Christian Lalanne

Mechanical Vibration & Shock

Random Vibration

Volume III

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Contents

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Introduction

The vibratory environment found in the majority of vehicles essentially consists of random vibrations. Each recording of the same phenomenon results in a signal different from the previous ones. Characterization of a random environment therefore requires an infinite number of measurements to cover all the possibilities. Such vibrations can only be analyzed statistically.

The first stage consists of defining the properties of the processes comprising all the measurements, making it possible to reduce the study to the more realistic measurement of single or several short samples. This means evidencing the stationary character of the process, making it possible to demonstrate that its statistical properties are conserved in time, then its ergodicity, with each recording representative of the entire process. As a result, only a small sample consisting of one recording has to be analysed (Chapter 1).

The value of this sample gives an overall idea of the severity of the vibration, but the vibration has a continuous frequency spectrum that must be determined in order to understand its effects on a structure. This frequency analysis is performed using the power spectral density (PSD) (Chapter 2) which is the ideal tool for describing random vibrations. This spectrum, a basic element for many other treatments, has numerous applications, the first being the calculation of the rms value of the vibration in a given frequency band (Chapter 3).

The practical calculation of the PSD, completed on a small signal sample, provides only an estimate of its mean value, with a statistical error that must be evaluated. Chapter 4 shows how this error can be evaluated according to the analysis conditions, how it can be reduced, before providing rules for the determination of the PSD.

The majority of signals measured in the real environment have a Gaussian distribution of instantaneous values. The study of the properties of such a signal is extremely rich in content (Chapter 5). For example, knowledge of the PSD alone gives access, without having to count the peaks, to the distribution of the maxima of a random signal (Chapter 6), and in particular to the response of a system with one degree-of-freedom, which is necessary to calculate the fatigue damage caused by the vibration in question (Volume 4). It is also used to determine the law of distribution of the largest peaks, in itself useful information for the pre-sizing of a structure (Chapter 7).

List of symbols

The list below gives the most frequent definition of the main symbols used in this book. Some of the symbols can have another meaning which will be defined in the text to avoid any confusion.

a	Threshold value of $\ell(t)$ or maximum of $\ell(t)$
A	Maximum of $A(t)$
$A(t)$	Envelope of a signal
b	Exponent
c	Viscous damping constant
$E_1(\ )$	First definition of error function
$E_2(\ )$	Second definition of error function
Erf	Error function
$E(\ )$	Expected function of ...
f	Frequency of excitation
$f_{samp.}$	Sampling frequency
f_{max}	Maximum frequency
f_0	Natural frequency
g	Acceleration due to gravity
G	Particular value of power spectral density
$G(\ )$	Power spectral density for $0 \le f \le \infty$
$\hat{G}(\ )$	Measured value of $G(\ )$
$G_{\ell u}(\ )$	Cross-power spectral density
h	Interval (f/f_0) or f_2/f_1
$h(t)$	Impulse response
$H(\ )$	Transfer function
i	$\sqrt{-1}$
k	Stiffness
K	Number of subsamples
ℓ	Value of $\ell(t)$
$\bar{\ell}$	Mean value of $\ell(t)$
$\bar{\ell}_N$	Average maximum of N_p peaks
ℓ_{rms}	Rms value of $\ell(t)$
$\ddot{\ell}_{rms}$	Rms value of $\ddot{\ell}(t)$
$\ell(t)$	Generalized excitation (displacement)
$\dot{\ell}(\ )$	First derivative of $\ell(t)$
$\ddot{\ell}(t)$	Second derivative of $\ell(t)$
L	Given value of $\ell(t)$
L_{rms}	Rms value of filtered signal
$L(\Omega)$	Fourier transform of $\ell(t)$

$\dot{L}(\Omega)$	Fourier transform of $\dot{\ell}(t)$
m	Mean
M	Number of points of PSD
M_a	Average number of maxima which exceeds threshold per unit time
M_n	Moment of order n
n	Order of moment or number of degrees of freedom
n_a	Average number of crossings of threshold a per unit time
n_a^+	Average number of crossings of threshold a with positive slope per unit time
n_0	Average number of zero-crossings per unit time
n_0^+	Average number of zero-crossings with positive slope per second (average frequency)
n_p^+	Average number of maxima per unit time
N	Number of curves or number of points of signal or numbers of dB
N_p	Number of peaks
N_a^+	Average numbers of crossings of threshold a with positive slope for given length of time
N_0^+	Average number of zero-crossings with positive slope for given length of time
N_p^+	Average number of positive maxima for given length of time
p( )	Probability density
$p_N(\ )$	Probability density of largest maximum over given duration
P	Probability
PSD	Power spectral density
q	$\sqrt{1-r^2}$
q_{max}^+	Probability that a maximum is positive
q_{max}^-	Probability that a maximum is negative
q( )	Probability density of maxima of $\ell(t)$
Q	Q factor (quality factor)
Q( )	Distribution function of maxima of $\ell(t)$
Q(u)	Probability that a maximum is higher than given threshold
r	Irregularity factor
rms	Root mean square (value)
r(t)	Temporal window
R	Slope in dB/octave
$R_{\ell u}$	Cross-correlation function between $\ell(t)$ and u(t)
R(f)	Fourier transform of r(t)
R(τ)	Auto-correlation function
s	Standard deviation
S_0	Value of constant PSD

S( )	Power spectral density for $-\infty \le f \le +\infty$
t	Time
T	Duration of sample of signal
T_a	Average time between two successive maxima
u	Ratio of threshold a to rms value ℓ_{rms} of $\ell(t)$
$\overline{u}_0$	Average of highest peaks
$u(t)$	Generalized response
$\dot{u}(t)$	First derivative of $u(t)$
$\ddot{u}(t)$	Second derivative of $u(t)$
v_{rms}	Rms value of $\dot{x}(t)$
x_{rms}	Rms value of $x(t)$
$\ddot{x}(t)$	Absolute acceleration of base of one-degree-of-freedom system
$\ddot{x}_{rms}$	Rms value of $\ddot{x}(t)$
$\ddot{x}_m$	Maximum value of $\ddot{x}(t)$
α	Risk of up-crossing
χ_n^2	Variable of chi-square with n degrees of freedom
δt	Time step
$\delta(\ )$	Dirac delta function
Δf	Frequency interval between half-power points or frequency step of the PSD
ΔF	Bandwidth of analysis filter
$\Delta \ell$	Interval of amplitude of $\ell(t)$
Δt	Time interval
ε	Statistical error or Euler's constant (0.577 215 664 90…)
$\gamma_{\ell u}$	Coherence function between $\ell(t)$ and $u(t)$
φ	Phase
μ_n	Central moment of order n
μ'_n	Reduced central moment of order n
π	3.141 592 65 ...
ρ	Correlation coefficient
τ	Delay
τ_m	Average time between two successive maxima
ω_0	Natural pulsation ($2\pi f_0$)
Ω	Pulsation of excitation ($2\pi f$)
ξ	Damping factor
ψ	Phase

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Chapter 1

Statistical properties of a random process

1.1. Definitions

1.1.1. *Random variable*

A random variable is a quantity whose instantaneous value cannot be predicted. Knowledge of the values of the variable before time t not does make it possible to deduce the value at the time t from it.

Example: the Brownian movement of a particle.

1.1.2. *Random process*

Let us consider, as an example, the acceleration recorded at a given point on the dial of a truck travelling on a good road between two cities A and B. For a journey, recorded acceleration obeys the definition of a random variable. The vibration characterized by this acceleration is said to be *random* or *stochastic*.

If n journeys are performed, one obtains as many different ${}^{i}\ell(t)$ curves, each recording having a random character.

We define as a *random process* or *stochastic process* the ensemble of the time functions $\left\{{}^{i}\ell(t)\right\}$ for t included between $-\infty$ and $+\infty$, this ensemble being able to be defined by statistical properties [JAM 47].

Random movements are not erratic in the common meaning of the term, but follow a well defined law. They have specific properties and can be described by a law of probability.

The principal characteristic of a random vibration is simultaneously to excite all the frequencies of a structure [TUS 67]. In distinction from sinusoidal functions, random vibrations are made up of a continuous range of frequencies, the amplitude of the signal and its phase varying with respect to time in a random fashion [TIP 77], [TUS 79]. So the random vibrations are also called *noise*.

Random functions are sometimes defined as a continuous distribution of sinusoids of all frequencies whose amplitudes and phases vary randomly with time [CUR 64] [CUR 88].

1.2. Random vibration in real environments

By its nature, the real vibratory environment is random [BEN 61a]. These vibrations are encountered:

– on road vehicles (irregularities of the roads),

– on aircraft (noise of the engines, aerodynamic turbulent flow around the wings and fuselage, creating non-stationary pressures etc) [PRE 56a],

– on ships (engine, swell etc),

– on missiles. The majority of vibrations encountered by military equipment, and in particular by the internal components of guided missiles, are random with respect to to time and have a continuous spectrum [MOR 55]: gas jet emitted with large velocity creates important turbulences resulting in acoustic noise which attacks the skin of the missile until its velocity exceeds Mach 1 approximately (or until it leaves the Earth's atmosphere) [ELD 61] [RUB 64] [TUS 79],

– in mechanical assemblies (ball bearings, gears etc), etc.

1.3. Random vibration in laboratory tests

Tests using random vibrations first appeared around 1955 as a result of the inability of sine tests to excite correctly equipment exhibiting several resonances [DUB 59] [TUS 73]. The tendency in standards is thus to replace the old swept sine tests which excite resonances one after the other by a random vibration whose effects are nearer to those of the real environment.

Random vibration tests are also used in a much more marginal way:

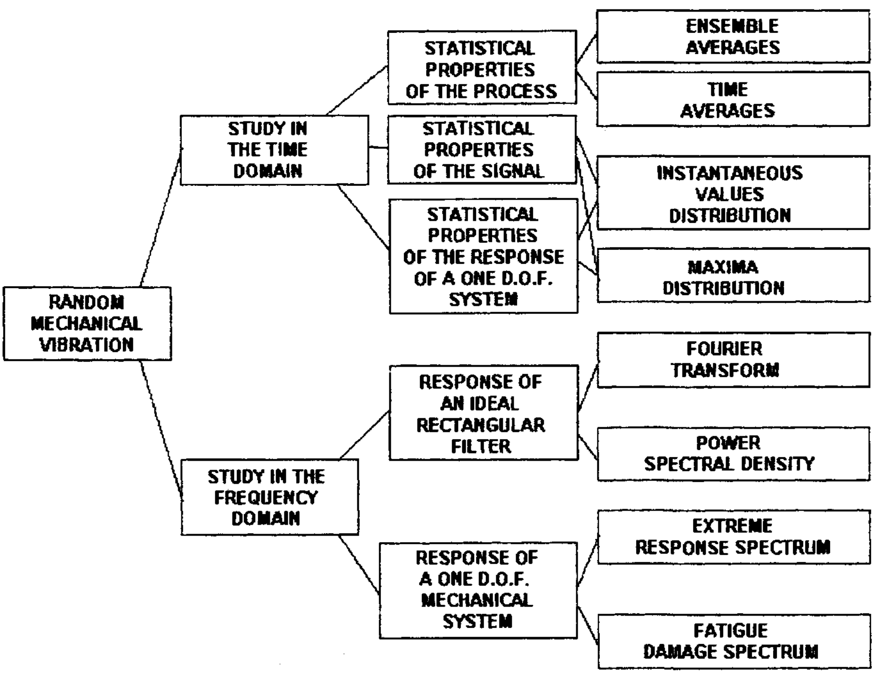
– to identify the structures (research of the resonance frequencies and measurement of Q factors), their advantage being that of shorter test duration,

– to simulate the effects of shocks containing high frequencies and difficult to replace by shocks of simple form.

1.4. Methods of analysis of random vibration

Taking into account their randomness and their frequency contents, these vibrations can be studied only using statistical methods applied to the signals with respect to time or using curves plotted in the frequency domain (spectra).

Table 1.1. *Analysis possibilities for random vibration*



One can distinguish schematically four ways of approaching analysis of random vibrations [CUR 64] [RAP 69]:

– analysis of the ensemble statistical properties of the process,

– methods of correlation,

– spectral analysis,

– analysis of statistical properties of the signal with respect to time.

The block diagram (Table 1.1) summarizes the main possibilities which will be considered in turn in what follows.

The parameters most frequently used in practice are:

– the rms value of the signal and, if it is the case, its variation as a function of time,

– the distribution of instantaneous accelerations of the signal with respect to time,

– the power spectral density.

1.5. Distribution of instantaneous values

1.5.1. *Probability density*

One of the objectives of the analysis of a random process is to determine the probability of finding extreme or peak values, or of determining the percentage of time that a random variable (acceleration, displacement etc) exceeds a given value [RUD 75]. Figure 1.1 shows a sample of a random signal with respect to time defined over duration T.

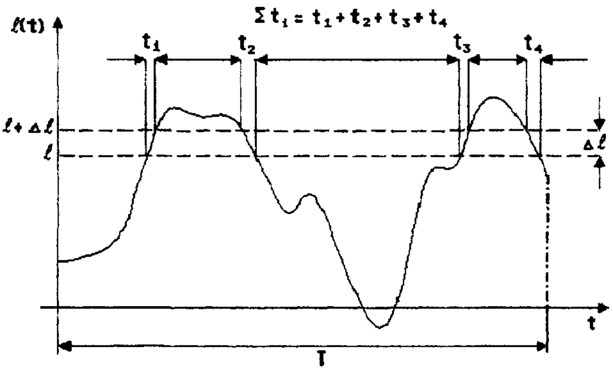


Figure 1.1. *Sample of random signal*

The probability that this function $\ell(t)$ is in the interval ℓ, $\ell + \Delta\ell$ is equal to the percentage of time during which it has values in this interval. This probability (or percentage of time) is expressed mathematically:

$$\text{Prob}[\ell < \ell(t) < \ell + \Delta\ell] = \sum_i \frac{t_i}{T} \qquad [1.1]$$

If this interval $\Delta\ell$ is small, a density function probability $p(\ell)$ is defined by:

$$\text{Pr ob}[\ell < \ell(t) < \ell + \Delta\ell] = p(\ell)\Delta\ell \qquad [1.2]$$

where:

$$p(\ell) = \frac{1}{T}\frac{\sum t_i}{\Delta\ell} \qquad [1.3]$$

To precisely define $p(\ell)$, it is necessary to consider very small intervals $\Delta\ell$ and of very long duration T, so that mathematically, the probability density function is defined by:

$$p(\ell) = \underset{\Delta\ell \to 0}{\text{limit}}\left[\underset{T \to \infty}{\text{limit}}\left(\frac{1}{T}\frac{\sum t_i}{\Delta\ell}\right)\right] \qquad [1.4]$$

1.5.2. *Distribution function*

Owing to the fact that $p(\ell)$ was given for the field of values of $\ell(t)$, the probability that the signal is inside the limits $a < \ell(t) < b$ is obtained by integration from [1.2]:

$$\text{Prob}\,[a < \ell(t) < b] = \int_a^b p(\ell)\, d\ell \qquad [1.5]$$

Since the probability that $\ell(t)$ within the limits $-\infty, +\infty$ is equal to 1 (absolutely certain event), it follows that

$$\int_{-\infty}^{+\infty} p(\ell)\, d\ell = 1 \qquad [1.6]$$

and the probability that ℓ exceeds a given level L is simply

$$\text{Prob}\,[L \le \ell(t)] = 1 - \int_L^{\infty} p(\ell)\, d\ell \qquad [1.7]$$

There exist electronic equipment and calculation programmes that make it possible to determine either *the distribution function*, or *the probability density function* of the instantaneous values of a real random signal $\ell(t)$. Figure 1.2 shows how one passes from the signal $\ell(t)$ to the probability density and the distribution function.

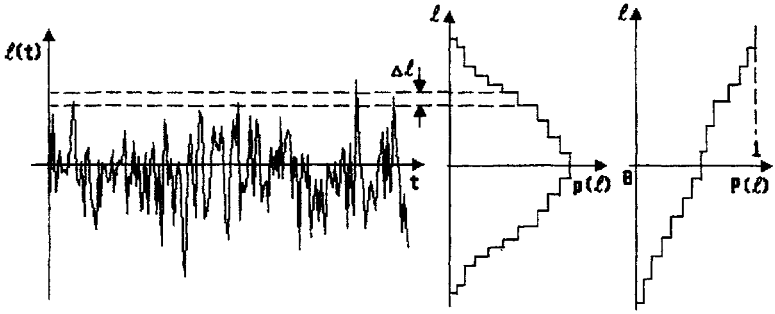


Figure 1.2. *Distribution of instantaneous values of the signal*

Among the mathematical laws representing the most usual probability densities, one can distinguish two particularly important in the field of random vibrations: Gauss's law and Rayleigh's law.

1.6. Gaussian random process

A *Gaussian random process* $\ell(t)$ is one such that the ensemble of the instantaneous values of $\ell(t)$ obeys a law of the form:

$$p[\ell(t)] = \frac{1}{s\sqrt{2\pi}} \exp\left\{-\frac{[\ell(t) - m]^2}{2\,s^2}\right\} \qquad [1.8]$$

where m and s are constants. The utility of the Gaussian law lies in the central limit theorem, which establishes that the sum of independent random variables follows a roughly Gaussian distribution whatever the basic distribution.

This the case for many physical phenomena, of parameters which result from a large number of independent and comparable fluctuating sources, and in particular the majority of vibratory random signals encountered in the real environment [BAN 78] [CRE 56] [PRE 56a].

A Gaussian process is fully determined by knowledge of the mean value m (generally zero in the case of vibratory phenomena) and of the standard deviation s.

Moreover, it is shown that:

– if the excitation is a Gaussian process, the response of a linear time-invariant system is also a Gaussian process [CRA 83] [DER 80];

– the vibration in part excited at resonance tends to be Gaussian.

For a strongly resonant system subjected to broad band excitation, the central limit theorem makes it possible to establish that the response tends to be Gaussian even if the input is not. This applies when the excitation is not a white noise, provided that it is a broad band process covering the resonance peak [NEW 75] (provided that the probability density of the instantaneous values of the excitation does not have too significant an asymmetry [MAZ 54] and that the structure is not very strongly damped [BAN 78] [MOR 55]).

In many practical cases, one is thus led to conclude that the vibration is stationary and Gaussian, which simplifies the problem of calculation of the response of a mechanical system (Volume 4).

1.7. Rayleigh distribution

Rayleigh distribution of which the probability probability has the form:

$$p(\ell) = \frac{\ell}{s^2} e^{-\frac{\ell^2}{2 s^2}} \qquad [1.9]$$

$(\ell \geq 0)$ is also an important law in the field of vibration for the representation of:

– variations in the instantaneous value of the envelope of a narrow band Gaussian random process,

– peak distribution in a narrow band Gaussian process.

Because of its very nature, the study of vibration would be very difficult if one did not have tools permitting limitation of analysis of the complete process, which comprises a great number of signals varying with time and of very great duration, using a very restricted number of samples of reasonable duration. The study of statistical properties of the process will make it possible to define two very useful concepts with this objective in mind: stationarity and ergodicity.

1.8. Ensemble averages: *'through the process'*

1.8.1. *n order average*

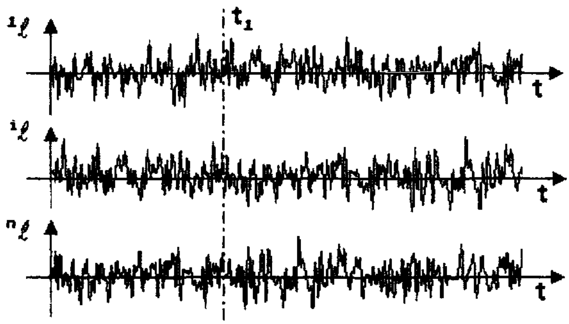


Figure 1.3. *'Through the process' study*

Let us consider N recordings of a random phenomenon varying with time ${}^i\ell(t)$ $[i \in (1, N)]$ for t varying from 0 to T (Figure 1.3). The ensemble of the curves ${}^i\ell(t)$ constitutes the process $\left\{{}^i\ell(t)\right\}$. A first possibility may consist in studying the distribution of the values of ℓ for $t = t_1$ given [JAM 47].

If we have (N) records of the phenomenon, we can calculate, for a given t_1, the mean [BEN 62] [BEN 63] [DAV 58] [JEN 68]:

$$\overline{\ell(t)} = \frac{{}^1\ell(t_1) + {}^2\ell(t_1) + \cdots + {}^n\ell(t_1)}{N} \qquad [1.10]$$

If the values ${}^i\ell(t)$ belong to an infinite discrete ensemble, the *moment of order n* is defined by:

$$E\left[\ell^n(t_1)\right] = \lim_{N \to \infty} \sum_{i=1}^{N} \frac{{}^i\ell^n(t_1)}{N} \qquad [1.11]$$

(E[ ] = mathematical expectation). By considering the ensemble of the samples at the moment t_1, the statistical nature of $\ell(t_1)$ can be specified by its probability density [LEL 76]:

$$p[\ell(t_1)] = \lim_{\Delta\ell \to 0} \frac{\text{Prob}[\ell \le \ell(t_1) \le \ell + \Delta\ell]}{\Delta\ell} \quad [1.12]$$

and by the moments of the distribution:

$$E[\ell^n(t_1)] = \int_{-\infty}^{\infty} \ell^n(t_1)\; p[\ell(t_1)]\; d\ell(t_1) \quad [1.13]$$

if the density $p[\ell(t_1)]$ exists and is continuous (or the distribution function). The moment of order 1 is the *mean* or *expected value*; the moment of order 2 is the *quadratic mean.*

For two random variables

The joint probability density is written:

$$p(\ell_1, t_1; \ell_2, t_2) = \lim_{\substack{\Delta\ell_1 \to 0 \\ \Delta\ell_2 \to 0}} \frac{\text{Prob}[\ell_1 \le \ell(t_1) \le \ell_1 + \Delta\ell_1;\ \ell_2 \le \ell(t_2) \le \ell_2 + \Delta\ell_2]}{\Delta\ell_1\ \Delta\ell_2} \quad [1.14]$$

and joint moments:

$$E[\ell(t_1)\ \ell(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} {}^{i}\ell(t_1)\ {}^{j}\ell(t_2)\ p[\ell(t_1), \ell(t_2)]\ d\ell(t_1)\ d\ell(t_2) \quad [1.15]$$

1.8.2. *Central moments*

The central moment of order n (with regard to the mean) is the quantity:

$$E\left\{[\ell(t_1) - m]^n\right\} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} [{}^{i}\ell(t_1) - m]^n \quad [1.16]$$

in the case of a discrete ensemble and, for $p(\ell)$ continuous:

$$E\left\{[\ell(t_1) - m]^n\right\} = \int_{-\infty}^{\infty} [\ell(t_1) - m]^n\ p[\ell(t_1)]\ d\ell(t_1) \quad [1.17]$$

1.8.3. *Variance*

The *variance* is the central moment of order 2

$$s^2_{\ell(t_1)} = E\left\{\left[\ell(t_1) - m\right]^2\right\} \qquad [1.18]$$

By definition:

$$s^2_{\ell(t_1)} = \int_{-\infty}^{\infty}\left[\ell(t_1) - m\right]^2 \, p\left[\ell(t_1)\right] \, d\ell(t_1) \qquad [1.19]$$

$$s^2_{\ell(t_1)} = \int_{-\infty}^{\infty}\ell^2(t_1) \, p\left[\ell(t_1)\right] \, d\ell(t_1)$$

$$-2\, m \underbrace{\int_{-\infty}^{\infty}\ell(t_1) \, p\left[\ell(t_1)\right] \, d\ell(t_1)}_{m} + m^2 \underbrace{\int_{-\infty}^{\infty} p\left[\ell(t_1)\right] \, d\ell(t_1)}_{1}$$

$$s^2_{\ell(t_1)} = E\left\{\left[\ell(t_1)\right]^2\right\} - 2\, m^2 + m^2$$

$$s^2_{\ell(t_1)} = E\left\{\left[\ell(t_1)\right]^2\right\} - m^2 \qquad [1.20]$$

1.8.4. *Standard deviation*

The quantity $s_{\ell(t_1)}$ is called the *standard deviation.* If the mean is zero,

$$s^2_{\ell(t_1)} = E\left\{\left[\ell(t_1)\right]^2\right\} \qquad [1.21]$$

When the mean m is known, an absolutely unbiased estimator of s^2 is $\sum \frac{\left({}^{i}\ell - m\right)^2}{N}$. When m is unknown, the estimator of s^2 is $\sum \frac{\left({}^{i}\ell - m'\right)^2}{N-1}$ where $m' = \frac{1}{N}\sum {}^{i}\ell$.

Example

Let us consider 5 samples of a random vibration $\ell(t)$ and the values of ℓ at a given time $t = t_1$ (Figure 1.4).

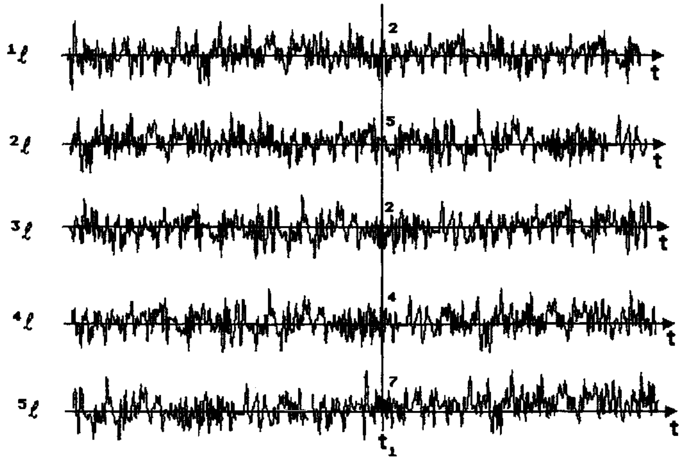


Figure 1.4. *Example of stochastic process*

If the exact mean m is known ($m = 4.2$ m/s^2 for example), the variance is estimated from:

$$s^2 = \frac{(2-4.2)^2 + (5-4.2)^2 + (2-4.2)^2 + (4-4.2)^2 + (7-4.2)^2}{5} \ (\text{m/s}^2)^2$$

$$s^2 = \frac{18.2}{5} = 3.64 \ (\text{m/s}^2)^2$$

If the mean m is unknown, it can be evaluated from

$$m' = \frac{1}{N} \sum {}^i\ell(t_1) = \frac{2+5+2+4+7}{5} = \frac{20}{5} = 4 \text{ m / s}^2$$

$$s^2 = \frac{18}{4} = 4.50 \ (\text{m/s}^2)^2$$

1.8.5. *Autocorrelation function*

Given a random process ${}^{i}\ell(t)$, the *autocorrelation function* is the function defined, in the discrete case, by:

$$R(t_1, t_1 + \tau) = \lim_{N \to \infty} \frac{1}{N} \sum_i {}^{i}\ell(t_1)\ {}^{i}\ell(t_1 + \tau) \qquad [1.22]$$

$$R(t_1, t_1 + \tau) = E\left[x(t_1) \,.\, x(t_1 + \tau)\right] \qquad [1.23]$$

or, for a continuous process, by:

$$R(\tau) = \int_{-\infty}^{\infty} x(t_1)\ x(t_1 + \tau)\ p\left[x(t_1)\right]\ dx(t_1) \qquad [1.24]$$

1.8.6. *Cross-correlation function*

Given the two processes $\{\ell(t)\}$ and $\{u(t)\}$ (for example, the excitation and the response of a mechanical system), the *cross-correlation function* is the function:

$$R_{\ell u}(t_1, t_1 + \tau) = E\left[\ell(t_1) \,.\, u(t_1 + \tau)\right] \qquad [1.25]$$

or

$$R(\tau) = \lim_{N \to \infty} \frac{1}{N} \sum_i {}^{i}\ell(t_1) \,.\, {}^{i}u(t_1 + \tau) \qquad [1.26]$$

The *correlation* is a number measuring the degree of resemblance or similarity between two functions of the same parameter (time generally) [BOD 72].

1.8.7. *Autocovariance*

Autocovariance is the quantity:

$$C(t_1, t_1 + \tau) = E\left\{\left[\ell(t_1) - \overline{\ell(t_1)}\right]\left[\ell(t_1 + \tau) - \overline{\ell(t_1 + \tau)}\right]\right\} \qquad [1.27]$$

$$C(t_1, t_1 + \tau) = R(t_1, t_1 + \tau) - \overline{\ell(t_1)}\ \overline{\ell(t_1 + \tau)} \qquad [1.28]$$

$C(t_1, t_1 + \tau) = R(t_1, t_1 + \tau)$ if the mean values are zero.

We have in addition:

$$R(t_1, t_2) = R(t_2, t_1) \qquad [1.29]$$

1.8.8. *Covariance*

One defines *covariance* as the quantity:

$$C_{\ell u} = E\left\{\left[\ell(t_1) - \overline{\ell(t_1)}\right]\left[u(t_1 + \tau) - \overline{u(t_1 + \tau)}\right]\right\} \qquad [1.30]$$

1.8.9. *Stationarity*

A phenomenon is *strictly stationary* if every moment of all orders and all the correlations are invariable with time t_1 [CRA 67] [JAM 47] [MIX 69] [PRE 90] [RAP 69] [STE 67].

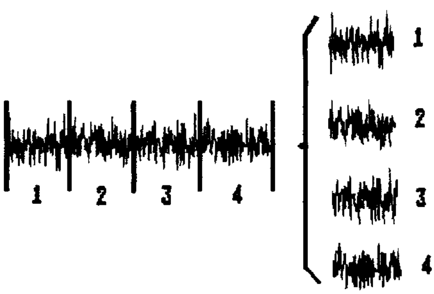


Figure 1.5. *Study of autostationarity*

The phenomenon is *wide-sense (*or *weakly) stationary* if only the mean, the mean square value and the autocorrelation are independent of time t_1 [BEN 58] [BEN 61b] [SVE 80].

If only one recording of the phenomenon $\ell(t)$ is available, one defines sometimes the autostationarity of the signal by studying the stationarity with n samples taken at various moments of the recording, by regarding them as samples obtained independently during n measurements (Figure 1.5).

One can also define *strong autostationarity* and *weak autostationarity*.

For a stationary process, the autocorrelation function is written:

$$R(\tau) = E\left[\ell(0)\ \ell(\tau)\right]$$

$$R(\tau) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} {}^{i}\ell(0)\ {}^{i}\ell(\tau) \qquad [1.31]$$

NOTES.

Based on this assumption, we have:

$$R(-\tau) = E\left\{\ell(0)\ \ell(-\tau)\right\}$$

$$R(-\tau) = E\left\{\ell(\tau)\ \ell(0)\right\}$$

$$R(-\tau) = R(\tau) \qquad [1.32]$$

(R is an even function of τ*) [PRE 90].*

$$R(0) = E\left\{\ell(0)\ \ell(0)\right\} = E\left\{\ell^2(t)\right\} \qquad [1.33]$$

$R(0)$ *is the ensemble mean square value at the arbitrary time* t

– $R(0) \geq |R(\tau)|$

We have

$$E\left\{\left[\ell(0) \pm \ell(\tau)\right]^2\right\} \geq 0$$

yielding

$$E\left\{\ell^2(0)\right\} \pm 2\ E\left\{\ell(0)\ \ell(\tau)\right\} + E\left\{\ell^2(\tau)\right\} \geq 0$$

$$R(0) \pm 2\ R(\tau) + R(0) \geq 0$$

and

$$R(0) \geq |R(\tau)| \qquad [1.34]$$

As for the cross-correlation function, it becomes, for a stationary process,

$$R_{\ell u}(\tau) = E\{\ell(0)\ \ell(\tau)\} = \lim_{N\to\infty} \frac{\sum_{i=1}^{N} {}^{i}\ell(0)\ {}^{i}\ell(\tau)}{N} \qquad [1.35]$$

Properties

1.

$$R_{\ell u}(-\tau) = R_{u\ell}(\tau) \qquad [1.36]$$

Indeed

$$R_{\ell u}(-\tau) = E\{\ell(0)\ u(-\tau)\}$$

$$R_{\ell u}(-\tau) = E\{\ell(\tau)\ u(0)\}$$

$$R_{\ell u}(-\tau) = E\{u(0)\ \ell(\tau)\}$$

$$R_{\ell u}(-\tau) = R_{u\ell}(\tau)$$

2. Whatever τ

$$R_{\ell u}(\tau) \le \sqrt{R_\ell(0)\ R_u(0)} \qquad [1.37]$$

1.9. Temporal averages: '*along the process*'

1.9.1. *Mean*

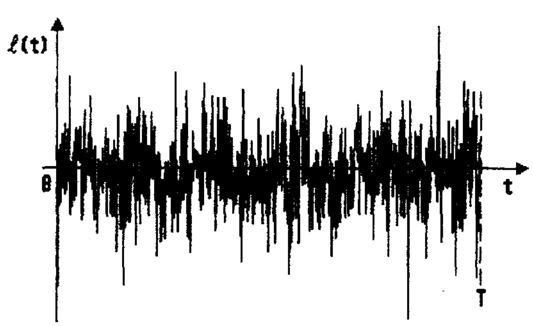


Figure 1.6. *Sample of random signal*

Let us consider a sample $\ell(t)$ of duration T of a recording. It can be interesting to study the statistical properties of the instantaneous values of the function $\ell(t)$. The first possibility is to consider the temporal mean of the instantaneous values of the recording.

We have:

$$\overline{\ell(t)} = \lim_{T \to \infty} \frac{1}{2\,T} \int_{-T}^{T} \ell(t)\, dt \qquad [1.38]$$

if this limit exists. This limit may very well not exist for some or for all the samples and, if it exists, it may depend on the selected sample $\ell(t)$; but it does not depend on time[1].

For practical reasons, one calculates in fact the mean value of the signal $\ell(t)$ over one finite duration T:

$$\overline{\ell(t)} = \frac{1}{T} \int_{0}^{T} \ell(t)\, dt \qquad [1.39]$$

1.9.2. *Quadratic mean – rms value*

The vibration $\ell(t)$ results in general in an oscillation of the mechanical system around its equilibrium position, so that the arithmetic mean of the instantaneous values can be zero if the positive and negative values are compensated. The arithmetic mean represents the signal poorly [RAP 69] [STE 67]. Therefore it is sometimes preferred to calculate the mean value of the absolute value of the signal

$$\overline{|\ell(t)|} = \frac{1}{T} \int_{0}^{T} |\ell(t)|\, dt \qquad [1.40]$$

and much more generally, by analogy with the measurement of the rms value of an electrical quantity, *the quadratic mean* (or *mean square value)* of the instantaneous values of the signal of which the square root is *the rms value.*

The rms value (root mean square value) $\ell_{rms} = \sqrt{\overline{\ell^2(t)}}$ is the simplest statistical characteristic to obtain. It is also most significant since it provides an order of magnitude of the intensity of the random variable.

1. One defines too $\overline{x(t)}$ from:

$$\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} x(t)\, dt \quad \text{ou} \quad \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, dt$$

If one can analyse the curve $\ell(t)$ by dividing the sample of duration T into N intervals of duration Δt_i $\left(i \in [1, N]\right)$, and if ℓ_i is the value of the variable during the interval of time Δt_i, the mean quadratic value is written:

$$\overline{\ell^2} = \frac{\ell_1^2\ \Delta t_1 + \cdots + \ell_i^2\ \Delta t_i + \cdots + \ell_N^2\ \Delta t_N}{T} \qquad [1.41]$$

with $T = \sum_{i=1}^{N} \Delta t_i$. If the intervals of time are equal to (Δt) and if N is the number of points characterizing the signal, $T = N\ \Delta t$ and:

$$\ell_{rms} = \frac{1}{N} \sum_i \ell_i^2$$

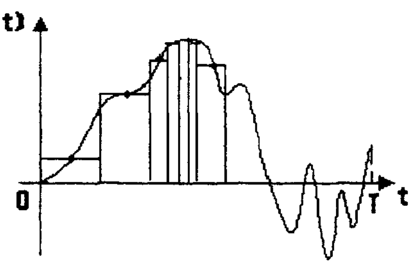


Figure 1.7. *Approximation to the signal*

If all Δt_i tend towards zero and if $N \to \infty$, the quadratic mean is defined by [BEN 63]:

$$\overline{\ell^2(t)} = \frac{1}{T} \int_0^T \ell^2(t)\ dt \qquad [1.42]$$

(or by $\frac{1}{2\,T} \int_{-T}^{T} \ell^2(t)\ dt$).

1.9.3. *Moments of order n*

As in the preceding paragraph, one also defines:

– moments of an order higher than 2; *the moment of order n* is expressed:

$$E\left\{\ell^n(t)\right\} = \overline{\ell^n(t)} = \lim_{T\to\infty} \frac{1}{2\,T} \int_{-T}^{T} \ell^n(t)\,dt \qquad [1.43]$$

– *central moments*: that of order n is defined by:

$$\mu_n = E\left\{\left[\ell(t) - \overline{\ell(t)}\right]^n\right\} = \lim_{T\to\infty} \frac{1}{2\,T} \int_{-T}^{T} \left[\ell(t) - \overline{\ell(t)}\right]^n dt \qquad [1.44]$$

For a signal made up of N points of mean $\bar{\ell}$:

$$\mu_n = \frac{1}{N} \sum_{i=1}^{N} \left(\ell_i - \bar{\ell}\right)^n$$

1.9.4. *Variance – standard deviation*

The central moment of order 2 is the *variance*, denoted by s_ℓ^2:

$$s_\ell^2 = E\left[\left(\ell - \bar{\ell}\right)^2\right] = \overline{\ell^2(t)} - \overline{\ell(t)}^2 \qquad [1.45]$$

s_ℓ is called the *standard deviation.*

Signal made up of N points:

$$s_\ell^2 = \frac{1}{N} \sum_{i=1}^{N} \left(\ell_i - \bar{\ell}\right)^2$$

1.9.5. *Skewness*

The central moment of order 3, denoted by μ_3, is sometimes reduced by division by s_ℓ^3:

$$\mu_3' = \frac{E\left\{\left[\ell(t) - \overline{\ell(t)}\right]^3\right\}}{s_\ell^3} \qquad [1.46]$$

One can show [GMU 68] that μ'_3 is characteristic of the symmetry of the probability density law $p(\ell)$ with regard to the mean $\overline{\ell(t)}$; for this reason, μ'_3 is sometimes called *skewness*.

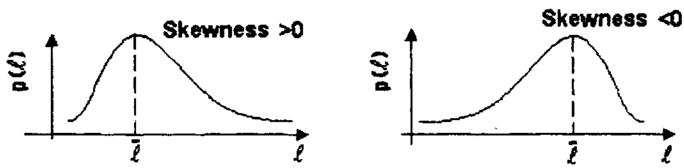


Figure 1.8. *Probability densities with non-zero skewness*

Signal made up of N points:

$$\mu'_3 = \frac{\sum_{i=1}^{N} \left(\ell_i - \bar{\ell}\right)^3}{N\, s_\ell^3}$$

$\mu'_3 = 0$ characterize a Gaussian process.

For $\mu'_3 > 0$, the probability density curve presents a peak towards the left and for $\mu'_3 < 0$, the peak of the curve is shifted towards the right.

1.9.6. *Kurtosis*

The central moment of order 4, reduced by division by s_ℓ^4, is also sometimes considered, for it makes it possible estimation of flatness of the probability density curve. It is often termed *kurtosis* [GUE 80].

$$\mu'_4 = \frac{E\left\{\left[\ell(t) - \overline{\ell(t)}\right]^4\right\}}{s_\ell^4} \qquad [1.47]$$

Signal made up of N points:

$$\mu'_4 = \frac{\sum_{i=1}^{N} \left(\ell_i - \bar{\ell}\right)^4}{N\, s_\ell^4}$$

$\mu'_4 = 3$ For a Gaussian process.

$\mu'_4 < 3$ Characteristic of a truncated signal or existence of a sinusoidal component ($\mu'_4 = 1.5$ for a pure sine).

$\mu'_4 > 3$ Presence of peaks of high value (more than in the Gaussian case).

