator of Eq. (10.132) and the equation becomes

$$dE_\theta = j\frac{\beta\eta \sin\theta I(z)dz}{4\pi r}e^{-j\beta r_1}. \tag{10.135}$$

Substituting Eq. (10.134) into Eq. (10.135) yields

$$dE_\theta = j\frac{\beta\eta \sin\theta I(z)dz}{4\pi r}e^{-j\beta(r-z\cos\theta)}. \tag{10.136}$$

The total far electric field is obtained by using superposition as

$$E_\theta = \int_{-h}^{h} dE_\theta. \tag{10.137}$$

Substituting Eq. (10.136) into Eq. (10.137) yields

$$E_\theta = j\frac{\beta\eta \sin\theta}{4\pi r}e^{-j\beta r}\int_{-h}^{h} I(z)e^{j\beta z\cos\theta}dz, \tag{10.138}$$

$$E_\theta = g(\theta)\int_{-h}^{h} I(z)e^{j\beta z\cos\theta}dz, \tag{10.139}$$

$$g(\theta) = j\frac{\beta\eta\sin\theta}{4\pi r}e^{-j\beta r}, \tag{10.140}$$

where $g(\theta)$ is the element factor. This element factor is produced by the differential current element. The second factor, $f(\theta)$ is defined as

$$f(\theta) = \int_{-h}^{h} I(z)e^{j\beta z\cos\theta}dz. \tag{10.141}$$

The second factor is known as space or pattern or array factor. This factor is dependent on the amplitude and the phase angle of the current at the antenna. Substituting Eqs. (10.130) and (10.131) into Eq. (10.141) yields

$$f(\theta) = I_m\int_{-h}^{0}\sin\beta(h+z)e^{j\beta z\cos\theta}dz + I_m\int_{0}^{h}\sin\beta(h-z)e^{j\beta z\cos\theta}dz. \tag{10.142}$$

Consider the following integral formulae to evaluate Eq. (10.142):

$$\int \sin(a+bx)e^{cx}dx = \frac{e^{cx}}{b^2+c^2}[c\sin(a+bx) - b\cos(a+bx)], \tag{10.143}$$

$$\int \sin(a-bx)e^{cx}dx = \frac{e^{cx}}{b^2+c^2}[c\sin(a-bx) + b\cos(a-bx)], \tag{10.144}$$

$$f_1(\theta) = \frac{1}{\beta^2+(j\beta\cos\theta)^2}[e^{j\beta z\cos\theta}\{j\beta\cos\theta sin(\beta h+\beta z) - \beta cos(\beta h+\beta z)\}]_{-h}^{0}, \tag{10.145}$$

$$f_1(\theta) = \frac{1}{\beta^2\sin^2\theta}[j\beta\cos\theta sin(\beta h) - \beta cos(\beta h) + \beta e^{-j\beta h\cos\theta}], \tag{10.146}$$

$$f_2(\theta) = \frac{1}{\beta^2+(j\beta\cos\theta)^2}[e^{j\beta z\cos\theta}\{j\beta\cos\theta sin(\beta h-\beta z) + \beta cos(\beta h-\beta z)\}]_{0}^{h}, \tag{10.147}$$

$$f_2(\theta) = \frac{1}{\beta^2\sin^2\theta}[-j\beta\cos\theta sin(\beta h) + \beta e^{j\beta h\cos\theta} - \beta cos(\beta h)]. \tag{10.148}$$

Equation (10.138) can be modified as

$$E_\theta = j\frac{\beta\eta\sin\theta}{4\pi r}e^{-j\beta r}\frac{I_m}{\beta^2\sin^2\theta}[\beta(e^{j\beta h\cos\theta} + e^{-j\beta h\cos\theta}) - 2\beta cos(\beta h)], \tag{10.149}$$

$$E_\theta = j\frac{\beta\eta\sin\theta}{4\pi r}e^{-j\beta r}\frac{I_m}{\beta^2\sin^2\theta}[2\beta\cos(\beta h\cos\theta) - 2\beta cos(\beta h)], \tag{10.150}$$

$$E_\theta = j\frac{\eta}{2\pi r}e^{-j\beta r}I_m\left[\frac{\cos(\beta h\cos\theta) - cos(\beta h)}{\sin\theta}\right], \quad (10.151)$$

$$E_\theta = j\frac{\eta}{2\pi r}e^{-j\beta r}I_m F(\theta), \quad (10.152)$$

$$F(\theta) = \left[\frac{\cos(\beta h\cos\theta) - cos(\beta h)}{\sin\theta}\right]. \quad (10.153)$$

The far-field magnetic field is

$$H_\phi = \frac{E_\theta}{\eta}. \quad (10.154)$$

Substituting Eq. (10.152) into Eq. (10.154) yields

$$H_\phi = j\frac{1}{2\pi r}e^{-j\beta r}I_m F(\theta). \quad (10.155)$$

The time-average complex Poynting vector is

$$\mathbf{S} = \frac{1}{2}\mathbf{E}\times\mathbf{H}^* = \frac{1}{2}E_\theta\mathbf{a}_\theta \times H_\phi{}^*\mathbf{a}_\phi. \quad (10.156)$$

Substituting Eqs. (10.152) and (10.155) into Eq. (10.156) yields

$$\mathbf{S} = \frac{\eta}{8\pi^2 r^2}I_m{}^2F^2(\theta)\mathbf{a}_r. \quad (10.157)$$

The total power radiated by the centre-fed antenna can be determined as

$$P_{rad} = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\mathbf{S}.\mathbf{a}_r r^2\sin\theta d\theta d\phi. \quad (10.158)$$

Substituting Eq. (10.157) into Eq. (10.158) and integrate yields

$$P_{rad} = \frac{\eta I_m{}^2}{4\pi}\int_{\theta=0}^{\pi}F^2(\theta)\sin\theta d\theta. \quad (10.159)$$

For a half-wave dipole, the value of the integration $\int_{\theta=0}^{\pi}F^2(\theta)\sin\theta d\theta = 1.218$. Equation (10.159) can be modified as

$$P_{rad} = \frac{120\pi I_m{}^2}{4\pi}\times 1.218 = 36.5I_m{}^2. \quad (10.160)$$

In terms of rms current, the radiation power can be expressed as

$$P_{rad} = 36.5(\sqrt{2}I_{rms})^2 = 73I_{rms}{}^2. \quad (10.161)$$

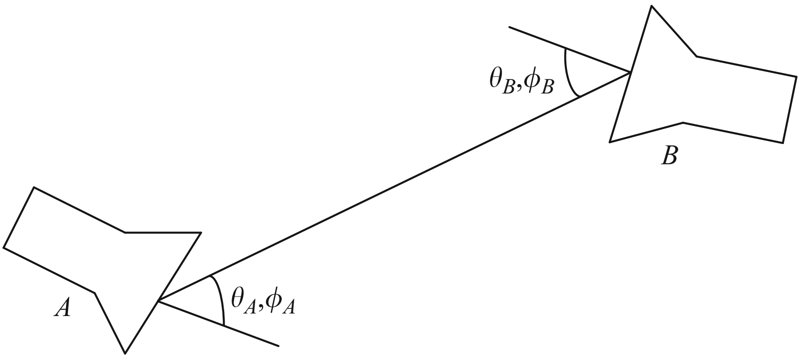

Fig. 10.5 Two coupled antennas

10.7 Friis Transmission Equation

It is difficult to calculate the gain when two antennas are coupled with each other. In 1946, H. T. Friis proposed an equation, which is capable to calculate the gain of couple antennas. Consider two antennas A and B are placed in the far-field region as shown in Fig. 10.5. Again, consider $G_t$is the gain of antenna A in the direction of antenna B. In this case, the average power density at the receiving terminals of antenna B is

$$S_{av} = \frac{P_t}{4\pi r^2} G_t. \tag{10.162}$$

The power received by antenna B is

$$P_r = S_{av} A_{er}. \tag{10.163}$$

According to Eq. (10.127), the following expressions can be written as

$$A_{er} = \frac{\lambda^2}{4\pi} G_r, \tag{10.164}$$

$$A_{et} = \frac{\lambda^2}{4\pi} G_t. \tag{10.165}$$

Substituting Eqs. (10.162) and (10.164) into Eq. (10.163) yields

$$P_r = \frac{P_t}{4\pi r^2} G_t \frac{\lambda^2}{4\pi} G_r, \tag{10.166}$$

$$\frac{P_r}{P_t} = G_t G_r \left(\frac{\lambda}{4\pi r} \right)^2. \tag{10.167}$$

Equation (10.165) is known as Friis transmission equation.

Equations (10.164) and (10.165) can be combined as

$$A_{er}A_{et} = \frac{\lambda^2}{4\pi}G_r\frac{\lambda^2}{4\pi}G_t, \tag{10.168}$$

$$\frac{A_{er}A_{et}}{G_rG_t\lambda^2} = \left(\frac{\lambda}{4\pi}\right)^2. \tag{10.169}$$

Substituting Eq. (10.169) into Eq. (10.167) yields

$$\frac{P_r}{P_t} = \frac{A_{et}A_{er}}{r^2\lambda^2}. \tag{10.170}$$

From Eq. (10.170), it is concluded that the given transmitted power, the received power, is directly proportional to the product of the effective areas of two antennas and inversely proportional to the square of the product of the wavelength and the separation distance.

The electric field intensity in the vicinity of antenna B can be determined as

$$S_{av} = \frac{1}{2}\frac{E^2}{\eta}. \tag{10.171}$$

Substituting Eq. (10.162) into Eq. (10.171) yields

$$\frac{P_t}{4\pi r^2}G_t = \frac{1}{2}\frac{E^2}{120\pi}, \tag{10.172}$$

$$E = \frac{\sqrt{60P_tG_t}}{r}. \tag{10.173}$$

Example 10.4 Two antennas are separated by a distance of 1200 m and each antenna has a directive gain of 20 db. A microwave link is established between the antennas at 300 MHz and a transmitted power of 600 W. Determine the received power and the electric field.

Solution The value of the directive gain can be determined as

$$10\log_{10}(G_{\text{D}}) = 20,$$

$$G_{\text{D}} = 10^2 = 100.$$

The separation distance is

$$r = 1200\text{ m}.$$

The value of the wavelength is

$$\lambda = \frac{3\times 10^8}{300\times 10^6} = 1\text{ m}.$$

From Eq. (10.166), the value of the received power can be determined as

$$\frac{P_r}{P_t} = {G_\mathrm{D}}^2\left(\frac{\lambda}{4\pi r}\right)^2,$$

where $G_\mathrm{D} = G_r G_t$,

$$P_r = 600\left(\frac{100 \times 1}{4\pi \times 1200}\right)^2 = 0.026\ \mathrm{W}$$

The electric field can be determined as

$$E = \frac{\sqrt{60 P_t G_\mathrm{t}}}{r} = \frac{\sqrt{60 \times 600 \times 100}}{1200} = 1.58\ \mathrm{V/m}.$$

Practice Problem 10.4 The separation distance between two identical antennas is 800 m. A microwave link is established between the antennas at 300 MHz and a transmitted power of 400 W. Determine the directive gain if the power received is 100 W.

10.8 Exercise Problems

10.1 The ratio of an infinitesimal distance to the wavelength of a Hertzian dipole is 0.22. Calculate the radiation resistance.

10.2 The radiation resistance and the wavelength of a Hertzian dipole antenna are found to be 10Ω and 0.12 m, respectively. Determine the differential length.

10.3 The infinitesimal distance and the radiation distance of a Hertzian dipole antenna are 0.02 m and 18Ω, respectively. Find the wavelength.

10.4 The effective aperture area is 70 % of the physical aperture area of a 1.2 m diameter parabolic reflector antenna. Calculate the directive gain at 6 GHz.

10.5 The directive gain and an effective aperture area of an antenna are found to be 2,500 and 1.12 m^2, respectively. Determine the frequency.

10.6 The separation distance between two antennas is 1,000 m and the directive gain of each antenna is 15 db. The power of 500 W is transmitted when a microwave link is established between the two antennas at 300 MHz. Calculate the received power and the electric field.

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Appendix A: Mathematical Formulae

A.1 Introduction

Mathematical formulae are very important to do in detail analysis of electromagnetic fields and waves. These formulae are mainly trigonometry, differentiation and integration. In this section, basic trigonometric formulae, derivatives and integration have been discussed.

A.2 Basic Trigonometric Formulae

The six basic trigonometric functions of an acute angle θ in a right angle triangle are defined as ratios between pairs of sides of the triangle. This acute angle θ is mentioned in Fig. A.1.