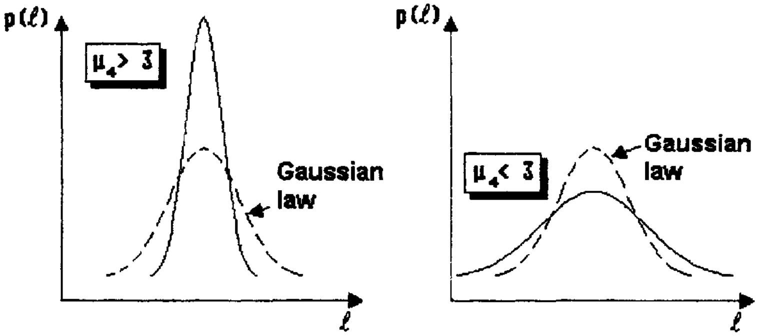


Figure 1.9. *Kurtosis influence on probability density*

Examples

1. Let us consider an acceleration signal sampled by a step of $\Delta t = 0.01$ s at 10 points (to facilitate calculation), each point representing the value of the signal for time interval Δt

$$\ddot{x}_{rms}^2 = \frac{\ddot{x}_1^2 + \ddot{x}_2^2 + \cdots + \ddot{x}_{10}^2}{T} \Delta t$$

$$T = N \ \Delta t = 10 \ . \ 0.01 \text{ s}$$

$$\ddot{x}_{rms}^2 = \frac{1^2 + 3^2 + 0^2 + (-1)^2 + (-3)^2 + (-2)^2 + (-3)^2 + (-1)^2 + 0^2}{0.1} 0.01$$

$$\ddot{x}_{rms}^2 = 3.8 \ (m/s^2)^2$$

and

$$\ddot{x}_{rms} = 1.95 \ m/s^2$$

This signal has as a mean

$$m = \frac{1 + 3 + 2 + \cdots + (-2) + 0}{0.1} 0.01 = -0.4 \ m/s^2$$

And for standard deviation s such that

$$s^2 = \frac{1}{T} \sum_i \left(\ddot{x}_i - m\right)^2 \Delta t_i$$

$$s^2 = \frac{1}{0.1}\Big[(1+0.4)^2 + (3+0.4)^2 + (2+0.4)^2 + (0+0.4)^2 + (-1+0.4)^2 + (-3+0.4)^2$$

$$+ (-2+0.4)^2 + (-3+0.4)^2 + (-1+0.4)^2 + (0+0.4)^2\Big] 0.01$$

$$s^2 = 3.64 \ (\text{m/s}^2)^2$$

$$s = 6.03 \ \text{m/s}^2$$

2. Let us consider a sinusoid $\ddot{x}(t) = \ddot{x}_m \sin(\Omega t + \varphi)$

$$\overline{\ddot{x}(t)} = 0$$

$$s^2 = \overline{\ddot{x}^2(t)} = \frac{\ddot{x}_m^2}{2}$$

$$\overline{\left|\ddot{x}(t)\right|} = \frac{2}{\pi} \ddot{x}_m \left(= \frac{2}{\pi}\sqrt{2}\ s \approx 0.9\ s \right)$$

(for a Gaussian distribution, $\overline{\left|\ddot{x}(t)\right|} \approx 0.798$ s).

1.9.7. *Temporal autocorrelation function*

We define in the time domain the autocorrelation function $R_\ell(\tau)$ of the calculated signal, for a given τ delay, of the product $\ell(t)\ \ell(t+\tau)$ [BEA 72] [BEN 58] [BEN 63] [BEN 80] [BOD 72] [JAM 47] [MAX 65] [RAC 69] [SVE 80].

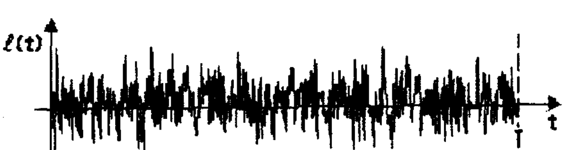


Figure 1.10. *Sample of random signal*

$$R_\ell(\tau) = E\left[\ell(t)\ \ell(t+\tau)\right] \quad [1.48]$$

$$R_\ell(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \ell(t)\ \ell(t+\tau)\ dt \quad [1.49]$$

The result is independent of the selected signal sample i. The *delay* τ being given, one thus creates, for each value of t, the product $\ell(t)$ and $\ell(t+\tau)$ and one calculates the mean of all the products thus obtained. The function $R_\ell(\tau)$ indicates the influence of the value of ℓ at time t on the value of the function ℓ at time $t+\tau$. Indeed let us consider the mean square of the variation between $\ell(t)$ and $\ell(t+\tau)$, i.e. $E\left\{\left[\ell(t)-\ell(t+\tau)\right]^2\right\}$, equal to:

$$E\left\{\left[\ell(t)-\ell(t+\tau)\right]^2\right\} = E\left[\ell^2(t)\right] + E\left[\ell^2(t+\tau)\right] - 2\ E\left[\ell(t).\ell(t+\tau)\right]$$

$$E\left\{\left[\ell(t)-\ell(t+\tau)\right]^2\right\} = 2\ R_\ell(0) - 2\ R_\ell(\tau) \quad [1.50]$$

One notes that the weaker the autocorrelation $R_\ell(\tau)$, the greater the mean square of the difference $\left[\ell(t)-\ell(t+\tau)\right]$ and, consequently, the less $\ell(t)$ and $\ell(t+\tau)$ resemble each other.

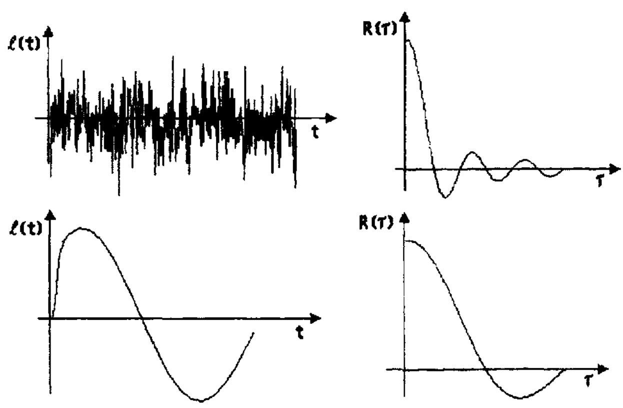


Figure 1.11. *Examples of autocorrelation functions*

The autocorrelation function measures the correlation between two values of $\ell(t)$ considered at different times t. If R_ℓ tends towards zero quickly when τ

becomes large, the random signal probably fluctuates quickly and contains high frequency components.

If R_ℓ tends slowly towards zero, the changes in the random function are probably very slow [BEN 63] [BEN 80] [RAC 69].

R_ℓ is thus a measurement of the degree of random fluctuation of a signal.

Discrete form

The autocorrelation function calculated for a sample of signal digitized with N points separated by Δt is equal, for $\tau = m\ \Delta t$, to [BEA 72]:

$$R_\ell(\tau) = \frac{1}{N - m} \sum_{i=1}^{N-m} \ell_i \cdot \ell_{i+m} \qquad [1.51]$$

Catalogues of correlograms exist allowing typological study and facilitating the identification of the parameters characteristic of a vibratory phenomenon [VIN 72]. Their use makes it possible to analyse, with some care, the composition of a signal (white noise, narrow band noise, sinusoids etc).

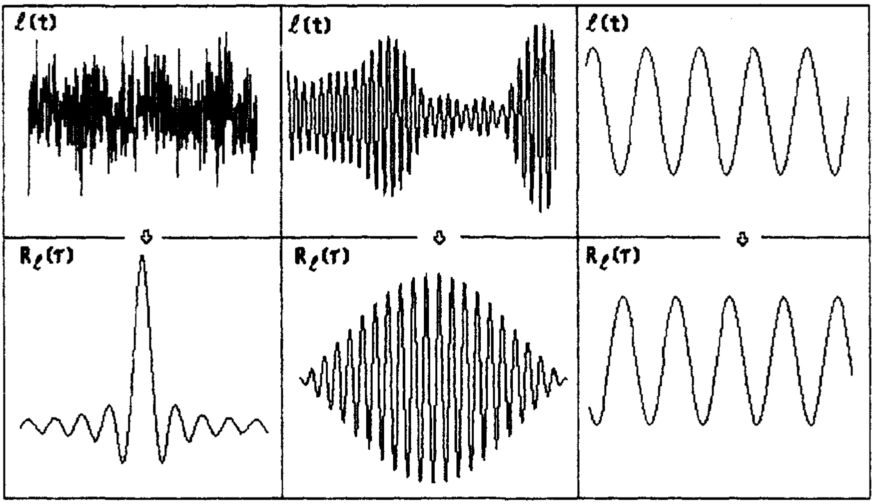


Figure 1.12. *Examples of autocorrelation functions*

Calculation of the autocorrelation function of a sinusoid

$$\ell(t) = \ell_m \sin(\Omega t)$$

$$R_\ell(\tau) = \frac{1}{T}\int_0^T \ell_m \sin \Omega t \sin \Omega(t+\tau)\, dt$$

$$R_\ell(\tau) = \frac{\ell_m^2}{2} \cos \Omega\tau \qquad [1.52]$$

The correlation function of a sinusoid of amplitude ℓ_m and angular frequency Ω is a cosine of amplitude $\frac{\ell_m^2}{2}$ and pulsation Ω. The amplitude of the sinusoid thus can, conversely, be deduced from the autocorrelation function:

$$\ell_m = \sqrt{2}\,\left[R_\ell(\tau)\right]_{max} \qquad [1.53]$$

1.9.8. *Properties of the autocorrelation function*

1. $R_\ell(0) = E\left[\ell^2(t)\right] = \overline{\ell^2(t)}$ = quadratic mean

$$R_\ell(0) = s^2 + \bar{\ell}^2 \qquad [1.54]$$

For a centered signal $(\bar{\ell} = 0)$, the ordinate at the origin of the autocorrelation function is equal to the variance of the signal.

2. The autocorrelation function is even [BEN 63] [BEN 80] [RAC 69]:

$$R_\ell(\tau) = R_\ell(-\tau) \qquad [1.55]$$

3.

$$\left|R_\ell(\tau)\right| < R_\ell(0) \quad \forall\ \tau \qquad [1.56]$$

If the signal is centered, $R_\ell(\tau) \to 0$ when $\tau \to \infty$. If the signal is not centered, $R_\ell(\tau) \to \bar{\ell}^2$ when $\tau \to \infty$.

4. It is shown that:

$$\frac{dR_\ell(\tau)}{d\tau} = E\left[\ell(t-\tau)\ \dot{\ell}(t)\right] \qquad [1.57]$$

$$\frac{d^2R_\ell(\tau)}{d\tau^2} = -E\left[\dot{\ell}(t)\ \dot{\ell}(t+\tau)\right] \qquad [1.58]$$

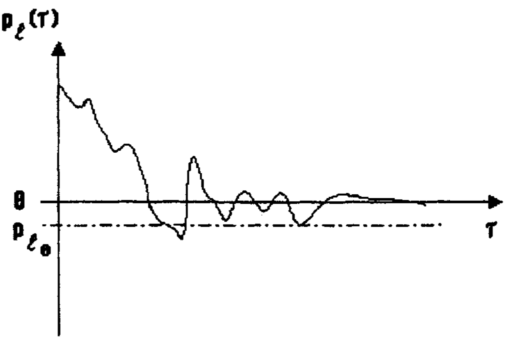


Figure 1.13. *Correlation coefficient*

NOTES.

1. *The autocorrelation function is sometimes expressed in the reduced form:*

$$\rho_\ell(\tau) = \frac{R_\ell(\tau)}{R_\ell(0)} \qquad [1.59]$$

or the normalized autocorrelation function [BOD 72] or the correlation coefficient

$\rho_\ell(\tau)$ *varies between* -1 *and* $+1$

$\rho_\ell = 1$ *if the signals are identical (superimposable)*

$\rho_\ell = -1$ *if the signals are identical in absolute value and of opposite sign.*

2. *If the mean* m *is not zero, the correlation coefficient is given by*

$$\rho_\ell(\tau) = \frac{R_\ell(\tau) - m^2}{R_\ell(0)}$$

1.9.9. *Correlation duration*

Correlation duration is the term given to a signal the value τ_0 of τ for which the function of reduced autocorrelation ρ_ℓ is always lower, in absolute value, than a certain value ρ_{ℓ_0}.

Correlation duration of:

– a wide-band noise is weak,

– a narrow band noise is large; in extreme cases, a sinusoidal signal, which is thus deterministic, has an infinite correlation duration.

This last remark is sometimes used to detect in a signal $\ell(t)$ a sinusoidal wave $s(t) = S \sin \Omega\, t$ embedded in a random noise $b(t)$:

$$\ell(t) = s(t) + b(t) \qquad [1.60]$$

The autocorrelation is written:

$$R_\ell(\tau) = R_S(\tau) + R_b(\tau) \qquad [1.61]$$

If the signal is centered, for τ sufficiently large, $R_b(\tau)$ becomes negligible so that:

$$R_\ell(\tau) = R_S(\tau) = \frac{S^2}{2} \cos \Omega\, \tau \qquad [1.62]$$

This calculation makes it possible to detect a sinusoidal wave of low amplitude embedded in a very significant noise [SHI 70a].

Examples of application of the correlation method [MAX 69]:

– determination of the dynamic characteristics of a systems,

– extraction of a periodic signal embedded in a noise,

– detection of periodic vibrations of a vibratory phenomenon,

– study of transmission of vibrations (cross-correlation between two points of a structure),

– study of turbubences,

– calculation of power spectral densities [FAU 69],

– more generally, applications in the field of signal processing, in particular in medicine, astrophysics, geophysics etc [JEN 68].

1.9.10. *Cross-correlation*

Let us consider two random functions $\ell(t)$ and $u(t)$; the *cross-correlation function* is defined by:

$$R_{\ell u}(\tau) = E[\ell(t)\ u(t+\tau)] = \lim_{T\to\infty} \frac{1}{2\,T} \int_{-T}^{T} \ell(t)\ u(t+\tau)\ dt \qquad [1.63]$$

The cross-correlation function makes it possible to establish the degree of resemblance between two functions of the same variable (time in general).

Discrete form [BEA 72]

If N is the number of sampled points and τ a delay such that $\tau = m\ \Delta t$, where Δt is the temporal step between two successive points, the cross-correlation between two signals ℓ and u is given by

$$R_{\ell u}(\tau) = \frac{1}{N-m} \sum_{i=1}^{N-m} \ell_i\ u_{i+m} \qquad [1.64]$$

1.9.11. *Cross-correlation coefficient*

Cross-correlation coefficient $\rho_{\ell u}(\tau)$ or *normalized cross-correlation function* or *normalized covariance* is the quantity [JEN 68]

$$\rho_{\ell u}(\tau) = \frac{R_{\ell u}(\tau)}{\sqrt{R_\ell(0)\ R_u(0)}} \qquad [1.65]$$

It is shown that:

$$-1 \le \rho_{\ell u}(\tau) \le 1$$

If $\ell(t)$ is a random signal *input* of a system and $u(t)$ the signal *response* at a point of this system, $\rho_{\ell u}(\tau)$ is characteristic of the degree of linear dependence of the signal u with respect to ℓ. At the limit, if $\ell(t)$ and $u(t)$ are independent, $\rho_{\ell u}(\tau) = 0$.

If the joint probability density of the random variables $\ell(t)$ and $u(t)$ is equal to $p(\ell, u)$, one can show that the cross-correlation coefficient $\rho_{\ell,u}$ can be written in the form:

$$\rho_{\ell u} = \frac{E\left[\left(\ell - m_\ell\right)\left(u - m_u\right)\right]}{s_\ell \; s_u} \qquad [1.66]$$

where m_ℓ, m_u, s_ℓ and s_u are respectively the mean values and the standard deviations of $\ell(t)$ and $u(t)$.

NOTE.

For a digitized signal, the cross-correlation function is calculated using the relation:

$$R_{\ell u}(m \, \Delta t) = \frac{1}{N} \sum_{p=1}^{N} \ell(p \, \Delta t) \; u\left[(p - m) \, \Delta t\right] \qquad [1.67]$$

1.9.12. *Ergodicity*

A process is known as *ergodic* if all the temporal averages exist and have the same value as the corresponding ensemble averages calculated at an arbitrary given moment [BEN 58] [CRA 67] [JAM 47] [SVE 80].

A ergodic process is thus necessarily stationary. One disposes in general only of a very restricted number of records not permitting experimental evaluation of the ensemble averages. In practice, one simply calculates the temporal averages by making the assumption that the process is stationary and ergodic [ELD 61].

The concept of ergodicity is thus particularly important. Each particular realization of the random function makes it possible to consider the statistical properties of the whole ensemble of the particular realizations.

NOTE.

A condition necessary and sufficient such that a stationary random vibration $\ell(t)$ is ergodic is that its correlation function satisfies the condition [SVE 80].

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \left(1 - \frac{\tau}{T}\right) R_\ell(\tau) \, d\tau = 0 \qquad [1.68]$$

where $R_\ell(\tau)$ is the autocorrelation function calculated from the centered variable $\ell(t) - m$.

1.10. Significance of the statistical analysis (ensemble or temporal)

Checking of stationarity and ergodicity should in theory be carried out before any analysis of a vibratory mechanical environment, in order to be sure that consideration of only one sample is representative of the whole process. Very often, for lack of experimental data and to save time, one makes these assumptions without checking (which is regrettable) [MIX 69] [RAC 69] [SVE 80].

1.11. Stationary and pseudo-stationary signals

We saw that the signal is known as stationary if the rms value as well as the other statistical properties remain constant over long periods of time.

In the real environment, this is not the case. The rms value of the load varies in a continuous or discrete way and gives the shape of signal known as *random pseudo-stationary*. For a road vehicle for example, variations are due to the changes in road roughness, to changes of velocity of the vehicle, to mass transfers during turns, to wind effect etc.

The temporal random function $\ell(t)$ is known as *quasi-stationary* if it can be divided into intervals of duration T sufficiently long compared with the characteristic correlation time, but sufficiently short to allow treatment in each interval as if the signal were stationary. Thus, the quasi-stationary random function is a function having characteristics which vary sufficiently slowly [BOL 84].

The study of the stationarity and ergodicity is an important stage in the analysis of vibration, but it is not in general sufficient; it in fact by itself alone does not make it possible to answer the most frequently encountered problems, for example the estimate of the severity of a vibration or the comparison of several stresses of this nature.

1.12 Summary chart of main definitions (Table 1.2) to be found on the next page.

Table 1.2 *Main definitions*

	Through the process (ensemble averages)	Along the process (temporal averages)
Moment of order n	$E[\ell^n(t_1)] = \lim \frac{1}{N}\sum_{i=1}^{N} {}^i\ell^n(t_1)$ $E[x^n(t_1)] = \int_{-\infty}^{+\infty} \ell^n(t_1)\, p[\ell(t_1)]\, d\ell(t_1)$	$E[\ell^n(t)] = \lim_{T\to\infty} \frac{1}{T}\int_0^T \ell^n(t)\, dt$
Central moment of order n	$E\{[\ell(t_1) - \overline{\ell(t_1)}]^n\} = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} [{}^i\ell(t_1) - \overline{\ell(t_1)}]^n$ $E[[\ell(t_1) - \overline{\ell(t_1)}]^n] = \int_{-\infty}^{+\infty} [\ell(t_1) - \bar{\ell}]^n\, p[\ell(t_1)]\, d\ell(t_1)$	$E[[\ell(t) - \bar{\ell}]^n] = \lim_{T\to\infty} \frac{1}{T}\int_0^T [\ell(t) - \bar{\ell}]^n\, dt$
Variance	$s^2 = E\{[\ell(t_1) - \overline{\ell(t_1)}]^2\}$	$s^2 = E\{[\ell(t) - \bar{\ell}]^2\} = \overline{\ell^2} - \bar{\ell}^2$
Autocorrelation	$R_\ell(t_1, t_1 + \tau) = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} {}^i\ell(t_1)\ {}^i\ell(t_1 + \tau)$ $R_\ell(\tau) = \int_{-\infty}^{+\infty} \ell(t_1)\ \ell(t_1 + \tau)\ p[\ell(t_1)]\, d\ell(t_1)$	$R_\ell(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \ell(t)\ \ell(t + \tau)\, dt$
Cross-correlation	$R_{\ell u}(t_1, t_1 + \tau) = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} {}^i\ell(t_1)\ {}^iu(t_1 + \tau)$ $R_{\ell u}(\tau) = \int_{-\infty}^{+\infty} \ell(t_1)\ u(t_1 + \tau)\ p[\ell(t_1)]\, d\ell(t_1)$	$R_{\ell u}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \ell(t)\ u(t + \tau)\, dt$
	Stationarity if all the averages of order n are independent of the selected time t_1.	Ergodicity if the temporal averages are equal to the ensemble averages.

Chapter 2

Properties of random vibration in the frequency domain

The frequency content of the random signal must produce useful information by comparison with the natural frequencies of the mechanical system which undergoes the vibration.

This chapter is concerned with power spectral density, with its properties, an estimate of statistical error necessarily introduced by its calculation and means of reducing it. Following chapters will show that this spectrum provides a powerful tool to enable description of random vibrations. It also provides basic data for many other analyses of signal properties.

2.1. Fourier transform

The Fourier transform of a non-periodic $\ell(t)$ signal, having a finite total energy, is given by the relationship:

$$L(\Omega) = \int_{-\infty}^{+\infty} \ell(t)\, e^{-i\Omega t}\, dt \qquad [2.1]$$

This expression is complex; it is therefore necessary in order to represent it graphically to plot:

– either the real and the imaginary part versus the angular frequency Ω,

– or the amplitude and the phase, versus Ω. Very often, one limits oneself to amplitude data. The curve thus obtained is called the *Fourier spectrum* [BEN 58].

The random signals are not of finite energy. One can thus calculate only the Fourier transform of a sample of signal of duration T by supposing this sample representative of the whole phenomenon. It is in addition possible, starting from the expression of $L(\Omega)$, to return to the temporal signal by calculation of the inverse transform.

$$\ell(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} L(\Omega)\, e^{i\Omega t}\, d\Omega \qquad [2.2]$$

One could envisage the comparison of two random vibrations (assumed to be ergodic) from their Fourier spectra calculated using samples of duration T. This work is difficult, for it supposes the comparison of four curves two by two, each transform being made up of a real part and an imaginary part (or amplitude and phase).

One could however limit oneself to a comparison of the amplitudes of the transforms, by neglecting the phases. We will see in the following paragraphs that, for reasons related to the randomness of the signal and the miscalculation which results from it, it is preferable to proceed with an average of the modules of Fourier transforms calculated for several signal samples (more exactly, an average of the squares of the amplitudes). This is the idea behind power spectral density.

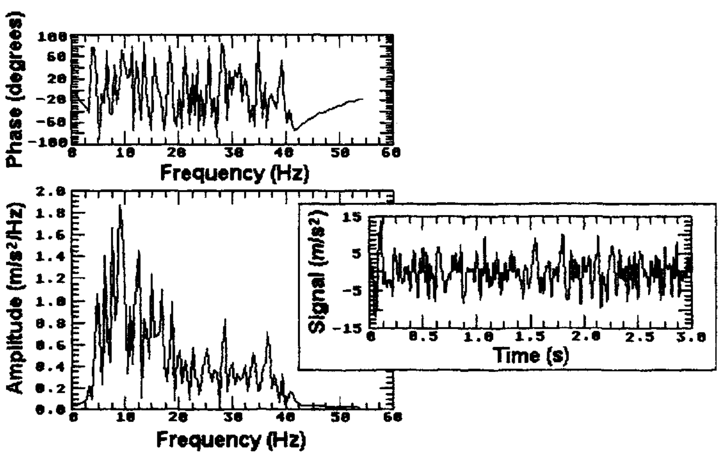


Figure 2.1. *Example of Fourier transform*

In an indirect way, the Fourier transform is thus very much used in the analysis of random vibrations.

2.2. Power spectral density

2.2.1. *Need*

The search for a criterion for estimating the severity of a vibration naturally results in examination of the following characteristics:

– The maximum acceleration of the signal: this parameter neglects the smaller amplitudes which can excite the system for a prolonged length of time,

– The mean value of the signal: this parameter has only a little sense as a criterion of severity, because negative accelerations are subtractive and the mean value is in general zero. If that is not the case, it does not produce information sufficient to characterize the severity of the vibration,

– The rms value: for a long time this was used to characterize the voltages in electrical circuits, the rms value is much more interesting data [MOR 55]:

- if the mean is zero, the rms value is in fact the standard deviation of instantaneous acceleration and is thus one of the characteristics of the statistical distribution,

- even if two or several signal samples have very different frequency contents, their rms values can be combined by using the square root of the sum of their squares.

This quantity is thus often used as a relative instantaneous severity criterion of the vibrations [MAR 58]. It however has the disadvantage of being global data and of not revealing the distribution of levels according to frequency, nevertheless very important. For this purpose, a solution can be provided by [WIE 30]:

– filtering the signal $\ell(t)$ using a series of rectangular filters of central frequency f and bandwidth Δf (Figure 2.2),

– calculating the rms value L_{rms} of the signal collected at the output of each filter.

The curve which would give L_{rms} with respect to f would be indeed a description of the spectrum of signal $\ell(t)$, but the result would be different depending on the width Δf derived from the filters chosen for the analysis. So, for a stationary noise, one filters the supposed broad band signal using a rectangular filter of filter width Δf, centered around a central frequency f_c, the obtained response having the aspect of a stable, permanent signal. Its rms value is more or less constant with time. If, by preserving its central frequency, one reduces the filter width Δf, maintaining its gain, the output signal will seem unstable, fluctuating greatly with time (as well as its rms value), and more especially so if Δf is weaker.

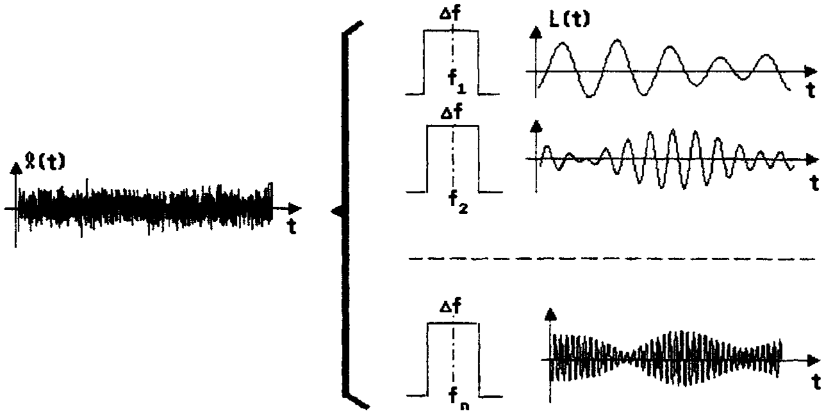


Figure 2.2. *Filtering of the random signal*

To obtain a characteristic value of the signal, it is thus necessary to calculate the mean over a much longer length of time, or to calculate the mean of several rms values for various samples of the signal. One in addition notes that the smaller Δf is, the more the signal response at the filter output has a low rms value [TIP 77].

To be liberated from the width Δf, one considers rather the variations of $\frac{L_{rms}^2}{\Delta f}$ with f. The rms value is squared by analogy with electrical power.

2.2.2. *Definition*

If one considers a tension $u(t)$ applied to the terminals of a resistance $R = 1\ \Omega$, passing current $i(t)$, the energy dissipated (Joule effect) in the resistance during time dt is equal to:

$$dE = R\ i^2(t)\ dt = i^2(t) \qquad [2.3]$$

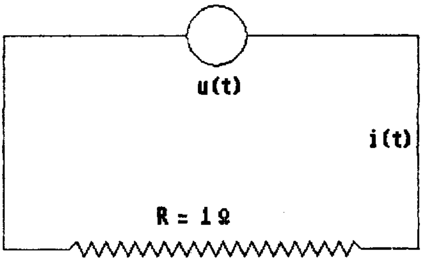


Figure 2.3. *Electrical circuit with source of tension and resistance*

The instantaneous power of the signal is thus:

$$P(t) = \frac{dE}{dt} = i^2(t) \tag{2.4}$$

and the energy dissipated during time T, between t and t + T, is written:

$$E_T = \int_t^{t+T} i^2(t)\, dt \tag{2.5}$$

$P(t)$ depends on time t (if i varies with t). It is possible to calculate a mean power in the interval T using:

$$P_{m_T} = \frac{1}{T} \int_t^{t+T} p(t)\, dt = \frac{1}{T} E_T \tag{2.6}$$

The total energy of the signal is therefore:

$$E = \int_{-\infty}^{+\infty} i^2(t)\, dt \tag{2.7}$$

and total mean power:

$$P_m = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} i^2(t)\, dt \tag{2.8}$$

By analogy with these calculations, one defines [BEN 58] [TUS 72] in vibration mechanics the *mean power* of an excitation $\ell(t)$ between $-T/2$ and $+T/2$ by:

$$P_m = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} \left| \ell_T(t) \right|^2 dt \tag{2.9}$$

where

$$\begin{cases} \ell_T = \ell(t) & \text{for } |t| \le T/2 \\ \ell_T = 0 & \text{for } |t| > T/2 \end{cases}$$

Let us suppose that the function $\ell_T(t)$ has as a Fourier transform $L_T(f)$. According to Parseval's equality,

$$\int_{-\infty}^{+\infty} |\ell_T|^2 \, dt = \int_{-\infty}^{+\infty} |L_T(f)|^2 \, df \qquad [2.10]$$

yielding, since [JAM 47]

$$\int_{-T/2}^{+T/2} |\ell_T|^2 \, dt = \int_{-\infty}^{+\infty} |\ell_T(t)|^2 \, dt \qquad [2.11]$$

$$P_m = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{+\infty} |L_T(f)|^2 \, df = \lim_{T\to\infty} \frac{2}{T} \int_0^{\infty} |L_T(f)|^2 \, df \qquad [2.12]$$

This relation gives the mean *power* contained in $\ell(t)$ when all the frequencies are considered. Let us find the mean power contained in a frequency band Δf. For that, let us suppose that the excitation $\ell(t)$ is applied to a linear system with constant parameters whose weighting function is $h(t)$ and the transfer function $H(f)$. The response $r_T(t)$ is given by the convolution integral

$$r_T(t) = \int_0^{\infty} h(\lambda) \, \ell_T(t - \lambda) \, d\lambda \qquad [2.13]$$

where λ is a constant of integration. The mean power of the response is written:

$$P_{m\,response} = \lim_{T\to\infty} \frac{2}{T} \int_0^T r_T^2(t) \, dt \qquad [2.14]$$

i.e., according to Parseval's theorem:

$$P_{m\,response} = \lim_{T\to\infty} \frac{2}{T} \int_0^T |R_T(f)|^2 \, df \qquad [2.15]$$

If one takes the Fourier transform of the two members of [2.13], one can show that:

$$R_T(f) = H(f) \, L_T(f) \qquad [2.16]$$

yielding

$$P_{m_{response}} = \lim_{T \to \infty} \frac{2}{T} \int_0^{\infty} |H(f)|^2 \, |L_T(f)|^2 \, df \qquad [2.17]$$

Examples

1. If $H(f) = 1$ for any value of f,

$$P_{m_{response}} = \lim_{T \to \infty} \int_0^{\infty} \frac{2|L_T(f)|^2}{T} \, df = P_{m_{input}} \qquad [2.18]$$

a result which *a priori* is obvious.

2. If $H(f) = 1$ for $0 \le f - \frac{\Delta f}{2} \le f \le f + \frac{\Delta f}{2}$

$H(f) = 0$ elsewhere

$$P_{m_{response}} = \lim_{T \to \infty} \int_{f-\Delta f/2}^{f+\Delta f/2} \frac{2\,|L_T(f)|^2}{T} \, df \qquad [2.19]$$

In this last case, let us set:

$$G_T(f) = \frac{2\,|L_T(f)|^2}{T} \qquad [2.20]$$

The mean power corresponding to the record $\ell_T(t)$, finite length T, in the band Δf centered on f, is written:

$$P_T(f, \Delta f) = \int_{f-\Delta f/2}^{f+\Delta f/2} G_T(f) \, df \qquad [2.21]$$

and total mean power in all the record

$$P(f, \Delta f) = \lim_{T \to \infty} \int_{f-\Delta f/2}^{f+\Delta f/2} G_T(f) \, df \qquad [2.22]$$

One terms *power spectral density* the quantity:

$$G(f) = \lim_{T \to \infty} \frac{P(f, \Delta f)}{\Delta f} \qquad [2.23]$$

In what follows, we will call this function *PSD*.

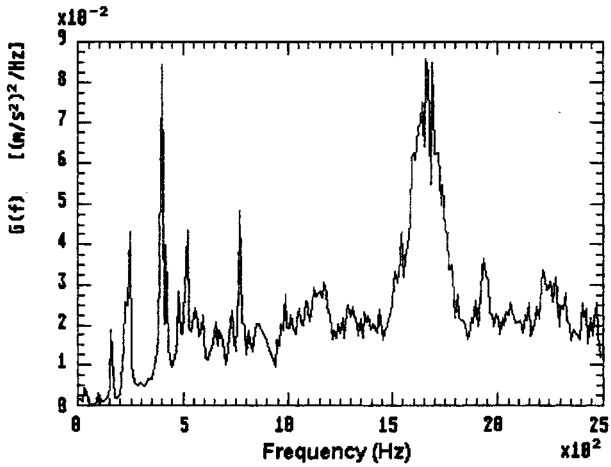


Figure 2.4. *Example of PSD (aircraft)*

NOTE.

By using the angular frequency Ω, we would obtain:

$$P(\Omega, \Delta\Omega) = \lim_{T \to \infty} \int_{\Omega - \Delta\Omega/\Omega}^{\Omega + \Delta\Omega/\Omega} G_T(\Omega)\, d\Omega \qquad [2.24]$$

with

$$G_T(\Omega) = \frac{2\,\left|L_T(\Omega)\right|^2}{2\,\pi\,T} \qquad [2.25]$$

Taking into account the above relations, and [2.10] in particular, the PSD $G(f)$ can be written [BEA 72] [BEN 63] [BEN 80]:

$$\boxed{G(f) = \lim_{\substack{\Delta f \to 0 \\ T \to \infty}} \frac{1}{T\,\Delta f} \int_0^T \ell_T^2(t, \Delta f)\, dt} \qquad [2.26]$$

where $\ell_T(t, \Delta f)$ is the part of the signal ranging between the frequencies $f - \Delta f/2$ and $f + \Delta f/2$. This relation shows that the PSD can be obtained by filtering the signal using a narrow band filter of given width, by squaring the response and by taking the mean of the results for a given time interval [BEA 72]. This method is used for analog computations.

The expression [2.26] defines theoretically the PSD. In practice, this relation cannot be respected exactly since the calculation of $G(f)$ would require an infinite integration time and an infinitely narrow bandwidth.

NOTES.

– *The function* $G(f)$ *is positive or zero whatever the value of* f.

– *The PSD was defined above for* f *ranging between* 0 *and infinity, which corresponds to the practical case. In a more general way, one could define* $S(f)$ *mathematically between* $-\infty$ *and* $+\infty$, *in such a way that*

$$S(-f) = S(f) \qquad [2.27]$$

– *The pulsation* $\Omega = 2\pi f$ *is sometimes used as variable instead of* f. *If* $G_\Omega(\Omega)$ *is the corresponding PSD, we have*

$$G(f) = 2\pi\, G_\Omega(\Omega) \qquad [2.28]$$

The relations between these various definitions of the PSD can be easily obtained starting from the expression of the rms value:

$$\ell_{rms}^2 = \int_{-\infty}^{+\infty} S(f)\, df = \int_0^{\infty} G(f)\, df = \int_0^{\infty} G_\Omega(\Omega)\, d\Omega = \int_{-\infty}^{+\infty} S(\Omega)\, d\Omega \qquad [2.29]$$

One then deduces:

$$G(f) = 2\, S(f) \qquad [2.30]$$

$$G(f) = 2\pi\, G_\Omega(\Omega) \qquad [2.31]$$

$$G(f) = 4\pi\, S_\Omega(\Omega) \qquad [2.32]$$

NOTE.

A sample of duration T *of a stationary random signal can be represented by a Fourier series, the term* a_i *of the development in an exponential Fourier series being equal to:*

$$a_i = \frac{2}{T} \int_{-T/2}^{+T/2} \ell(t) \begin{cases} \sin \dfrac{2\pi k t}{T} \\ \cos \dfrac{2\pi k t}{T} \end{cases} dt \qquad [2.33]$$

The signal $\ell(t)$ *can be written in complex form*

$$\ell(t) = \sum_{k=-\infty}^{\infty} c_k \, e^{2 \pi i k \frac{t}{T}} \qquad [2.34]$$

where $c_k = \frac{1}{2} \left(\alpha_i - \beta_i \, i\right)$

The power spectral density can also be defined from this development in a Fourier series. It is shown that [PRE 54] [RAC 69] [SKO 59] [SVE 80]

$$G(f) = \lim_{T \to \infty} \frac{a_i^2}{2 \, \Delta f} \qquad [2.35]$$

The power spectral density is a curve very much used in the analysis of vibrations:

– either in a direct way, to compare the frequency contents of several vibrations, to calculate, in a given frequency band, the rms value of the signal, to transfer a vibration from one point in a structure to another, ...

– or as intermediate data, to evaluate certain statistical properties of the vibration (frequency expected, probability density of the peaks of the signal, number of peaks expected per unit time etc).

NOTE.

The function $G(f)$*, although termed power, does not have the dimension of it. This term is often used because the square of the fluctuating quantity appears often in the expression for the power, but it is unsuitable here [LAL 95]. So it is often preferred to name it 'acceleration spectral density' or 'acceleration density' [BOO 56] or 'power spectral density of acceleration' or 'intensity spectrum' [MAR 58].*

2.3. Cross-power spectral density

From two samples of random signal records $\ell_1(t)$ and $\ell_2(t)$, one defines the cross-power spectrum by

$$G_{\ell_1 \ell_2}(f) = \lim_{T \to \infty} \frac{2 \, L_1^*(f) \, L_2(f)}{T} \qquad [2.36]$$

if the limit exists, L_1 and L_2 being respectively the Fourier transforms of $\ell_1(t)$ and $\ell_2(t)$ calculated between 0 and T.

2.4. Power spectral density of a random process

The PSD was defined above for only one function of time $\ell(t)$. Let us consider the case here where the function of time belongs to a random process, where each function will be noted ${}^{i}\ell(t)$. A sample of this signal of duration T will be denoted by ${}^{i}\ell_T(t)$, and its Fourier transform ${}^{i}L_T(f)$. Its PSD is

$$ {}^{i}G_T(f) = \frac{2\left|{}^{i}L_T(f)\right|^2}{T} \qquad [2.37] $$

By definition, the PSD of the random process is, over time T, equal to:

$$ \boxed{G_T(f) = \frac{\sum_{i=1}^{n} {}^{i}G_T(f)}{n}} \qquad [2.38] $$

n being the number of functions ${}^{i}\ell(t)$ and, for T infinite,

$$ G(f) = \lim_{T \to \infty} G_T(f) \qquad [2.39] $$

If the process is stationary and ergodic, the PSD of the process can be calculated starting from several samples of one recording only.

2.5. Cross-power spectral density of two processes

As previously, one defines the cross-power spectrum between two records of duration T each one taken in one of the processes by:

$$ {}^{i}G_T(f) = \frac{2\ {}^{i}L_1^*\ {}^{i}L_2}{T} \qquad [2.40] $$

The cross-power spectrum of the two processes is, over T,

$$ G_T(f) = \frac{\sum_{i=1}^{n} {}^{i}G_T(f)}{n} \qquad [2.41] $$

and, for T infinite,

$$G(f) = \lim_{T \to \infty} G_T(f) \qquad [2.42]$$

2.6. Relation between PSD and correlation function of a process

It is shown that, for a stationary process [BEN 58] [BEN 80] [JAM 47] [LEY 65] [NEW 75]:

$$G(f) = 2 \int_{-\infty}^{+\infty} R(\tau)\, e^{-2 \pi i f \tau}\, d\tau \qquad [2.43]$$

$R(\tau)$ being an even function of τ, we have:

$$G(f) = 4 \int_0^{\infty} R(\tau) \cos(2 \pi f \tau)\, d\tau \qquad [2.44]$$

If we take the inverse transform of $G(f)$ given in [2.43], it becomes:

$$R(\tau) = \frac{1}{2} \int_{-\infty}^{+\infty} G(f)\, e^{2 \pi i f \tau}\, df \qquad [2.45]$$

i.e., since $G(f)$ is an even function of f [LEY 65]:

$$\boxed{R(\tau) = \int_0^{\infty} G(f) \cos(2 \pi f \tau)\, df} \qquad [2.46]$$

and

$$R(0) = \overline{\ell^2(t)} = \int_0^{\infty} G(f)\, df = (\text{rms value})^2 \qquad [2.47]$$

NOTE.