In Fig. A.1, the terms 'adj', 'opp', and 'hyp', stand for adjacent, opposite and hypotenuse respectively. The following basic trigonometric functions can be written as:

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\operatorname{cosec}\theta = \frac{\text{hyp}}{\text{opp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

In addition, the following relations may be written as:

$$\cos(-\theta) = \cos\theta \qquad \sin(-\theta) = -\sin\theta$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad \sec^2\theta = 1 + \tan^2\theta \qquad \operatorname{cosec}^2\theta = 1 + \cot^2\theta$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} \qquad \sin\theta = \frac{1}{\operatorname{cosec}\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Md. A. Salam, *Electromagnetic Field Theories for Engineering*,
DOI 10.1007/978-981-4585-66-8, © Springer Science+Business Media Singapore 2014

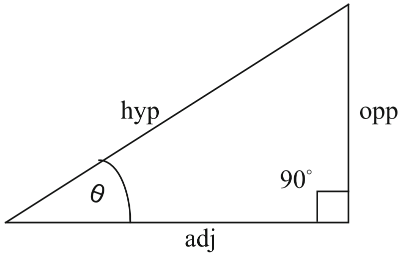

Fig. A.1 Triangle with acute angle

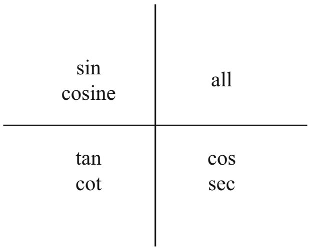

Fig. A.2 Basic trigonometric functions

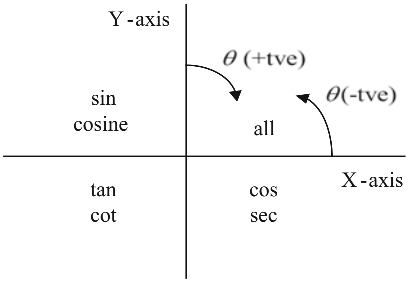

Fig. A.3 Rotations of the angles

A.3 Trigonometric Formulae

Four quadrants with basic trigonometric functions are shown in Fig. A.2. Figure A.2 has been drawn based on the sentence in quotes, 'all students take calculus', before entering in any engineering branch. Here, all represents all, the first letter of students represents sin, the first letter of take represents tan and the first letter of calculus represents cos. The basic trigonometric functions mentioned in each quadrant are considered positive.

The sign of the trigonometric functions can be determined by considering Fig. A.3. For positive Î¸, the starting point would be from the Y-axis and will rotate in the clockwise direction. Whereas, for negative Î¸, the starting point would be from the X-axis and will rotate in the anti-clockwise direction. From Fig. A.3, the following formulae can be determined as:

$$\sin\left(90^0 \pm \theta\right) = \cos\theta \qquad \cos\left(90^0 \pm \theta\right) = \mp\sin\theta \qquad \tan\left(90^0 \pm \theta\right) = -\cot\theta$$

$$\sin\left(180^0 \pm \theta\right) = \mp\sin\theta \qquad \cos\left(180^0 \pm \theta\right) = -\cos\theta \qquad \tan\left(180^0 \pm \theta\right) = \tan\theta$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)]$$

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos^2\alpha = \frac{1}{2}[1+\cos 2\alpha]$$

$$\sin^2\alpha = \frac{1}{2}[1-\cos 2\alpha]$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} \qquad \sin\alpha = \frac{e^{j\alpha} - e^{-j\alpha}}{2j} \qquad \cos\alpha = \frac{e^{j\alpha} + e^{-j\alpha}}{2}$$

$$e^{j\alpha} = \cos\alpha + j\sin\alpha \qquad \sinh\alpha = \frac{e^{\alpha} - e^{-\alpha}}{2} \qquad \cosh\alpha = \frac{e^{\alpha} + e^{-\alpha}}{2}$$

$$\sinh j\theta = j\sin\theta \qquad \cosh j\theta = \cos\theta$$

$$\sinh(\alpha \pm \beta) = \sinh\alpha\cosh\beta \pm \cosh\alpha\sinh\beta$$

$$\cosh(\alpha \pm \beta) = \cosh\alpha\cosh\beta \pm \sinh\alpha\sinh\beta$$

$$\sinh(\alpha \pm j\beta) = \sinh\alpha\cos\beta \pm j\cosh\alpha\sin\beta$$

$$\cosh(\alpha \pm j\beta) = \cosh\alpha\cos\beta \pm j\sinh\alpha\sin\beta$$

$$\cosh^2\theta - \sinh^2\theta = 1$$

$$\operatorname{sech}^2\theta + \tanh^2\theta = 1$$

A.4 Derivative and Integral Formulae

$$\frac{d}{dx}x^n = nx^{n-1} \qquad \frac{d}{dx}x^{xt} = te^{xt}$$

$$\frac{d}{dx}\cos\omega x = -\omega\sin\omega x \qquad \frac{d}{dx}\sin\omega x = \omega\cos\omega x$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \qquad \frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}(1) = 0 \qquad \frac{d}{dx}\text{In}x = \frac{1}{x}$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\sinh x = \cosh x$$

$$\frac{d}{dx}\cosh x = \sinh x \qquad \frac{d}{dx}e^x = e^x$$

A.5 Exponential and Logarithmic Formulae

$$e^{\alpha}.e^{\beta} = e^{(\alpha+\beta)} \qquad e^0 = 1$$

$$\frac{e^{\alpha}}{e^{\beta}} = e^{(\alpha-\beta)} \qquad (e^{\alpha})^{\beta} = e^{\alpha\beta}$$

$$\log_e(xy) = \log_e x + \log_e y \qquad \log_e\left(\frac{x}{y}\right) = \log_e x - \log_e y$$

$$\log_e(x^n) = n\log_e x \qquad \log_e(1) = 0$$

A.6 Integral Formulae

$$\int \frac{dx}{x} = \ln x + C \qquad \int e^x dx = e^x + C \qquad \int a^x dx = \frac{a^x}{\text{Ina}} + C$$

$$\int \sin x dx = -\cos x + C \qquad \int \cos x dx = \sin x + C \qquad \int \sec^2 x dx = \tan x + C$$

$$\int \ln x dx = x \ln x - x + C \qquad \int \sin ax dx = -\frac{\cos ax}{a} + C \qquad \int \cos ax dx = \frac{\sin ax}{a} + C$$

$$\int \tan ax dx = \frac{1}{a}\ln \sec ax + C \qquad \int e^{ax}\sin bx dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$$

$$\int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C \qquad \int \sinh ax dx = \frac{1}{a}\cosh ax + C$$

$$\int \cosh ax dx = \frac{1}{a}\sinh ax + C$$

Appendix B: Answers to Practice and Exercise Problems

Chapter 1

Practice Problems

1.1 $E = 1482.38$ V/m

1.2 $H = 49.74$ A/m

1.3 $E = 0.043$ V/m

Exercise Problems

1.1 $E = 221.92$ V/m

1.2 $\varepsilon_r = 8$

1.3 $J = 1.57 \times 10^9$ A/m^2

1.4 $E = 9.60 \times 10^{-3}$ V/m

Chapter 2

Practice Problems

2.1 (i) $|\mathbf{R}_a| = 7.14$, (ii) $|\mathbf{R}_s| = 4.12$

2.2 $\mathbf{a}_{\mathbf{R}_s} = 0.14\mathbf{a}_x - 0.95\mathbf{a}_y + 0.27\mathbf{a}_z$

2.3 $\theta = 40.19°$

2.4 $|\mathbf{A} \times \mathbf{B}| = 19.75$

2.5 $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = 70$

2.6 $|\mathbf{R}_{12}| = 4.69$

2.7 $\mathbf{A} = (2\rho\cos^2\phi - 3\rho\sin^2\phi)\mathbf{a}_\rho - 5\rho\cos\phi\sin\phi\mathbf{a}_\phi + z\mathbf{a}_z$

2.8 $\rho = 2.62, \phi = 61.77°, z = 2.4$

2.9 $\mathbf{A} = r\sin^2\theta\cos^2\phi\mathbf{a}_r + r\sin\theta\cos\theta\cos^2\phi\mathbf{a}_\theta - r\sin\theta\cos\phi\sin\phi\mathbf{a}_\phi$

DOI 10.1007/978-981-4585-66-8,

2.10 $Q(x = 1.53, y = -2.65, z = -2.57),\ Q(\rho = 3.06, \phi = 60^0, z = -2.65)$

2.11 $\nabla \mathbf{V} = 2yx\mathbf{a}_x + (x^2 + 4yz)\mathbf{a}_y + 2y^2\mathbf{a}_z$

2.12 (i) $\mathbf{E} = 2.4\mathbf{a}_x + 14\mathbf{a}_y + 10\mathbf{a}_z$V/m, (ii) $\mathbf{a_E} = 1.4\mathbf{a}_x + 0.81\mathbf{a}_y + 0.58\mathbf{a}_z$

2.13 $\mathbf{E} = -2\mathbf{a}_\rho + 1.28\mathbf{a}_\phi + 0.64\mathbf{a}_z$V/m, $a_\mathbf{E} = -0.81\mathbf{a}_\rho + 0.52\mathbf{a}_\phi + 0.26\mathbf{a}_z$

2.14 $div\mathbf{A} = 22$

2.15 $\nabla \cdot \mathbf{A} = -5.39, \nabla \cdot \mathbf{B} = 7.16$

2.16 $\nabla \times \mathbf{A} = 13.15\mathbf{a}_x - 3.95\mathbf{a}_y + 15.05\mathbf{a}_z$

Exercise Problems

2.1 (i) $|\mathbf{R}_a| = 18.41$, (ii) $|\mathbf{R}_s| = 12.69$

2.2 $\mathbf{a}_{\mathbf{R}_a} = 0.36\mathbf{a}_x + 0.91\mathbf{a}_y - 0.18\mathbf{a}_z$

2.3 (i) $|\mathbf{R}_a| = 25.34$, (ii) $|\mathbf{R}_s| = 5.92$

2.4 $\mathbf{A} \cdot \mathbf{B} = 19$

2.5 $\theta = 113.90°$

2.6 $V_p = 3.94$

2.7 $p = 8$

2.8 $\mathbf{B} \times \mathbf{C} = -2\mathbf{a}_x - 7\mathbf{a}_y - 4\mathbf{a}_z,\ \mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = -8$

2.9 $(\mathbf{A} + \mathbf{B}) \times (\mathbf{B} - \mathbf{C}) = -23\mathbf{a}_x - 13\mathbf{a}_y + 21\mathbf{a}_z$

2.10 $q = -3, p = 2$

2.11 $|\mathbf{R}_{12}| = 4.58$

2.12 (i) $\mathbf{R}_{AB} = -4\mathbf{a}_x - 4\mathbf{a}_y - 2\mathbf{a}_z$, (ii) $\mathbf{R}_{BC} = 3\mathbf{a}_x + 4\mathbf{a}_y + 3\mathbf{a}_z$,
(iii) $\mathbf{r}_C = 0.17\mathbf{a}_x + 0.51\mathbf{a}_y + 0.84\mathbf{a}_z$

2.13 (i) $\mathbf{A} = 2\mathbf{a}_x + 7\mathbf{a}_y - 10\mathbf{a}_z$, (ii) $\mathbf{a}_A = -0.11\mathbf{a}_x + 0.61\mathbf{a}_y - 0.78\mathbf{a}_z$,
(iii) $\mathbf{a}_{PQ} = -0.71\mathbf{a}_x + 0.71\mathbf{a}_z$

2.14 $\mathbf{a}_{PQ} = 0.87\mathbf{a}_x - 0.13\mathbf{a}_y + 0.48\mathbf{a}_z$

2.15 $\mathbf{A} \times \mathbf{B} = 7\mathbf{a}_\rho + 7\mathbf{a}_\phi + 7\mathbf{a}_z$

2.16 $\mathbf{A} = \cos\phi\mathbf{a}_\rho + \cot\phi\cos\phi\mathbf{a}_\phi$

2.17 $\mathbf{A} = (z\cot\phi\cos\phi + \tan\phi\sin\phi)\mathbf{a}_\rho + (-z\cos\phi + \sin\phi)\mathbf{a}_\phi$

2.18 $\rho = 3.2, \phi = 51.35°, z = 1.2$

2.19 $\mathbf{A} = \frac{2}{r}\mathbf{a}_r + \frac{2}{r}\cot\theta\mathbf{a}_\theta + \frac{2}{r}\cot\phi\cos ec\,\theta\mathbf{a}_\phi$

2.20 $P(x = -1.11, y = -1.33, z = -1),\ P(\rho = 1.73, \phi = 50.15^0, z = -1)$

2.21 $\mathbf{E} = 3\mathbf{a}_x + 2\mathbf{a}_y - 4\mathbf{a}_z,\ \mathbf{a_E} = 0.56\mathbf{a}_x + 0.37\mathbf{a}_y - 0.74\mathbf{a}_z$

2.22 $\mathbf{E} = 2\mathbf{a}_\rho + 0.87\mathbf{a}_\phi - 2\mathbf{a}_z,\ \mathbf{a_E} = 0.68\mathbf{a}_\rho + 0.29\mathbf{a}_\phi - 0.68\mathbf{a}_z$

2.23 $div\mathbf{A} = y$

2.24 $\nabla \cdot \mathbf{A} = 6z^2\cos\phi + 2z\cos\phi - \sin 2\phi$

2.25 $\nabla \cdot \mathbf{A} = 3 + r\sin\phi\cot\theta + \frac{\cos\phi}{r\sin\theta}$