These relations, named 'Wiener-Khinchine relations', can be expressed in terms of the angular frequency Ω in the form [BEN 58] [KOW 69] [MIX 69]:

$$G(\Omega) = \frac{2}{\pi} \int_0^{\infty} R(\tau) \cos(\Omega \tau)\, d\tau \qquad [2.48]$$

$$R(\tau) = \int_0^{\infty} G(\Omega) \cos(\Omega \tau)\, d\tau \qquad [2.49]$$

2.7. Quadspectrum – cospectrum

The cross-power spectral density $G_{\ell u}(f)$ can be written in the form [BEN 80]:

$$G_{\ell u}(f) = 2 \int_{-\infty}^{+\infty} R_{\ell u}(\tau)\, e^{-2\pi i f \tau}\, d\tau = C_{\ell u}(f) - i\, Q_{\ell u}(f) \qquad [2.50]$$

where the function

$$C_{\ell u}(f) = 2 \int_{-\infty}^{+\infty} R_{\ell u}(\tau) \cos(2\,\pi\, f\, \tau)\, d\tau \qquad [2.51]$$

is the *cospectrum* or *coincident spectral density,* and where

$$Q_{\ell u}(f) = 2 \int_{-\infty}^{+\infty} R_{\ell u}(\tau) \sin(2\,\pi\, f\, \tau)\, d\tau \qquad [2.52]$$

is the *quadspectrum* or *quadrature spectral density function.*

We have:

$$R_{\ell u}(\tau) = \frac{1}{2} \int_0^{\infty} G_{\ell u}(f)\, e^{2\pi i f \tau}\, df + \frac{1}{2} \int_0^{\infty} G^*_{\ell u}(f)\, e^{-2\pi i f \tau}\, df \qquad [2.53]$$

$$R_{\ell u}(\tau) = \int_0^{\infty} \left[C_{\ell u}(f) \cos(2\,\pi\, f\, \tau) + Q_{\ell u}(f) \sin(2\,\pi\, f\, \tau)\right] df \qquad [2.54]$$

$$\boxed{G_{\ell u}(f) = \left|G_{\ell u}(f)\right|\, e^{-i\,\theta_{\ell u}(f)}} \qquad [2.55]$$

$$\left|G_{\ell u}(f)\right| = \sqrt{C_{\ell u}^2 + Q_{\ell u}^2(f)} \qquad [2.56]$$

$$\theta_{\ell u}(f) = \operatorname{Arc\,tan}\left(\frac{Q_{\ell u}(f)}{C_{\ell u}(f)}\right) \qquad [2.57]$$

2.8. Definitions

2.8.1. *Broad-band process*

A broad-band process is a random stationary process whose power spectral density $G(\Omega)$ has significant values in a frequency band or a frequency domain which is rigorously of the same order of magnitude as the central frequency of the band [PRE 56a].

Figure 2.5. *Wide-band process*

Such processes appear in pressure fluctuations on the skin of a missile rocket (jet noise and turbulence of supersonic boundary layer).

2.8.2. *White noise*

When carrying out analytical studies, it is now usual to idealize the wide-band process by considering a uniform spectral density $G(f) = G_0$.

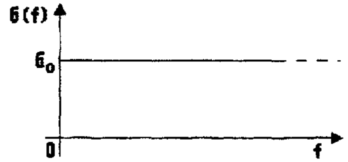


Figure 2.6. *White noise*

A process having such a spectrum is named *white noise* by analogy with white light which contains the visible spectrum.

An ideal white noise, which is supposed to have a uniform density at all frequencies, is a theoretical concept, physically unrealizable, since the area under the curve would be infinite (and therefore also the rms value). Nevertheless, model ideal white noise is often used to simplify calculations and to obtain suitable orders of magnitude of the solution, in particular for the evaluation of the response of a one-degree-of-freedom system to wide-band noise. This response is indeed primarily produced by the values of the PSD in the frequency band ranging between the half-power points. If the PSD does not vary too much in this interval, one can compare it at a first approximation to that of a white noise of the same amplitude. It should however be ensured that the results of this simplified analysis do indeed provide a correct approximation to that which would be obtained with physically attainable excitation.

2.8.3. *Band-limited white noise*

One also uses in the calculations the spectra of band-limited white noises, such as that in Figure 2.7, which are correct approximations to many realizable random processes on exciters.

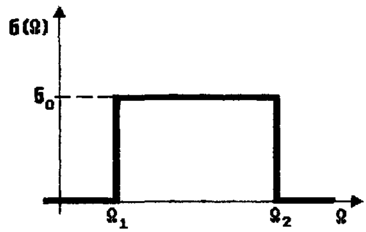


Figure 2.7. *Band-limited white noise*

2.8.4. *Narrow-band process*

A narrow-band process is a random stationary process whose PSD has significant values in one frequency band only or a frequency domain whose width is small compared with the value of the central frequency of the band [FUL 62].

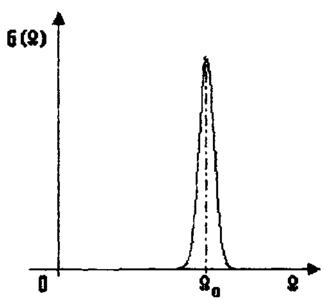


Figure 2.8. *PSD of narrow-band noise*

The signal as a function of time $\ell(t)$ looks like a sinusoid of angular frequency Ω_0, with amplitude and phase varying randomly. There is only one peak between two zero crossings.

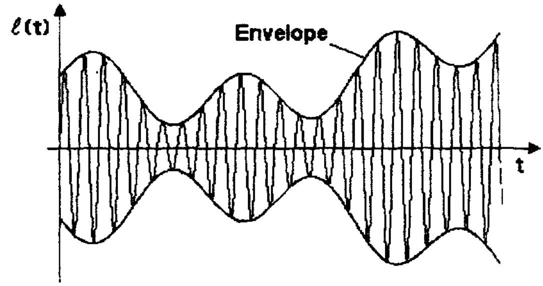


Figure 2.9. *Narrow-band noise*

It is of interest to consider *individual cycles* and *envelopes,* whose significance we will note later on.

If the process is Gaussian, it is possible to calculate from $G(\Omega)$ the expected frequency of the cycles and the probability distribution of the points on the envelope

These processes relate in particular to the response of low damped mechanical systems, when the excitation is a broad-band noise.

2.8.5. *Pink noise*

A *pink noise* is a vibration of which the power spectral density amplitude is of inverse proportion to the frequency.

2.9. Autocorrelation function of white noise

The relation [2.45] can be also written, since $G(f) = 4\,\pi\,S(\Omega)$ [BEN 58] [CRA 63]:

$$R(\tau) = \int_{-\infty}^{+\infty} S(\Omega)\, e^{i\,\Omega\,\tau}\, d\Omega \qquad [2.58]$$

If $S(\Omega)$ is constant equal to S_0 when Ω varies, this expression becomes:

$$R(\tau) = 2\,\pi\,S_0 \int_{-\infty}^{+\infty} e^{2\,\pi\,i\,f\,\tau}\, df \qquad [2.59]$$

where the integral is the Dirac delta function $\delta(\tau)$, such as:

$$\begin{cases} \delta(\tau) \to \infty & \text{when } \tau \to 0 \\ \delta(\tau) = 0 & \text{when } \tau = 0 \\ \int_{0^-}^{0^+} \delta(\tau)\, d\tau = 1 \end{cases} \qquad [2.60]$$

yielding

$$R(\tau) = 2\,\pi\, S_0\, \delta(\tau) \qquad [2.61]$$

NOTE.

If the PSD is defined by $G(\Omega)$ *in (0, ∞), this expression becomes*

$$R(\tau) = \pi\, \frac{G_0}{2}\, \delta(\tau) \qquad [2.62]$$

whilst, for $G(f) \in (0, \infty)$*:*

$$R(\tau) = \frac{1}{2}\, G_0\, \delta(\tau) \qquad [2.63]$$

For $\tau = 0$, $R \to \infty$. *Knowing that* $R(0)$ *is equal to the square of the rms value, the property of the white noise is verified (infinite rms value).*

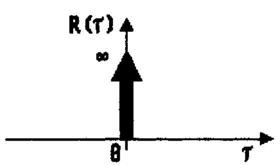


Figure 2.10. *Autocorrelation of a white noise*

It is noted in addition that the correlation is zero between two arbitrary times.

An ideal white noise thus has an infinite intensity, but has no correlation whatever between past and present [CRA 63].

2.10. Autocorrelation function of band-limited white noise

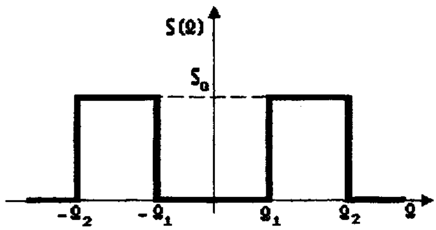


Figure 2.11. *Band-limited white noise*

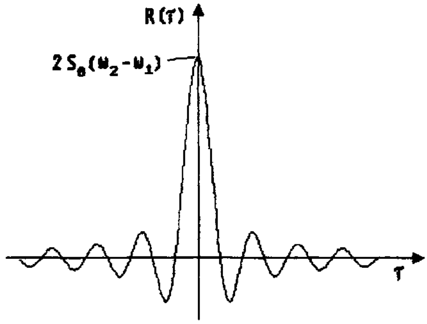


Figure 2.12. *Autocorrelation of band-limited white noise*

From the definition [2.58], we have, if $S(\Omega) = S_0$ [FUL 62],

$$R(\tau) = 2 S_0 \int_0^\infty e^{i \Omega \tau} \, d\Omega \tag{2.64}$$

$$R(\tau) = 2 S_0 \int_{\Omega_1}^{\Omega_2} \cos(\Omega \tau) \, d\Omega \tag{2.65}$$

$$\boxed{R(\tau) = 2 S_0 \frac{\sin \Omega_2 \tau - \sin \Omega_1 \tau}{\tau}} \tag{2.66}$$

$R(\tau)$ can be also written

$$R(\tau) = \frac{4 S_0}{\tau} \cos\left(\frac{\Omega_1 + \Omega_2}{2} \tau\right) \sin\left(\frac{\Omega_2 - \Omega_1}{2} \tau\right) \tag{2.67}$$

The rms value, given by [BEN 61a]

$$\ell_{rms} = \sqrt{R_\ell(0)} = \sqrt{2 S_0 (\Omega_2 - \Omega_1)} \tag{2.68}$$

is finite. If τ tends towards zero, $R(\tau) \to 2 S_0 (\Omega_2 - \Omega_1)$ (square of the rms value). The correlation between the past and the present is nonzero, at least for small intervals. When the bandwidth is widened, the above case obtains.

NOTES.

1. *The result obtained for a white noise process is demonstrated from this particular case when* $\Omega_1 = 0$ *and* $\Omega_2 \to \infty$*; indeed, if* $\Omega_2 = 0$,

$$R(\tau) = \frac{4\,S_0}{\tau} \cos\frac{\Omega_2\,\tau}{2} \sin\frac{\Omega_2\,\tau}{2} = \frac{2\,S_0}{\tau} \sin\Omega_2\,\tau$$

If $\Omega_2 \to \infty$ *[2.61],*

$$R(\tau) \to 2\,\pi\,S_0\,\delta(\tau)$$

Conversely, if $R(\tau)$ *has this value,*

$$S(\Omega) = \frac{1}{2\,\pi} \int_{-\infty}^{+\infty} 2\,\pi\,S_0\,\delta(\tau)\,e^{-i\,\Omega\,\tau}\,d\tau = S_0$$

2. *If we set* $\Omega_1 = \Omega_0 - \dfrac{\Delta\Omega}{2}$ *and* $\Omega_2 = \Omega_0 + \dfrac{\Delta\Omega}{2}$, $R(\tau)$ *can be written [COU 70]:*

$$R(\tau) = \frac{4\,S_0}{\tau} \cos(\Omega_0\,\tau)\,\sin\left(\frac{\tau\,\Delta\Omega}{2}\right) \qquad [2.69]$$

If $\tau \to 0$,

$$R(0) \to 2\,S_0\,\Delta\Omega$$

yielding

$$\boxed{\rho = \frac{R(\tau)}{R(0)} = \frac{2}{\tau\,\Delta\Omega} \cos\Omega_0\tau\,\sin\frac{\tau\,\Delta\Omega}{2}} \qquad [2.70]$$

3. *If, in practice, the noise is defined only for the positive frequencies, the expressions [2.66] and [2.68] become*

$$R(\tau) = G_0 \frac{\sin\Omega_2\,\tau - \sin\Omega_1\,\tau}{\tau} \qquad [2.71]$$

$$\ell_{rms} = \sqrt{R_\ell(0)} = \sqrt{G_0(\Omega)(\Omega_2 - \Omega_1)} = \sqrt{G_0(f)(f_2 - f_1)} \qquad [2.72]$$

2.11. Peak factor

The *peak factor* or *peak ratio* or *crest factor* F_p of a signal can be defined as the ratio of its maximum value (positive or negative) to its standard deviation (or to its rms value). For a sinusoidal signal, this ratio is equal to $\sqrt{2}$ (≈ 1.414). For a signal made up of periodic rectangular waveforms, it equals 1 while for saw tooth waveforms, it is approximately equal to 1.73.

In the case of a random signal, the probability of finding a peak of given amplitude is an increasing function of the duration of the signal. The peak factor is thus undefined and extremely large. Such a signal will thus necessarily have peaks which will be truncated because of the limitation of the dynamics of the analyser. From it will result an error in the PSD calculation.

Let us consider a random signal $\ell(t)$ of rms value ℓ_{rms}. If the signal filtered in a filter of width Δf has its values truncated higher than ℓ_0, the calculated PSD is equal to

$$G'(f) = \frac{\overline{\ell'^2_{\Delta f}(f)}}{\Delta f} \text{ instead of } G(f) = \frac{\ell^2_{\Delta f}(f)}{\Delta f}$$

Let us set:

$$F_p = \frac{\ell_0}{\ell_{rms \Delta f}} \quad [2.73]$$

The error will thus be, at frequency f,

$$e = 100\left[1 - \frac{G'(f)}{G(f)}\right] \quad [2.74]$$

with

$$\frac{G'(f)}{G(f)} = \frac{\overline{\ell'^2_{\Delta f}(f)} / \Delta f}{\overline{\ell_0^2(f)} / \Delta f} = \frac{\overline{\ell'^2_{\Delta f}(f)}}{\overline{\ell_0^2(f)}} = \rho \quad [2.75]$$

It is shown that [PIE 64], for a Gaussian signal, the error varies according to the peak factor F_p according to the law

$$e = 100\,(1 - \rho) \quad [2.76]$$

where

$$\rho = 2\ F_p^2 \int_{F_p}^{\infty} p(x)\ dx + 2 \int_0^{F_p} p(x)\ dx - 2\ F_p\ p(x) \qquad [2.77]$$

and

$$p(x) = \frac{1}{\sqrt{2\ \pi}} \exp\left(-\frac{x^2}{2}\right) \qquad [2.78]$$

The variations of the error e according to F_p are represented in Figure 2.13.

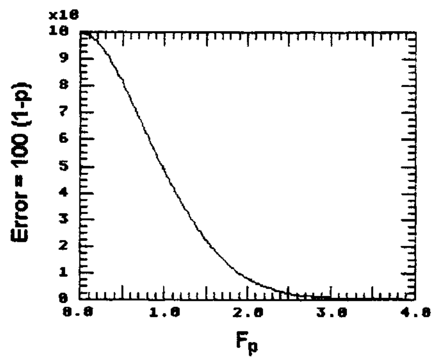


Figure 2.13. *Error versus peak factor (according to [PIE 64])*

The calculation of ρ can be simplified if it is noted that:

$$\frac{1}{\sqrt{2\ \pi}} \int_{-\infty}^{-F_p} e^{-\frac{x^2}{2}}\ dx + \frac{1}{\sqrt{2\ \pi}} \int_{-F_p}^{F_p} e^{-\frac{x^2}{2}}\ dx + \frac{1}{\sqrt{2\ \pi}} \int_{F_p}^{\infty} e^{-\frac{x^2}{2}}\ dx = 1$$

and that the probability density is symmetrical about the y-axis:

$$\int_{F_p}^{\infty} p(x)\ dx = \frac{1}{2} - \int_0^{F_p} p(x)\ dx$$

ρ is then written:

$$\rho = 2\ F_p^2 \left[\frac{1}{2} - \int_0^{F_p} p(x)\ dx\right] + 2 \int_0^{F_p} p(x)\ dx - 2\ F_p\ p(F_p)$$

$$\rho = 2\left(1 - F_p^2\right)\int_0^{F_p} p(x)\,dx + F_p^2 - 2\,F_p\,p(F_p) \qquad [2.79]$$

The integral $\int_0^{F_p} p(x)\,dx$ can be calculated using the error function, knowing that this function can be defined in the form [LAL 94] (Appendix A4.1):

$$\mathrm{Erf}_2(F_p) = \frac{1}{\sqrt{2\pi}}\int_0^{F_p} e^{-\frac{x^2}{2}}\,dx \qquad [2.80]$$

2.12. Standardized PSD/density of probability analogy

Standardized PSD is the term given to the quantity [WAN 45]:

$$G_N = \frac{G(\Omega)}{\ell_{rms}^2} \qquad [2.81]$$

It is noticed that the standardized PSD and the probability density function have common properties:

– they are nonnegative functions,

– they have an unit area under the curve,

– if we set $R(\Omega) = \int_0^{\Omega} G_N(u)\,du$, $R(\Omega)$ increases in a monotonous way from zero ($\Omega = 0$) to 1 (for Ω infinite). $R(\Omega)$ can thus be regarded as the analogue of the distribution function of $G(\Omega)$.

2.13. Spectral density as a function of time

In practice, the majority of the physical processes are, to a certain degree, nonstationary, i.e. their statistical properties vary with time. Very often however, the excitation is clearly nonstationary over a long period of time, but, for small intervals, which are still long with respect to the time of response of the dynamic system, the excitation can be regarded as stationary. Such a process is known as '*quasi-stationary*'. It can be analysed for two aspects [CRA 83]:

– study of the stationary parts by calculation of PSD whose parameters are functions slowly variable with time,

– study of the long-term behaviour, described for example by a cross probability distribution for the parameters slowly variable with PSD.

The non-stationary process can also be of short duration. This is particularly the case of a mechanical oscillator at rest suddenly exposed to a stationary random excitation; there is a phase of transitory response, therefore nonstationary. Many studies have been conducted on this last subject [CHA 72] [HAM 68] [PRI 67] [ROB 71] [SHI 70b]. Various solutions were obtained, among those of T.K. Caughey and H.J. Stumpf [CAU 61] (Volume 4), R.L. Barnoski and J.R. Maurer [BAR 69] and Y.K. Lin [LIN 67]. Other definitions also were proposed for PSD of nonstationary phenomena [MAR 70].

2.14. Relation between PSD of excitation and response of a linear system

One can easily show that [BEN 58] [BEN 62] [BEN 63] [BEN 80] [CRA 63]:

– if the excitation is a random stationary process, the response of a linear system is itself stationary,

– if the excitation is ergodic, the response is also ergodic.

Let us consider one of the functions ${}^{i}\ell(t)$ of a process (whether stationary or not); the response of a linear system can be written:

$$ {}^{i}u(t) = \int_0^\infty h(\lambda)\ {}^{i}\ell(t-\lambda)\ d\lambda \qquad [2.82] $$

yielding:

$$ {}^{i}u(t_1)\ {}^{i}u(t_2) = \int_0^\infty h(\lambda)\ {}^{i}\ell(t_1-\lambda)\ d\lambda\ \int_0^\infty h(\mu)\ {}^{i}\ell(t_2-\mu)\ d\mu $$

$$ {}^{i}u(t_1)\ {}^{i}u(t_2) = \int_0^\infty \int_0^\infty h(\lambda)\ h(\mu)\ {}^{i}\ell(t_1-\lambda)\ {}^{i}\ell(t_2-\mu)\ d\lambda\ d\mu \qquad [2.83] $$

Ensemble average

$$ R_u(t_1, t_2) = E\left[u(t_1)\ u(t_2)\right] \qquad [2.84] $$

$$ R_u(t_1, t_2) = \int_0^\infty \int_0^\infty h(\lambda)\ h(\mu)\ R_\ell(t_1-\lambda, t_2-\mu)\ d\lambda\ d\mu \qquad [2.85] $$

where

$$ R_\ell(t_1-\lambda, t_2-\mu) = E\left[\ell(t_1-\lambda)\ \ell(t_2-\mu)\right] \qquad [2.86] $$

Example of a stationary process

In this case,

$$R_u(t_1, t_2) = R_u(0, t_2 - t_1) = R_u(t_2 - t_1) = R_u(\tau)$$

and

$$R_u(\tau) = \int_0^\infty \int_0^\infty h(\lambda)\ h(\mu)\ R_\ell(\tau, \lambda - \mu)\ d\lambda\ d\mu \qquad [2.87]$$

In addition, we have seen that [2.43]:

$$G(f) = 2 \int_{-\infty}^{+\infty} R(\tau)\, e^{-2\pi i f \tau}\, d\tau$$

The PSD of the response can be calculated from this expression [CRA 63] [JEN 68]:

$$G_u(f) = 2 \int_{-\infty}^{+\infty} R_u(\tau)\, e^{-2\pi i f \tau}\, d\tau$$

$$G_u(f) = 2 \int_{-\infty}^{+\infty} e^{-2\pi i f \tau} \left[\int_0^\infty \int_0^\infty h(\lambda) h(\mu) R_\ell(\tau + \lambda - \mu) d\lambda\, d\mu \right] d\tau$$

$$G_u(f) = 2 \int_0^\infty h(\lambda)\, e^{2\pi i f \lambda}\, d\lambda \int_0^\infty h(\mu)\, e^{-2\pi i f \mu}\, d\mu$$

$$\int_{-\infty}^{+\infty} R_\ell(\tau + \lambda - \mu)\, e^{-2\pi i f (\tau+\lambda-\mu)}\, d\tau$$

$$G_u(f) = H^*(f)\ \ H(f)\ \ G_\ell(f)$$

$$\boxed{G_u(f) = \left|H(f)\right|^2\ G_\ell(f)} \qquad [2.88]$$

Depending on the angular frequency, this expression becomes:

$$G_u(\Omega) = \left|H(j\,\Omega)\right|^2\ G_\ell(\Omega) \qquad [2.89]$$

NOTE.

This result can be found starting with a Fourier series development of the excitation $\ell(t)$. *Let us set* $u(t)$ *as the response at a point of the system. With each frequency* f_j, *the response is equal to* H_j *times the input (*H_j *= a real number). So*

$u(t)$ *can also be expressed in the form of a Fourier series, each term of* $u(t)$ *being equal to the corresponding term of* $\ell(t)$ *modified by the factor* H_j *and phase* φ_j*:*

$$u(t) = \sum_j u_j H_j \sin\left(\frac{2\pi j}{T} t + \varphi_j\right) \quad [2.90]$$

i.e.

$$u(t) = \sum_j u_j H_j \left(\cos\varphi_j \sin\frac{2\pi j t}{T} + \sin\varphi_j \cos\frac{2\pi j t}{T}\right) \quad [2.91]$$

The rms value of $u(t)$ *is equal to*

$$u_{rms}^2 = \frac{1}{T}\sum_{j=1}^{\infty} u_j^2 \ H_j^2 \left(\frac{T}{2}\cos^2\varphi_j + \frac{T}{2}\sin^2\varphi_j\right) \quad [2.92]$$

When $T \to \infty$,

$$u(t) \to \int_0^{\infty} G_\ell(f)\, H^2(f)\, df \quad [2.93]$$

Knowing that, if $G_u(f)$ *is the PSD of the response,* $u_{rms}^2 = \int_0^{\infty} G_u(f)\, df$, *it becomes*

$$G_u(f) = H^2(f)\, G_\ell(f) \quad [2.94]$$

This method can be used for the measurement of the transfer function of a structure undergoing a pseudo-random vibration (random vibration of finite duration, possibly repeated several times). The method consists of applying white noise of duration T *to the material, in measuring the response at a point and in determining the transfer function by term to term calculation of the ratio of the input and output coefficients of the Fourier series development.*

2.15. Relation between PSD of the excitation and cross-power spectral density of the response of a linear system

$${}^{i}u(t) = \int_0^{\infty} h(\lambda)\ {}^{i}\ell(t-\lambda)\, d\lambda$$

$${}^{i}u(t)\ {}^{i}u(t+\tau) = \int_0^{\infty} h(\lambda)\ {}^{i}\ell(t)\ {}^{i}\ell(t+\tau-\lambda)\ d\lambda$$

If the process is stationary, the ensemble average is:

$$R_{\ell u}(\tau) = \int_0^{\infty} h(\lambda)\, R_\ell(\tau - \lambda)\, d\lambda$$

and the cross-spectrum:

$$G_u(f) = 2 \int_{-\infty}^{+\infty} e^{-2\pi i f \tau} \left[\int_0^{\infty} h(\lambda) R_\ell(\tau - \lambda) d\lambda \right] d\tau$$

$$G_{\ell u}(f) = \int_{-\infty}^{+\infty} h(\lambda) e^{-2\pi i f \lambda}\, d\lambda \left[2 \int_{-\infty}^{\infty} R_\ell(\tau - \lambda) e^{-2\pi i f (\tau - \lambda)}\, d\tau \right]$$

$$\boxed{G_{\ell u}(f) = H(f)\, G_\ell(f)} \qquad [2.95]$$

NOTE.

If we set:

$$G_{\ell u}(f) = |A(f)|\, e^{i\varphi(f)}$$

the transfer function $H(f)$ *can be written, knowing that the PSD* $G_\ell(f)$ *is a real function*

$$\begin{cases} |H(f)| = \dfrac{|A(f)|}{G_\ell(f)} \\ \psi(f) = \varphi(f) \end{cases} \qquad [2.96]$$

2.16. Coherence function

The coherence function between two signals $\ell(t)$ and $u(t)$ is defined by [BEN 63] [BEN 78] [BEN 80] [ROT 70]:

$$\gamma_{\ell u}^2(f) = \frac{|G_{\ell u}(f)|^2}{G_{\ell\ell}(f)\, G_{uu}(f)} \qquad [2.97]$$

This function is a measure of the effect of input on response of a system.

In an ideal case,

$$G_{\ell u} = H(f)\, G_{\ell\ell}$$

and

$$\gamma_{\ell u}^2 = 1$$

$\gamma_{\ell u}$ is in addition zero if the signals $\ell(t)$ and $u(t)$ are completely uncorrelated. In general, $\gamma_{\ell u}$ lies between 0 and 1 for the following reasons:

– presence of noise in measurements,

– nonlinear relation between $\ell(t)$ and $u(t)$,

– the response $u(t)$ is due to other inputs than $\ell(t)$.

Let us consider the case where noise exists only with the response. Setting G_{vv} the PSD of the response without noise and G_{nn} that of the noise alone, it becomes:

$$G_{uu}(f) = G_{vv} + G_{nn}$$

where

$$G_{vv}(f) = |H(f)|^2 \, G_{\ell\ell}(f)$$

$$G_{vv} = \left|\frac{G_{\ell u}}{G_{\ell\ell}}\right|^2 G_{\ell\ell} = \gamma_{\ell u}^2 \, G_{uu} \qquad [2.98]$$

yielding

$$\boxed{\gamma_{\ell u}^2 = \frac{G_{vv}}{G_{uu}}} \qquad [2.99]$$

The quantity $\gamma_{\ell u}^2 \, G_{uu}$ is named *coherent ouput power spectrum.*

$$\gamma_{\ell u}^2 = \frac{G_{uu} - G_{nn}}{G_{uu}} = 1 - \frac{G_{nn}}{G_{uu}} \qquad [2.100]$$

If the noise is present only on the input, we set $\ell(t) = a(t) + m(t)$ where $a(t)$ is the pure signal and $m(t)$ the noise. We have in the same way, for the PSD,

$$G_{\ell\ell}(f) = G_{aa}(f) + G_{mm}(f)$$

$$G_{uu}(f) = |H(f)|^2 \, G_{aa}$$

$$G_{\ell u}(f) = H(f) \, G_{aa}$$

$$\gamma_{\ell u}^2 \ G_{\ell\ell} = G_{aa} \qquad [2.101]$$

$$\boxed{\gamma_{\ell u}^2 = \frac{G_{aa}}{G_{\ell\ell}} = 1 - \frac{G_{mm}}{G_{\ell\ell}}} \qquad [2.102]$$

2.17. Effects of truncation of peaks of acceleration signal

The example below makes it possible to highlight the influence of a truncation of the peaks of a random acceleration signal on its power spectral density.

2.17.1. *Acceleration signal selected for study*

The signal considered is a sample of duration 1 second of a white noise over a bandwidth 10 – 2500 Hz, of rms value $\ddot{x}_{rms} = 49.9$ m/s^2 (PSD amplitude: 1 (m/s^2)^2/Hz, sampling frequency: 8192 Hz). This signal was truncated with various acceleration values: $\pm 5\ \ddot{x}_{rms}$, $\pm 4.5\ \ddot{x}_{rms}$ ..., until $\pm 0.5\ \ddot{x}_{rms}$.

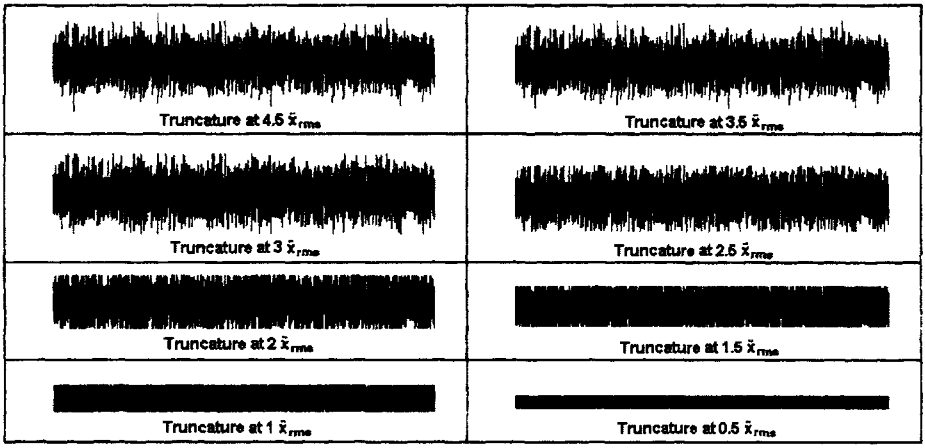


Figure 2.14. *Truncated signals*

2.17.2. *Power spectral densities obtained*

The spectral densities of these signals were calculated between 10 Hz and 10000 Hz. We observe from the PSD (Figure 2.15) that:

– truncation causes the amplitude of the PSD to decrease uniformly in the defined bandwidth (between 10 Hz and 2500 Hz);

– this reduction is only sensitive if one clips the peaks below $2\,\ddot{x}_{rms}$ approximately;

– truncation increases the amplitude of the PSD beyond its specified bandwidth (2500 Hz). This effect is related to the mode of truncation selected (clean cut-off at the peaks and no nonlinear attenuation, which would smooth out the signal in the zone concerned).

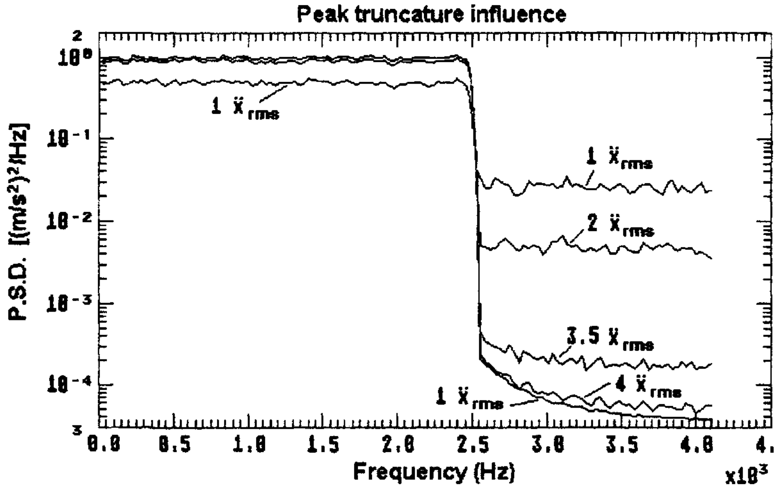


Figure 2.15. *PSD of the truncated signals*

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Chapter 3

Rms value of random vibration

3.1. Rms value of a signal as function of its PSD

We saw that [2.26]:

$$G(f) = \lim_{\substack{\Delta f \to 0 \\ T \to \infty}} \frac{1}{T\, \Delta f} \int_0^T \ell_T^2(t, \Delta f)\, dt$$

$G(f)$ is the square of the rms value of the signal filtered by a filter Δf whose width tends towards zero, centered around f. To obtain the total rms value ℓ_{rms} of the signal, taking into account all the frequencies, it is thus necessary to calculate

$$\ell_{rms}^2 = \int_{0^-}^{\infty} G(f)\, df \qquad [3.1]$$

The notation 0^- means that integration is carried out in a frequency interval covering $f = 0$, while 0^+ indicates that the interval does not include the limiting case $f = 0$. In a given frequency band f_1, f_2 $(f_2 > f_1)$,

$$\ell_{rms}^2 = \int_{f_1}^{f_2} G(f)\, df \qquad [3.2]$$

The square of the rms value of the signal in a limited frequency interval f_1, f_2 is equal to the area under the curve $G(f)$ in this interval. In addition:

$$\int_{0^+}^{\infty} G(f)\, df = s_\ell^2 \qquad [3.3]$$

where

s_ℓ^2 is the variance of the signal without its continuous component and

s_ℓ is the standard deviation of the signal.

Figure 3.1. *Non-zero lower limit of the PSD*

In addition,

$$\int_{0^-}^{0^+} G(f)\, df = (\bar{\ell})^2 \qquad [3.4]$$

$\bar{\ell}$ is the mean value of the signal. We thus have:

$$\ell_{rms}^2 = s_\ell^2 + (\bar{\ell})^2 \qquad [3.5]$$

Lastly, for $f \neq 0$, we have:

$$\int_{f^-}^{f^+} G(f)\, df = 0 \qquad [3.6]$$

A purely random signal does not have a discrete frequential component.

NOTE.

The mean value $\bar{\ell}$ corresponding to the continuous component of the signal can originate in:

– shift due to the measuring equipment, the mean value of the signal being actually zero. This component can be eliminated, either by centring the signal ℓ before the calculation of its PSD, or by calculating the PSD between $f_1 = 0 + \varepsilon$ *and* f_2 *(ε being a positive constant different from zero, arbitrarily small),*

– permanent acceleration, constant or slowly variable, corresponding to a rigid body movement of the vehicle (for example, static acceleration in phase propulsion of a launcher using propellant). One often dissociates this by filtering such static acceleration of vibrations which are superimposed on it, the consideration of static and dynamic phenomena being carried out separately. It is however important to be able to identify and measure these two parameters, in order to be able to study the combined effects of them, for example during calculations of fatigue strength, if necessary (using the Goodman or Gerber rule, ... cf. Volume 4) or of reaction to extreme stress.

Static and dynamic accelerations are often measured separately by different sensors, vibration measuring equipment not always covering DC the component. Except for particular cases, we will always consider in what follows the case of zero mean signals. We know that, in this case, the rms value of the signal is equal to its standard deviation.

Obtained by calculation of a mean square value, the power spectral density is an incomplete description of the signal $\ell(t)$. There is loss of information on phase. Two signals of comparable nature and of different phases will have the same PSD.

Example

Let us consider a stationary random acceleration $\ddot{x}(t)$ having an uniform power spectral density given by:

$$G_{\ddot{x}}(f) = 0.0025 \ (\text{m/s}^2)^2/\text{Hz}$$

in the frequency domain ranging between $f_1 = 10$ Hz and $f_2 = 1\,000$ Hz, and zero elsewhere.

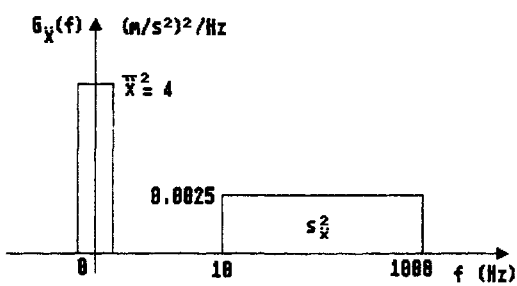

Figure 3.2. *PSD of a signal having a continuous component*

Let us suppose in addition that the continuous component of the signal is equal to $\bar{\ddot{x}} = 2\ \text{m/s}^2$. Let us calculate the rms value and the standard deviation of the signal. The mean square value of the signal is given by the relation [3.5]:

$$\overline{\ddot{x}^2} = \bar{\ddot{x}}^2 + s_{\ddot{x}}^2 = \int_{0^-}^{\infty} G_{\ddot{x}}(f)\ df$$

$$s_{\ddot{x}}^2 = \int_{0^+}^{\infty} G_{\ddot{x}}(f)\ df$$

$$s_{\ddot{x}}^2 = \int_{0}^{1000} 0.0025\ df$$

$$s_{\ddot{x}}^2 = (1000 - 10)\ 0.0025 = 2.475\ (\text{m/s}^2)^2$$

$$\bar{\ddot{x}}^2 = 4\ (\text{m/s}^2)^2$$

yielding the mean square value

$$\overline{\ddot{x}^2} = 4 + 2.475 = 6.475\ (\text{m/s}^2)^2$$

and the rms value

$$\sqrt{\overline{\ddot{x}^2}} \approx 2.545\ \text{m/s}^2$$

while the standard deviation is equal to $s_{\ddot{x}} = 1.573\ \text{m/s}^2$

The random signals are in general centered before the calculation of the spectral density, so that $\overline{\ddot{x}^2} = s_{\ddot{x}}^2$.

3.2. Relations between PSD of acceleration, velocity and displacement

Let us set as $\ell(t)$ a random signal with Fourier transform $L(f)$; by definition, we have:

$$L(f) = \int_{-\infty}^{+\infty} \ell(t)\ e^{-2\pi i f t}\ dt \qquad [3.7]$$

and

$$\ell(t) = \int_{-\infty}^{+\infty} L(f)\ e^{2\pi i f t}\ df \qquad [3.8]$$

yielding

$$\dot{\ell}(t) = \frac{d\ell}{dt} = \int_{-\infty}^{+\infty} 2\,\pi\,i\,f\,L(f)\,e^{2\pi i f t}\,dt$$

$$\dot{\ell}(t) = \int_{-\infty}^{+\infty} \dot{L}(f)\,e^{2\pi i f t}\,dt \qquad [3.9]$$

By identification, it becomes:

$$\dot{L}(f) = 2\,\pi\,i\,f\,L(f) \qquad [3.10]$$

The conjugate expressions of $L(f)$ and of $\dot{L}(f)$ are obtained by replacing i by $-i$. If $G_\ell(f)$ and $G_{\dot{\ell}}(f)$ are respectively the PSD of $\ell(t)$ and of $\dot{\ell}(t)$, one thus obtains, since these quantities are functions of the products $L^*(f)\,L(f)$ and $\dot{L}^*(f)\,\dot{L}(f)$ [LEY 65] [LIN 67],

$$G_{\dot{\ell}}(f) = 4\,\pi^2\,f^2\,G_\ell(f) \qquad [3.11]$$

yielding

$$\dot{\ell}^2_{rms} = \int_0^{+\infty} G_{\dot{\ell}}(f)\,df \qquad [3.12]$$

$$\boxed{\dot{\ell}^2_{rms} = \int_0^{+\infty} (2\,\pi\,f)^2\,G_\ell(f)\,df} \qquad [3.13]$$

and, in the same way,

$$\boxed{\ddot{\ell}^2_{rms} = \int_0^{+\infty} (2\,\pi\,f)^4\,G_\ell(f)\,df} \qquad [3.14]$$

NOTES.