2.26 $\nabla \cdot \mathbf{A} = -1$

2.27 $\nabla \cdot \mathbf{A} = 3.09$

2.28 $\nabla \times \mathbf{A} = 13.15\mathbf{a}_x - 3.95\mathbf{a}_y + 15.05\mathbf{a}_z$

Chapter 3

Practice Problems

3.1 (i) $\mathbf{R}_{12} = \mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z$, (ii) $|\mathbf{R}_{12}| = 3.74$, (iii) $\mathbf{a}_{12} = 0.27\mathbf{a}_x + 0.53\mathbf{a}_y + 0.81\mathbf{a}_z$, (iv) $\mathbf{F}_{12} = -(9.37\mathbf{a}_x + 18.02\mathbf{a}_y + 27.54\mathbf{a}_z) \times 10^{-3} N$

3.2 $\mathbf{E} = -8.65\mathbf{a}_x - 8.65\mathbf{a}_y - 8.65\mathbf{a}_z$ V/m

3.3 $\oint_S D \cdot \partial s = 39$

3.4 $\oint_S D \cdot \partial s = -12.6$

3.5 $E_\rho = \frac{Q}{2\pi\rho\varepsilon_0}$

3.6 $W = -19$ J

3.7 $W = -1.32$ J

3.8 (i) $V = 25.98$ V, (ii) $\mathbf{E} = -12.99\mathbf{a}_\rho - 7.5\mathbf{a}_\phi - 51.96\mathbf{a}_z$ V/m, (iii) $\mathbf{D} = -132.81z^2 \sin\phi\mathbf{a}_\rho - 132.81z^2 \cos\phi\mathbf{a}_\phi - 265.62\rho z \sin\phi\mathbf{a}_z$ pC/m^3, (iv) $\rho_v = -402.56$ pC/m^3

3.9 $V = 4.05$ kV

3.10 $V = 0.0324$ V

3.11 $V = 12.65$ V

3.12 (i) $\mathbf{E}_{2t} = \mathbf{E}_{1t} = 5\mathbf{a}_x - 8\mathbf{a}_y$, (ii) $\mathbf{E}_{2n} = -1.29\mathbf{a}_z$, (iii) $\mathbf{E}_2 = 5\mathbf{a}_x - 8\mathbf{a}_y - 1.29\mathbf{a}_z$, (iv) $\alpha_1 = 17.64°$, $\alpha_2 = 7.79°$

Exercise Problems

3.1 $\mathbf{F}_{12} = 0.074\mathbf{a}_x + 0.37\mathbf{a}_y + 0.14\mathbf{a}_z N$

3.2 (i) $\mathbf{R}_{12} = 0.5\mathbf{a}_x + 5\mathbf{a}_y + 1.5\mathbf{a}_z$, (ii) $|\mathbf{R}_{12}| = 5.24$, (iii) $\mathbf{a}_{12} = 0.095\mathbf{a}_x + 0.95\mathbf{a}_y + 0.29\mathbf{a}_z$, (iv) $\mathbf{F}_{12} = 0.7\mathbf{a}_x + 6.96\mathbf{a}_y + 2.13\mathbf{a}_z N$

3.3 $\mathbf{E} = 0.19\mathbf{a}_x + 0.15\mathbf{a}_y + 0.30\mathbf{a}_z$ V/m

3.4 $\mathbf{E} = 3.86\mathbf{a}_x - 13.51\mathbf{a}_y + 1.93\mathbf{a}_z$ V/m

3.5 $\oint_S D \cdot \partial s = c(c = 8k)$

3.6 $\oint_S D \cdot \partial s = 37.70$

3.7 $\oint_S D \cdot \partial s = 30$

3.8 $\oint_S D \cdot \partial s = -201.06$

3.9 $W = 6$ J

3.10 $W = 32.5$ J

3.11 $W = -5$ J

3.12 $W = -945$ J

3.13 (i) $V = 45$ V, (ii) $\mathbf{E} = -16\mathbf{a}_\rho - 72\mathbf{a}_\phi + 3\mathbf{a}_z$ V/m,
(iii) $\mathbf{a}_E = -0.22\mathbf{a}_x - 0.98\mathbf{a}_y + 0.038\mathbf{a}_z$,
(iv) $\mathbf{D} = -17.71y^3\mathbf{a}_\rho - 53.12xy^2\mathbf{a}_\phi + 26.56\mathbf{a}_z$ pC/m^3,
(v) $\rho_v = -637.44$ pC/m^3

3.14 (i) $V = 5.53$ V, (ii) $\mathbf{E} = -1.53\mathbf{a}_\rho - 1.29\mathbf{a}_\phi - 4\mathbf{a}_z$ V/m,
(iii) $\mathbf{D} = -17.17\sin\phi\mathbf{a}_\rho - 17.17\cos\phi\mathbf{a}_\phi - 17.17z\mathbf{a}_z$ pC/m^3,
(iv) $\rho_v = -4.14$ pC/m^3

3.15 (i) $V = 16.45$ V, (ii) $\mathbf{E} = -17.32\mathbf{a}_\rho - 5\mathbf{a}_\phi + 0.29\mathbf{a}_z$V/m,
(iii) $\mathbf{D} = -88.54r\sin\theta\mathbf{a}_r - 44.27r\cos\theta\mathbf{a}_\theta + 8.85\frac{\sin\phi}{\sin\theta}\mathbf{a}_\phi$pC/m^3,
(iv) $\rho_v = -40.9$ pC/m^3

3.16 $V(r) = 421.37$ kV

3.17 $V(r) = 4.84$ kV

3.18 $V(r) = 0.86$ V

3.19 $V(r) = 14.89$ V

3.20 $V(r) = 3.41$ V

3.21 (i) $\mathbf{E}_{2t} = \mathbf{E}_{1t} = 2\mathbf{a}_x + 3\mathbf{a}_y$, (ii) $\mathbf{E}_{2n} = 0.56\mathbf{a}_z$, (iii) $\mathbf{E}_2 = 2\mathbf{a}_x + 3\mathbf{a}_y + 0.56\mathbf{a}_z k$ V/m, (iv) $\alpha_1 = 15.5°$, $\alpha_2 = 8.83°$

3.22 (i) $\mathbf{E}_{2t} = \mathbf{E}_{1t} = 3\mathbf{a}_x + 5\mathbf{a}_y$, (ii) $\mathbf{E}_{2n} = -1.43\mathbf{a}_z$, (iii) $\mathbf{E}_2 = 3\mathbf{a}_x + 5\mathbf{a}_y - 1.43\mathbf{a}_z k$ V/m, (iv) $\alpha_1 = 18.93°$, $\alpha_2 = 13.78°$