1. *These relations use an integral with respect to the frequency between* 0 *and* $+\infty$. *In practice, the PSD is calculated only for one frequency interval* $(f_1,\ f_2)$. *The initial frequency* f_1 *is a function of the duration of the sample selected; this duration being necessarily limited,* f_1 *cannot be always taken as low as would be desirable.*

The limit f_2 *is if possible selected sufficiently large so that all the frequency content is described. It is not always possible for certain phenomena, if only because of the measuring equipment. A value often used is, for example, 2000 Hz. However,* *the integral necessary for the evaluation of* $\ddot{\ell}^2_{rms}$ *includes a term in* f^4 *which makes it very sensitive to the high frequencies.*

In the calculation of all the expressions utilizing $\ddot{\ell}^2_{rms}$, *as will be the case of the irregularity parameter r which we will define later, the result could be spoilt having considerable error in the event of a inappropriate choice of the limits* f_1 *and* f_2.

J. Schijve [SCH 63] considers that the high frequency/small amplitude peaks have little influence on the fatigue suffered by the materials and proposes to limit integration to approximately 1000 Hz (for vibratory environments of aircraft type).

2. *It is known that the rms value of a sinusoidal acceleration signal is related to the corresponding velocity and the displacement by*

$$\ddot{\ell} = 2\,\pi\,f\,\dot{\ell} = (2\,\pi\,f)^2\,\ell \qquad [3.15]$$

These relationships apply at first approximation to the rms values of a very narrow band random signal of central frequency f.

This makes it possible to demonstrate differently the relations [3.13] and [3.14]. The PSD of a signal $\ddot{\ell}(t)$ *is indeed calculated while filtering* $\ddot{\ell}(t)$ *using a filter of width* Δf *whose central frequency varies in the definition interval of the PSD, the result being squared and divided by* Δf *for each point of the PSD. One thus obtains [CUR 64] [DEE 71] [HIM 59]:*

$$G_{\dot{\ell}} = (2\,\pi\,f)^2\,G_\ell \qquad [3.16]$$

$$G_{\ddot{\ell}} = (2\,\pi\,f)^2\,G_{\dot{\ell}} = (2\,\pi\,f)^4\,G_\ell \qquad [3.17]$$

yielding [OSG 69] [OSG 82]:

$$\dot{\ell}^2_{rms} = \int_0^\infty \frac{G_{\ddot{\ell}}(f)}{(2\,\pi\,f)^2}\,df \qquad [3.18]$$

and

$$\ell^2_{rms} = \int_0^\infty \frac{G_{\ddot{\ell}}(f)}{(2\,\pi\,f)^4}\,df \qquad [3.19]$$

One can deduce from these relations the rms value of the displacement of a very narrow band noise [BAN 78] [OSG 69]:

$$\ell_{rms} = \frac{\ddot{\ell}_{rms}}{4\,\pi^2 f^2} \qquad [3.20]$$

3.3. Graphical representation of PSD

We will consider here the most frequent case where the vibratory signal to analyse is an acceleration. The PSD is the subject of four general presentations:

– the first with the frequency on the x axis (Hz), the amplitude of the PSD on the y axis $[(m/s^2)^2/Hz]$, the points being regularly distributed by frequency (constant filter width Δf throughout the whole range of analysis);

– the second, met primarily in acoustics problems, uses an analysis in the $\frac{1}{n}$th octave, the filter width being thus variable with the frequency; one finds more often in this case the ordinates expressed in decibels (dB). The number of decibels is then given, by:

$$n_{dB} = 10 \log \frac{G}{G_0} \qquad [3.21]$$

where

G is the amplitude of the measured PSD.
G_0 is a reference value, selected equal to 10^{-12} $(m/s^2)^2/Hz$ in general,

or, if we consider the rms value in each band of analysis, by

$$n_{dB} = 20 \log \frac{a}{a_0} \qquad [3.22]$$

where

a = rms value of the signal in the selected band of analysis,
a_0 = reference value of (10^{-6} m/s^2);

– sometimes, the analysis in $\frac{1}{n}$th octave is carried out by indicating in ordinates the rms value obtained in each filter. For a noise whose PSD varies little with the frequency (close to white noise), the rms value obtained varies with the bandwidth of the filter,

– the relationships [3.17] show that the PSD can also be plotted on a four-coordinate nomographic grid on which can be directly read the PSD value for acceleration, velocity and displacement [HIM 59].

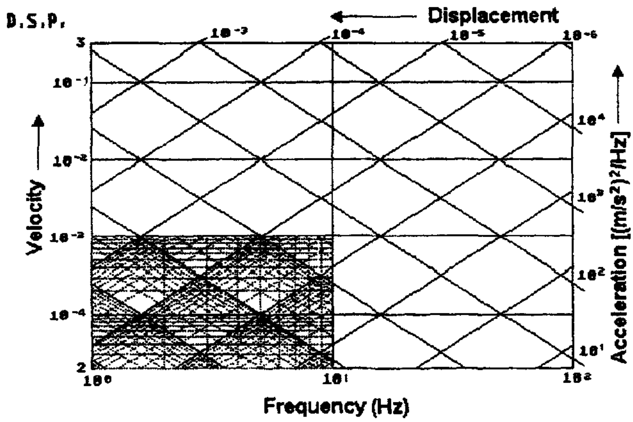

Figure 3.3. *Four-coordinate representation [HIM 59]*

3.4. Practical calculation of acceleration, velocity and displacement rms values

3.4.1. *General expressions*

The rms values of acceleration, velocity and displacement are more particularly useful for evaluation of feasibility of a specified random vibration on a test facility (electrodynamic shaker or hydraulic vibration machine). Control in a general way being carried out from a PSD of acceleration, we will in this case temporarily abandon the generalized co-ordinates. We saw that the rms value $\ddot{x}_{rms}$ of a random vibration $\ddot{x}(t)$ of PSD $G(f)$ is equal to:

$$\ddot{x}_{rms}^2 = \int_0^\infty G(f)\, df$$

The rms velocity and displacement corresponding to this signal of acceleration are respectively given by:

$$v_{rms}^2 = \int_0^\infty \frac{G(f)}{(2\pi f)^2}\, df \qquad [3.23]$$

and

$$x_{rms}^2 = \int_0^\infty \frac{G(f)}{(2\pi f)^4}\, df \qquad [3.24]$$

In the general case where the PSD $G(f)$ is not constant, the calculation of these three parameters is made by numerical integration between the two limits f_1 and f_2 of the definition interval of $G(f)$. When $G(f)$ can be represented by a succession of horizontal or arbitrary slope straight line segments, it is possible to obtain analytical expressions.

3.4.2. *Constant PSD in frequency interval*

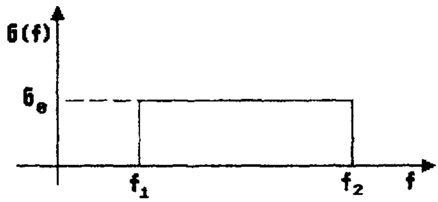

Figure 3.4. *Constant PSD between two frequencies*

In this very simple case where the PSD is constant between f_1 and f_2, $G(f) = G_0$, yielding:

$$\ddot{x}_{rms} = \sqrt{G_0 \left(f_2 - f_1\right)} \qquad [3.25]$$

$$v_{rms}^2 = \frac{G_0}{4\,\pi^2} \int_{f_1}^{f_2} \frac{df}{f^2}$$

$$v_{rms} = \frac{1}{2\,\pi} \sqrt{G_0 \left(\frac{1}{f_1} - \frac{1}{f_2}\right)} \qquad [3.26]$$

$$x_{rms} = \frac{1}{4\,\pi^2} \sqrt{\frac{G_0}{3} \left(\frac{1}{f_1^3} - \frac{1}{f_2^3}\right)} \qquad [3.27]$$

NOTES.

The rms displacement x_{rms} *can be also written as a function of rms acceleration:*

$$x_{rms} = \frac{1}{4\,\pi^2}\sqrt{\frac{1}{3}\,\frac{\ddot{x}_{rms}^2}{f_2 - f_1}\left(\frac{1}{f_1^3} - \frac{1}{f_2^3}\right)}$$

$$\frac{x_{rms}}{\ddot{x}_{rms}} = \frac{1}{4\,\pi^2\sqrt{3}}\left[\frac{f_1^2 + f_1\,f_2 + f_2^2}{f_1^3\,f_2^3}\right]^{\frac{1}{2}} \qquad [3.28]$$

If $f_1 << f_2$ *[CRE 56]*

$$x_{rms} = \frac{\ddot{x}_{rms}}{4\,\pi^2\,f_1\,\sqrt{3\,f_1\,f_2}} \qquad [3.29]$$

3.4.3. *PSD comprising several horizontal straight line segments*

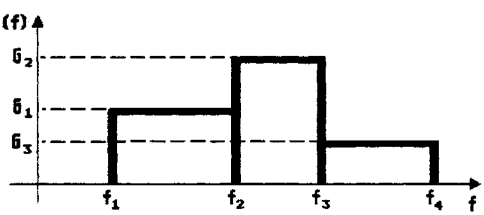

Figure 3.5. *PSD comprising horizontal segments*

We then have [SAN 63]:

$$\ddot{x}_{rms} = \sqrt{\sum_i G_i\left(f_{i+1} - f_i\right)} \qquad [3.30]$$

$$v_{rms} = \frac{1}{2\,\pi}\sqrt{\sum_i G_i\left(\frac{1}{f_i} - \frac{1}{f_{i+1}}\right)} \qquad [3.31]$$

$$x_{rms} = \frac{1}{4\,\pi^2}\sqrt{\sum_i \frac{G_i}{3}\left(\frac{1}{f_i^3} - \frac{1}{f_{i+1}^3}\right)} \qquad [3.32]$$

3.4.4. *PSD defined by a linear segment of arbitrary slope*

It is essential in this case to specify in which scales the segment of straight line is plotted.

Linear-linear scales

Between the frequencies f_1 and f_2, the PSD $G(f)$ obeys $G(f) = a\, f + b$, where a and b are constants such that, for $f = f_1$, $G = G_1$ and for $f = f_2$, $G = G_2$, yielding $a = \dfrac{G_2 - G_1}{f_2 - f_1}$ and $b = \dfrac{f_1\, G_2 - f_2\, G_1}{f_1 - f_2}$.

$$\ddot{x}_{rms}^2 = \int_{f_1}^{f_2} G(f)\, df = \int_{f_1}^{f_2} (a\, f + b)\, df$$

$$\boxed{\ddot{x}_{rms} = \sqrt{\frac{(f_2 - f_1)(G_2 + G_1)}{2}}} \qquad [3.33]$$

$$v_{rms}^2 = \frac{1}{4\,\pi^2}\left[a \ln\frac{f_2}{f_1} - b\left(\frac{1}{f_2} - \frac{1}{f_1}\right)\right]$$

$$\boxed{v_{rms} = \frac{1}{2\,\pi}\sqrt{\frac{G_2 - G_1}{f_2 - f_1}\ln\left(\frac{f_2}{f_1}\right) + \frac{G_1}{f_1} - \frac{G_2}{f_2}}} \qquad [3.34]$$

$$x_{rms}^2 = \frac{1}{16\,\pi^4}\int_{f_1}^{f_2}\frac{a\, f + b}{f^4}\, df$$

$$\boxed{x_{rms} = \frac{1}{4\,\pi^2\, f_1\, f_2}\sqrt{\frac{G_2 - G_1}{2}(f_1 + f_2) + \frac{1}{3}\left(\frac{G_1}{f_1} - \frac{G_2}{f_2}\right)\left(f_1^2 + f_1\, f_2 + f_2^2\right)}} \qquad [3.35]$$

Linear-logarithmic scales

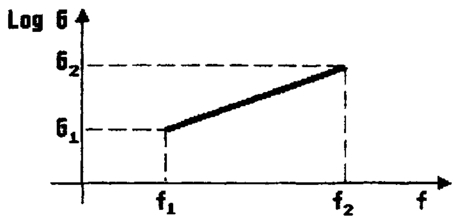

Figure 3.6. *Segment of straight line in lin-log scales*

The PSD can be expressed analytically in the form:

$$\ln G = a\,f + b$$

where $a = \dfrac{\ln G_2 - \ln G_1}{f_2 - f_1}$ and $b = \dfrac{f_1 \ln G_2 - f_2 \ln G_1}{f_1 - f_2}$.

$$\ddot{x}_{rms}^2 = \int_{f_1}^{f_2} e^{a\,f+b}\,df$$

$$\boxed{\ddot{x}_{rms} = \sqrt{\frac{e^b}{a}\left(e^{a\,f_2} - e^{a\,f_1}\right)}} \qquad [3.36]$$

(if $a \neq 0$, i.e. if $G_2 \neq G_1$)

$$\boxed{v_{rms}^2 = \frac{1}{4\,\pi^2}\int_{f_1}^{f_2} \frac{e^{a\,f}}{f^2}\,df + \frac{e^b}{4\,\pi^2}\int_{f_1}^{f_2}\frac{df}{f^2}} \qquad [3.37]$$

Knowing that $\displaystyle\int \frac{e^{a\,f}}{f^2}\,df = -\frac{e^{a\,f}}{f} + a\int\frac{e^{a\,f}}{f}\,df$, this integral can be calculated by a development in series (Appendix A4.2):

$$\int \frac{e^{a\,f}}{f}\,df = \ln|f| + \frac{a\,f}{1!} + \frac{a^2}{2}\,\frac{f^2}{2!} + \cdots + \frac{a^n}{n}\,\frac{f^n}{n!} + \cdots$$

In the same way,

$$\boxed{x_{rms}^2 = \frac{e^b}{16\,\pi^4} \int_{f_1}^{f_2} \frac{e^{a\,f}}{f^4}\, df} \qquad [3.38]$$

The integral is calculated like above, from (Appendix A4.2):

$$\frac{e^{a\,f}}{f^4}\, df = -\frac{e^{a\,f}}{3\,f^3} + \frac{a}{3} \int \frac{e^{a\,f}}{f^3}\, df$$

$$\int \frac{e^{a\,f}}{f^4}\, df = -\frac{e^{a\,f}}{3\,f^3} - \frac{a}{6}\frac{e^{a\,f}}{f^2} - \frac{a^2}{6}\frac{e^{a\,f}}{f} + \frac{a^3}{6} \int \frac{e^{a\,f}}{f}\, df$$

Particular case where $G_2 = G_1$

In this case,

$$\ddot{x}_{rms} = \sqrt{e^b\,(f_2 - f_1)} = \sqrt{G_1\,(f_2 - f_1)} \qquad [3.39]$$

$$v_{rms} = \frac{1}{2\,\pi} \sqrt{G_1 \left(\frac{1}{f_1} - \frac{1}{f_2}\right)} \qquad [3.40]$$

$$x_{rms} = \frac{1}{4\,\pi^2} \sqrt{\frac{G_1}{3} \left(\frac{1}{f_1^3} - \frac{1}{f_2^3}\right)} \qquad [3.41]$$

Logarithmic-linear scales

In these scales, the segment of straight line has as an analytical expression:

$$G = a \ln f + b$$

with $a = \dfrac{G_2 - G_1}{\ln f_2 - \ln f_1}$ and $b = \dfrac{G_2 \ln f_1 - G_1 \ln f_2}{\ln f_1 - \ln f_2}$.

$$\ddot{x}_{rms}^2 = a\,(f_2 \ln f_2 - f_1 \ln f_1) + (f_2 - f_1)(b + a) \qquad [3.42]$$

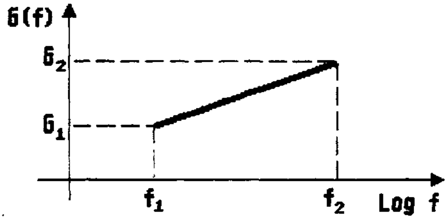

Figure 3.7. *Segment of straight line in log-lin scales*

$$v_{rms}^2 = \frac{a}{4\pi^2}\left(\frac{\ln f_1}{f_1} - \frac{\ln f_2}{f_2}\right) + \left(\frac{1}{f_1} - \frac{1}{f_2}\right)\left(\frac{a+b}{4\pi^2}\right) \qquad [3.43]$$

$$x_{rms}^2 = \frac{a}{48\pi^4}\left(\frac{\ln f_1}{f_1^3} - \frac{\ln f_2}{f_2^3}\right) - \frac{a+3b}{144\pi^4}\left(\frac{1}{f_2^3} - \frac{1}{f_1^3}\right) \qquad [3.44]$$

Logarithmic-logarithmic scales

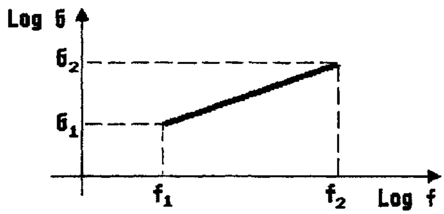

Figure 3.8. *Segment of straight line in logarithmic scales*

The PSD is such that:

$$\ln G(f) = \ln G_1 + b\left(\ln f - \ln f_1\right)$$

whence

$$G(f) = G_1\left(\frac{f}{f_1}\right)^b$$

The constant b is calculated from the co-ordinates of the point f_2, G_2:

$$b = \frac{\ln \frac{G_2}{G_1}}{\ln \frac{f_2}{f_1}}$$

Rms acceleration [PRA 70]:

$$\ddot{x}_{rms}^2 = \int_{f_1}^{f_2} G_1 \left(\frac{f}{f_1} \right)^b df$$

If $b \neq -1$:

$$\ddot{x}_{rms} = \sqrt{\frac{f_2 \, G_2 - f_1 \, G_1}{b+1}} \qquad [3.45]$$

If $b = -1$:

$$\ddot{x}_{rms}^2 = f_1 \, G_1 \int_{f_1}^{f_2} \frac{df}{f}$$

$$\ddot{x}_{rms} = \sqrt{f_1 \, G_1 \ln \frac{f_2}{f_1}} \qquad [3.46]$$

$$v_{rms}^2 = \frac{G_1}{4 \pi^2 f_1^b} \int_{f_1}^{f_2} f^{b-2} \, df$$

If $b \neq -1$:

$$v_{rms} = \frac{1}{2\pi} \sqrt{\frac{G_1}{(b-1) f_1} \left[\left(\frac{f_2}{f_1} \right)^{b-1} - 1 \right]} = \frac{1}{2\pi} \sqrt{\frac{1}{b-1} \left(\frac{G_2}{f_2} - \frac{G_1}{f_1} \right)} \qquad [3.47]$$

If $b = 1$:

The parameter b can be equal to 1 only if $\frac{G_2}{G_1} = \frac{f_2}{f_1}$, the commonplace case $G_1 = G_2$ and $f_1 = f_2$ being excluded. On this assumption, $G = G_1 \frac{f}{f_1}$ and

$$v_{rms} = \frac{1}{2\pi}\sqrt{\frac{G_1}{f_1}\ln\frac{f_2}{f_1}} \qquad [3.48]$$

$$x_{rms}^2 = \int_{f_1}^{f_2} \frac{1}{16\pi^4 f^4}\frac{G_1}{f_1^b} f^b \, df$$

If $b \neq 3$:

$$x_{rms} = \frac{1}{4\pi^2}\sqrt{\frac{1}{b-3}\frac{G_1}{f_1^3}\left[\left(\frac{f_2}{f_1}\right)^{b-3} - 1\right]} = \frac{1}{4\pi^2}\sqrt{\frac{1}{b-3}\left(\frac{G_2}{f_2^3} - \frac{G_1}{f_1^3}\right)} \qquad [3.49]$$

If $b = 3$:

$$x_{rms} = \frac{1}{4\pi^2}\sqrt{\frac{G_1}{f_1^3}\ln\frac{f_2}{f_1}} \qquad [3.50]$$

In logarithmic scales, a straight line segment is sometimes defined by three of the four values corresponding to the co-ordinates of the first and the last point, supplemented by the slope of the segment. The slope R, expressed in dB/octave, can be calculated as follows:

– the number N of dB is given by

$$N = 10 \log_{10} \frac{G_2}{G_1} \qquad [3.51]$$

– the number of octaves n between f_1 and f_2 is, by definition, such as $\frac{f_2}{f_1} = 2^n$, yielding:

$$n = \frac{\log_{10} \frac{f_2}{f_1}}{\log_{10} 2}$$

and

$$R = 10 \log_{10}(2) \frac{\log_{10} G_2/G_1}{\log_{10} f_2/f_1} \qquad [3.52]$$

$$R = 10 \log_{10}(2) b \approx 3.01\, b \qquad [3.53]$$

Let us set $\alpha = 10 \log_{10}(2)$. It becomes, by replacing b by $\frac{R}{\alpha}$ in the preceding expressions [CUR 71]:

$$\ddot{x}_{rms}^2 = \frac{\alpha}{R+\alpha}\left(f_2\, G_2 - f_1\, G_1\right)$$

If $R \neq -\alpha$:

$$\ddot{x}_{rms}^2 = \frac{\alpha\, f_1\, G_1}{R+\alpha}\left[\left(\frac{f_2}{f_1}\right)^{\frac{R}{\alpha}+1} - 1\right] \qquad [3.54]$$

This can be also written [SAN 66]:

$$\ddot{x}_{rms}^2 = \frac{f_2\, G_2}{\frac{R}{\alpha}+1}\left(1 - \frac{f_1\, G_1}{f_2\, G_2}\right) = \frac{f_2\, G_2}{\frac{R}{\alpha}+1}\left[1 - \left(\frac{f_1}{f_2}\right)^{1+\frac{R}{\alpha}}\right] \qquad [3.55]$$

or [OSG 82]:

$$\ddot{x}_{rms}^2 = \frac{\alpha\, G_2}{\alpha + R}\left[f_2 - f_1\left(\frac{f_1}{f_2}\right)^{\frac{R}{\alpha}}\right] \qquad [3.56]$$

Reference [SAN 66] gives this expression for an increasing slope and, for a decreasing slope,

$$\ddot{x}_{rms}^2 = \frac{f_1\, G_1}{\frac{R}{\alpha}-1}\left[1 - \left(\frac{f_2}{f_1}\right)^{1-\frac{R}{\alpha}}\right] \qquad [3.57]$$

if $R \neq -\alpha$, or

$$\ddot{x}_{rms}^2 = \frac{\alpha\, f_2\, G_2}{R+\alpha}\left[1 - \left(\frac{f_1}{f_2}\right)^{\frac{R}{\alpha}+1}\right] \qquad [3.58]$$

For $R = -\alpha$:

$$\ddot{x}_{rms}^2 = f_1\, G_1 \ln\frac{f_2}{f_1} = f_2\, G_2 \ln\frac{f_2}{f_1} \qquad [3.59]$$

If $R \neq \alpha$:

$$v_{rms}^2 = \frac{\alpha}{4\,\pi^2\,(R-\alpha)}\frac{G_1}{f_1}\left[\left(\frac{f_2}{f_1}\right)^{\frac{R-\alpha}{\alpha}} - 1\right] = \frac{\alpha}{4\,\pi^2\,(R-\alpha)}\frac{G_2}{f_2}\left[1-\left(\frac{f_1}{f_2}\right)^{\frac{R-\alpha}{\alpha}}\right] \qquad [3.60]$$

For $R = \alpha$:

$$v_{rms}^2 = \frac{G_1}{4\,\pi^2\, f_1}\ln\frac{f_2}{f_1} = \frac{G_2}{4\,\pi^2\, f_2}\ln\frac{f_2}{f_1} \qquad [3.61]$$

If $R \neq 3\alpha$:

$$x_{rms}^2 = \frac{\alpha\, G_1}{16\,\pi^4\, f_1^3\,(R-3\alpha)}\left[\left(\frac{f_2}{f_1}\right)^{\frac{R-3\alpha}{\alpha}} - 1\right] = \frac{\alpha\, G_2}{16\,\pi^4\, f_2^3\,(R-3\alpha)}\left[1-\left(\frac{f_2}{f_1}\right)^{\frac{R-3\alpha}{\alpha}}\right] \qquad [3.62]$$

For $R = 3\alpha$:

$$x_{rms}^2 = \frac{1}{16\,\pi^4}\frac{G_1}{f_1^3}\ln\frac{f_2}{f_1} = \frac{1}{16\,\pi^4}\frac{G_2}{f_2^3}\ln\frac{f_2}{f_1} \qquad [3.63]$$

Figures 3.9, 3.10 and 3.11 respectively show $\frac{\ddot{x}_{rms}^2}{f_1\, G_1}$, $\frac{f_1\, v_{rms}^2}{G_1}$ and $\frac{f_1^3\, x_{rms}^2}{G_1}$ versus $\frac{f_2}{f_1}$, for different values of R.

Abacuses of this type can be used to calculate the rms value of $\ddot{x}$, v or x from a spectrum made up of straight line segments on logarithmic scales [HIM 64].

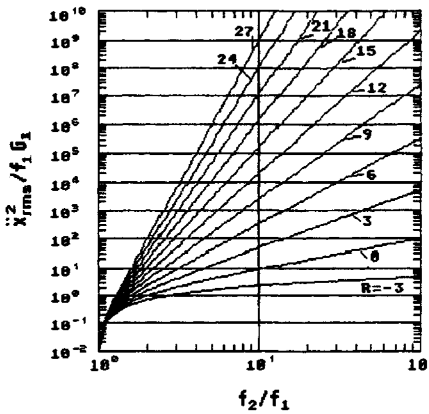

Figure 3.9. *Reduced rms acceleration versus* f_2/f_1 *and* R

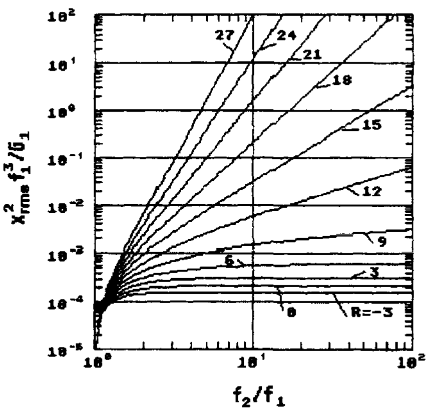

Figure 3.10. *Reduced rms velocity versus* f_2/f_1 *and* R

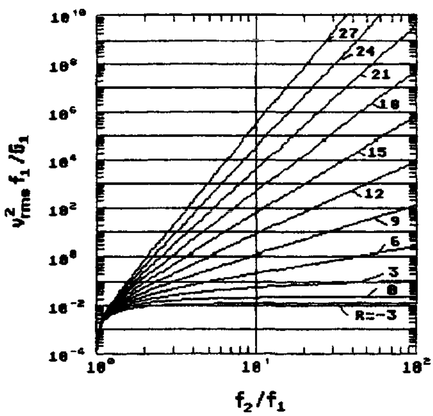

Figure 3.11. *Reduced rms Displacement versus* f_2/f_1 *and* R

3.4.5. *PSD comprising several segments of arbitrary slopes*

Whatever the scales chosen, the rms value of a PSD made up of several straight line segments of arbitrary slope will be such as in [OSG 82] [SAN 63]:

$$\ddot{x}_{rms} = \sqrt{\sum_i \ddot{x}_{i\ rms}^2} \qquad [3.64]$$

$x_{i\ rms}^2$ being calculated starting from the relations above. In the same way:

$$v_{rms} = \sqrt{\sum_i v_{i\ rms}^2} \qquad [3.65]$$

and

$$x_{rms} = \sqrt{\sum_i x_{i\ rms}^2} \qquad [3.66]$$

3.5. Case: periodic signals

It is known that any periodic signal can be represented by a Fourier series in accordance with:

$$\ell(t) = L_0 + \sum_{i=1}^{n} L_n \sin\left(2\ \pi\ n\ f_1\ t + \phi_n\right) \qquad [3.67]$$

Power spectral density [2.26]:

$$G(f) = \lim_{\substack{T \to \infty \\ \Delta f \to 0}} \frac{1}{T\ \Delta f} \int_0^T \ell_T^2(t, \Delta f)\ dt$$

is zero for $f \neq f_n$ (with $f_n = n\ f_1$) and infinite for $f = f_n$ since the spectrum of $\ell(t)$ is a discrete spectrum, in which each component L_n has zero width Δf.

If one wishes to standardize the representations and to be able to define the PSD of a periodic function, so that the integral $\int_0^\infty G(f)\ df$ is equal to the mean square value of $\ell(t)$, one must consider that each component is related to Dirac delta function, the area under the curve of this function being equal to the mean square value of the component. With this definition,

$$G(f) = \sum_{n=0}^{\infty} \ell_{rms_n}^2\ \delta(f - nf_1) \qquad [3.68]$$

where ℓ^2_{rmsn} is the mean square value of the n[th] harmonic $\ell_n(t)$ defined by

$$\ell^2_{rmsn} = \frac{1}{T_n} \int_0^{T_n} \ell_n^2(t)\, dt \qquad [3.69]$$

$$T_n = \frac{1}{n\, f_1} \qquad [3.70]$$

(n = 1, 2, 3, ...). $\ell_n(t)$ is the value of the n[th] component and

$$\ell^2_{rms0} = \frac{1}{T} \int_0^{T} \ell_0^2(t)\, dt = \left(\bar{\ell}\right)^2$$

where T is arbitrary and $\bar{\ell}$ is the mean value of the signal $\ell(t)$. The Dirac delta function $\delta\left(f - n\ f_1\right)$ at the frequency f_n is such that:

$$\int_{f_n - \varepsilon}^{f_n + \varepsilon} \delta\left(f - f_n\right)\, df = 1 \qquad [3.71]$$

and

$$\delta\left(f - f_n\right) = 0 \qquad [3.72]$$

for $f \neq f_n$ ($\varepsilon =$ positive constant different from zero, arbitrarily small). The definition of the PSD in this particular case of a periodic signal does not require taking the limit for infinite T, since the mean square value of a periodic signal can be calculated over only one period or a whole number of periods.

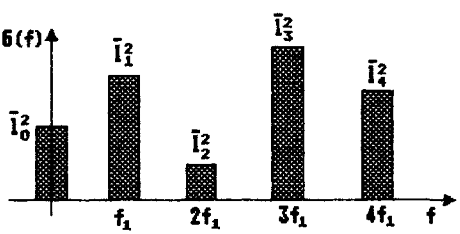

Figure 3.12. *PSD of a periodic signal*

The chart of the PSD of a periodic signal is that of a discrete spectrum, the amplitude of each component being proportional to the area representing its mean square value (and not its amplitude).

We have, with the preceding notations, relationships of the same form as those obtained for a random signal:

$$\int_{0^-}^{\infty} G(f)\, df = \sum_{n=0}^{\infty} \overline{\ell_n^2} = \overline{\ell^2} = s_\ell^2 + \left(\bar{\ell}\right)^2 \qquad [3.73]$$

$$\int_{0^+}^{\infty} G(f)\, df = \sum_{n=1}^{\infty} \overline{\ell_n^2} = s_\ell^2 \qquad [3.74]$$

$$\int_{0^-}^{0^+} G(f)\, df = \left(\bar{\ell}\right)^2 \qquad [3.75]$$

and, between two frequencies f_i and f_j ($f_i = i\, f_1 - \varepsilon$, $f_j = j\, f_1 + \varepsilon$, i and j integers, $j > i$):

$$\int_{i\, f_1 - \varepsilon}^{j\, f_1 + \varepsilon} G(f)\, df = \sum_{n=i}^{j} \overline{\ell_n^2} = \overline{\ell^2} \qquad [3.76]$$

Lastly, if for a random signal, we had:

$$\int_{f^-}^{f^+} G(f)\, df = 0 \qquad [3.77]$$

we have here:

$$\int_{f^-}^{f^+} G(f)\, df = \begin{cases} \overline{\ell_n^2} & \text{for } f = n\, f_1 \\ 0 & \text{for } f \neq n\, f_1 \text{ et } f \neq 0 \end{cases} \qquad [3.78]$$

The area under the PSD at a given frequency is either zero, or equal to the mean square value of the component if $f = n\, f_1$ (whereas, for a random signal, this area is always zero).

3.6. Case: periodic signal superimposed onto random noise

Let us suppose that:

$$\ell(t) = a(t) + p(t) \qquad [3.79]$$

$a(t)$ = random signal, of PSD $G_a(f)$ defined in [2.26]

$p(t)$ = periodic signal, of PSD $G_p(f)$ defined in the preceding paragraph.

The PSD of $\ell(t)$ is equal to:

$$G_\ell(f) = G_a(f) + G_p(f) \qquad [3.80]$$

$$G_\ell(f) = G_a(f) + \sum_{n=0}^{\infty} \overline{\ell_n^2}\ \delta(f - f_n) \qquad [3.81]$$

where

$f_n = n\ f_1$

n = integer $\in (0, \infty)$

f_1 = fundamental frequency of the periodic signal

$\overline{\ell_n^2}$ = mean square value of the n^{th} component $\ell_n(t)$ of $\ell(t)$

The rms value of this composite signal is, as previously, equal to the square root of the area under $G_\ell(f)$.

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Chapter 4

Practical calculation of power spectral density

Analysis of random vibration is carried out most of the time by supposing that it is stationary and ergodic. This assumption makes it possible to replace a study based on the statistical properties of a great number of signals by that of only one sample of finite duration T. Several approaches are possible for the calculation of the PSD of such a sample.

4.1. Sampling of the signal

Sampling consists in transforming a vibratory signal continuous at the outset by a succession of sample points regularly distributed in time. If δt is the time interval separating two successive points, the sampling frequency is equal to $f_{samp.} = 1/\delta t$. So that the digitized signal is correctly represented, it is necessary that the sampling frequency is sufficiently high compared to the largest frequency of the signal to be analysed.

A too low sampling frequency can thus lead to an aliasing phenomenon, characterized by the appearance of frequency components having no physical reality.

Example

Figure 4.1 thus shows a component of frequency 70 Hz artificially created by the sampling of 200 points/s of a sinusoidal signal of frequency 350 Hz.

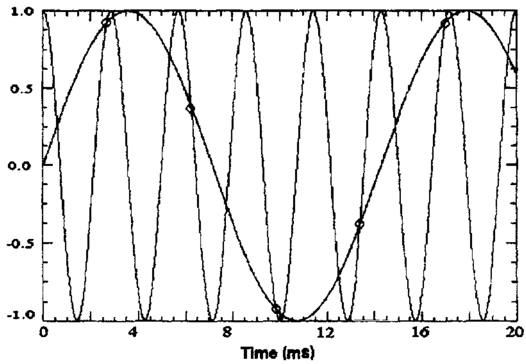

Figure 4.1. *Highlighting of the aliasing phenomenon due to under-sampling*

Shannon's theorem indicates that *if a function contains no frequencies higher than* f_{max} *Hz, it is completely determined by its ordinates at a series of points spaced* $1/2\,f_{max}$ *seconds apart* [SHA 49].

Given a signal which one wishes to analyse up to the frequency f_{max}, it is thus appropriate, to avoid aliasing

– to filter it using a low-pass filter in order to eliminate frequencies higher than f_{max} (the high frequency part of the spectrum which can have a physical reality or noise),

– to sample it with a frequency at least equal to $2\,f_{max}$ [CUR 87] [GIL 88] [PRE 90] [ROT 70].

NOTE.
$f_{Nyquist} = f_{samp.}/2$ *is called Nyquist frequency.*

In practice however, the low-pass filters are imperfect and filter incompletely the frequencies higher than the wanted value. Let us consider a low-pass filter having a decrease of 120 dB per octave beyond the desired cut-out frequency (f_{max}). It is considered that the signal is sufficiently attenuated at – 40 dB. It thus should be considered that the true contents of the filtered signal extend to the frequency corresponding to this attenuation (f_{-40}), calculated as follows.

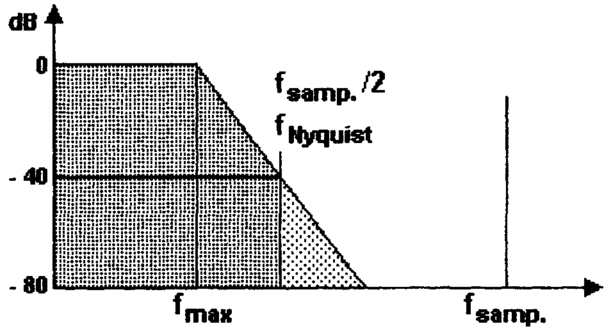

Figure 4.2. *Taking account of the real characteristics of the low-pass filter for the determination of the sampling frequency*

An attenuation of 120 dB per octave means that

$$-120 = \frac{10 \log \frac{A_1}{A_0}}{\log \frac{f_{-40}}{f_{max}}} \log 2$$

where A_0 and A_1 are respectively the amplitudes of the signal not attenuated (frequency f_{max}) and attenuated at –40 dB (frequency f_{-40}).

Yielding

$$-120 = \frac{10 \frac{-40}{10}}{\log \frac{f_{-40}}{f_{max}}} \log 2$$

and

$$\frac{f_{-40}}{f_{max}} = 10^{\frac{\log 2}{3}} \approx 1.26$$

The frequency f_{-40} being the greatest frequency of the signal requires that, according to Shannon's theorem, $f_{samp.} = 2\, f_{-40}$, yielding

$$\frac{f_{samp.}}{f_{max}} \approx 2.52$$

The frequency f_{-40} is also the Nyquist frequency $f_{Nyquist}$.

This result often resulted in the belief that Shannon's theorem imposes a sampling frequency at least equal to 2.6 times the highest frequency of the signal to be analysed.

We will use this value in the followings paragraphs.

4.2. Calculation of PSD from rms value of filtered signal

The theoretical relation [2.26] which would assume one infinite duration T and a zero analysis bandwidth Δf is replaced by the approximate relation [KEL 67]:

$$G(f) = \frac{1}{T\, \Delta f} \int_0^T \ell_{\Delta f}^2(f, t)\, dt = \frac{\overline{\ell_{\Delta f}^2}}{\Delta f} \qquad [4.1]$$

where $\overline{\ell_{\Delta f}^2}$ is the mean square value of the sample of finite duration T, calculated at the output of a filter of central frequency f and non zero width Δf [MOR 56].

NOTE.

Given a random vibration $\ell(t)$ *of white noise type and a perfect rectangular filter, the result of filtering is a signal having a constant spectrum over the width of the filter, zero elsewhere [CUR 64].*

The result can be obtained by multiplying the PSD G_0 *of the input* $\ell(t)$ *by the square of the transmission characteristic of the filter (frequency-response characteristic) at each frequency (transfer function, defined as the ratio of the amplitude of the filter response to the amplitude of the sinewave excitation as a function of the frequency. If this ratio is independent of the excitation amplitude, the filter is said to be linear).*

In practice, the filters are not perfectly rectangular. The mean square value of the response is equal to G_0 *multiplied by the area squared under the transfer function of the filter. This surface is defined as the 'rms bandwidth of the filter'.*

If the PSD of the signal to be analysed varies with the frequency, the mean square response of a perfect filter divided by the width Δf *of the filter gives a point on the PSD (mean value of the PSD over the width of the filter). With a real filter, this approximate value of the PSD is obtained by considering the ratio of the mean square value of the response to the rms bandwidth of the filter* Δf, *defined by [BEN 62], [GOL 53] and [PIE 64]:*

$$\Delta f = \int_0^\infty \left| \frac{H(f)}{H_{max}} \right|^2 dt \qquad [4.2]$$

where $H(f)$ *is the frequency response function of the (narrow) band-pass filter used and* H_{max} *its maximum value.*

4.3. Calculation of PSD starting from Fourier transform

The most used method consists in considering expression [2.39]:

$$G_{\ell\ell}(f) = \lim_{T \to \infty} \frac{2}{T} E\left[\left| L(f, T) \right|^2 \right] \qquad [4.3]$$

NOTES.

1. *Knowing that the discrete Fourier transform can be written [KAY 81]*

$$L(m, T) = \frac{T}{N} \sum_{j=0}^{N-1} \ell_j \exp\left(-i \frac{2 \pi j m}{N} \right) \qquad [4.4]$$

the expression of the PSD can be expressed for calculation in the form [BEN 71] [ROT 70]:

$$G(m\, \Delta f) = \frac{2}{N} \left| \sum_{j=0}^{N-1} \ell_j \exp\left(-i \frac{2 \pi j m}{N} \right) \right|^2 \qquad [4.5]$$

where $0 < m \le M$ *and* $\ell_j = j\, \delta t$.

2. *The calculation of the PSD can also be carried out by using relation [2.26], by evaluating the correlation in the time domain and by carrying out a Fourier transformation (Wiener-Khintchine method) (correlation analysers) [MAX 86].*

The calculation data are in general the following:

– the maximum frequency of the spectrum,

– the number of points of the PSD (or the frequency step Δf),

– the maximum statistical error tolerated.

4.3.1. *Maximum frequency*

Given an already sampled signal (frequency $f_{samp.}$) and taking into account the elements of paragraph 4.1, the PSD will be correct only for frequencies lower than $f_{max} = f_{samp.} / 2.6$.

4.3.2. *Extraction of sample of duration T*

Two approaches are possible for the calculation of the PSD:

– to suppose that the signal is periodic and composed of the repetition of the sample of duration T,

– to suppose that the signal has zero values at all the points outside the time time corresponding to the sample.

These two approaches are equivalent [BEN 75]. In both cases, one is led to isolate by truncation a part of the signal, which amounts to applying to it a rectangular temporal window $r(t)$ of amplitude 1 for $0 \le t \le T$ and zero elsewhere.

If $\ell(t)$ is the signal to be analysed, the Fourier transform is thus calculated in practice with $f(t) = \ell(t)\ r(t)$.

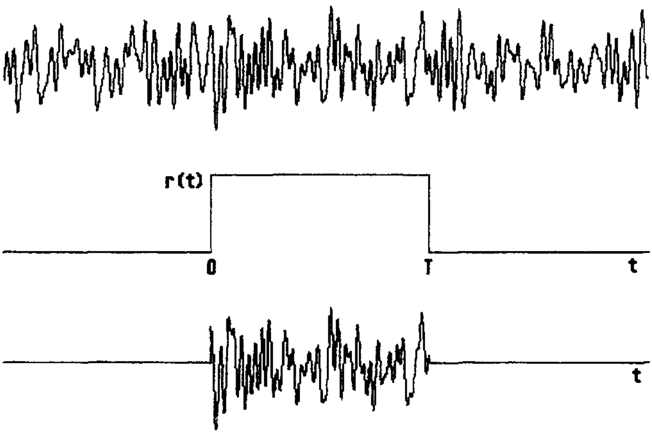

Figure 4.3. *Application of a temporal window*

In the frequency domain, the transform of a product is equal to the convolution of the Fourier transforms $L(\Omega)$ and $R(\Omega)$ of each term:

$$F(\Omega) = \int_0^{\Omega} L(\omega)\; R(\Omega - \omega)\; d\omega \qquad [4.6]$$

(ω is a variable of integration).

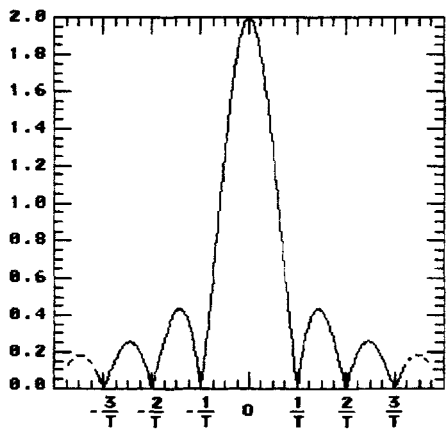

Figure 4.4. *Fourier transform of a rectangular waveform*

The Fourier transform of a rectangular temporal window appears as a principal central lobe surrounded by small lobes of decreasing amplitude (cf. Volume 2, Chapter 1). The transform cancels out regularly for Ω a multiple of $\frac{2\,\pi}{T}$ (i.e. a frequency f multiple of $\frac{1}{T}$). The effect of the convolution is to widen the peaks of the spectrum, the resolution, consequence of the width of the central lobe, not being able to better $\Delta f = \frac{1}{T}$.

The expression [4.6] shows that, for each point of the spectrum of frequency Ω (multiple of $\frac{2\,\pi}{T}$), the side lobes have a parasitic influence on the calculated value of the transform (*leakage*). To reduce this influence and to improve the precision of calculation, their amplitude needs to be reduced.

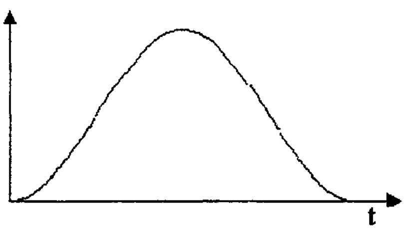

Figure 4.5. *The Hanning window*

This result can be obtained by considering a modified window which removes discontinuities of the beginning and end of the rectangular window in the time domain.

Many shapes of temporal windows are used [BLA 91] [DAS 89] [JEN 68] [NUT 81].

One of best known and the most used is the Hanning window, which is represented by a versed sine function (Figure 4.5):

$$r(t) = \frac{1}{2}\left(1 - \cos\frac{2\,\pi\,t}{T}\right) \qquad [4.7]$$

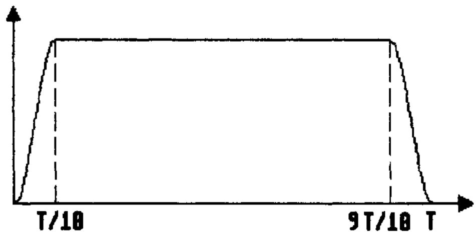

Figure 4.6. *The Bingham window [BIN 67]*

This shape is only sometimes used to constitute the rising and decaying parts of the window (Bingham window, Figure 4.6).

Weighting coefficient of the window is the term given to the percentage of rise time (equal to the decay time) of the total length T of the window. This ratio cannot naturally exceed 0.5, corresponding to the case of the previously defined Hanning window.

Examples of windows

The advantages of the various have been discussed in the literature [BIN 67] [NUT 81]. These advantages are related to the nature of the signal to be analysed. Actually, the most important point in the analysis is not the type of window, but rather the choice of the bandwidth [JEN 68]. The Hanning window is nevertheless recommended.

The replacement of the rectangular window by a more smoothed shape modifies the signal actually treated through attenuation of its ends, which results in a reduction of the rms duration of the sample and in consequence in a reduction of the resolution, depending on the width of the central lobe.

One should not forget to correct the result of the calculation of the PSD to compensate for the difference in area related to the shape of the new window. Given a temporal window defined by $r(t)$, having $R(f)$ for Fourier transform, the area intervening in the calculation of the PSD is equal to:

$$Q = \int_{-\infty}^{+\infty} \left|R(f)\right|^2 \, df \qquad [4.8]$$

From Parseval's theorem, this expression can be written in a form utilizing the N points of the digitized signal:

$$\boxed{Q = \frac{1}{N} \sum_{i=0}^{N-1} r_i^2} \qquad [4.9]$$

Table 4.1. *The principal windows*

Window type	Definition	Compensation factor
Bingham (Figure 4.6)	$r(t) = \frac{1}{2}\left\{1 + \cos\left[\frac{10\,\pi\left(t - 9\,T/10\right)}{T}\right]\right\}$ for $0 \le t \le \frac{T}{10}$ and $\frac{9\,T}{10} \le t \le T$	1/0.875
Hamming	$r(t) = 0.08 + 0.46\left(1 - \cos\frac{2\,\pi\,t}{T}\right)$ for $0 \le t \le T$	1/0.3974
Hanning	$r(t) = \frac{1}{2}\left[1 - \cos\left(\frac{2\,\pi\,t}{T}\right)\right]$ for $0 \le t \le T$	1/0.375
Parzen	$r(t) = 1 - 6\left(\frac{2\,t}{T} - 1\right)^2 + 6\left\lvert\frac{2\,t}{T} - 1\right\rvert^3$ for $\frac{T}{4} \le t \le \frac{3\,T}{4}$ $r(t) = 2\left[1 - \left\lvert\frac{2\,t}{T} - 1\right\rvert\right]^3$ for $0 \le t \le \frac{T}{4}$ and $\frac{3\,T}{4} \le t \le T$	1/0.269643
Flat top	$r(t) = 1 - 1.933\cos\left(\frac{2\,\pi}{T}t\right) + 1.286\cos\left(2\frac{2\,\pi}{T}t\right) - 0.388\cos\left(3\frac{2\,\pi}{T}t\right) + 0.032\cos\left(4\frac{2\,\pi}{T}t\right)$ for $0 \le t \le T$	1/3.7709265
Kaiser-Bessel	$r(t) = 1 - 1.24\cos\left(\frac{2\,\pi}{T}t\right) + 0.244\cos\left(2\frac{2\,\pi}{T}t\right) - 0.00305\cos\left(3\frac{2\,\pi}{T}t\right)$ for $0 \le t \le T$	1/1.798573

The multiplicative compensation factor to apply in order to take account of the difference between this area and the unit area of a rectangular window are thus equal to 1/Q [DUR 72].

4.3.3. *Averaging*

We attempt in the calculations to obtain the best possible resolution with the data at our disposal, which results in trying to plot the PSD with the smallest possible frequency step. For a sample of duration T, this step cannot be lower than 1/T. With this resolution, the precision obtained is unacceptable. Several solutions are possible:

– to carry out several measurements of the phenomenon, to calculate the PSD of each sample of duration T and to proceed to an average of the obtained spectra,

– if only one sample of duration T is available, to voluntarily limit the resolution by accepting an analysis step Δf larger than 1/T and to carry out an averaging [BEN 71]:

- either by calculating the average of several frequential components close to the considered spectrum component, separated by intervals 1/T, when the noise to be analysed can be comparable to a white noise. If the average is carried out on K PSD, the average obtained is assigned to the central frequency of an interval of width equal to K/T (which characterizes the effective resolution of the PSD thus calculated),

- or by dividing up the initial sample of duration T into K subsamples (or blocks) of duration $\Delta T = T/K$ which will be used to calculate K spectra of resolution $1/\Delta T$ and their average [BAR 55] [MAX 81]:

$$\frac{1}{K}\sum_i G_i(f)$$

The results of these two approaches are identical for given duration T and given resolution [BEN 75]. It is the last procedure which is the most used. The window, rectangular or not, is applied to each block.

4.3.4. *Addition of zeros*

The smallest interval Δf between two points of the PSD is related to the duration of the block considered by at least $\Delta f = \frac{1}{\Delta T}$. The calculation of the PSD is carried out at M points with distances of Δf between 0 and $f_{samp.}/2$ ($f_{samp.}$ = sampling frequency of the signal). As long as this condition is observed, it is said that the components of the spectrum are *statistically independent*.

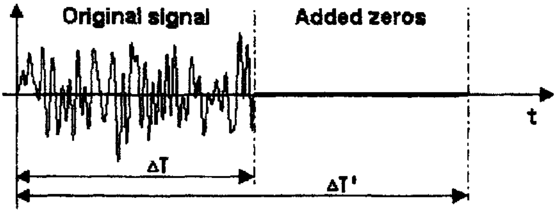

Figure 4.7. *Addition of zeros at end of the signal sample*

One can however add components to the spectrum to obtain a more smoothed curve by artificially increasing the number of points using zeros placed at the end of the block (leading to a new duration ΔT' > ΔT).

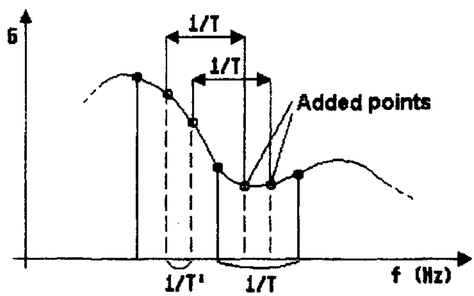

Figure 4.8. *The addition of zeros increases the number of points of the PSD*

Although the components added are no longer statistically independent, the validity of each individual component remains whole.

The additional points of the PSD thus obtained lie between the original points corresponding to the duration ΔT and are on the continuous theoretical curve.

The resolution is unchanged. All the components have an equal validity in the analysis [ENO 69]. One should attach no importance particularly to the components spaced out at $\dfrac{1}{\Delta T}$, except that they constitute an ensemble of independent components. An equivalent unit could be selected by considering the points at the frequencies $(1+\delta f)/T$, $(2+\delta f)/T$ etc where δf is an increment ranging between 0 and 1 [BEN 75].

4.4. FFT

J.W. Cooley and J. Tukey [COO 65] developed in 1965 a method named *Fast Fourier Transform* or *FFT* making it possible to reduce considerably the computing time of the Fourier transforms.

A FFT analyser functions with a number of points which is [MAX 86]:

– a power of 2 for the Cooley-Tukey algorithms and those which derive from them,

– a product of integral powers of prime numbers (Vinograd's algorithm).

With the Cooley-Tukey's algorithm, the computing time of the transform of a signal defined by N points is proportional to $N \log_2 N$ instead of the theoretically necessary value N^2.

Table 4.2. *Speed ratio for FFT computation*

Number of Points	Number of points of the Fourier transform	Speed ratio $\dfrac{N}{\log_2 N}$
256	128	32
512	256	56.9
1024	512	102.4
2048	1024	186.2
4096	2048	341.3

Calculations of PSD are done today primarily using the FFT, which also has applications for the calculation of coherence functions (square of the amplitude) [CAR 73] and of convolutions. This algorithm, which is based in practice on the discrete Fourier transform, leads to a frequency sampling of the Fourier transform and thus of the PSD.

NOTES.

1. *Whilst in theory equivalent, the FFT and the method using the correlation can lead in practice to different results, which can be explained by the non cognisance of the theoretical assumptions due to the difficulties of producing the analysers [MAX 86]. J. Max, M. Diot and R. Bigret showed that a correlation analyser presents a certain number of advantages such as:*

– a greater flexibility in the choice of the frequency sampling step, facilitating the analysis of the periodic signals,

– a choice more adapted to the conditions of analysis of the signal.

2. *When these algorithms are used to calculate the Fourier transform of a shock, one should not forget to multiply the result by the duration* T *of the treated signal.*

4.5. Particular case of a periodic excitation

The PSD of a periodic excitation was defined by [3.68]:

$$G'(f) = \sum_{n=0}^{\infty} \overline{\ell_n^2}\, \delta(f - f_n) \qquad [4.10]$$

The PSD of such an excitation being characterized by very narrow bands centered on the frequencies f_n, the calculation of $G'(f)$ supposes that $\ell(t)$ is analysed in sufficiently narrow filters Δf. The PSD is approximated by:

$$G'(f) = \sum_{n=0}^{\infty} \overline{\ell_n^2}\, \delta\left(f - f_n \pm \frac{\Delta f}{2}\right) \qquad [4.11]$$

$\overline{\ell_n^2}$ can be obtained either by direct calculation of:

$$\overline{\ell_n^2} = \frac{1}{T} \int_0^T L_n^2 \sin^2 2\pi f_n \quad dt = \frac{1}{2} L_n^2 \qquad [4.12]$$

with $T = \dfrac{1}{f_n}$ or $T = \dfrac{k}{f_n}$, i.e. by calculation of the mean value:

$$\overline{\ell_n} = \frac{1}{T} \int_0^T L_n \sin 2\pi f_n \quad dt = \frac{2}{\pi} L_n \qquad [4.13]$$

T having the same definition. It is noted that:

$$\overline{\ell_n^2} = \frac{\pi^2}{8} \left(\bar{\ell}\right)^2 \qquad [4.14]$$

T must be multiple of $\frac{1}{f_n}$. If it is not the case, the error is all the weaker since the number of selected periods is larger. For a periodic excitation, the measurment or calculation accuracy is only related to the selected width Δf of the chosen filter (the signal being periodic and thus determinist, there is no error of statistical origin related to the choice of T).

4.6. Statistical error

4.6.1. *Origin*

Let us consider a stationary random signal whose PSD we wish to determine. The characteristic of such a signal being precisely to vary in a random way, the PSD obtained is different according to the moment at which it is calculated.

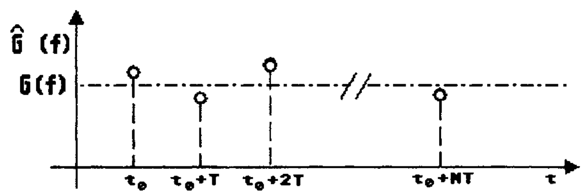

Figure 4.9. *Estimates of the PSD for various signal samples*

Let us consider the PSD $\hat{G}(f)$ evaluated at frequency f starting from a sample of duration T chosen successively between the times t_0 and $t_0 + T$, then $t_0 + T$ and $t_0 + 2$ Tetc.

The values of $\hat{G}(f)$ thus calculated are all different from each other and different also from the exact value $G(f)$. We have:

$$G(f) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \hat{G}_i(f) = E\left[\hat{G}(f)\right] \qquad [4.15]$$

The true PSD is thus the mean value of the quantities $\hat{G}(f)$ estimated at various times, when their number tends towards the infinity. One could also define the standard deviation $\hat{s}$ of $\hat{G}(f)$. For N values,

$$\hat{s} = \sqrt{\frac{1}{N-1}\left\{\sum_{i=1}^{N}\hat{G}_i(f) - \frac{1}{N}\left[\sum_{i=1}^{N}\hat{G}_i(f)\right]^2\right\}} \qquad [4.16]$$

and, for $N \to \infty$

$$s = \lim_{N\to\infty} \hat{s} \qquad [4.17]$$

s being the true standard deviation for a measurement $\hat{G}(f)$, is a description of uncertainty of this measure. In practice, one will make only one calculation of $\hat{G}(f)$ at the frequency f and one will try to estimate the error carried out according to the conditions of the analysis.

4.6.2. *Definition*

Statistical error or *normalized rms error* is the quantity defined by the ratio:

$$\varepsilon = \frac{s_{\ell^2_{\Delta f}}}{\overline{\ell^2_{\Delta f}}} \qquad [4.18]$$

(variation coefficient) where $\overline{\ell^2_{\Delta f}}$ is the mean square value of the signal filtered in the filter of width Δf (quantity proportional to $\hat{G}(f)$) and $s_{\ell^2_{\Delta f}}$ is the standard deviation of the measurement of $\overline{\ell^2_{\Delta f}}$ related to the error introduced by taking a finite duration T.

NOTE.

We are interested here in the statistical error related to calculation of the PSD. One makes also an error of comparable nature during the calculation of other quantities such as coherence, transfer function etc (cf. paragraph 4.12).

4.7. Statistical error calculation

4.7.1. *Distribution of measured PSD*

If the ratio $\varepsilon = \frac{s_{\ell^2_{\Delta f}}}{\ell^2_{\Delta f}}$ is small, one can ensure with a high confidence level that a measurement of the PSD is close to the true average [NEW 75]. If on the contrary ε is large, the confidence level is small. We propose below to calculate the confidence level which can be associated with a measurement of the PSD when ε is known. The analysis is based on an assumption concerning the distribution of the measured values of the PSD.

The measured value of the mean square z^2 of the response of a filter Δf to a random vibration is itself a random variable. It is supposed in what follows that z^2 can be expressed as the sum of the squares of a certain number of Gaussian random variables statistically independent, zero average and of the same variance:

$$z^2 = \frac{1}{T}\left[\int_0^{T/n} \ddot{x}^2(t)\,dt + \int_{T/n}^{2T/n} \ddot{x}^2(t)\,dt + \cdots + \int_{T(1-1/n)}^{T} \ddot{x}^2(t)\,dt\right] \qquad [4.19]$$

One can indeed think that z^2 satisfies this assumption, but one cannot prove that these terms have an equal weight or that they are statistically independent. One notes however in experiments [KOR 66] that the measured values of z^2 roughly have the distribution which would be obtained if these assumptions were checked, namely a chi-square law, of the form:

$$\chi^2 = \chi_1^2 + \chi_2^2 + \chi_3^2 + \cdots + \chi_n^2 \qquad [4.20]$$

If it can be considered that the random signal follows a Gaussian law, it can be shown ([BEN 71] [BLA 58] [DEN 62] [GOL 53] [JEN 68] [NEW 75]) that measurements $\hat{G}(f)$ of the true PSD $G(f)$ are distributed as $G(f)\,\frac{\chi_n^2}{n}$ where χ_n^2 is the chi-square law with n degrees of freedom, mean n and variance 2 n (if the mean value of each independent variable is zero and their variance equal to 1 [BLA 58] [PIE 64]).

Figure 4.10 shows some curves of the probability density of this law for various values of n. One notices that, when n grows, the density approaches that of a normal law (consequence of the central limit theorem).

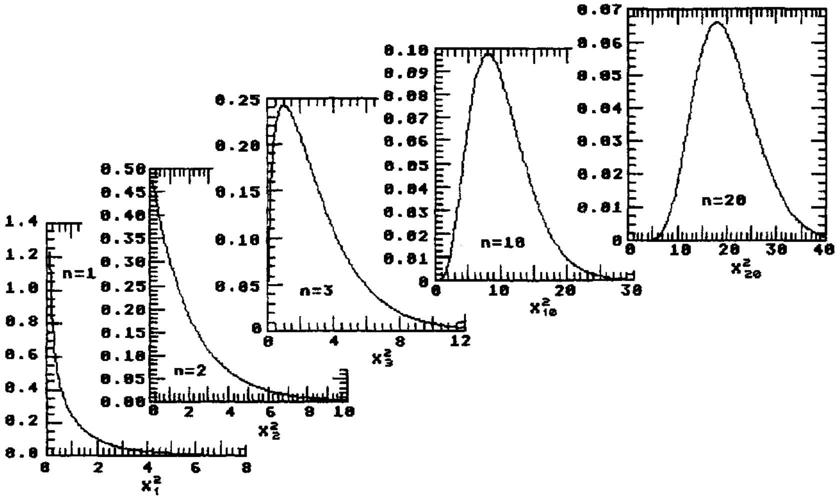

Figure 4.10. *Probability density: the chi-square law*

NOTE.

Some authors [OSG 69] consider that measurements $\hat{G}(f)$ *are distributed more like* $G(f)\,\dfrac{\chi^2_{n-1}}{n-1}$*, basing themselves on the following reasoning. From the values* $X_1, X_2, X_3, \cdots, X_n$ *of a normally distributed population, of mean* m *(unknown value) and standard deviation* s*, one can calculate*

$$\chi^2 = \frac{\left(X_1 - \overline{X}\right)^2 + \left(X_2 - \overline{X}\right)^2 + \left(X_3 - \overline{X}\right)^2 + \cdots + \left(X_n - \overline{X}\right)^2}{s^2} \qquad [4.21]$$

where

$$\overline{X} = \frac{\sum X_1}{n} \qquad [4.22]$$

(mean of the various values taken by variable X *by each of the* n *elements). Let us consider the reduced variable*

$$U_i = \frac{X_i - \overline{X}}{s} \qquad [4.23]$$

The variables U_i *are no longer independent, since there is a relationship between them: according to a property of the arithmetic mean, the algebraic sum of the deviations with respect to the mean is zero, therefore* $\sum\left(X_i - \overline{X}\right) = 0$, *and consequently,* $\dfrac{\sum\left(x_i - \overline{x}\right)}{s} = 0$ *yielding:*

$$\sum U_i = 0$$

In the sample of size n, *only* $n - 1$ *data are really independent, for if* $n - 1$ *variations are known, the last results from this. If there is* $n - 1$ *independent data, there are also* $n - 1$ *degrees of freedom.*

The majority of authors however agree to consider that it is necessary to use a law with n *degrees of freedom. This dissension has little incidence in practice, the number of degrees of freedom to be taken into account being necessarily higher than 90 so that the statistical error remains, according to the rules of the art, lower than 15% approximately.*

4.7.2. *Variance of measured PSD*

The variance of $\hat{G}(f)$ is given by:

$$s_{\hat{G}(f)}^2 = \text{var}\left[\frac{\hat{G}(f)\ \chi_n^2}{n}\right]$$

$$s_{\hat{G}(f)}^2 = \left[\frac{G(f)}{n}\right]^2 \text{var}\left[\chi_n^2\right] \qquad [4.24]$$

However the variance of a chi-square law is equal to twice the number of degrees of freedom:

$$\text{Var}\left(\chi^2\right) = 2\ n \qquad [4.25]$$

yielding

$$s_{\hat{G}(f)}^2 = 2\,\frac{G^2(f)}{n^2}\,n = 2\,\frac{G^2(f)}{n} \qquad [4.26]$$

The mean of this law is equal to n.

4.7.3. *Statistical error*

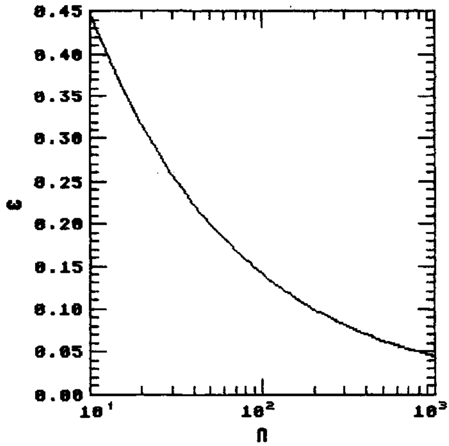

Figure 4.11. *Statistical error as function of the number of dof*

$$\overline{\hat{G}(f)} = \frac{G(f)}{n}\,\overline{\chi^2} = \frac{G(f)}{n}\,n$$

$$\hat{G}(f) = G(f) \qquad [4.27]$$

The statistical error is thus such as:

$$\varepsilon^2 = \frac{s^2_{G(f)}}{\overline{G(f)}^2} = \frac{2}{n}$$

$$\boxed{\varepsilon = \sqrt{\frac{2}{n}}} \qquad [4.28]$$

ε is also termed '*standard error*'.

4.7.4. *Relationship between number of degrees of freedom, duration and bandwidth of analysis*

This relation can be obtained, either using a series expansion of $E\left\{\left[\hat{G}(f) - G(f)\right]^2\right\}$, or starting from the autocorrelation function.

From a series expansion:

It is shown that [BEN 61b] [BEN 62]:

$$E\left\{\left[\hat{G}(f) - G(f)\right]^2\right\} \approx \underbrace{\frac{G^2(f)}{T\,\Delta f}}_{\text{variability}} + \underbrace{\frac{\Delta f^2}{576}\left[G''(f)\right]^2}_{\text{bias}} \qquad [4.29]$$

Except when the slope of the PSD varies greatly with Δf, the bias is in general negligible. Then

$$\varepsilon^2 = \frac{E\left\{\left[\hat{G}(f) - G(f)\right]^2\right\}}{G^2(f)} \approx \frac{1}{T\,\Delta f}$$

This relation is a good approximation as long as ε is lower than approximately 0.2 (i.e. for $T\,\Delta f > 25$).

$$\boxed{\varepsilon \approx \frac{1}{\sqrt{T\,\Delta f}}} \qquad [4.30]$$

The error is thus only a function of the duration T of the sample and of the width Δf of the analysis filter (always assumed ideal [BEA 72] [BEN 63] [NEW 75]).

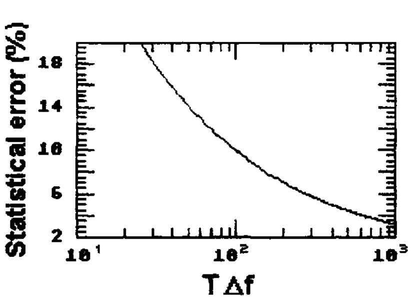

Figure 4.12. *Statistical error*

Figure 4.12 shows the variations of this quantity with the product $T\,\Delta f$. The number of events n represented by a record of white noise type signal, duration T, filtered by a filter of width Δf, is thus, starting from [4.28]:

$$n = 2\,\Delta f\,T \qquad [4.31]$$

Definition

The quantity $n = 2\,\Delta f\,T$ is called *number of degrees of freedom* (dof).

From the autocorrelation function

Let us consider $\ell(t)$ a vibratory signal response collected at the output of a filter of width Δf. The mean square value of $\ell(t)$ is given by [COO 65]:

$$\lambda_{\Delta f}^2 = \frac{1}{T}\int_0^T \ell^2(t)\,dt$$

Setting $\overline{\ell_{\Delta f}^2}$ the measured value of $\lambda_{\Delta f}^2$, we have, by definition:

$$\varepsilon = \frac{\left[\overline{\left(\lambda_{\Delta f}^2 - \overline{\ell_{\Delta f}^2}\right)^2}\right]^{1/2}}{\overline{\ell_{\Delta f}^2}} \qquad [4.32]$$

$$\varepsilon = \frac{\sqrt{\overline{\lambda_{\Delta f}^4 - 2\,\lambda_{\Delta f}^2\,\overline{\ell_{\Delta f}^2} + \left(\overline{\ell_{\Delta f}^2}\right)^2}}}{\overline{\ell_{\Delta f}^2}} = \frac{\sqrt{\overline{\lambda_{\Delta f}^4} - \left(\overline{\ell_{\Delta f}^2}\right)^2}}{\overline{\ell_{\Delta f}^2}}$$

However, we can write:

$$\overline{\lambda_{\Delta f}^4} - \left(\overline{\ell_{\Delta f}^2}\right)^2 = \frac{1}{T}\int_0^T \ell^2(u)\,du \,.\, \frac{1}{T}\int_0^T \ell^2(v)\,dv - \left(\overline{\ell_{\Delta f}^2}\right)^2$$

$$\overline{\lambda_{\Delta f}^4} - \left(\overline{\ell_{\Delta f}^2}\right)^2 = \frac{1}{T^2}\int_0^T du \int_0^T \left[\overline{\ell^2(u)\,\ell^2(v)} - \left(\overline{\ell_{\Delta f}^2}\right)^2\right] dv$$

i.e., while setting $t = u$ and $\tau = v - u = v - t$,

$$\overline{\lambda_{\Delta f}^4} - \left(\overline{\ell_{\Delta f}^2}\right)^2 = \frac{1}{T^2} \int_0^T dt \int_{-t}^{T-t} \left[\overline{\ell^2(t)\, \ell^2(t+\tau)} - \left(\overline{\ell_{\Delta f}^2}\right)^2 \right] d\tau$$

yielding

$$\overline{\lambda_{\Delta f}^4} - \left(\overline{\ell_{\Delta f}^2}\right)^2 = \frac{2}{T^2} \int_0^T dt \int_{-t}^{T-t} \left(\overline{\ell_{\Delta f}^2}\right)^2 \rho^2(\tau)\, d\tau$$

where $\rho(\tau)$ is the autocorrelation coefficient. Given a narrow band random signal, we saw that the coefficient ρ is symmetrical with regard to the axis $\tau = 0$ and that ρ decrease when $|\tau|$ becomes large. If T is sufficiently large, as well as the majority of the values of t:

$$\varepsilon^2 = \frac{\overline{\lambda_{\Delta f}^4} - \left(\overline{\ell_{\Delta f}^2}\right)^2}{\left(\overline{\ell_{\Delta f}^2}\right)^2} = \frac{2}{T^2} \int_0^T dt \int_{-\infty}^{+\infty} \rho^2(\tau)\, d\tau$$

yielding the *standardized variance* ε^2 [BEN 62]:

$$\varepsilon^2 = \frac{2}{T} \int_{-\infty}^{+\infty} \rho^2(\tau)\, d\tau = \frac{4}{T} \int_0^{\infty} \rho^2(\tau)\, d\tau \qquad [4.33]$$

$$\boxed{\varepsilon = 2 \sqrt{\frac{1}{T} \int_0^{\infty} \rho^2(\tau)\, d\tau}} \qquad [4.34]$$

Particular cases

1. Rectangular band-pass filter

We saw [2.70] that in this case [MOR 58]:

$$\rho(\tau) = \frac{\cos 2\pi f_0 \tau \ \sin \pi \Delta f \tau}{\pi \tau \Delta f}$$

yielding

$$\varepsilon^2 \approx \frac{4}{T} \int_0^{\infty} \frac{\cos^2 2\pi f_0 \tau \ \sin^2 \pi \Delta f \tau}{\pi^2 \tau^2 \Delta f^2} d\tau$$

$$\varepsilon^2 \approx \frac{1}{T\,\Delta f}$$

and [BEN 62] [KOR 66] [MOR 63]:

$$\boxed{\varepsilon \approx \frac{1}{\sqrt{T\,\Delta f}}} \qquad [4.35]$$

> **Example**
>
> For ε to be lower than 0.1, it is necessary that the product $T\,\Delta f$ be greater than 100, which can be achieved, for example, either with T = 1 s and Δf = 100 Hz, or with T = 100 s and Δf = 1 Hz. We will see, later on, the incidence of these choices on the calculation of the PSD.

2. Resonant circuit

For a resonant circuit:

$$\rho(\tau) = \cos 2\,\pi\,f_0\,\tau \;\; e^{-\pi\,\tau\,\Delta f}$$

yielding

$$\varepsilon^2 \approx \frac{4}{T}\int_0^\infty \cos^2 2\,\pi\,f_0\,\tau \;\; e^{-2\,\pi\,\tau\,\Delta f}\,d\tau$$

$$\varepsilon^2 \approx \frac{2}{T}\int_0^\infty e^{-2\,\pi\,\tau\,\Delta f}\,d\tau$$

$$\boxed{\varepsilon \approx \frac{1}{\sqrt{\pi\,T\,\Delta f}}} \qquad [4.36]$$

4.7.5. *Confidence interval*

Uncertainty concerning $\hat{G}(f)$ can also be expressed in term of confidence interval. If the signal $\ell(t)$ has a roughly Gaussian probability density function, the distribution of $\frac{\hat{G}(f)}{G(f)}$, for any f, is the same as $\frac{\chi^2}{n}$. Given an estimate $\hat{G}(f)$ obtained from a signal sample, for $n = 2\,\Delta f\,T$ events, the confidence interval in which the true PSD $G(f)$ is located is, on the confidence level $(1-\alpha)$:

$$\boxed{\frac{n\,\hat{G}(f)}{\chi^2_{n,\,1-\alpha/2}} \le G(f) \le \frac{n\,\hat{G}(f)}{\chi^2_{n,\,\alpha/2}}} \qquad [4.37]$$

where $\chi^2_{n,\,\alpha/2}$ and $\chi^2_{n,\,1-\alpha/2}$ have n degrees of freedom. Table 4.3 gives some values of $\chi^2_{n,\,\alpha}$ according to the number of degrees of freedom n for various values of α.

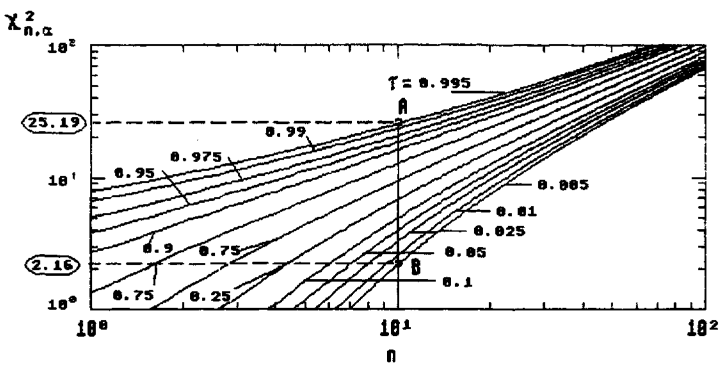

Figure 4.13. *Values of $\chi^2_{n,\,\alpha}$ with respect to the number of degrees of freedom and of α*

Figure 4.13 represents graphically the function χ^2_α with respect to n, parameterized by the probability α.

> **Example**
>
> 99% of the values lie between 0.995 and 0.005. One reads from Figure 4.13, for $n = 10$, that the limits are $\chi^2 = 25.2$ and 2.16.

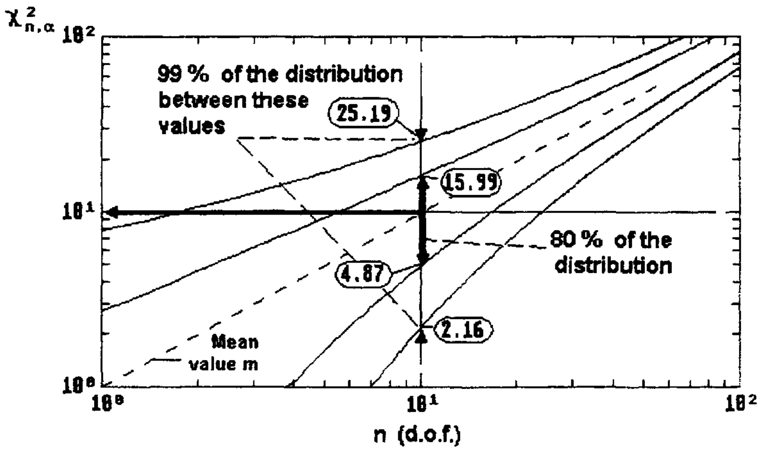

Figure 4.14. *Example of use of the curves* $\chi^2_{n,\alpha}(n)$

Example

Figure 4.14 shows how in a particular case these curves can be used to evaluate numerically the limits of the confidence interval defined by the relation [4.37].

Let us set n = 10. One notes from this Figure that 80% of the values are within the interval 4.87 and 15.99 with mean value m = 10. If the true value of the mean of the calculated PSD S_0 is m, it cannot be determined exactly, nevertheless it is known that

$$\frac{4.87}{10} < \frac{S_0}{m} < \frac{15.99}{10}$$

$$2.05\ S_0 > m > 0.625\ S_0$$

Table 4.3. *Values of* $\chi^2_{n,\alpha}$ *as function of the number* N *of dof [SPI 74]*

	$\chi^2_{n,\alpha}$												
N \ α	**0.995**	**0.99**	**0.975**	**0.95**	**0.90**	**0.75**	**0.50**	**0.25**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**
1	7.88	6.63	5.02	3.84	2.71	1.32	0.455	0.102	0.0158	0.0039	0.0010	0.0002	0.0000
2	10.6	9.21	7.38	5.99	4.61	2.77	1.39	0.575	0.211	0.103	0.0506	0.0201	0.0100
3	12.8	11.3	9.35	7.81	6.25	4.11	2.37	1.21	0.584	0.352	0.216	0.115	0.072
4	14.9	13.3	11.1	9.49	7.78	5.39	3.36	1.92	1.06	0.711	0.484	0.297	0.207
5	16.7	15.1	12.8	11.1	9.24	6.63	4.35	2.67	1.61	1.15	0.831	0.554	0.412
6	18.5	16.8	14.4	12.6	10.6	7.84	5.35	3.45	2.20	1.64	1.24	0.872	0.676
7	20.3	18.5	16.0	14.1	12.0	9.04	6.35	4.25	2.83	2.17	1.69	1.24	0.989
8	22.0	20.1	17.5	15.5	13.4	10.2	7.34	5.07	3.49	2.73	2.18	1.65	1.34
9	23.6	21.7	19.0	16.9	14.7	11.4	8.34	5.90	4.17	3.33	2.70	2.09	1.73
10	25.2	23.2	20.5	18.3	16.0	12.5	9.34	6.74	4.87	3.94	3.25	2.56	2.16
11	26.8	24.7	21.9	19.7	17.3	13.7	10.3	7.58	5.58	4.57	3.82	3.05	2.60
12	28.3	26.2	23.3	21.0	18.5	14.8	11.3	8.44	6.30	5.23	4.40	3.57	3.07
13	29.8	27.7	24.7	22.4	19.8	16.0	12.3	9.30	7.04	5.89	5.01	4.11	3.57
14	31.3	29.1	26.1	23.7	21.1	17.1	13.3	10.2	7.79	6.57	5.63	4.66	4.07
15	32.8	30.6	27.5	25.0	22.3	18.2	14.3	11.0	8.55	7.26	6.26	5.23	4.60
16	34.3	32.0	28.8	26.3	23.5	19.4	15.3	11.9	9.31	7.96	6.91	5.81	5.14
17	35.7	33.4	30.2	27.6	24.8	20.5	16.3	12.8	10.1	8.67	7.56	6.41	5.70
18	37.2	34.8	31.5	28.9	26.0	21.6	17.3	13.7	10.9	9.39	8.23	7.01	6.26
19	38.6	36.2	32.9	30.1	27.2	22.7	18.3	14.6	11.7	10.1	8.91	7.63	6.84

20	40.0	37.6	34.2	31.4	28.4	23.8	19.3	15.5	12.4	10.9	9.59	8.26	7.43
21	41.4	38.9	35.5	32.7	29.6	24.9	20.3	16.3	13.2	11.6	10.3	8.90	8.03
22	42.8	40.3	36.8	33.9	30.8	26.0	21.3	17.2	14.0	12.3	11.0	9.54	8.64
23	44.2	41.6	38.1	35.2	32.0	27.1	22.3	18.1	14.8	13.1	11.7	10.2	9.26
24	45.6	43.0	39.4	36.4	33.2	28.2	23.3	19.0	15.7	13.8	12.4	10.9	9.89
25	46.9	44.3	40.6	37.7	34.4	29.3	24.3	19.9	16.5	14.6	13.1	11.5	10.5
26	48.3	45.6	41.9	38.9	35.6	30.4	25.3	20.8	17.3	15.4	13.8	12.2	11.2
27	49.6	47.0	43.2	40.1	36.7	31.5	26.3	21.7	18.1	16.2	14.6	12.9	11.8
28	51.0	48.3	44.5	41.3	37.9	32.6	27.3	22.7	18.9	16.9	15.3	13.6	12.5
29	52.3	49.6	45.7	42.6	39.1	33.7	28.3	23.6	19.8	17.7	16.0	14.3	13.1
30	53.7	50.9	47.0	43.8	40.3	34.8	29.3	24.5	20.6	18.5	16.8	15.0	13.8
40	66.8	63.7	59.3	55.8	51.8	45.6	39.3	33.7	29.1	26.5	24.4	22.2	20.7
50	79.5	76.2	71.4	67.5	63.2	56.3	49.3	42.9	37.7	34.8	32.4	29.7	28.0
60	92.0	88.4	83.3	79.1	74.4	67.0	59.3	52.3	46.5	43.2	40.5	37.5	35.5
70	104.2	100.4	95.0	90.5	85.5	77.6	69.3	61.7	55.3	51.7	48.8	45.4	43.3
80	116.3	112.3	106.6	101.9	96.6	88.1	79.3	71.1	64.3	60.4	57.2	53.5	51.2
90	128.3	124.1	118.1	113.1	107.6	98.6	89.3	80.6	73.3	69.1	65.6	61.8	59.2
100	140.2	135.8	129.6	124.3	118.5	109.1	99.3	90.1	82.4	77.9	74.2	70.1	67.3

More specific tables or curves were published to provide directly the value of the limits [DAR 72] [MOO 61] [PIE 64]. For example, Table 4.4 gives the confidence interval defined in [4.37] for three values of $1 - \alpha$ [PIE 64].