Chapter 4

Practice Problems

4.1 (i) $V = 4.5$ V, (ii) $\mathbf{E} = -9\mathbf{a}_\rho - 7.79\mathbf{a}_\phi - 15\mathbf{a}_z$ V/m, (iii) $\nabla^2 V \neq 0$

4.2 $\mathbf{E} = -\frac{V_0}{\rho r}\mathbf{a}_\phi$

4.3 $V\left(a, \frac{b}{2}\right) = 110.77$ V

Exercise Problems

4.1 (i) $V = 3$ V, (ii) $\mathbf{E} = -\left(2xy\mathbf{a}_x + x^2\mathbf{a}_y - 2z\mathbf{a}_z\right)$ V/m, (iii) $\nabla^2 V \neq 0$

4.2 (i) $V = 6.35$ V, (ii) $\mathbf{E} = -2.72\mathbf{a}_x - 0.37\mathbf{a}_y - 4\mathbf{a}_z$ V/m, (iii) $\nabla^2 V \neq 0$

4.3 (i) $V = 3.54$ V, (ii) $\mathbf{E} = 7.07\mathbf{a}_\rho - 3.54\mathbf{a}_\phi - 0.71\mathbf{a}_z$ V/m, (iii) $\nabla^2 V \neq 0$

4.4 (i) $V = -1.61$ V, (ii) $\mathbf{E} = 3.21\mathbf{a}_r + 1.92\mathbf{a}_\theta + 4.33\mathbf{a}_\phi$ V/m, (iii) $\nabla^2 V \neq 0$

4.5 $\mathbf{E} = -23.3\mathbf{a}_x$ V/m

4.6 $\mathbf{E} = -16.84\mathbf{a}_\rho$ V/m

4.7 $\mathbf{E} = -\frac{160}{r^2}\mathbf{a}_r$ V/m

4.8 $V\left(a, \frac{b}{2}\right) = 30.74$ kV

Chapter 5

Practice Problems

5.1 (i) $\mathbf{J} = 2\mathbf{a}_\rho + 1.64\mathbf{a}_\phi$ A/m^2, (ii) $I = 37.7$ A

5.2 (i) $\mathbf{J} = 1.49\mathbf{a}_r + 1.53\mathbf{a}_\theta$ A/m^2, (ii) $I = 7.79$ A

5.3 10 Ω

5.4 $T_r = 1.11$ s

5.5 $C = 0.044$ μF, $Q = 34.32$ μC

5.6 $a = 1.98$ mm

5.7 (i) $C = 16.84 \times 10^{-18}$ F, (ii) $Q = 0.0013$ pC, (iii) $D = 0.05 \times 10^{-5}$ C/m^2,
(iv) $E_1 = 18823.88$ V/m $E_2 = 11294.33$ V/m,
(v) $V_1 = 37.65$ V, $V_2 = 39.53$ V

Exercise Problems

5.1 (i) $\mathbf{J} = 18\mathbf{a}_x + 7.5\mathbf{a}_z$ A/m^2, (ii) 40 A

5.2 (i) $\mathbf{J} = 8\mathbf{a}_x + 10\mathbf{a}_y + 6\mathbf{a}_z$ A/m^2, (ii) 756 A

5.3 (i) $\mathbf{J} = 1.5\mathbf{a}_\rho - 2.12\mathbf{a}_\phi$ A/m^2, (ii) $I = 345.58$ A

5.4 (i) $\mathbf{J} = -2.56\mathbf{a}_r + 2\mathbf{a}_\theta$ A/m^2, (ii) $I = 2.81$ A

5.5 $R = 2.74\Omega$

5.6 $R = 19.06\Omega$

5.7 $\rho = 25.57$ C/m^3

5.8 $T_r = 0.34$ s

5.9 (i) $C = 0.212$μF, (ii) $Q = 424$ μC

5.10 $C = 0.18$ μF

5.11 $b = 6.32$ mm

5.12 (i) $C = 5.73 \times 10^{-17}$ F, (ii) $Q = 5.73 \times 10^{-15}$ C,
(iv) $E_1 = 17396.91$ V/m, $E_2 = 8698.46$ V/m,
(v) $V_1 = 43.49$ V, $V_2 = 56.54$ V

Chapter 6

Practice Problems

6.1 $\mathbf{H} = -0.12\mathbf{a}_x$ A/m

6.2 $\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_5 = 20\mathbf{a}_y$ A/m

6.3 $\mathbf{J} = -3\mathbf{a}_x - 9\mathbf{a}_y + \mathbf{a}_z$ A/m^2

6.4 $\mathbf{J} = 0.722\mathbf{a}_\rho + 0.419\mathbf{a}_\phi + 0.167\mathbf{a}_z$ A/m^2

6.5 $\mathbf{J} = 0.58\mathbf{a}_r - 4.15\mathbf{a}_\theta + \mathbf{a}_\phi$ A/m^2

6.6 $\mathbf{B} = -y\mathbf{a}_x + x\mathbf{a}_y - y\mathbf{a}_z$ Wb/m^2

6.7 $\mathbf{M} = 54.05$ A/m

6.8 $\mathbf{H} = 78.95$ A/m

6.9 $N = 14$ turns

6.10 $I = 86.2$ A

6.11 $I = 16.8$ A

6.12 $L = 1.11$ μH

Exercise Problems

6.1 $\mathbf{H} = -0.036\mathbf{a}_z$ A/m

6.2 $\mathbf{H} = 0.05\mathbf{a}_x$ A/m

6.3 $\mathbf{J} = -213\mathbf{a}_x - 60\mathbf{a}_y - 42\mathbf{a}_z$ A/m^2

6.4 $\mathbf{J} = -17\mathbf{a}_x - 7\mathbf{a}_z$ A/m^2

6.5 $\mathbf{J} = -0.739\mathbf{a}_r - 0.18\mathbf{a}_z$ A/m^2

6.6 $\mathbf{J} = -0.767\mathbf{a}_\rho + 1.85\mathbf{a}_\phi - 0.12\mathbf{a}_z$ A/m^2

6.7 $\mathbf{J} = 0.32\mathbf{a}_r + 0.63\mathrm{a}_\phi$ A/m^2

6.8 $\mathbf{B} = qe^{-px}\sin qy\mathbf{a}_z$ Wb/m^2

6.9 $\mathbf{B} = 4y^2\mathbf{a}_x + 6xz\mathbf{a}_y$ Wb/m^2

6.10 (i) $\chi = 34$, (ii) $\mathbf{H} = 1591.55$ A/m, (iii) $\mathbf{M} = 30239.44$ A/m