Table 4.4. *Confidence limits for the calculation of a PSD [PIE 64]*

Degrees of freedom n	Confidence interval limits relating to a measured power spectral density $\hat{G}(f) = 1$					
	$(1-\alpha) = 0.90$		$(1-\alpha) = 0.95$		$(1-\alpha) = 0.99$	
	Lower limit	Higher limit	Lower limit	Higher limit	Lower limit	Higher limit
10	0.546	2.54	0.483	3.03	0.397	4.63
15	0.599	2.07	0.546	2.39	0.457	3.26
20	0.637	1.84	0.585	2.08	0.500	2.69
25	0.662	1.71	0.615	1.90	0.532	2.38
30	0.685	1.62	0.637	1.78	0.559	2.17
40	0.719	1.51	0.676	1.64	0.599	1.93
50	0.741	1.44	0.699	1.54	0.629	1.78
75	0.781	1.34	0.743	1.42	0.680	1.59
100	0.806	1.28	0.769	1.35	0.714	1.49
150	0.833	1.22	0.806	1.27	0.758	1.37
200	0.855	1.19	0.826	1.23	0.781	1.31
250	0.870	1.16	0.847	1.20	0.800	1.27
300	0.877	1.15	0.855	1.18	0.820	1.25
400	0.893	1.13	0.877	1.15	0.840	1.21
500	0.901	1.11	0.885	1.14	0.855	1.18
750	0.917	1.09	0.909	1.11	0.877	1.15
1000	0.934	1.08	0.917	1.09	0.893	1.12
5000	0.971	1.03	0.962	1.04	0.952	1.05
Multiply the lower and higher limits in the table by the measured value $\hat{G}(f)$ to obtain the limits of the confidence interval of the true value $G(f)$.						

NOTE.

When $n \geq 30$, $\sqrt{2\,\chi_n^2}$ *follows a law close to a Gaussian law of mean* $\sqrt{2\,n-1}$ *and standard deviation 1 (Fisher's law). Let* x *be a normal reduced variable and* α *a value of the probability such that*

$$\text{Prob}\big[|x| < k(\alpha)\big] = 1 - \alpha \qquad [4.38]$$

where k *is a constant function of the probability* α.

For example:

α	*90%*	*95%*	*99%*
$k(\alpha)$	*1.645*	*1.960*	*2.58*

We have

$$\text{Prob}\left[\sqrt{2\,n-1} - k(\alpha) \leq \sqrt{2\,\chi_n^2} \leq \sqrt{2\,n-1} + k(\alpha)\right] \qquad [4.39]$$

yielding the approximate value of the limits of χ_n^2

$$\text{Prob}\left[\frac{\left[\sqrt{2\,n-1} - k(\alpha)\right]^2}{2} \leq \chi_n^2 \leq \frac{\left[\sqrt{2\,n-1} + k(\alpha)\right]^2}{2}\right] = 1 - \alpha \qquad [4.40]$$

and that of the confidence interval limits of $\dfrac{G(f)}{\hat{G}(f)}$ *(since the probability of* $\dfrac{\hat{G}}{G}$ *is the same as that of* $\dfrac{\chi_n^2}{n}$*):*

$$\text{Prob}\left[\frac{2\,n}{\left[\sqrt{2\,n-1} + k(\alpha)\right]^2} \leq \frac{G(f)}{\hat{G}(f)} \leq \frac{2\,n}{\left[\sqrt{2\,n-1} - k(\alpha)\right]^2}\right] = 1 - \alpha \qquad [4.41]$$

For large values of n $[n \geq 120]$, *i.e. for* ε *small, it is shown that the chi-square law tends towards the normal law and that the distribution of the values of* $\hat{G}(f)$ *can itself be approximated by a normal law of mean* n *and standard deviation* $\sqrt{2\,n}$ *(law of large numbers). In this case,*

$$\text{Prob}\left[n - k(\alpha)\sqrt{2\,n} \le \chi_n^2 \le n + k(\alpha)\sqrt{2\,n}\right] = 1 - \alpha \qquad [4.42]$$

yielding

$$\text{Prob}\left[\frac{n}{n + k(\alpha)\sqrt{2\,n}} \le \frac{G(f)}{\hat{G}(f)} \le \frac{n}{n - k(\alpha)\sqrt{2\,n}}\right] = 1 - \alpha \qquad [4.43]$$

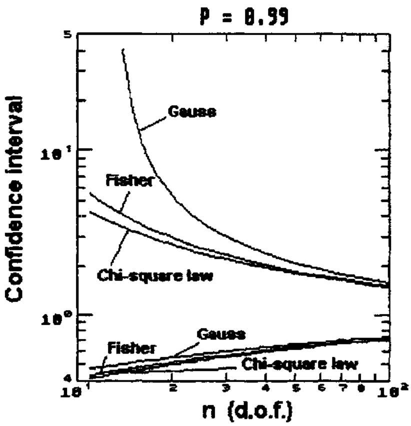

Figure 4.15. *Confidence interval for* P = 0.99

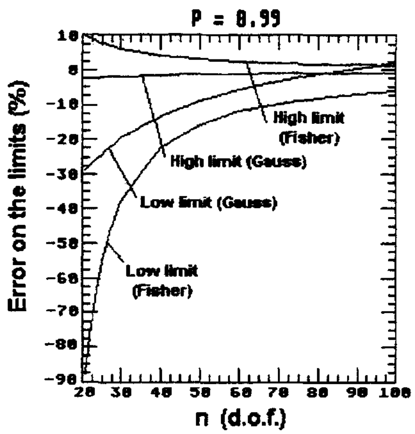

Figure 4.16. *Error related to the use of the Gauss or Fisher laws*

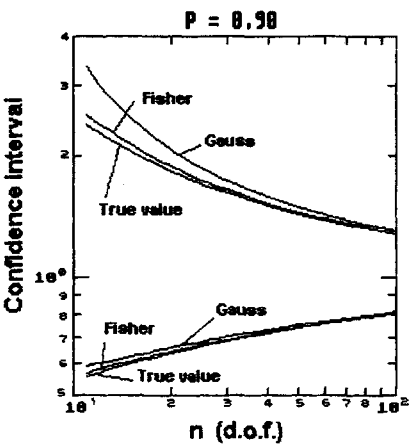

Figure 4.17. *Confidence interval for* P = 0.90

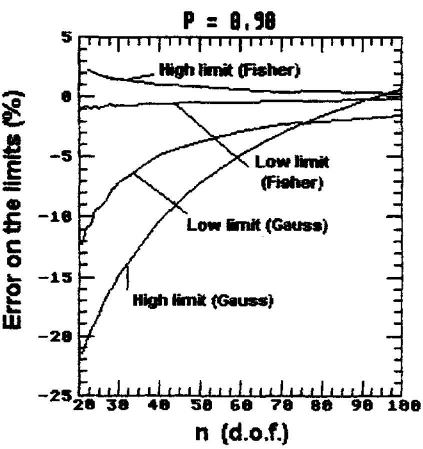

Figure 4.18. *Error related to the use of the Gauss or Fisher laws*

Figures 4.15 to 4.18 provide, for a confidence level of 99%, and then 90%:

– variations in the confidence interval limits depending to the number of degrees of freedom n*, obtained using an exact calculation (chi-square law), by considering the Fisher and Gauss assumptions,*

– the error made using each one of these simplifying assumptions.

These curves show that the Fisher assumption constitutes an approximation acceptable for n *greater than 30 approximately (according to the confidence level), with relatively simple analytical expressions for the limits.*

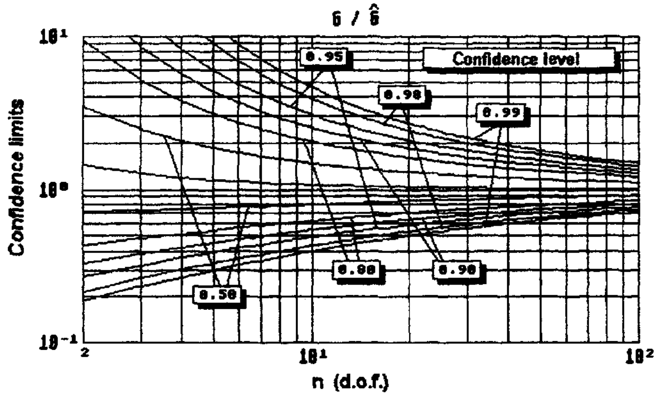

Figure 4.19. *Confidence limits* ($\hat{G}/G$)

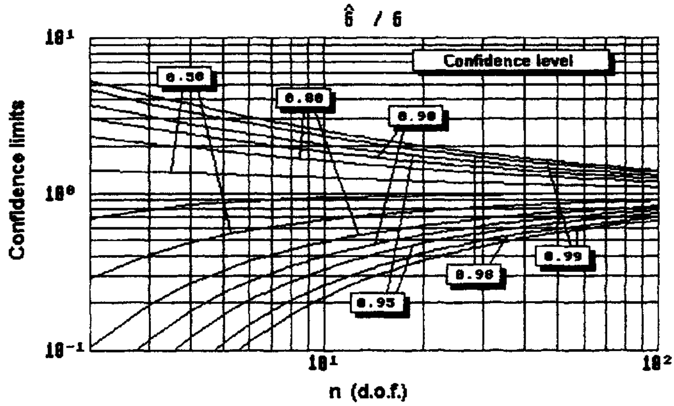

Figure 4.20. *Confidence limits* ($\hat{G}/G$) *[MOO 61]*

The ratio $\frac{G}{\hat{G}}$ (or $\frac{\hat{G}}{G}$, depending on the case) is plotted in Figures 4.19 and 4.20 with respect to n, for various values of the confidence level.

Example

Let us suppose that a PSD level $\hat{G} = 2$ has been measured with a filter of width $\Delta f = 2.5$ Hz and from a signal sample of duration $T = 10$ s. The number of degrees of freedom is $n = 2\, T\, \Delta f = 50$ (yielding $\varepsilon = \frac{1}{\sqrt{T\, \Delta f}} = 0.2$). Table 4.4 gives, for $1 - \alpha = 0.90$:

$$0.741\,\hat{G} < G < 1.44\,\hat{G}$$

i.e. $1.482 < G < 2.88$ if $\hat{G} = 2$. Reading from the curves in Figure 4.21, for $n = 50$,

$\frac{\hat{G}}{G} < 0.69$ on the confidence level 5%,

$\frac{\hat{G}}{G} < 1.35$ on the confidence level 95%.

With a confidence level of 90%, we thus have:

$$0.69 < \frac{\hat{G}}{G} < 1.35$$

i.e.

$$\frac{\hat{G}}{1.35} < G < \frac{\hat{G}}{0.69}$$

$$0.74\,\hat{G} < G < 1.44\,\hat{G}$$

$$1.48 < G < 2.88$$

For $\varepsilon \le 0.1$ [PIE 64], we can see that the relative error between the true PSD and the calculated PSD lies between $\pm\, s_{\hat{G}}$ with a confidence level of 68%, i.e. that during approximately 68% of the time, the exact PSD lies between $\hat{G}(f) \pm s_{\hat{G}}$:

$$\left|G(f) - \hat{G}(f)\right| < s_{\hat{G}} \qquad [4.44]$$

From this inequality, can be written [PIE 64]:

$$\frac{\hat{G}(f)}{1+\varepsilon} < G(f) < \frac{\hat{G}(f)}{1-\varepsilon} \qquad [4.45]$$

The confidence limits on the 68% level are plotted in Figure 4.21 for n ranging between 2 and 1000, then ranging between 20 and 1000.

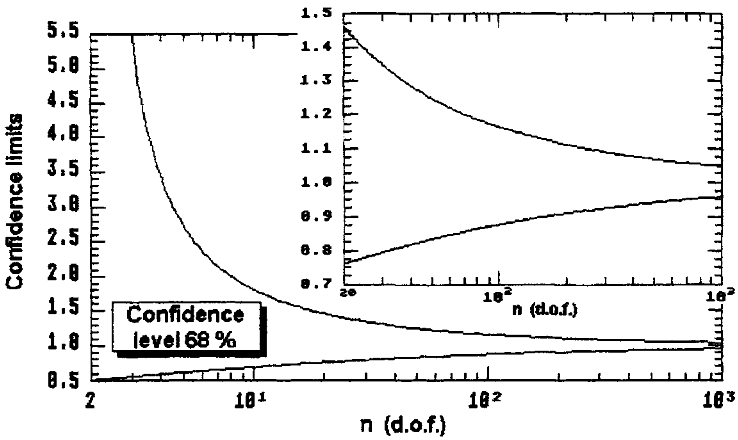

Figure 4.21. *Confidence limits at the 68% level*

NOTE.

At confidence level $1 - \alpha = 68\%$, *the expressions [4.37] and [4.45] show that*

$$\begin{cases} \dfrac{1}{1+\varepsilon} = \dfrac{n}{\chi^2_{n,\,1-\alpha/2}} \\ \dfrac{1}{1-\varepsilon} = \dfrac{n}{\chi^2_{n,\,\alpha/2}} \end{cases} \qquad [4.46]$$

yielding

$$\begin{cases} \varepsilon = \dfrac{\chi^2_{n,\,1-\alpha/2}}{n} - 1 \\ \varepsilon = 1 - \dfrac{\chi^2_{n,\,\alpha/2}}{n} \end{cases} \qquad [4.47]$$

where, if $\varepsilon \le 0.2$, $\varepsilon \approx \dfrac{1}{\sqrt{T\,\Delta f}}$, *one deduces:*

$$\chi^2_{n,\,1-\alpha/2} + \chi^2_{n,\,\alpha/2} = 2\,n \qquad [4.48]$$

This expression is applicable for any n *for confidence level 68% and any* α *when* n *is large.*

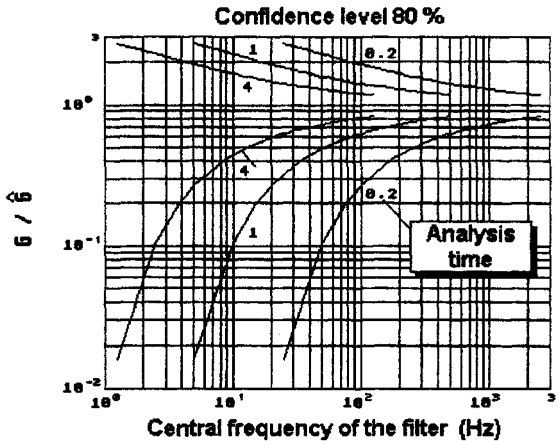

Figure 4.22. $\hat{G}/G$ *as function of frequency of filter and length of analysis [CUR 64]*

Figure 4.22 shows the variations of:

$$\frac{G}{\hat{G}} = \frac{\text{true PSD (large T)}}{\text{measured PSD}}$$

with respect to the central frequency of the filter, for various lengths of analysis, at the confidence level 80% and for a ratio $\dfrac{\text{central frequency}}{\Delta f} = 10$ [CUR 64].

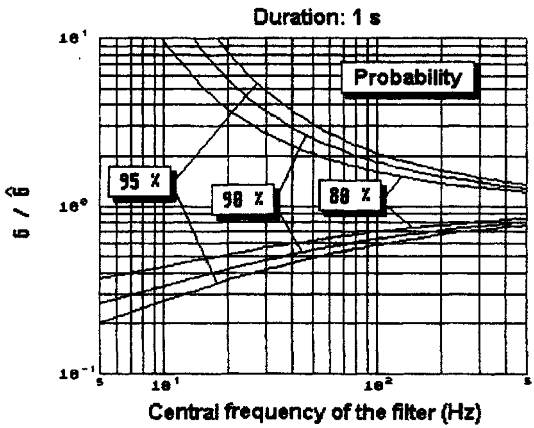

Figure 4.23. $\hat{G}/G$ *as function of frequency of the filter and probability*

Figure 4.23 is parameterized, in the same axes, by the probability.

Figures 4.22 and 4.23 are deduced from Figure 4.21 as follows: for a given f, $\Delta f = \frac{f}{10}$ is calculated, then, for a given T, $n = 2\ \Delta f\ T$, yielding $\frac{\hat{G}}{G}$ and $\frac{G}{\hat{G}}$.

Example

We want to calculate a PSD with a statistical error less than 17.5% at confidence level 95%. At this level 95%, we have ± 1.96 times the standard error. The standard error should thus not exceed:

$$\varepsilon = \frac{17.5}{1.96} = 8.94\,\%$$

Knowing ε, the calculation conditions can be chosen from

$$\varepsilon = \frac{1}{\sqrt{T\,\Delta f}} = 8.94\ 10^{-2}.$$

4.7.6. *Expression for statistical error in decibels*

While dividing, in [4.40], $\sqrt{2\,\chi_n^2}$ by its mean value, i.e. χ_n^2 by $\dfrac{2\,n-1}{2}$, it becomes

$$\text{Prob}\left[\frac{\left[\sqrt{2\,n-1}-k(\alpha)\right]^2}{2\,\dfrac{2\,n-1}{2}} \le \frac{\chi_n^2}{\dfrac{2\,n-1}{2}} \le \frac{\left[\sqrt{2\,n-1}+k(\alpha)\right]^2}{2\,\dfrac{2\,n-1}{2}}\right] = 1-\alpha \qquad [4.49]$$

The error can be evaluated from $\dfrac{\hat{G}}{G}$, i.e. $\dfrac{\chi_n^2}{\dfrac{2\,n-1}{2}}$, in the form

$$\varepsilon_{dB} = 10\,\log_{10}\left[\frac{\chi_n^2}{(2\,n-1)/2}\right] \qquad [4.50]$$

It is raised, according to n, by

$$\varepsilon_{dB} = 10\,\log_{10}\frac{\left[\sqrt{2\,n-1}+k(\alpha)\right]^2}{2\,n-1}$$

$$\varepsilon_{dB} = 10\,\log_{10}\left[1+\frac{2\,k(\alpha)}{\sqrt{2\,n-1}}+\frac{k^2(\alpha)}{2\,n-1}\right] \qquad [4.51]$$

Figure 4.24 shows the variations of ε_{dB} with the number of degrees of freedom n.

If $k(\alpha)=1$, there is a 68.27% chance that the measured value is in the interval $\pm 1\ s_{\hat{G}(f)}$ and an 84.13% chance that it is lower than $1\ s_{\hat{G}(f)}$. Then:

$$\varepsilon_{dB} = 10\,\log_{10}\left[1+\frac{2}{\sqrt{2\,n-1}}+\frac{1}{2\,n-1}\right]$$

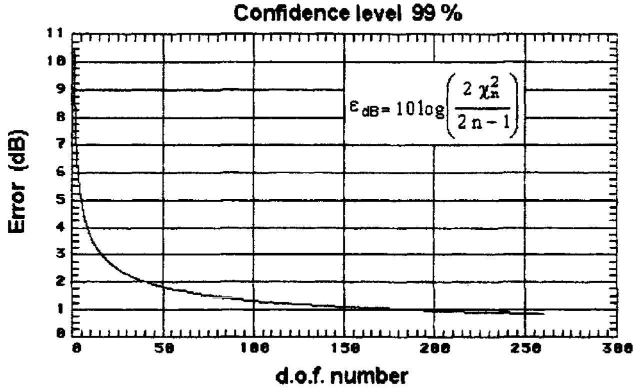

Figure 4.24. *Statistical error in dB*

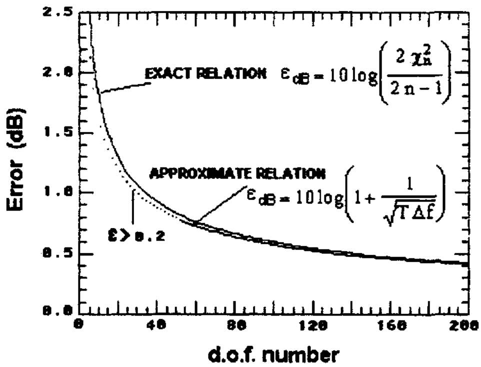

Figure 4.25. *Statistical error approximation*

$$\varepsilon_{dB} \approx 10\,\log_{10}\left[1+\frac{2}{\sqrt{2\,n-1}}\right] \approx 10\,\log_{10}\left(1+\sqrt{\frac{2}{n}}\right)$$

$$\boxed{\varepsilon_{dB} = 10\,\log\left(1+\frac{1}{\sqrt{T\,\Delta f}}\right)} \qquad [4.52]$$

The curves in Figure 4.25 allow comparison of the exact relation [4.50] with the approximate relation [4.52]: the approximation is good for n > 50.

Example

If it is required that $\varepsilon = \pm 0.5$ dB, i.e. that $\varepsilon = \pm 12.2\ \%$, it is necessary, at confidence level 84%, that $T\,\Delta f = 67.17$, or that the number of degrees of freedom is equal to $n = 2\ T\ \Delta f \approx 135$.

If $\Delta f = 24$ Hz, $T = \dfrac{1}{(0.122)^2\ \Delta f} 10^4 = 2.8$ s. At confidence level 90%, the variations of the PSD are, in the interval [BAN 78]:

n	Lower limit (dB)	Upper limit (dB)
50	-1.570	1.329
100	-1.077	0.958
250	-0.665	0.617

4.7.7. *Statistical error calculation from digitized signal*

Let N be the number of sampling points of the signal $\ddot{x}(t)$ of duration T,

M the number of points in frequency of the PSD

$f_{samp.}$ the sampling frequency of the signal

f_{max} the maximum frequency of the PSD, lower or equal to $\dfrac{f_{samp.}}{2.6}$

(modified Shannon's theorem, paragraph 4.1)

δt the time interval between two points.

We have:

$$T = N\ \delta t \qquad [4.53]$$

$$\Delta f = \frac{f_{samp.}}{2\,M} \qquad [4.54]$$

NOTE.

M points separated by an interval Δf *lead to a maximum frequency* $f_{MAX} = M\ \Delta f = \frac{f_{samp.}}{2}$. *To fulfill the condition of paragraph 4.3.1, it is necessary to limit in practice the useful field of the PSD to* $f_{max} \le \frac{f_{samp.}}{2.6}$.

If we need a PSD calculated based on M points, we need at least $\Delta N = 2\ M$ points per block. Since the signal is composed of N points, we will cut up it into $K = \frac{N}{2\ M}$ blocks of duration $\Delta T = \frac{T}{K}$.

Knowing that $f_{samp.} = \frac{1}{\delta t}$:

$$\Delta f = \frac{1}{2\ M\ \delta t}$$

yielding

$$\varepsilon = \frac{1}{\sqrt{T\ \Delta f}} = \sqrt{\frac{2\ M\ \delta t}{N\ \delta t}}$$

i.e.

$$\boxed{\varepsilon = \sqrt{\frac{2\ M}{N}}} \qquad [4.55]$$

Example

N = 32 768 points M = 512 points T = 64 s

yielding

2M = 1 024 points per sample

$$K = \frac{N}{2\,M} = 32 \text{ samples (of 2 s)}$$

$$\Delta f = \frac{K}{T} = \frac{32}{64} = 0.5\,\text{Hz} \qquad f_{samp.} = \frac{N}{T} = \frac{32768}{64} = 512 \text{ points/s}$$

$$\varepsilon = \sqrt{\frac{2 \times 512}{32\,768}} = 0.1768$$

Even if $M\,\Delta f = 512 \;\; 0.5\text{ Hz} = 256\text{ Hz}$, we must have, in practice,

$$f_{max} \le \frac{f_{samp.}}{2.6} = \frac{512}{2.6} \approx 197\text{ Hz}.$$

4.8. Overlapping

4.8.1. *Utility*

One can carry out an overlapping of blocks for three reasons:

– to limit the loss of information related to the use of a window on sequential blocks, which results in ignorance of a significant part of the signal because of the low values of the window at its ends [GAD 87];

– to reduce the length of analysis (interesting for real time analyses) [CON 95];

– to reduce the statistical error when the duration T of the signal sample cannot be increased. We saw that this error is related to the number of blocks taken in the sample of duration T. If all the blocks are sequential, the maximum number K of blocks of fixed duration ΔT (arising from the frequency resolution desired) is equal to the integer part of $T/\Delta T$ [WEL 67]. An overlapping makes it possible to increase this number of blocks whilst preserving their size ΔT.

Overlapping rate

The *overlapping rate* R is the ratio of the duration of the block overlapped by the following block over the total duration of the block.

This rate is in general limited to the interval between 0 and 0.75.

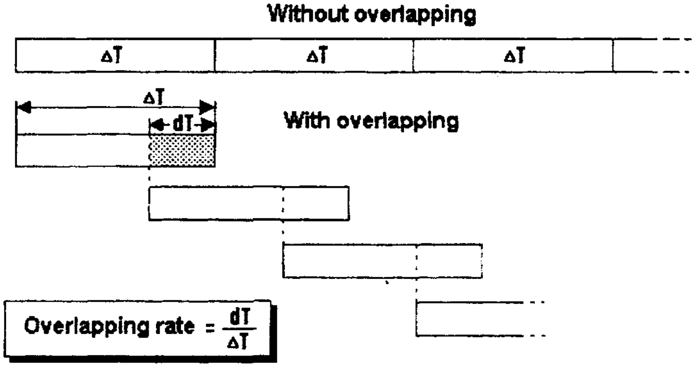

Figure 4.26. *Overlapping of blocks*

Overlapping in addition makes it possible to minimize the influence of the side lobes of the windows [CAR 80] [NUT 71] [NUT 76].

4.8.2. *Influence on number of degrees of freedom*

Let N be the number of points of the signal sample, N' (> N) the number of points necessary to respect the desired statistical error with K blocks of size ΔN ($N' = K\ \Delta N$). The difference $N' - N$ must be distributed over $K - 1$ possible overlappings [NUT 71]:

$$N' - N = (K - 1)\ R\ \Delta N$$

yielding

$$R = \frac{N' - N}{\Delta N\ (K - 1)} = \frac{N' - N}{N' - \Delta N} \qquad [4.56]$$

For R to be equal to 0.5 for example, it is necessary that $N' = 2\ N - \Delta N$.

Overlapping modifies the number of degrees of freedom of the analysis since the blocks cannot be regarded any more as independent and noncorrelated. The estimated value of the PSD no longer obeys a one chi-square law. The variance of the PSD measured from an overlapping is less than that calculated from contiguous blocks [WEL 67]. R. Potter and J. Lortscher [POT 78] showed however that, when K is sufficiently large, the calculation could still be carried out on the assumption of non overlapping, on the condition the result could still be corrected by a reduction factor depending on the type of window and the selected overlapping rate. The correlation as a function of overlapping can be estimated using the coefficient:

$$c(R) = \frac{\int_0^{\Delta T} r(t)\, r\left[t + (1 - R)\, \Delta T\right] dt}{\int_0^{\Delta T} r^2(t)\, dt} \qquad [4.57]$$

Table 4.5. *Reduction factor*

Window	Correlation coefficient C			Coefficient μ	
	R = 25%	R = 50%	R = 75%	R = 50%	R = 75%
Rectangle	0.25000	0.50000	0.75000	0.66667	0.36364
Bingham	0.17143	0.45714	0.74286	0.70524	0.38754
Hamming	0.02685	0.23377	0.70692	0.90147	0.47389
Hanning	0.00751	0.16667	0.65915	0.94737	0.51958
Parzen	0.00041	0.04967	0.49296	0.999509	0.67071
Flat signal	0.00051	-0.01539	0.04553	0.99953	0.99540
Kaiser-Bessel	0.00121	0.07255	0.53823	0.98958	0.62896

4.8.3. *Influence on statistical error*

When the blocks are statistically independent, the number of degrees of freedom is equal to $n = 2\,K = 2\,T\,\Delta f$ whatever the window. With overlappings of K blocks, the effective number of blocks to consider in order to calculate the statistical error is given [HAR 78] [WEL 67]:

– for R = 50 % by:

$$K_{50} = \frac{1}{\dfrac{1 + 2\,c_{50\%}^2}{K} - \dfrac{2\,c_{50\%}^2}{K^2}} \approx \frac{K}{1 + 2\,c_{50\%}^2} = \mu_{50}\,K \qquad [4.58]$$

– for R = 75% by:

$$K_{75\%} = \frac{1}{\dfrac{1 + 2\,c_{75\%}^2 + 2\,c_{50\%}^2 + 2\,c_{25\%}^2}{K} - 2\,\dfrac{c_{75\%}^2 + c_{50\%}^2 + 3\,c_{25\%}^2}{K^2}}$$

$$K_{75\%} \approx \frac{K}{1 + 2\,c_{75\%}^2 + 2\,c_{50\%}^2 + 2\,c_{25\%}^2} = \mu_{75}\,K \qquad [4.59]$$

(the approximation being acceptable for $K > 10$). Under these conditions, the statistical error is no longer equal to $1/\sqrt{K}$, but to:

$$\varepsilon = \frac{1}{\sqrt{\mu\,K}} \qquad [4.60]$$

The coefficient μ being less than 1, the statistical error is, for a given K, all the larger as overlapping is greater. But with an overlapping, the total duration of the treated signal is smaller, which makes it possible to carry out more quickly the analyses in real time (control of the test facilities). The time saving can be calculated from [4.56]:

$$R = \frac{N'-N}{N'-\Delta N} = \frac{T'-T}{T'-\Delta T}$$

(ΔT =duration of a block). To avoid a confusion of notations, we will let T_O be the duration of the signal to be treated with an overlapping and T the duration without overlapping. We then have:

$$R = \frac{N'-N}{N'-\Delta N} = \frac{T - T_O}{T - \Delta T}$$

yielding

$$T_O = T\,(1 - R) + R\,\Delta T \qquad [4.61]$$

Since $R < 1$ and $\Delta T << T$, we have in general $T_O \approx T\,(1 - R)$. The time saving is thus approximately equal to $\dfrac{T_O}{T} \approx (1 - R)$.

Example

$T = 25$ s, $\Delta f = 4$ Hz (i.e. $K = T\,\Delta f = 100$), $\varepsilon_0 = 0.1$ (without overlapping).

With $R = 0.75$ and a Hanning window, $\mu \approx 0.52$; yielding $\varepsilon = 1/\sqrt{0.52 \times 25 \times 4} \approx 0.139$. But this result is obtained for a signal of duration $T_0 \approx (1 - 0.75)\, 25 \approx 6.25$ s.

If we consider now a sample of given duration T, overlapping makes it possible to define a greater number of blocks. This K' number can be deducted from [4.56]:

$$N' = \frac{N - R\,\Delta N}{1 - R}$$

yielding, if $N' = K'\,\Delta N$

$$K' = \frac{K - R}{1 - R} \qquad [4.62]$$

The increase in the number of blocks makes it possible to reduce the statistical error which becomes equal to:

$$\varepsilon = \frac{1}{\sqrt{\mu \dfrac{K - R}{1 - R}}} \approx \sqrt{\frac{1 - R}{\mu\,K}} = \varepsilon_0 \sqrt{\frac{1 - R}{\mu}} \qquad [4.63]$$

Example

With the data of the above example, the statistical error would be equal to

$$\varepsilon \approx \varepsilon_0 \sqrt{\frac{1 - 0.75}{0.52}} \approx 0.693\,\varepsilon_0 = 0.0693.$$

4.8.4. *Choice of overlapping rate*

The calculation of the PSD uses the square of the signal values to be analysed. In this calculation the square of the function describing the window for each block thus intervenes in an indirect way, by taking account of the selected overlapping rate R. For a linear average, this leads to an effective weighting function $r_{rms}(t)$ such as [GAD 87]:

$$r_{rms}^2(R) = \frac{1}{K} \sum_{i=1}^{K} r^2 \left[t - i(1-R)T\right] \qquad [4.64]$$

where T is the duration of the window used (duration of the block),

i is the number of the window in the sum,

K is the number of windows at time t.

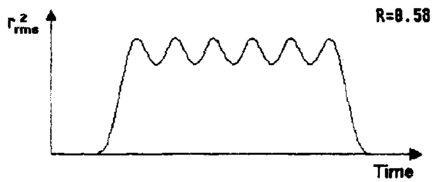

Figure 4.27. *Ripple on the Hanning window (R = 0.58)*

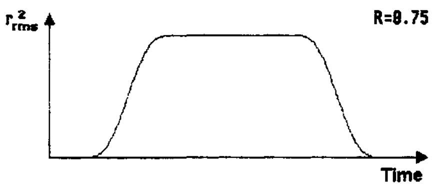

Figure 4.28. *Hanning window for* R = 0.75

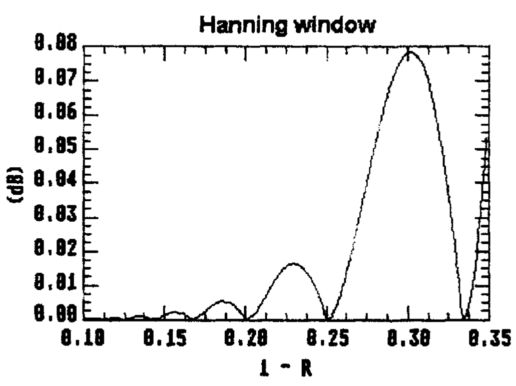

Figure 4.29. *Ripple amplitude versus* 1 – R

With the Hanning window, one of the most used, it can be observed (Figure 4.27) that there is a ripple on $r_{rms}^2(t)$, except when $1-R$ is of the form $1/p$ where p is an integer equal to or higher than 3 (Figure 4.28). The ripple has a negligible amplitude when $1-R$ is small (lower than $1/3$) [CON 95] [GAD 87]. This property can be observed in Figure 4.29, which represents the variations of the ratio of the maximum and minimum amplitudes of the ripple (in dB) with respect to $1-R$.

This remark makes it possible to justify the use, in practice, of an overlapping equal to 0.75 which guarantees a constant weighting on a broad part of the window (the other possible values, 2/3, 3/4, 4/5, etc..., are less used, because they do not lead as 3/4 to a integer number of points when the block size is a power of two).

4.9. Calculation of PSD for given statistical error

4.9.1. *Case: digitalization of signal is to be carried out*

Given a vibration $\ell(t)$, one sets out to calculate its power spectral density between 0 and f_{max} with M points (M must be a power of 2), for a statistical error not exceeding a selected value ε. The procedure is summarized in Table 4.6 [BEA 72] [LEL 73] [NUT 80].

Table 4.6. *Computing process of a PSD starting from a non-digitized signal*

The signal of total duration T (to be defined) will be cut out in K blocks of unit duration ΔT, under the following conditions:	
$f_{samp.} \geq 2.6\, f_{max}$	Condition to avoid the aliasing phenomenon (modified Shannon's theorem).
$f_{Nyquist} = \frac{f_{samp.}}{2}$	Nyquist frequency [PRE 90].
$\Delta f = \frac{f_{Nyquist}}{M}$	Interval between two points of the PSD (this interval limits the possible precision of the analysis starting from the PSD).
$\delta t = \frac{1}{f_{samp.}}$	Temporal step (time interval between two points of the signal), if the preceding condition is observed.
$\Delta N = 2\, M$	Number of points per block.
$N = \frac{2\, M}{\varepsilon^2}$	Minimum number of signal points to analyse in order to respect the statistical error.
$T = N\, \delta t$	Minimum total duration of the sample to be treated.
$K = \frac{N}{2\, M}$	Number of blocks.
$\Delta T = \frac{T}{K} \left(= 2\, M\, \delta t = \frac{1}{\Delta f}\right)$	Duration of one block.

Calculation of $\frac{1}{\Delta T}\left\|L(f)\right\|^2$ for each point of the PSD, where $f = m\,\Delta f$ $(0 < m \le M)$	Calculation from the FFT of each block.
$\frac{1}{K}\sum_{i=1}^{K}\frac{1}{\Delta T}\left\|L_i(f)\right\|^2$	Averaging of the spectra obtained for each of the K blocks (stationary and ergodic process)

With these conditions, the maximum frequency of the PSD computed is equal to $f'_{max} = f_{Nyquist}$. But it is preferable to consider the PSD only in the interval $(0,\ f_{max})$.

NOTE.

It is supposed here that the signal has frequency components greater than f_{max} *and that it was thus filtered by a low-pass filter to avoid aliasing. If it is known that the signal has no frequency beyond* f_{max}, *this filtering is not necessary and* $f'_{max} = f_{max}$.

4.9.2. *Case: only one sample of an already digitized signal is available*

If the signal sample of duration T has already been digitized with N points, one can use the value of the statistical error to calculate the number of points M of the PSD (i.e. the frequency interval Δf), which is thus no longer to be freely selected (but it is nevertheless possible to increase the number of points of the PSD by overlapping and/or addition of zeros).

Table 4.7. *Computing process of a PSD starting from an already digitized signal*

Data: The digitized signal, f_{max} and ε.	
$f'_{max} = \frac{f_{samp.}}{2}$	Theoretical maximum frequency of the PSD (see preceding note).
$f_{max} = \frac{f_{samp.}}{2.6}$	Practical maximum frequency.
$\delta t = \frac{1}{f_{samp.}}$	Temporal step (time interval between two points of the signal).

$N = \dfrac{T}{\delta t}$	Number of signal points of duration T.
$M = \dfrac{N\,\varepsilon^2}{2}$	Number of points of the PSD necessary to respect the statistical error (one will take the number immediately beneath that equal to the power of 2).
$f_{Nyquist} = \dfrac{f_{samp.}}{2}$	Nyquist frequency.
$\Delta f = \dfrac{f_{Nyquist}}{M}$	Interval between two points of the PSD.
$\Delta N = 2\,M$	Number of points per block.
$K = \dfrac{N}{2\,M}$	Number of blocks. Etc

If the number of points M of the PSD to be plotted is itself imposed, it would be necessary to have a signal defined by N' points instead of N given points ($N < N'$). One can avoid this difficulty in two complementary ways:

– either by using an overlapping of the blocks (of $2\,M$ points). One will set the overlapping rate R equal to 0.5 and 0.75 while taking smallest of these two values (for a Hanning window) which satisfies the inequality:

$$\sqrt{\frac{1-R}{\mu}\,\frac{2\,M}{N}} \le \varepsilon$$

When it is possible, overlapping chosen in this manner makes it possible to use K' blocks with $K' = \dfrac{N'}{2\,M}$, where [4.56] $N' = \dfrac{N - 2\,M\,R}{1-R}$;

– or, if overlapping does not sufficiently reduce the statistical error, by fixing this rate at 0.75 to benefit as much as possible from its effect and then to evaluate the size of the blocks which would make it possible, with this rate, to respect the statistical error, using:

$$\sqrt{\frac{1-R}{\mu}\,\frac{\Delta N}{N}} \le \varepsilon$$

The value ΔN thus obtained is lower than the number $2\,M$ necessary to obtain the desired resolution on the PSD. Under these conditions, the number of items used for the calculation of the PSD is equal to:

$$N' = \frac{N - 0.75\,\Delta N}{1 - 0.75}$$

and the numbers of blocks to $K' = N'/\Delta N$. One can then add zeros to each block to increase the number of calculation points of the PSD and to make it equal to 2 M. For each block, this number is equal to $\frac{2\ M\ K' - N'}{K'}$. This is however only an artifice, the information contained in the initial signal not evidently increasing with the addition of zeros.

4.10. Choice of filter bandwidth

4.10.1. *Rules*

It is important to recall that the precision of calculation of the PSD depends, for given T, on the width Δf of the filter used [RUD 75]. The larger the width Δf of the filter is, the smaller the statistical error ε and the better the precision of calculation of $G(f)$. However, this width cannot be increased limitlessly [MOO 61]. The larger Δf is, the less the details on the curve obtained, which is smoothed. The resolution being weaker, the narrow peaks of the spectrum are not shown any more [BEN 63]. A compromise must thus be found.

Figure 4.30 shows as an example three spectral curves obtained starting from the same vibratory signal with three widths of filter (3.9 Hz, 15.625 Hz and 31.25 Hz). These curves were plotted without the amplitude being divided by Δf, as is normally the case for a PSD.

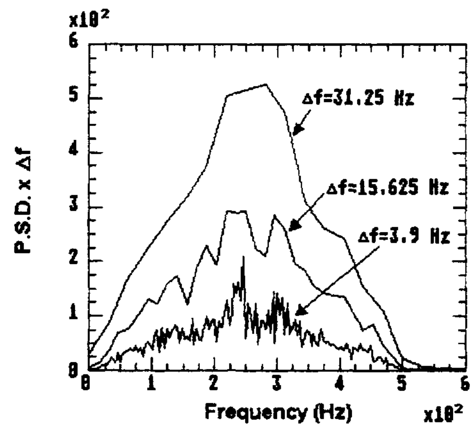

Figure 4.30. *Influence of width of filter*

One observes in these conditions, the area under the curve calculated for $\Delta f = 15.625$ Hz is approximately half of that obtained for $\Delta f = 31.25$ Hz. In the case of a true PSD, division by Δf gives the same area for whatever the value of Δf.

We note in addition on these curves that the spectrum obtained for $\Delta f = 15.6$ Hz is very much smoothed; in particular, the peak observed for $\Delta f = 3.9$ Hz has disappeared. To choose the value of Δf, it would be necessary to satisfy two requirements:

1. The filter should not be broader than aquarter of the width of the narrowest resonance peak expected [BEN 61b] [BEN 63] [FOR 64] [MOO 61] [WAL 81];

2. The statistical error should remain small, with a value not exceeding approximately 15%.

If the first condition is observed, the precision of the PSD calculation is proportional to the width of the filter. If, on the contrary, resonances are narrower than the filter, the precision of the estimated PSD is proportional to the width of the resonance of the specimen and not to the width of the filter. To solve this problem, C.T. Morrow [MOR 58], and then R.C. Moody [MOO 61] suggested making two analyses, by using the narrowest filter first of all to emphasize resonances, then by making a second analysis with a broader filter in order to improve the precision of the PSD estimate.

Other more complicated techniques have been proposed (H. Press and J.W. Tukey for example [BLA 58] [NEW 75]).

4.10.2. *Bias error*

Let us consider a random signal $\ell(t)$ with a constant PSD (white noise) $G_\ell(f) = G_{\ell 0}$ applied to a linear system with transfer function of one-degree-of-freedom [PIE 93] [WAL 81]:

$$H(f) = \frac{1}{\sqrt{\left[1-\left(\frac{f}{f_0}\right)^2\right]^2+\left(\frac{f}{Q\, f_0}\right)^2}} \qquad [4.65]$$

(f_0 being the natural frequency and Q the quality factor of the system). The response $u(t)$ of this system has the following PSD:

$$G_u(f) = \left|H(f)\right|^2 \, G_\ell(f)$$

$$G_u(f) = \frac{G_\ell(f)}{\left[1 - \left(\frac{f}{f_0}\right)^2\right]^2 + \left(\frac{f}{Q\, f_0}\right)^2} \qquad [4.66]$$

Let us analyse this PSD, which presents a peak at $f = f_0$, using a rectangular filter of width ΔF centered on f_c, with transfer function [FOR 64]:

$$\begin{cases} H_A = 1 & \text{for } f_c - \frac{\Delta F}{2} \le f \le f_c + \frac{\Delta F}{2} \\ H_A = 0 & \text{elsewhere} \end{cases} \qquad [4.67]$$

We propose to calculate the *bias error* made over the width between the half-power points of the peak of the PSD response and on the amplitude of this peak when using an analysis filter ΔF of nonzero width. For given f_c, the PSD calculated with this filter has a value of:

$$G_F(f_c) = \frac{1}{\Delta F} \int_0^\infty |H_A|^2\, G_u(f)\, df \qquad [4.68]$$

i.e.

$$G_F(f_c) = \frac{1}{\Delta F} \int_{f_c - \Delta F/2}^{f_c + \Delta F/2} \frac{G_{\ell 0}}{\left[1 - \left(\frac{f}{f_0}\right)^2\right]^2 + \left(\frac{f}{Q\, f_0}\right)^2}\, df \qquad [4.69]$$

It is known [LAL 94] (Volume 4, Appendix A3) that the integral $A = \int \frac{dh}{\left(1 - h^2\right)^2 + h^2/Q^2}$ is equal to:

$$A = \frac{1}{8\sqrt{1-\xi^2}} \ln \frac{h^2 + 2\, h\sqrt{1-\xi^2} + 1}{h^2 - 2\, h\sqrt{1-\xi^2} + 1} + \frac{1}{4\,\xi}\left(\text{Arc}\tan\frac{h + \sqrt{1-\xi^2}}{\xi} + \text{Arc}\tan\frac{h - \sqrt{1-\xi^2}}{\xi}\right)$$

Consequently,

$$\Delta F \frac{G_F(f_c)}{G_{\ell 0}} = \frac{f_0}{8\sqrt{1-\xi^2}} \left[\ln \frac{h^2 + 2\,h\sqrt{1-\xi^2}+1}{h^2 - 2\,h\sqrt{1-\xi^2}+1} \right]_{h_1}^{h_2}$$

$$+ \frac{1}{4\,\xi} \left(\text{Arc}\tan \frac{h+\sqrt{1-\xi^2}}{\xi} + \text{Arc}\tan \frac{h-\sqrt{1-\xi^2}}{\xi} \right)_h^{h_2} \qquad [4.70]$$

$$\frac{G_F(f_c)}{G_{\ell 0}} \approx \frac{f_0\,Q}{2\,\Delta F} \left[\text{Arc}\tan 2Q \frac{f_c - f_0 + \frac{\Delta F}{2}}{f_0} - \text{Arc}\tan 2Q \frac{f_c - f_0 - \frac{\Delta F}{2}}{f_0} \right] \qquad [4.71]$$

For $f_c = f_0$:

$$\frac{G_F(f_0)}{G_{\ell 0}} \approx \frac{f_0\,Q}{2\,\Delta F} \left[\text{Arc}\tan\left(Q \frac{\Delta F}{f_0} \right) - \text{Arc}\tan\left(Q \frac{(-\Delta F)}{f_0} \right) \right] \qquad [4.72]$$

i.e.

$$\frac{G_F(f_0)}{G_{\ell 0}} \approx \frac{f_0\,Q}{\Delta F}\,\text{Arc}\tan \frac{Q\,\Delta F}{f_0} \qquad [4.73]$$

However, by definition, the bandwidth between the half-power points is equal to $\Delta f = \frac{f_0}{Q}$, yielding:

$$\frac{G_F(f_0)}{G_{\ell 0}} \approx \frac{f_0^2}{\Delta f\,\Delta F}\,\text{Arc}\tan \frac{\Delta F}{f} \qquad [4.74]$$

At the half-power points, the calculated spectrum has a value:

$$G_F(f_c) = \frac{1}{2}\,G_F(f_0) \qquad [4.75]$$

where $f_c = f_0 + \frac{\Delta f_F}{2}$. One deduce that:

$$\frac{G_F}{G_{\ell_0}} = Q \frac{f_0}{2\,\Delta F} \left[\operatorname{Arc\,tan} \frac{\Delta f_F + \Delta F}{\Delta f} - \operatorname{Arc\,tan} \frac{\Delta f_F - \Delta F}{\Delta f} \right] \qquad [4.76]$$

From [4.74], [4.75] and [4.76], it becomes:

$$\boxed{\operatorname{Arc\,tan} \frac{\Delta F}{\Delta f} = \left[\operatorname{Arc\,tan} \frac{\Delta f_F + \Delta F}{\Delta f} - \operatorname{Arc\,tan} \frac{\Delta f_F - \Delta F}{\Delta f} \right]} \qquad [4.77]$$

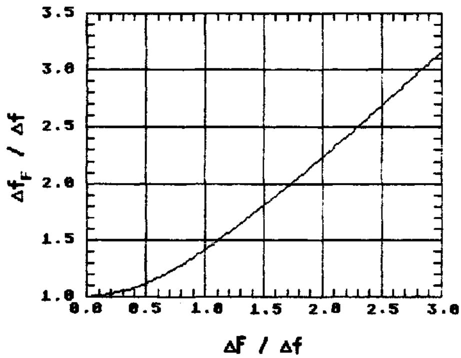

Figure 4.31. *Width of the peak at half-power versus width of the analysis filter (according to [FOR 64])*

The curve in Figure 4.31 gives the variations of $\frac{\Delta f_F}{\Delta f}$ against $\frac{\Delta F}{\Delta f}$ after numerical resolution. It is noted that the measured value Δf_f of the width of the peak at half-power is obtained with an error lower than 10% so long as the width of the analysis filter is less than half the true value Δf.

Setting $x = \frac{\Delta F}{\Delta f}$ and $y = \frac{\Delta f_F}{\Delta f}$.

$$\operatorname{Arc\,tan}\ x = \operatorname{Arc\,tan}(y + x) - \operatorname{Arc\,tan}(y - x)$$

Knowing that $\operatorname{Arc\,tan}\ a - \operatorname{Arc\,tan}\ b = \operatorname{Arc\,tan} \frac{a - b}{1 + a\,b}$, we have:

$$\operatorname{Arc\,tan}\ x = \operatorname{Arc\,tan} \frac{2\,x}{1 + y^2 - x^2}.$$

This yields $x = \dfrac{2\,x}{1 + y^2 - x^2}$ and $y^2 = x^2 + 1$, i.e.:

$$\boxed{\frac{\Delta f_F}{\Delta f} = \left[\left(\frac{\Delta F}{\Delta f}\right)^2 + 1\right]^{1/2}} \qquad [4.78]$$

In addition, the peak of the PSD occurs for $f = f_0$:

$$^{P}G = Q^2 = \left(\frac{f_0}{\Delta f}\right)^2 \qquad [4.79]$$

yielding the relationship between the measured value of the peak and the true value:

$$\boxed{\frac{G_F(f_0)}{^{P}G} = \frac{\Delta f}{\Delta F}\ \mathrm{Arc\,tan}\frac{\Delta F}{\Delta f}} \qquad [4.80]$$

Figure 4.32 shows the variations of this ratio versus $\Delta F/\Delta f$. If $\Delta F = \dfrac{1}{4}\,\Delta f$ according to the rule previously suggested,

$$\frac{\Delta f_F}{\Delta f} = \sqrt{1 + \left(\frac{1}{4}\right)^2} = 1.0308$$

and

$$\frac{G_F}{^{P}G} = 4\ \mathrm{Arc\,tan}\frac{1}{4} \approx 0.98$$

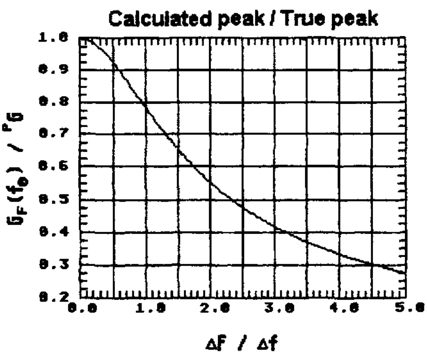

Figure 4.32. *Amplitude of peak versus filter width*

Under these conditions, the error is about 3% of Δf and 2% of the peak.

Example

Let us consider a one-degree-of-freedom system of natural frequency $f_0 = 100$ Hz and quality factor $Q = 10$, excited by a white noise. The error of measure of the PSD response peak is given by the curve in Figure 4.32.

If $\Delta F = 5$ Hz:

$$\frac{Q}{f_0}\Delta F = \frac{10}{100} 5$$

$$\frac{Q}{f_0}\Delta F = \frac{1}{2}$$

yielding:

$$\frac{G_F}{P_G} = 0.92$$

For $f_0 = 50$ Hz and $Q = 10$, one would obtain similarly $\frac{Q}{f_0}\Delta F = \frac{10}{50}5 = 1$ and $\frac{G_F}{P_G} = 0.78$.

4.10.3. *Maximum statistical error*

When the phenomenon to be analysed is of short duration, it can be difficult to obtain a good resolution (small ΔF) whilst preserving an acceptable statistical error.

Example

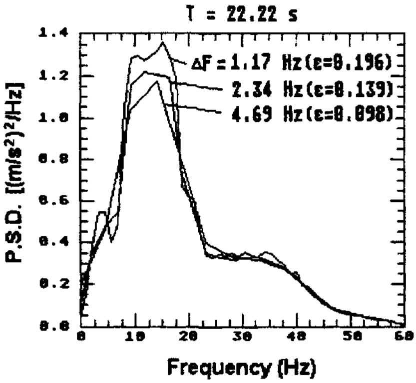

Figure 4.33. *Influence of analysis filter width for a sample of given duration*

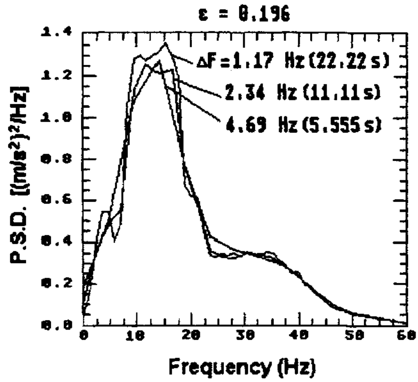

Figure 4.34. *Influence of analysis filter width for constant statistical error*

Figure 4.33 shows, as an example, the PSDs of same signal duration 22.22 seconds, calculated respectively with ΔF equal to 4.69 Hz; 2.34 Hz and 1.17 Hz (i.e. with a statistical error equal to 0.098, 0.139 and 0.196).

We observe that, the more detailed the curve (ΔF small), the larger the statistical error. Although the duration of the sample is longer than 20 s, a resolution of the order of one Hertz can be obtained only with an error close to 20%.

A constant statistical error with different durations T and widths ΔF can lead to appreciably different results. Figure 4.34 shows three calculated PSD of the same signal all three for $\varepsilon = 19.6\,\%$, with respectively:

$$T = 22.22 \text{ s} \quad \text{and} \quad \Delta F = 1.17 \text{ Hz}$$

$$T = 11.11 \text{ s} \quad \text{and} \quad \Delta F = 2.34 \text{ Hz}$$

$$T = 5.555 \text{ s} \quad \text{and} \quad \Delta F = 4.69 \text{ Hz}$$

The choice of ΔF must thus be a compromise between the resolution and the precision. In practice, one tries to comply with the two following rules: ΔF less than a quarter of the width of the narrowest peak of the PSD, which limits the width measurement error of the peak and its amplitude to less than 3%, and a statistical error less than 15% (which corresponds to a number of degrees of freedom n equal to approximately 90). Certain applications (calculation of random transfer functions for example) can justify a lower value of the statistical error.

Taking into account the importance of these parameters, the filter width used for the analysis and the statistical error should always be specified on the power spectral density curves.

4.10.4. *Optimum bandwidth*

A.G. Piersol [PIE 93] defines the optimum bandwidth ΔF_{op} as the value of ΔF minimizing the total mean square error, sum of the squares of the bias error and of the statistical error:

$$\varepsilon^2 = \varepsilon_{bias}^2 + \varepsilon_{stat}^2 \qquad [4.81]$$

The bias error calculated from [4.80] is equal to:

$$\varepsilon_{bias} = \frac{\Delta f}{\Delta F} \text{Arc}\tan\left(\frac{\Delta F}{\Delta f}\right) - 1 \qquad [4.82]$$

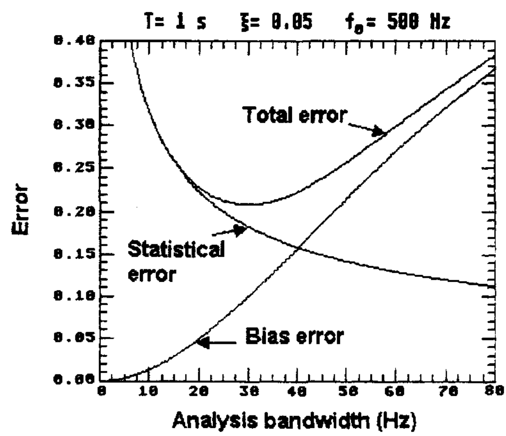

Figure 4.35. *Total mean square error*

where Δf is the width between the half-power points of the peak. Whence:

$$\varepsilon^2 \approx \left[\frac{\Delta f}{\Delta F}\ \text{Arc}\tan\left(\frac{\Delta F}{\Delta f}\right) - 1\right]^2 + \frac{1}{\Delta F\ T} \qquad [4.83]$$

Figure 4.35 shows the variations of bias error, statistical error and ε with ΔF.

Error ε has a minimum at $\Delta F = \Delta F_{op}$. The optimum bandwidth ΔF_{op} is thus obtained by cancelling the derivative of ε^2 with respect to ΔF. This research is carried out numerically.

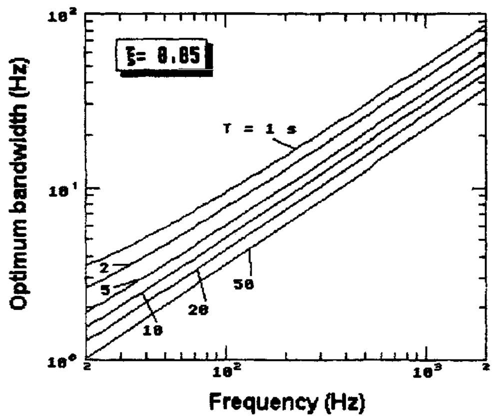

Figure 4.36. *Optimum bandwidth versus peak frequency and duration of the sample*

The curves in Figure 4.36 show ΔF_{op} versus f_0, for $\xi = 0.05$ and for some values of duration T.

If $\Delta F/\Delta f < 0.4$, the bias error can be approximated by:

$$\varepsilon_{bias} \approx -\frac{1}{3}\left(\frac{\Delta F}{\Delta f}\right)^2 \qquad [4.84]$$

Then,

$$\varepsilon^2 \approx \frac{\Delta F^4}{9\ \Delta f^4} + \frac{1}{T\ \Delta F} \qquad [4.85]$$

Whence, by cancelling the derivative,

$$\Delta F_{op} \approx \left(\frac{9\ \Delta f^4}{4\ T} \right)^{1/5} \approx 2\ \frac{(\xi\ f_0)^{4/5}}{T^{1/5}} \qquad [4.86]$$

Figure 4.37 shows the error made versus the natural frequency f_0, for various values of T.

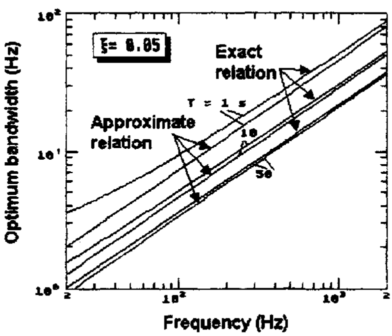

Figure 4.37. *Comparison of approximate and exact relations for calculation of optimum bandwidth*

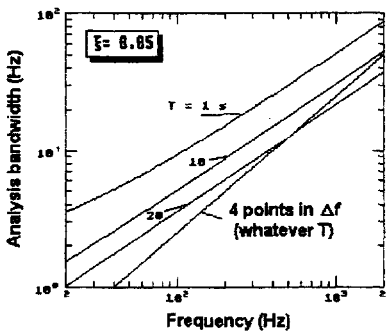

Figure 4.38. *Comparison of the optimum bandwidth with the standard rules*

It can be interesting to compare the values resulting from these calculations with the standard rules which require four points in the half-power interval (Figure 4.38). It is noted that this rule of four points leads generally to a smaller bandwidth in general. The method of calculation of optimum width must be used with prudence, for it can lead to a much too large statistical error (Figure 4.39, plotted for $\xi = 0.05$).

To confine this error to the low resonance frequencies, A.G. Piersol [PIE 93] suggested limiting the optimum band to 2.5 Hz, which leads to the curves in Figure 4.40.

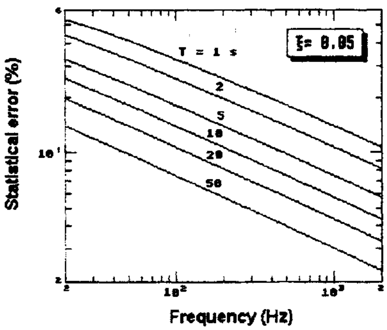

Figure 4.39. *Statistical error obtained using the optimum bandwidth*

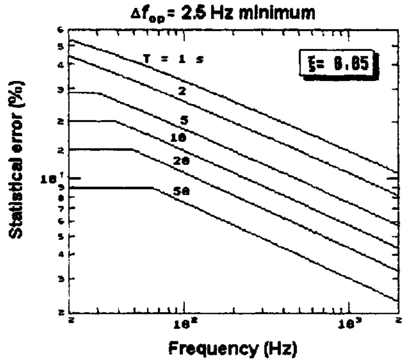

Figure 4.40. *Statistical error obtained using the optimum bandwidth limited to 2.5 Hz*

By plotting the variations of $\Delta f / \Delta F_{op}$, one can also evaluate, with respect to f_0, the number of points in Δf which determines this choice of ΔF_{op}, in order to compare this number with the four points of the empirical rule. Figures 4.41 and 4.42 show the results obtained, for several values of T, with and without limitation of the ΔF_{op} band.

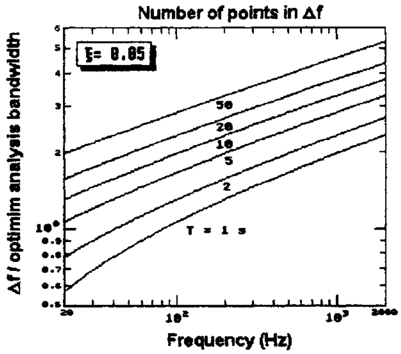

Figure 4.41. *Number of points in* Δf *resulting from choice of optimum bandwidth*

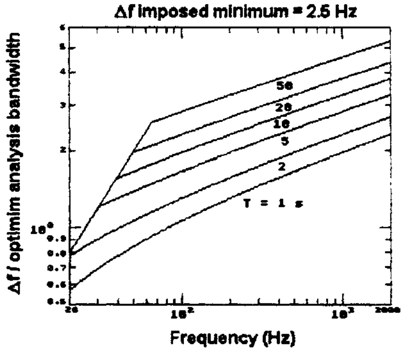

Figure 4.42. *Number of points in* Δf *resulting from the choice of optimum bandwidth limited to 2.5 Hz*

4.11. Probability that the measured PSD lies between ± one standard deviation

We saw that the approximate relation $\varepsilon = \dfrac{1}{\sqrt{T\,\Delta f}}$ is acceptable as long as $\varepsilon < 0.20$. In this same range, the error on the measured PSD $\hat{G}$ (or on $\hat{G}/G$) has a roughly Gaussian distribution [MOO 61] [PRE 56a]. Let us set $\hat{s} = s_{\hat{G}}$ to simplify the notations. The probability that the measured PSD is false by a quantity greater than $\theta\,\hat{s}$ (error in the positive sense) is [MOR 58]:

$$P = \frac{1}{\hat{s}\,\sqrt{2\,\pi}} \int_{\theta\,\hat{s}}^{\infty} e^{-\frac{a^2}{2\,\hat{s}^2}}\, da \qquad [4.87]$$

If we set $v = \dfrac{a}{\hat{s}}$, P takes the form:

$$P = \frac{1}{\sqrt{2\,\pi}} \int_{\theta}^{\infty} e^{-v^2/2}\, dv$$

Knowing that:

$$\operatorname{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \qquad [4.88]$$

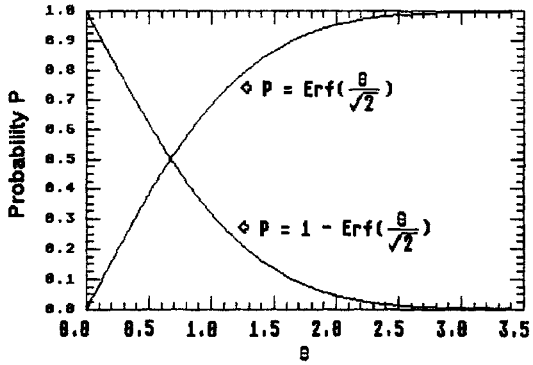

Figure 4.43. *Probability that the measured PSD lies between ± 1 standard deviation*

P can be also written, to facilitate its numerical calculation (starting from the approximate expressions given in Appendix A4):

$$P = \frac{1}{2}\left[1 - \operatorname{Erf}\left(\frac{\theta}{\sqrt{2}}\right)\right] \qquad [4.89]$$

The probability of a negative error is identical. The probability of an error outside the range $\pm\,\theta\,\hat{s}$ is thus equal to:

$$\boxed{P = 1 - \operatorname{Erf}\left(\frac{\theta}{\sqrt{2}}\right)} \qquad [4.90]$$

Example

$\theta = 1$ $\qquad P = 68.26\,\%$

$\theta = 2$ $\qquad P = 95\%$

4.12. Statistical error: other quantities

The statistical error related to the estimate of the mean and mean square value is, according to the case, given by [BEN 80]:

Table 4.8. *Statistical error of the mean and the mean square value*

	Mean estimation	Error	Estimate of the mean square value	Error
Ensemble averages	$\frac{1}{N}\sum_{i=1}^{N} x_i$	$\frac{s_x}{\mu_x\,\sqrt{N}}$	$\frac{1}{N}\sum_{i=1}^{N} x_i^2$	$\sqrt{\frac{2}{N}}$
Temporal averages	$\frac{1}{T}\int_0^T x(t)\,dt$	$\frac{s_x}{\mu_x\,\sqrt{2\,T\,\Delta f}}$	$\frac{1}{T}\int_0^T x^2(t)\,dt$	$\frac{1}{\sqrt{T\,\Delta f}}$

Calculations of the quantities defined in this chapter are carried out in practice on samples of short duration T, subdivided into K blocks of duration ΔT [BEN 71] [BEN 80], by using filters of non-zero width Δf. These approximations lead to the errors in Table 4.9.

Table 4.9. *Other statistical errors*

	Quantity Z	Error ε_r
Direct PSD	$\hat{G}_{xx}$ or $\hat{G}_{yy}$	$\dfrac{1}{\sqrt{T\,\Delta f}}$
Cross PSD	$\left\lvert\hat{G}_{xy}(f)\right\rvert$	$\dfrac{1}{\left\lvert\rho_{xy}(f)\right\rvert\,\sqrt{T\,\Delta f}}$
Coherence	$\hat{\gamma}_{xy}(f)$	$\dfrac{\sqrt{2}\left[1-\rho_{xy}^2(f)\right]}{\left\lvert\rho_{xy}(f)\right\rvert\,\sqrt{T\,\Delta f}}$
	$\hat{G}_{vv}(f)=\hat{\gamma}_{xy}^2(f)\ \hat{G}_{yy}(f)$	$\dfrac{\sqrt{2-\rho_{xy}^2(f)}}{\left\lvert\rho_{xy}(f)\right\rvert\,\sqrt{T\,\Delta f}}$
Transfer function	$\left\lvert\hat{H}_{xy}(f)\right\rvert$	$\dfrac{\sqrt{1-\rho_{xy}^2(f)}}{\left\lvert\rho_{xy}(f)\right\rvert\,\sqrt{2\,T\,\Delta f}}$
	$\left\lvert\hat{H}_{xy}(f)\right\rvert^2$	$\dfrac{\sqrt{2}\,\sqrt{1-\rho_{xy}^2(f)}}{\left\lvert\rho_{xy}(f)\right\rvert\,\sqrt{T\,\Delta f}}$

These expressions can be used with an estimated value $\hat{\rho}$ of the correlation coefficient instead of ρ (unknown); one then obtains approximate values of ε_r, when ε_r is small (i.e. $\varepsilon_r < 0.20$), which can be limited at the 95% confidence level using:

$$\hat{Z}\left(1-2\,\varepsilon_r\right)\le Z\le\hat{Z}\left(1+2\,\varepsilon_r\right)$$

where Z is the true value of the parameter and $\hat{Z}$ its estimated value.

Figure 4.44 shows the variations of the error made during the calculation of the transfer function $\left\lvert\hat{H}_{xy}(f)\right\rvert$, given by:

$$\varepsilon_r \approx \frac{\left(1-\rho_{xy}^2\right)^{1/2}}{\left|\rho_{xy}\right| \sqrt{2\ T\ \Delta f}} \qquad [4.91]$$

for various values of $n_d = T\ \Delta f$.

$$\hat{H}(f) = \frac{G_{xy}(f)}{G_x(f)} = \left|\hat{H}(f)\right| e^{j\,\hat{\phi}(f)}$$

$\hat{H}(f)$ = measured transfer function
$H(f)$ = exact function [BEN 63] [GOO 57].

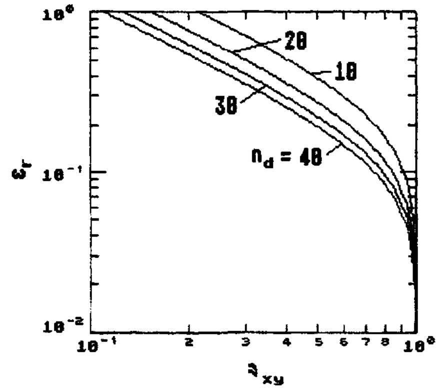

Figure 4.44. *Statistical error related to the calculation of the transfer function*

It is shown that, if

$$P = \text{Prob}\left[\left|\frac{\hat{H}(f) - H(f)}{H(f)}\right| < \sin \varepsilon_r \ \text{ and } \left|\hat{\phi}(f) - \phi(f)\right| < \varepsilon_r\right]$$

$$P = 1 - \left[\frac{1 - v_{xy}^2(f)}{1 - v_{xy}^2(f)\cos^2\varepsilon}\right]^n \qquad [4.92]$$

where n is the number of degrees of freedom, equal to $2\ T\ \Delta f$.

$$n = \frac{\ln(1-P)}{\ln\left(\dfrac{1-\nu_{xy}^2}{1-\nu_{xy}^2\cos^2\varepsilon_r}\right)} \qquad [4.93]$$

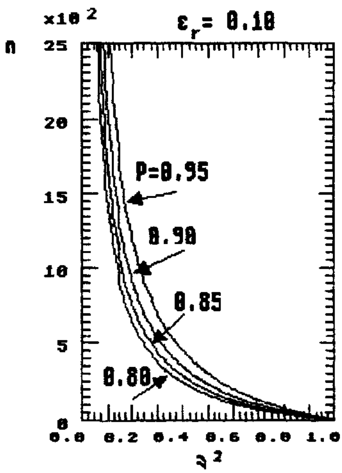

Figure 4.45. *Number of dof necessary for the statistical error on the transfer function to be lower than 0.10 with probability* P

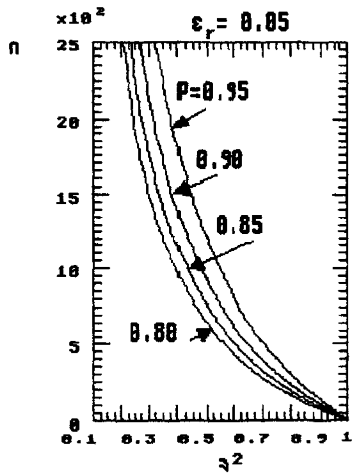

Figure 4.46. *Number of dof necessary for the statistical error on the transfer function to be lower than 0.05 with probability* P

The statistical error resulting from the calculation of the autocorrelation R_x is given by [VIN 72]:

$$\varepsilon_x = \frac{1}{\sqrt{2\,T\,\Delta f}}\left[1+\frac{R_x^2(0)}{R_x^2(\tau)}\right]^{1/2} \qquad [4.94]$$

A reasonable value of T for the calculation of R_x is $T = \dfrac{1}{\Delta f\ \varepsilon_x^2}$. For the cross-correlation R_{xy}:

$$\varepsilon_{xy} = \frac{1}{\sqrt{2\,T\,\Delta f}}\left[1+\frac{R_{xy}^2(0)}{R_{xy}^2(\tau)}\right]^{1/2} \qquad [4.95]$$

4.13. Generation of random signal of given PSD

4.13.1. *Method*

The method of generation of a random signal varying with time of given duration T from a PSD of maximum frequency f_{max} includes the following stages:

– calculation of the temporal step $\delta t = \dfrac{1}{f_{samp.}} = \dfrac{1}{2.6\ f_{max}}$,

– choice of the number M of points of definition of the PSD (power of two),

– calculation of the number of signal points: $N = \dfrac{T}{\delta t}$,

– possibly, modification of N (and thus of the duration) and/or of M in order to respect a maximum statistical error ε_0 (for a future PSD calculation of the generated signal), starting from the relation $\dfrac{M}{N} \le \dfrac{\varepsilon_0^2}{2}$, maintaining M equal to a power of two,

– calculation of the frequency interval between 2 points of the PSD $\Delta f = \dfrac{f_{samp.}}{2\,M}$,

– for each M points of the PSD, calculation at every time $t = k\ \delta t$ (k = constant integer between 1 and N) of a 'sinusoid'

- of the form: ${}^m x(t) = {}^m x_{max} \sin\left(2\ \pi\ f\ t + \varphi_m\right)$,
- of duration T,
- of frequency $f_m = m\ df$ (m integral such that $1 \le m \le M$),
- of amplitude $\sqrt{2\ G\left(f_m\right)\ \Delta f}$ [where $G\left(f_m\right)$ is the value of the given PSD at the frequency f_m, the amplitude of a sinusoid being equal to twice its rms value],
- of random phase φ_m, whose expression is a function of the specified distribution law for the instantaneous values of the signal,

– sum of the M sinusoids at each time.

4.13.2. *Expression for phase*

4.13.2.1. *Gaussian law*

It is shown that one can obtain a normal distribution of the signal's instantaneous values when the phase is equal to [KNU 98]

$$\varphi_m = 2\pi\sqrt{-2\ln r_1}\cos(2\pi r_2) \qquad [4.96]$$

or

$$\varphi_m = 2\pi\sqrt{-2\ln r_1}\sin(2\pi r_2) \qquad [4.97]$$

In these expressions, r_1 and r_2 are two random numbers obeying a rectangular distribution in the interval [0, 1].

Definition

A random variable r has an uniform or rectangular distribution in the interval [a, b] if its probability density obeys

$$p(r) = \begin{cases} \dfrac{1}{b-a} & \text{for } a \le r \le b \\ 0 & \text{for } r < a \text{ or } r > b \end{cases} \qquad [4.98]$$

If a random variable is uniformly distributed about [0, 1], the variable $y = a + (b - a)\, r$ is uniformly distributed about [a, b], having a mean of $\dfrac{a+b}{2}$ and standard deviation $s = \dfrac{b-a}{2\sqrt{3}}$.

4.13.2.2. *Other laws*

We want here to create a signal whose instantaneous values obey a given distribution law $F(X)$. This function being nondecreasing, the probability that $x \le X$ is equal to [DAH 74]:

$$P(x \le X) = P[F(x) \le F(X)] \qquad [4.99]$$

Let us set $F(x) = r$ where r is a random variable uniformly distributed about $[0,1]$. It then becomes:

$$P[F(x) \le F(X)] = P[r \le F(X)] \qquad [4.100]$$

From definition of the uniform distribution, $P(r \le R) = R$ where R is an arbitrary number between 0 and 1, yielding $P(x \le X) = P[r \le F(X)] = F(X)$. To create a signal of distribution $F(X)$, it is necessary thus that:

$$F(x) = r \qquad [4.101]$$

The problem can also be solved by setting:

$$F(x) = 1 - r \qquad [4.102]$$

Examples

1. Signal of exponential distribution: the distribution is defined by (Appendix A1)

$$F(X) = 1 - e^{-\lambda X} \qquad [4.103]$$

From [4.102],

$$1 - e^{-\lambda x} = 1 - r \qquad [4.104]$$

yielding

$$x = -\frac{\ln r}{\lambda} \qquad [4.105]$$

and

$$\varphi_m = -2\pi \frac{\ln r}{\lambda} \qquad [4.106]$$

2. Signal with Weibull distribution: from (Appendix A.1)

$$F(X) = \begin{cases} 1 - \exp\left[-\left(\dfrac{X - \varepsilon}{\nu - \varepsilon}\right)^{\alpha}\right] & X > \varepsilon \\ 0 & X < \varepsilon \end{cases} \qquad [4.107]$$

it is shown in a similar way that we must have:

$$x = \varepsilon + (\nu - \varepsilon)\left(-\ln r\right)^{1/\alpha} \qquad [4.108]$$

$$\varphi_m = 2\,\pi\left[\varepsilon + (\nu - \varepsilon)\left(-\ln r\right)^{1/\alpha}\right] \qquad [4.109]$$

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Chapter 5

Properties of random vibration in the time domain

5.1. Averages

5.1.1. *Mean value*

The mean value of a random vibration $\ell(t)$ calculated over duration T,

$$m = \frac{1}{T} \int_0^T \ell(t) \, dt \qquad [5.1]$$

is related to the difference between the positive and negative areas ranging between the curve $\ell(t)$ and the time axis [GRE 81].

The mean m of a centered signal is zero, so this parameter cannot be used by itself alone to correctly evaluate the severity of the excitation.

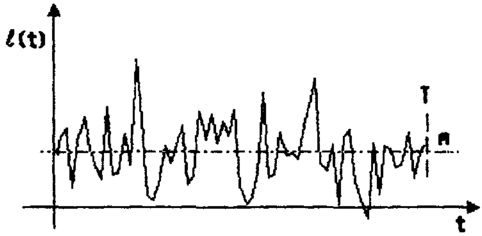


Figure 5.1. *Random vibration with non-zero mean*

The mean value is equal to the absolute value of the parallel shift of the Ot axis necessary to cancel out this difference. A signal $\ell(t)$ of mean m, can be written:

$$\ell(t) = m + \ell^{*}(t) \qquad [5.2]$$

where $\ell^{*}(t)$ is a centered signal. This mean value is in general a static component which can be due to the weight of the structure, to the manoeuvrings of an aircraft, to the thrust of a missile in phase propulsion etc. In practice, one often considers this mean to be zero.