6.11 $\mathbf{M} = 6.73$ A/m

6.12 $\phi = 1.37 \times 10^{-4}$ Wb, $B = 0.15$ Wb/m^2, $H = 220.59$ A/m

6.13 $I = 2.55$ A

6.14 $\Re_t = 26.74 \times 10^6$ At/Wb, $I = 4.46$ A

6.15 $L = 33.66$ μH

Chapter 7

Practice Problems

7.1 $V_{ind} = -2$ V

7.2 $I_d = 2.66 \times 10^{-6} \cos 10^4 t$ A, $J_d = 1.33 \times 10^{-3} \cos 10^4 t$ A/m

7.3 $\mathbf{D} = \frac{-2}{10^{12}} \alpha \sin\left(10^{12} t - \alpha x\right) \mathbf{a}_z$ C/m^2, $\mathbf{E}_z = -1.33 \times 10^{-6} \alpha \sin\left(10^{12} t - \alpha x\right)$

7.4 $\phi_m = 0.9 \times 10^{-4}$ wb, $N_2 = 265$ turns

7.5 $\mathbf{Q} = \mathrm{Re}\left[\mathbf{Q_s} e^{j(-3x-65°)}\right]$, $\mathbf{R_s} = -9 \sin\left(\omega t + \frac{x}{2}\right) \mathbf{a}_x + 12 \cos\left(\omega t - \frac{\pi x}{5}\right) \mathbf{a}_y$

7.6 $\omega = \frac{12\eta}{\mu_o 5}$, $\eta = 218\ \Omega$

Exercise Problems

7.1 $V_{ind} = 5.12 \cos 1000t$ V

7.2 $\phi_m = 0.081$ wb , $V_{ind} = 243$ V

7.3 $V_{ind} = 6.4$ V

7.5 $V_{ind} = 1.96$ V

7.6 $I_d = 2.78 \times 10^{-9} \cos\ 314t$ A, $J_d = 6.95 \times 10^{-6} \cos\ 314t$ A/m^2

7.7 $J_C = 125 \cos\ 314t$ mA/m^2 $J_d = -1.11 \times 10^{-6} \sin\ 314t$ mA/m^2

7.8 $\mathbf{D} = 2.5 \times 10^{-7} \beta \cos\left(10^8 t - \beta x\right) \mathbf{a}_y \mathrm{C/m^2}$,
$\mathbf{E} = 0.083 \beta \cos\left(10^8 t - \beta x\right)$ V/m,
$\mathbf{B} = 8.3 \times 10^{-10} \beta^2 \cos\left(10^8 t - \beta x\right) \mathbf{a}_z$ Wb/m^2

7.9 $\phi_m = 16 \times 10^{-6}$ wb, $V_2 = 1.51$ V

7.10 $\phi_m = 8.1 \times 10^{-6}$ wb, $N_2 = 236$

7.11 $\mathbf{R} = -15 \sin \omega t\ \mathbf{a}_x + 4 \cos(\omega t + j35°)\ \mathbf{a}_y$

7.12 $\mathbf{E}_s = 5e^{j(4x-80)}\ \mathbf{a}_x + 12e^{-j(4x+15)}\ \mathbf{a}_y$

Chapter 8

Practice Problems

8.1 (i) $v = 293.11 \times 10^6$ m/s, (ii) $R_O = 284.31\ \Omega$, (iii) $\rho = -0.128$,
(iv) $\lambda = 293.11 \times 10^3$ m, (v) $\beta = 21.44 \times 10^{-6}$ rad/m,
(vi) $V_o^+ = 252.29 \sin 6283t$ $V_o^- = -32.29 \sin 6283t$

8.2 $C = 34.27$ µF/m, $L = 18.47 \times 10^{-3}$ H/m

8.3 $C = 5.15$ pF/m, $L = 37.20$ nH/m, $R = 5.95\ \Omega$/m

8.4 (i) $\alpha = 47.70$ Np/m, (ii) $\beta = 32.51$ rad/m

8.5 $\alpha = 0.21$ Np/m

8.6 (i) $Z_O = 712.32 \angle -6.8\ \Omega$, (ii) $\gamma = 0.00363 + j0.0287$,
(iii) $Z_{in} = 701.33 \angle 67.40°$

8.7 (i) $Z_o = 77.78\ \Omega$, (ii) $\beta l = 1.91$ rad, (iii) $Z_{sc} = -j22.44\ \Omega$

8.8 $\rho = 0.67 \angle 32.27^\circ$ $S = 5.06$ $\beta l = 30^\circ$ $Z_{in} = 27.05 \angle 11.15^\circ$

8.9 (i) $\rho = 0.45 \angle 31.97^\circ$, (ii) $S = 2.64$, (iii) $P_{av}(incident) = 14.55$ W, (iv) $P_{av}(reflected) = 2.95$ W, (v) $P_{net} = 11.61$ W

Exercise Problems

8.1 (i) $v = 10 \times 10^6$ m/s, (ii) $R_O = 40\ \Omega$, (iii) $\rho = -0.14$, (iv) $\lambda = 10 \times 10^3$ m, (v) $\beta = 628.3 \times 10^{-6}$ rad/m, (vi) $V_o^+ = 116.3 \sin 6283\, t$ $V_o^- = 16.3 \sin 6283\, t$

8.2 (i) $v = 338 \times 10^3$ m/s, (ii) $R_O = 118.3\ \Omega$, (iii) $\rho = 0.039$, (iv) $\lambda = 338$ m, (v) $\beta = 0.019$ rad/m, (vi) $V_o^+ = 116.3 \sin 6283\, t$

8.3 (i) $\alpha = 91.12$ dB/m, (ii) $\beta = 5.23$ rad/m

8.4 $C = 2.55$ pF/m, $L = 14.32 \times 10^{-9}$ H/m

8.5 (i) $C = 62.5$ pF/m, (ii) $L = 100$ nH/m

8.6 (i) $\alpha = 45$ dB/m, (ii) $\beta = 56.73$ rad/m

8.7 $P(z) = 50$ W/m

8.8 $\alpha = 0.2$ Np/m

8.9 (i) $Z_O = 1734.94 \angle -2.0\ \Omega$, (ii) $\gamma = 0.00088 + j0.03265$, (iii) $Z_{in} = 4007 \angle 81.47^\circ$

8.10 (i) $Z_O = 1146 \angle -5.78\ \Omega$, (ii) $\gamma = 0.00272 + j0.02506$, (iii) $Z_{in} = 4964 \angle 33.72^\circ$

8.11 (i) $Z_o = 61.64\ \Omega$, (ii) $\beta = 0.249$ rad/m, (iii) $\beta l = 1.990$ rad, $Z_{sc} = -j138.2\ \Omega$

8.12 (i) $Z_o = 70.71\ \Omega$, (ii) $\beta = 0.191$ rad/m, (iii) $\beta l = 2.866$ rad, $Z_{sc} = -j19.99\ \Omega$

8.13 (i) $Z_o = 244.95 \angle -5^\circ$, (ii) $\beta = 7.850$ rad/m

8.14 $\rho = 0.48 \angle 47.10^\circ$, $S = 2.85$, $Z_{in} = 19.65 \angle -85.49^\circ\ \Omega$

8.15 (i) $\rho = 0.36 \angle 56.92^\circ$, (ii) $S = 2.13$, (iii) $P_{av}(incident) = 18.41$ W, (iv) $P_{av}(reflected) = 2.39$ W, (v) $P_{net} = 16.02$ W

Chapter 9

Practice Problems

9.1 (i) $\eta = 240\ \Omega$, (ii) $\varepsilon_r = 2.6$ $\mu_r = 1.07$, (iii) $\beta = 0.56$ rad/m, (iv) $\lambda = 11.22$ m

9.2 (i) $\beta = 2$ rad/m, propagation in the $a_z$direction, (ii) $\omega = 188.5 \times 10^6$ rad/s, (iii) $v = 94.25 \times 10^6$ m/s, (iv) $\varepsilon_r = 10.13$ $\eta = 118.45\ \Omega$

9.3 $\beta = 0.63$ rad/m, E(z,t) $= 200 \cos(\omega t - 0.63z)\, a_y$ V/m,$\eta = 125.67\ \Omega$, H(z,t) $= 1.59 \cos(\omega t - 0.63z)\, a_x$ A/m

9.4 (i) $\beta = 0.28$ rad/m, (ii) $\tan\delta = 0.228$

9.5 (i) $\alpha = 9.16 \times 10^5$ Np/m, (ii) $\beta = 9.16 \times 10^5$ rad/m, (iii) $\eta = 0.036 \angle 45^\circ\ \Omega$ (iv) $\lambda = 0.96 \times 10^{-5}$ m, (v) $v = 41169.35$ m/s

9.6 (i) $\alpha = 0.14$ Np/m, (ii) $\beta = 0.14$ rad/m, (iii) $\eta = 647.21 \angle 45^\circ\ \Omega$, (iv) $\lambda = 44.88$ m, (v) $v = 2.28 \times 10^9$ m/s, (vi) $\delta = 7.28$ m

9.7 (i) $\eta = 80.36\ \Omega\ \ \varepsilon_r = 22$, (ii) $\omega = 0.255 \times 10^8\ \ f = 4.07$ MHz, (iii) $\mathrm{H} = 1.87 \sin\ (\omega t - 0.43)\ \mathrm{a}_x$ A/m

9.8 (i) $\eta_1 = 377\ \Omega$, (ii) $\eta_2 = 177.72\ \Omega$, (iii) $P_i = 3.32\ a_z$ W/m^2, (iv) $P = -\ 0.36\ \ P_r = -\ 0.\ 406\ a_z$ W/m^2, (v) $P_2 = 2.91\ a_z$ W/m^2

Exercise Problems

9.1 (i) $\beta_o = 2.09$ rad/m, (ii) $\lambda_o = 3$ m, (iii) $\mathbf{H}\,(x, y, z, t) = 0.4\,\mathbf{a}_y$ A/m

9.2 (i) $\beta = 1.5$ Rad/m, (ii) $\omega = 25.13 \times 10^7$ rad/s, (iii) $v = 16.76 \times 10^7$ m/s, (iv) $\mu_r \varepsilon_r = 3.2$

9.3 $\theta = 53.13°$

9.4 (i) $\beta = 0.33$ rad/m, (ii) $\lambda = 19.04$ m, (iii) $\mathrm{E}(z, t) = 0.66\ \cos\left(10^8 t - 0.33z\right)\ \mathrm{a}_x$ V/m

9.5 (i) $\beta = 502.65$ rad/m, (ii) $\tan\delta = 0.22$

9.6 (i) $\beta = 0.41$ rad/m, (ii) $\alpha = 0.0012$ Np/m

9.7 (i) $\gamma = \alpha + j\beta = 0.08 + j2.5\ \mathrm{m}^{-1}$, (ii) $\beta = 2.5$ rad/m, (iii) $\lambda = 2.51$ m, (iv) $v = 0.4 \times 10^8$ m/s, (v) $\delta = 12.5$ m

9.8 (i) $\eta = 150\ \Omega$, (ii) $\mu_r = 2.53$, (iii), $\beta = 4.19$ rad/m, (iv) $\lambda = 1.5$ m

9.9 (i) $\beta = 3$ rad/m, (ii) $\omega = 25.13 \times 10^6$ rad/s, (iii) $v = 8.38 \times 10^7$ m/s, (iv) $\varepsilon_r = 12.85\ \ \eta = 105.31\ \Omega$, (v) $\mathbf{H} = 0.019\, e^{-j3z}\, a_y - j0.047\, e^{-j3z}\, \mathbf{a}_x$ A/m

9.10 (i) $\beta = 0.8$ rad/m, $\sqrt{\varepsilon_r} = 5.76$, (ii) $\eta = 157.08\ \Omega$, (iii) $P_{av} = 2\ a_x$ W/m^2

9.11 (i) $\beta_1 = 0.21$, (ii) $\beta_2 = 0.51$, (iii) $\eta_1 = 377\ \Omega$, (iv) $\eta_2 = 153.91\ \Omega$, (v) $\rho = -\ 0.42\ \ \mathrm{T} = 0.58$

9.12 (i) $\eta_1 = 377\ \Omega$, (ii) $\eta_2 = 217.66\ \Omega$, (iii) $P_i = 1.19\ a_z$ W/m^2, (iv) $\rho = -\ 0.27\ \ P_r = -\ 0.09\ a_z$ W/m^2, (v) $P_2 = 1.1$ W/m^2

Chapter 10

Practice Problems

10.1 $\frac{dz}{\lambda} = 0.08$

10.2 $dz = 0.61$ m

10.3 $A_e = 5.75$ m

10.4 $G_D = 5026.55$

Exercise Problems

10.1 $R_{rad} = 38.22\ \Omega$

10.2 $dz = 0.013$ m

10.3 $\lambda = 0.13$ m

10.4 $G_{db} = 37.95$

10.5 $f = 4$ GHz

10.6 $P_r = 0.16$ W, $E = 2.6$ V/m

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Md. A. Salam, *Electromagnetic Field Theories for Engineering*,
DOI 10.1007/978-981-4585-66-8, © Springer Science+Business Media Singapore 2014