5.1.2. *Mean quadratic value; rms value*

The rms value is calculated from the mean quadratic value of the instantaneous values of the signal. The dispersion of the signal around its mean is characterized by:

$$s^2 = \frac{1}{T}\int_0^T [\ell(t) - m]^2 \, dt = \frac{1}{T}\int_0^T \ell^{*2}(t) \, dt \qquad [5.3]$$

$$s^2 = \ell_{rms}^2 - m^2$$

It is pointed out that s^2 is the variance of the distribution of the instantaneous values of $\ell(t)$ and that s is the standard deviation. Two signals having very different frequency contents, corresponding to very dissimilar temporal forms, can have the same mean quadratic value. In this expression, the rms value takes into account the totality of the frequencies of the signal.

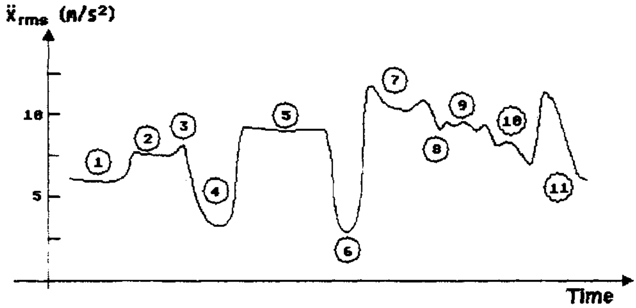
If the mean m is zero, the standard deviation s is equal to the rms value of the signal $\ell(t)$.

NOTE.

On the assumption of zero mean, one can however note a difference between the standard deviation and the rms value when the latter is calculated starting from the power spectral density, which does not necessarily cover the whole of the frequential contents of the signal, in particular beyond 2000 Hz (value often selected as upper limit of the analysis band). The rms value is then lower than the standard deviation. The comparison of the two values makes it possible to evaluate the importance of the neglected range.

Example

Vibratory environment on an aircraft, represented by acceleration as function of time:



1. Taxi
2. Takeoff
3. Climb
4. Cruise at high altitude
5. Maximum velocity at high altitude
6. Cruise at low altitude
7. Maximum velocity at low altitude
8. Climbing turn
9. Deceleration
10. Landing approach
11. Touchdown

Figure 5.2. *Rms acceleration recorded on a aircraft during flight [KAT 65]*

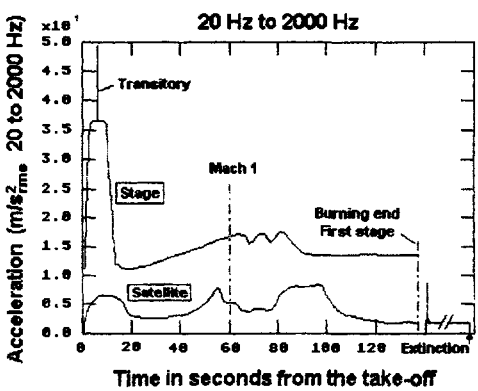


Figure 5.3. *Rms value of vibrations measured on a satellite during launch*

It is very useful to plot the variations of the rms value against time (sliding mean on n points), in order to:

– choose the time intervals over which the rms value varies little: each corresponding phase can then be characterized by a PSD,

– study the very short duration phenomena (nonstationary phenomena). The analysis for example measure the number of times that the rms value crosses a given threshold with respect to the amplitude of this threshold (rms value of the total signal or of the response of a one-degree-of-freedom mechanical system of constant Q factor, generally equal to 10, whose natural frequency varies in the useful frequency band) [KEL 61].

The variation of the rms value with time has also been used as a monitoring tool the correct operation of rotating machinery based on a statistical study of their vibratory behaviour [ALL 82] [PAR 82].

5.2. Statistical properties of instantaneous values of random signal

The analysis of the statistical properties of the instantaneous values of a random signal $\ell(t)$ is based primarily on the work of S.O. Rice [RIC 44] and of S.H. Crandall [CRA 63]. One is more particularly interested in the study of the probability density of the instantaneous values of the signal and in that of the peaks (positive and negative maximum amplitude).

This study results in considering simultaneously at a given time $\ell(t)$ and its derivatives $\dot{\ell}(t)$ and $\ddot{\ell}(t)$ which respectively represent the value of the signal, its slope and its curvature at the time t. These parameters are in particular associated with a multidimensional normal probability density function of the form [BEN 58]:

$$p(\ell_1, \ell_2, \cdots, \ell_n) = (2\pi)^{-\frac{n}{2}} |M|^{-\frac{1}{2}} \exp\left(-\frac{1}{2\,|M|} \sum_{i,j=1}^{n} M_{ij}\, \ell_i\, \ell_j\right) \qquad [5.4]$$

for the research of the distribution law of the peak values.

5.2.1. *Distribution of instantaneous values*

The distribution of the instantaneous values of the parameter describing the random phenomenon can very often be represented by a Gaussian law [MOR 75]. There can of course be particular cases where this assumption is not justified, for example, for vibrations measured on the axle of a vehicle whose suspension has just compressed an elastic thrust after deflection of the dampers (non linear behaviour in compression only).

5.2.2. *Properties of derivative process*

Let us consider a stationary random vibration $\ell(t)$ and its derivative $\dot{\ell}(t)$, defined by:

$$\dot{\ell}(t) = \lim_{\Delta t \to 0} \left[\frac{\ell(t+\Delta t) - \ell(t)}{\Delta t} \right] \qquad [5.5]$$

With the condition that

$$\lim_{\Delta t \to 0} E\left\{ \left[\frac{\ell(t+\Delta t) - \ell(t)}{\Delta t} - \dot{\ell}(t) \right]^2 \right\} = 0 \qquad [5.6]$$

Average value of the derivative process

This is

$$E[\dot{\ell}] = \lim_{\Delta t \to 0} E\left[\frac{\ell(t+\Delta t) - \ell(t)}{\Delta t} \right] \qquad [5.7]$$

If the process is stationary,

$$E[\ell(t+\Delta t)] = E[\ell(t)]$$

yielding

$$E[\dot{\ell}] = 0 \qquad [5.8]$$

NOTE.

The autocorrelation $R_\ell(\tau)$ *presents an absolute maximum for* $\tau = 0$. *One thus has* $R''_\ell(0) < 0$.

$$E[\ell\,\dot{\ell}] = E[\ell(t)\,\dot{\ell}(t)] = \lim_{\Delta t \to 0} E\left[\ell(t)\,\frac{\ell(t+\Delta t) - \ell(t)}{\Delta t} \right] \qquad [5.9]$$

$$E[\ell\,\dot{\ell}] = R'_\ell(0) \qquad [5.10]$$

The derivative of the autocorrelation function of a derivable process is:

– continuous and derivable at any point,

– even.

It is thus cancelled for $\tau = 0$, yielding

$$E[\ell \, \dot{\ell}] = 0 \tag{5.11}$$

There is no correlation between a stationary process $\ell(t)$ and the derivative process $\dot{\ell}(t)$ (whatever the distribution law).

Mean square of the derivative

$$E\left\{\left[\frac{\ell(t+\Delta t) - \ell(t)}{\Delta t}\right]^2\right\} = 2\,\frac{R_\ell(0) - R_\ell(-\Delta t)}{(\Delta t)^2}$$

$$E\left\{[\dot{\ell}(t)]^2\right\} = -R''_\ell(0) \tag{5.12}$$

A stationary process $\ell(t)$ is thus derivable in the mean square sense if and only if its correlation function $R_\ell(\tau)$ contains a continuous second derivative.

Correlation function of the process and its derivative

1. By definition [1.48], $R_{\dot{\ell}\dot{\ell}}(\tau) = E[\dot{\ell}(t)\ \dot{\ell}(t+\tau)]$

$$R_{\dot{\ell}\dot{\ell}}(\tau) = \lim_{\substack{\Delta t_1 \to 0 \\ \Delta t_2 \to 0}} E\left[\frac{\ell(t+\Delta t_1) - \ell(t)}{\Delta t_1}\ \frac{\ell(t+\tau+\Delta t_2) - \ell(t+\tau)}{\Delta t_2}\right]$$

$$R_{\dot{\ell}\dot{\ell}}(\tau) = \lim_{\substack{\Delta t_1 \to 0 \\ \Delta t_2 \to 0}} \frac{R_\ell(\tau + \Delta t_2 - \Delta t_1) - R_\ell(\tau - \Delta t_1) - R_\ell(\tau + \Delta t_2) - R_\ell(\tau)}{\Delta t_1\ \Delta t_2}$$

$$R_{\dot{\ell}\dot{\ell}}(\tau) = \lim_{\Delta t_1 \to 0} \frac{R''_\ell(t - \Delta t_1) - R'_\ell(\tau)}{\Delta t_1}$$

$$R_{\ddot{\ell}\ell}(\tau) = -\frac{d^2 R_\ell(\tau)}{d\tau^2} = -R''_\ell(\tau) \quad [5.13]$$

2. $R_{\ell\dot{\ell}}(\tau) = E[\ell(t)\, \dot{\ell}(t+\tau)]$

$$R_{\ell\dot{\ell}}(\tau) = \lim_{\Delta t \to 0} E\left[\ell(t)\, \frac{\ell(t+\tau+\Delta t) - \ell(t+\tau)}{\Delta t}\right]$$

$$R_{\ell\dot{\ell}}(\tau) = \lim_{\Delta t \to 0} \left[\frac{R_\ell(\tau+\Delta t) - R_\ell(\tau)}{\Delta t}\right]$$

$$R_{\ell\dot{\ell}}(\tau) = \frac{dR_\ell(\tau)}{d\tau} = R'_\ell(\tau) \quad [5.14]$$

In the same way:

$$R_{\dot{\ell}\ell}(\tau) = -\frac{dR_\ell(\tau)}{d\tau} = -R'_\ell(\tau) \quad [5.15]$$

In a more general way, if $\ell^{(m)}$ and $u^{(n)}$ are the m[th] derivative processes of $\ell(t)$ and n[th] of $u(t)$, one has, if the successive derivatives exist,

$$\boxed{R_{\ell^{(m)}u^{(n)}} = (-1)^m \frac{d^{m+n} R_{\ell u}(\tau)}{d\tau^{m+n}}} \quad [5.16]$$

Variance of the derivative process

$$E[\dot{\ell}^2] = R_{\dot{\ell}\dot{\ell}}(0) = -R''(0) \quad [5.17]$$

Since $E[\dot{\ell}] = 0$, the variance $s_{\dot{\ell}}^2$ is equal to

$$s_{\dot{\ell}}^2 = E[\dot{\ell}^2] - [E(\dot{\ell})]^2 \quad [5.18]$$

Power spectral density of the derivative process

By definition [2.45]:

$$R_{\dot{\ell}}(\tau) = \int_{-\infty}^{+\infty} S_{\dot{\ell}}(\Omega)\ e^{i\Omega t}\ d\Omega$$

Knowing that:

$$R_{\dot{\ell}}(\tau) = -R''_{\ell}(\tau)$$

$$R_{\dot{\ell}}(\tau) = -\int_{-\infty}^{+\infty} (i\,\Omega)^2\ S_{\ell}(\Omega)\ e^{i\Omega\tau}\ d\Omega \qquad [5.19]$$

This yields

$$\boxed{S_{\dot{\ell}}(\Omega) = \Omega^2\ S_{\ell}(\Omega)} \qquad [5.20]$$

$$E\left[\dot{\ell}^2\right] = \int_{-\infty}^{+\infty} S_{\dot{\ell}}(\Omega)\ d\Omega = \int_{-\infty}^{+\infty} \Omega^2\ S_{\ell}(\Omega)\ d\Omega \qquad [5.21]$$

and of the same way [MOR 56] [NEW 75] [SVE 80]:

$$E\left[\ddot{\ell}^2\right] = \int_{-\infty}^{+\infty} S_{\ddot{\ell}}(\Omega)\ d\Omega = \int_{-\infty}^{+\infty} \Omega^4\ S_{\ell}(\Omega)\ d\Omega \qquad [5.22]$$

$$\boxed{S_{\ddot{\ell}}(\Omega) = \Omega^4\ S_{\ell}(\Omega)} \qquad [5.23]$$

$$R_{\ddot{\ell}\ddot{\ell}}(\tau) = -R_{\dot{\ell}\dot{\ell}}(\tau) = \frac{d^4 R_{\ell}(\tau)}{d\tau^4} \qquad [5.24]$$

NOTES.

The autocorrelation functions of the derivative processes of $\ell(t)$ depend only on τ. The derivatives of a stationary process are stationary functions. However, the integral of a stationary function is not necessarily stationary.

The result obtained shows the existence of a transfer function $H(\Omega)$ between $\ell(t)$ and its derivatives:

$$S_{\dot{\ell}}(\Omega) = |H(\Omega)|^2\ S_{\ell}(\Omega) \qquad [5.25]$$

$$S_{\ddot{\ell}}(\Omega) = |H(\Omega)|^4\ S_{\ell}(\Omega) \qquad [5.26]$$

where

$$H(\Omega) = i\,\Omega$$

5.2.3. *Number of threshold crossings per unit time*

Let us consider a stationary and ergodic random vibration $\ell(t)$; and $p(\ell)$, the probability density function of the instantaneous values of $\ell(t)$. Let us seek to determine the number of times per unit time n_a^+ the signal crosses a threshold chosen *a priori* with a positive slope.

Let us set n_a the number of occasions per unit time that the signal crosses the interval a, a + da with a positive or negative arbitrary slope, da being an very small interval corresponding to the time increment dt. We have, on average,

$$n_a^+ = \frac{n_a}{2} \qquad [5.27]$$

Let us set n_0^+ the number of occasions per unit time that the signal crosses the threshold $a = 0$ with a positive slope (n_0^+ gives an indication of the average frequency of the signal). Let us set finally $\dot{\ell}(t)$ the derivative of the process $\ell(t)$ and b the value of $\dot{\ell}(t)$ when $\ell = a$. Let us suppose that the time interval dt is sufficiently small that the variation of the signals between t and t + dt is linear. To cross the threshold a, the process must have a velocity $\dot{\ell}(t)$ greater than $\dfrac{a - \ell(t)}{dt}$.

The probability of crossing is related to the joint probability density $p(\ell, \dot{\ell})$ between ℓ and $\dot{\ell}$. Given a threshold a, the probability that:

$$a < \ell(t) \le a + da$$

and [5.28]

$$b < \dot{\ell}(t) \le b + db$$

is thus, in a time unit,

$$p(a,b)\, da\, db = P\left[a < \ell(t) \le a + da,\; b < \dot{\ell}(t) \le b + db\right] \qquad [5.29]$$

Setting t_a the time spent in the interval da:

$$t_a = \frac{da}{|b|} \qquad [5.30]$$

(t_a being a primarily positive quantity). The number of passages per unit time in the interval a, a + da for $\dot{\ell}(t) = b$ is thus:

$$\frac{p(a,\, b)\ da\ db}{t_a} = |b|\ p(a,\, b)\ da \qquad [5.31]$$

and the average total number of crossings of the threshold a, per unit time, for all the possible values of $\dot{\ell}(t)$ is written:

$$\boxed{n_a = \int_{-\infty}^{+\infty} |b|\ p(a,\, b)\ db = 2\ n_a^+} \qquad [5.32]$$

where

$$n_a^+ = \left| \int_0^{\infty} p(\ell,\, \dot{\ell})\ d\ell\ d\dot{\ell} \right|_{\ell=a} \qquad [5.33]$$

This expression is sometimes called the *Rice formula.* The only assumption considered is that of the stationarity. One deduces some, for $a = 0$,

$$\boxed{n_0 = 2\ n_0^+ = \int_{-\infty}^{+\infty} |b|\ p(a,\, b)\ db} \qquad [5.34]$$

and

$$\frac{n_a}{n_0} = \frac{n_a^+}{n_0^+} = \frac{\int_{-\infty}^{+\infty} |b|\ p(a,\, b)\ db}{\int_{-\infty}^{+\infty} |b|\ p(0,\, b)\ db} \qquad [5.35]$$

These expressions can be simplified since the signals $\ell(t)$ and $\dot{\ell}(t)$ are statistically independent:

$$p(\ell,\, \dot{\ell}) = p(\ell)\ \pi(\dot{\ell}) \qquad [5.36]$$

Then,

$$n_a = p(a) \int_{-\infty}^{+\infty} |b|\ \pi(b)\ db \qquad [5.37]$$

and

$$\frac{n_a}{n_0} = \frac{n_a^+}{n_0^+} = \frac{p(a)}{p(0)} \qquad [5.38]$$

Lastly, if $\pi(\dot{\ell})$ is an even function of b,

$$\pi(b) = \pi(-b)$$

yielding

$$n_a = 2\ p(a) \int_0^{\infty} b\ \pi(b)\ db \qquad [5.39]$$

Particular case

If the function $\dot{\ell}(t)$ has instantaneous values distributed according to a Gaussian law, zero mean and variance $\dot{\ell}^2_{rms}$, such that

$$\pi(\dot{\ell}) = \frac{1}{\dot{\ell}_{rms}\ \sqrt{2\pi}}\ e^{-\frac{\dot{\ell}^2}{2\,\dot{\ell}^2_{rms}}} \qquad [5.40]$$

it comes, starting from [5.37]:

$$n_a = p(a) \int_{-\infty}^{+\infty} \frac{|b|}{\dot{\ell}_{rms}\ \sqrt{2\pi}}\ e^{-\frac{b^2}{2\,\dot{\ell}^2_{rms}}}\ db \qquad [5.41]$$

or since $\pi(\dot{\ell})$ is even,

$$n_a = \frac{2\ \dot{\ell}_{rms}}{\sqrt{2\pi}}\ p(a) \qquad [5.42]$$

If the instantaneous acceleration is itself distributed according to a Gaussian law $(0,\ \ell_{rms})$:

$$p(\ell) = \frac{1}{\ell_{rms}\ \sqrt{2\pi}}\ e^{-\frac{\ell^2}{2\,\ell^2_{rms}}} \qquad [5.43]$$

$$p(\ell,\dot{\ell}) = \frac{1}{2\ \pi\ \ell_{rms}\ \dot{\ell}_{rms}}\ e^{-\frac{1}{2}\left(\frac{\ell^2}{\ell^2_{rms}}+\frac{\dot{\ell}^2}{\dot{\ell}^2_{rms}}\right)} \qquad [5.44]$$

and [LEY 65] [LIN 67] [NEW 75] [PRE 56a] [THR 64] [VAN 75]:

$$n_a = \frac{1}{\pi}\ \frac{\dot{\ell}_{rms}}{\ell_{rms}}\ e^{-\frac{a^2}{2\,\ell^2_{rms}}} \qquad [5.45]$$

$$n_0 = \frac{1}{\pi} \frac{\dot{\ell}_{rms}}{\ell_{rms}} \qquad [5.46]$$

$$n_a = n_0 \, e^{-\frac{a^2}{2 \ell_{rms}^2}} \qquad [5.47]$$

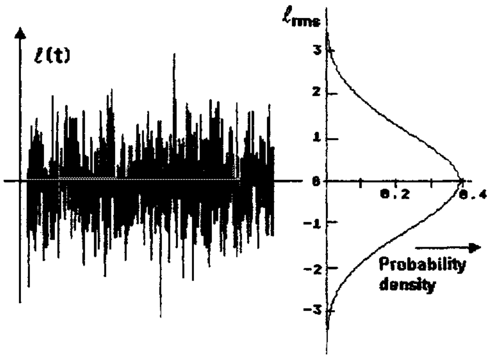


Figure 5.4. *Probability density of instantaneous values of a random signal*

Since

$$\ell_{rms}^2 = \int_0^\infty G(\Omega) \, d\Omega = R(0) \qquad [5.48]$$

$$\dot{\ell}_{rms}^2 = \int_0^\infty \Omega^2 \, G(\Omega) \, d\Omega = -R''(0) \left[-R^{(2)}(0)\right] \qquad [5.49]$$

$$\ddot{\ell}_{rms}^2 = \int_0^\infty \Omega^4 \, G(\Omega) \, d\Omega = R^{(4)}(0) \qquad [5.50]$$

there results [DEE 71] [VAN 75]:

$$n_a = 2 \, n_a^+ = \frac{1}{\pi} \left[\frac{\int_0^\infty \Omega^2 \, G(\Omega) \, d\Omega}{\int_0^\infty G(\Omega) \, d\Omega} \right]^{\frac{1}{2}} \exp\left(- \frac{a^2}{2 \, \ell_{rms}^2} \right) \qquad [5.51]$$

n_a is the mean number of crossings of the threshold a per unit time.

n_a^+ is the mean number of crossings of the threshold a with positive slope and per unit time.

5.2.4. *Average frequency*

Let us set [PAP 65] [PRE 56b]:

$$n_0 = \frac{1}{\pi}\left[\frac{\int_0^\infty \Omega^2 \, G(\Omega)\, d\Omega}{\int_0^\infty G(\Omega)\, d\Omega}\right]^{\frac{1}{2}} = \frac{1}{\pi}\sqrt{-\frac{R^{(2)}(0)}{R(0)}} \qquad [5.52]$$

Depending on f, n_0 becomes [BEN 58] [BOL 84] [CRA 63] [FUL 61] [HUS 56] [LIN 67] [POW 58] [RIC 64] [SJÖ 61] [SWA 63]:

$$\boxed{n_0 = 2\, n_0^+ = 2\left[\frac{\int_0^\infty f^2 \, G(f)\, df}{\int_0^\infty G(f)\, df}\right]^{\frac{1}{2}}} \qquad [5.53]$$

The quantity n_0^+ (*average* or *expected* frequency) can be regarded as the frequency at which energy is most concentrated in the spectrum (apparent frequency of the spectrum).

Band-limited white noise

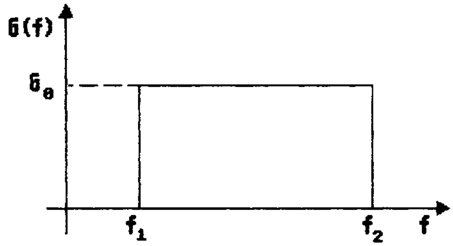


Figure 5.5. *PSD of a band-limited white noise*

If the PSD is defined by

$$\begin{cases} G(f) = G_0 & \text{for } f_1 \le f \le f_2 \\ G(f) = 0 & \text{elsewhere} \end{cases},$$

we have

$$n_0^+ = \left[\frac{f_2^3 - f_1^3}{3\,(f_2 - f_1)} \right]^{\frac{1}{2}} \qquad [5.54]$$

$$n_0^+ = \sqrt{\frac{f_1^2 + f_1\, f_2 + f_2^2}{3}} \qquad [5.55]$$

Ideal low-pass filter

If $f_1 = 0$,

$$n_0^+ = \frac{f_2}{\sqrt{3}} = 0.577\ f_2 \qquad [5.56]$$

Case of a narrow band noise

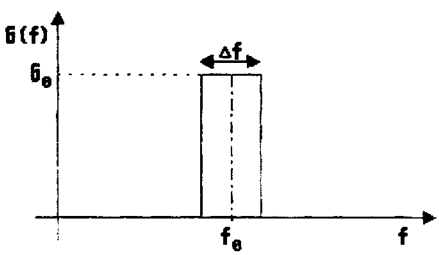


Figure 5.6. *PSD of a narrow band noise*

Let us consider a random vibration of constant PSD $G(\Omega) = G_0$ in the interval $\Delta\Omega$, zero elsewhere. We have [COU 70] [NEW 75]:

$$\dot{\ell}_{rms}^2 = \int_0^\infty (2\,\pi\,f)^2\ G(\Omega)\ d\Omega$$

$$\ddot{\ell}_{rms}^2 = \int_0^\infty (2\pi f)^4 \, G(\Omega) \, d\Omega$$

$$\dot{\ell}_{rms}^2 = G_0 \int_0^\infty \Omega^2 \, d\Omega = G_0 \, \omega_0^2 \, \Delta\Omega \qquad [5.57]$$

$$\ddot{\ell}_{rms}^2 = G_0 \int_0^\infty \Omega^4 \, d\Omega = G_0 \, \omega_0^4 \, \Delta\Omega \qquad [5.58]$$

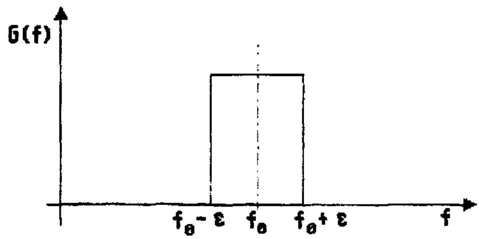


Figure 5.7. *PSD of a narrow band noise*

Let us set $\Delta f = 2\,\varepsilon$. We have $f_1 = f_0 - \varepsilon$ and $f_2 = f_0 - \varepsilon$, yielding

$$n_0^+ = \left[\frac{(f_0 - \varepsilon)^2 + (f_0 - \varepsilon)(f_0 + \varepsilon) + (f_0 + \varepsilon)^2}{3} \right]^{\frac{1}{2}}$$

and

$$\boxed{n_0^+ = f_0 \sqrt{1 + \frac{\varepsilon^2}{3\, f_0^2}} = f_0 \sqrt{1 + \frac{1}{12}\left(\frac{\Delta f}{f_0}\right)^2}} \qquad [5.59]$$

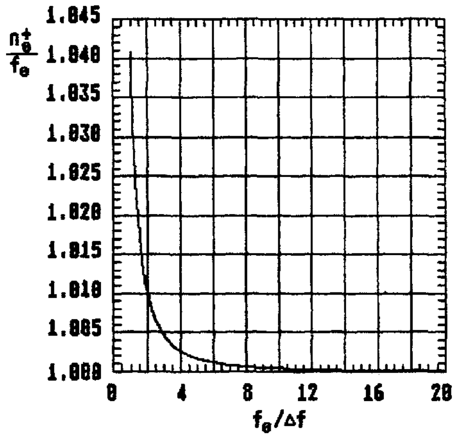


Figure 5.8. *Ratio of average frequency/central frequency of a narrow band noise*

n_0^+ tends towards f_0 when Δf tends towards zero. For whatever f_0, n_0^+ is equal to or higher than f_0.

In the case of the response of a linear slightly damped one-degree-of-freedom system, n_0^+ will be thus in general close to the natural frequency f_0 of the system.

5.2.5. *Threshold level crossing curves*

Threshold level crossing curves give, depending on the threshold a, the number of crossings of this threshold with positive slope. These curves can be plotted:

– either from the time history signal by effective counting of the crossings with positive slope over a duration T. For a given signal, the result is deterministic,

– or from the power spectral density of the vibration, by supposing that the distribution of the instantaneous values of the signal follows a Gaussian law to zero mean. One obtains here the expected value of the number of threshold crossings a over the duration T [LEA 69] [RIC 64]:

$$N_a^+ = n_a^+ \, T = n_0^+ \, T \, e^{-\frac{a^2}{2 \ell_{rms}^2}} \qquad [5.60]$$

with n_0^+= expected frequency defined in [5.53]:

$$n_0^{+2} = \frac{\int_0^\infty f^2\, G(f)\, df}{\int_0^\infty G(f)\, df}$$

The knowledge of $G(f)$ makes it possible to calculate n_0^+ and ℓ_{rms}, then to plot N_a^+ as a function of the threshold value a. In practice, one generally represents a with respect to N_a^+, the first value of N_a^+ being higher or equal to 1. For $N_a^+ = 1$,

$$\boxed{a_0 = \ell_{rms}\sqrt{2 \ln N_0^+} = \ell_{rms}\sqrt{2 \ln n_0^+ T}} \qquad [5.61]$$

a_0 is, on average, the strongest value of the signal observed over a duration T.

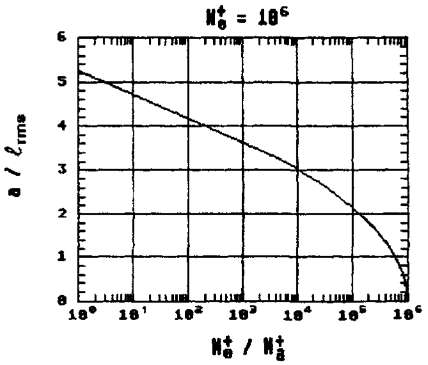


Figure 5.9. *Example of threshold level crossing curve for a Gaussian signal*

The curve in Figure 5.9 shows the variations of $\frac{a}{\ell_{rms}}$ with respect to $\frac{N_a^+}{N_0^+}$, plotted starting from the expression:

$$\frac{a}{\ell_{rms}} = \sqrt{2\,\frac{N_a^+}{N_0^+}}$$

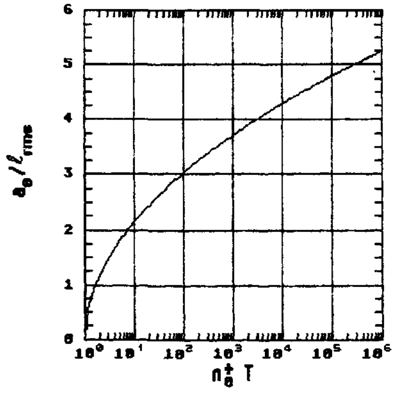


Figure 5.10. *Largest peak, on average, over a given duration*

The variations of $\dfrac{a_0}{\ell_{rms}}$ as function of the product $n_0^+ T$ are represented in Figure 5.10:

$$\frac{a_0}{\ell_{rms}} = \sqrt{2 \ln n_0^+ T}$$

It is observed that it is possible to obtain, in very realistic situations, combinations of n_0^+ and T such that the ratio $\dfrac{a_0}{\ell_{rms}}$ is equal to or higher than 5. For this, it is necessary that:

$$n_0^+ T \ge e^{25/2}$$

$$n_0^+ T \ge 2.7\ 10^5$$

For $T = 600$ s it is necessary that $n_0^+ \ge 447$ Hz

$T = 3600$ s $n_0^+ \ge 74.5$ Hz

$T = 4$ hours $n_0^+ \ge 18.6$ Hz

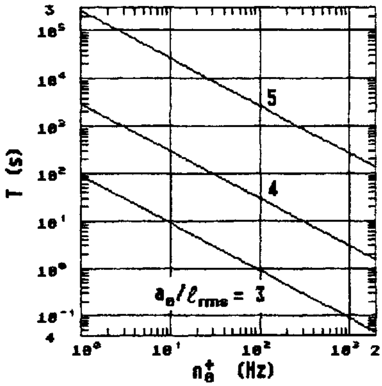


Figure 5.11. *Time necessary to obtain, on average, a given maximum level, versus the average frequency*

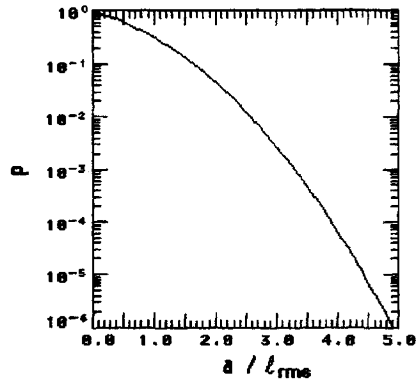


Figure 5.12. *Probability of crossing a given threshold, versus the threshold value*

Figure 5.11 indicates the duration T necessary to obtain a given ratio a_0/ℓ_{rms}, as function of n_0^+.

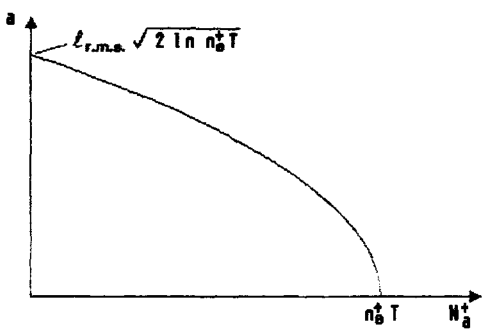


Figure 5.13. *Noteworthy points on the threshold level crossings curve*

For a = 0,

$$N_a^+ = N_0^+ = n_0^+ \, T$$

The probability that the signal crosses the level a with a positive slope and that $a \le \ell \le a + \Delta a$ is equal to $N_a^+ \, \Delta a / N_0^+ \, \ell_{rms}$.

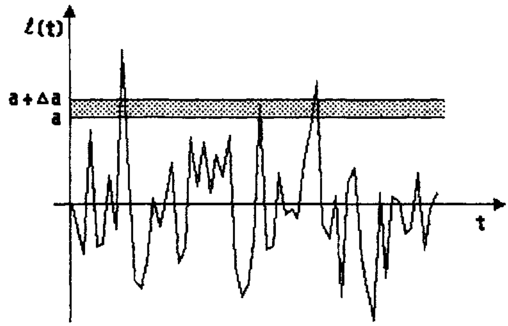


Figure 5.14. *Values of the signal in the interval* a, a + da, *after crossing threshold a with positive slope*

The probability that $\ell(t)$ is higher than a is equal to:

$$P = \frac{1}{N_0^+ \, \ell_{rms}} \int_a^{\infty} N_a^+ \, da \qquad [5.62]$$

$$P = \frac{1}{\ell_{rms}} \int_a^{\infty} e^{-\frac{a^2}{2\,\ell_{rms}^2}} \, da$$

Knowing that the error function can be written:

$$\operatorname{Erf}\left(\frac{u}{\sqrt{2}}\right) = \sqrt{\frac{2}{\pi}} \int_0^u e^{-\frac{u^2}{2}} \, du \qquad [5.63]$$

and that:

$$\int_{-\infty}^{+\infty} e^{-\frac{u^2}{2}} \, du = \sqrt{2\,\pi}$$

resulting in, if $u = a/\ell_{rms}$,

$$P = \left[\sqrt{\frac{\pi}{2}} - \int_0^u e^{-\frac{u^2}{2}} \, du\right] \qquad [5.64]$$

This yields, after standardization:

$$P(u > \frac{a}{\ell_{rms}}) = \left[1 - Erf\left(\frac{a}{\ell_{rms}\sqrt{2}}\right)\right] \quad [5.65]$$

Figure 5.12 shows the variations of $P(u > a/\ell_{rms})$ for a/ℓ_{rms} ranging between 0 and 5.

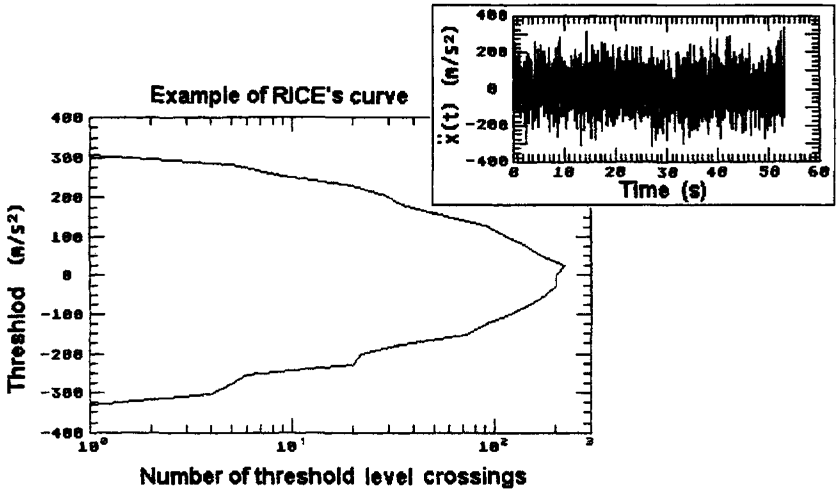


Figure 5.15. *Example of threshold level crossing curve*

Example

Let us consider a random acceleration defined over a duration T = 1 hr by its PSD G(f):

$G(f) = G_1 = 0.1\ m^2s^{-4}/Hz$ from 10 Hz to 50 Hz

$G(f) = G_2 = 0.2\ m^2s^{-4}/Hz$ from 50 Hz to 100 Hz

$G(f) = 0$ elsewhere

$\ddot{x}_{rms}^2 = (50 + 10)\ 0.1 + (100 - 50)\ 0.2 = 14\ (m / s^2)^2$

$$\ddot{x}_{rms} = 3.74 \text{ m/s}^2$$

From [5.53]:

$$n_0^{+2} = \frac{0.1\,(50^3 - 10^3) + 0.2\,(100^3 - 50^3)}{3\,\ddot{x}_{rms}^2}\ \text{Hz}^2$$

$$n_0^+ = 66.8 \text{ Hz}$$

and [5.60]:

$$N_a^+ = 66.8\ 3600\ e^{-\frac{a^2}{2\,14}} = 2.4\,10^5\ e^{-\frac{a^2}{28}}$$

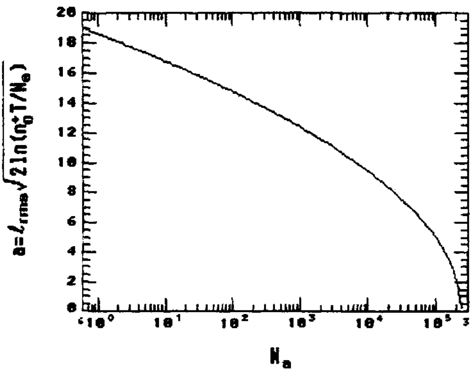


Figure 5.16. *Example of curve threshold level crossings*

The threshold which is only exceeded once on average over the duration T has an amplitude

$$a_0 = 3.74\sqrt{2\ \ln 66.8\ 3600} = 18.62 \text{ m/s}^2$$

These threshold level crossings curves were used to compare the severity of several random vibrations [KAZ 70], to evaluate their damage potential or to reduce the test duration. This method can be justified if the treated signal is the stress applied to a part of a structure, with just one reserve, which is the non immediate relation between the number of peaks and the number of threshold level crossings; it is not, on the other hand, usable starting from the input signal of acceleration. The threshold crossings curve of the excitation $\ddot{x}(t)$ is not representative of the damage undergone by a part which responds at its natural frequency with its Q factor. In random mode, one cannot directly associate a peak of the excitation with a peak of the response.

NOTES.

1. *All the relations of the preceding paragraphs can be applied either to the vibration input on the specimen, or to the response of the specimen.*

2. *ONERA proposed, in 1961 [COU 66], a method of calculation of the PSD* G(f) *of a stationary and Gaussian random signal starting from the average number of zero level crossings, its derivative and the rms value of the signal. The process can be extended to non Gaussian processes.*

5.3. Moments

Many important statistical properties of the signal considered (excitation or response) can be obtained directly from the power spectral density $G(\Omega)$ and in particular *the moments* [VAN 79].

Definition

Given a random signal $\ell(t)$, the *moment of order* n (close to the origin) is the quantity:

$$M_n = E\left\{\left[\frac{d^{n/2}\ell(t)}{dt^{n/2}}\right]^2\right\} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\left[\frac{d^{n/2}\ell(t)}{dt^{n/2}}\right]^2 dt \qquad [5.66]$$

(if the derivative exists). The moment of order zero is none other than the square of the rms value ℓ_{rms}:

$$M_0 = E\left[\ell^2(t)\right] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\ell^2(t)\,dt = \overline{\ell^2(t)} = \ell_{rms}^2$$

$$M_0 = R(0) = \int_0^\infty G(\Omega)\,d\Omega = \int_0^\infty G(f)\,df$$

The moment of order two is equal to:

$$M_2 = E\left[\left(\frac{d\ell}{dt}\right)^2\right] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\left(\frac{d\ell}{dt}\right)^2 dt = \overline{\dot{\ell}^2(t)} \qquad [5.67]$$

However, by definition,

$$R(\tau) = E[\ell(t)\ \ell(t+\tau)]$$

$$R(\tau) = \int_0^\infty G(\Omega)\ \cos \Omega\tau\ d\Omega \tag{5.68}$$

If we set:

$$S(\tau) = E[\dot{\ell}(t)\ \dot{\ell}(t+\tau)] \tag{5.69}$$

$$S(\tau) = \int_0^\infty \Omega^2\ G(\Omega)\ \cos \Omega\tau\ d\Omega = -\frac{d^2R(\tau)}{d\tau^2} \tag{5.70}$$

(if $\ddot{R}$ exists). We have, for $\tau = 0$,

$$S(\tau) = \int_0^\infty \Omega^2\ G(\Omega)\ d\Omega \tag{5.71}$$

In the same way, if:

$$T(\tau) = [\ddot{\ell}(t)\ \ddot{\ell}(t+\tau)] \tag{5.72}$$

$$T(\tau) = \frac{d^4R(\tau)}{d\tau^4} = \int_0^\infty \Omega^4\ G(\Omega)\ \cos \Omega\tau\ d\tau \tag{5.73}$$

it becomes, for $\tau = 0$,

$$T(0) = \int_0^\infty \Omega^4\ G(\Omega)\ d\Omega \tag{5.74}$$

yielding [KOW 69]:

$$M_2 = -\ddot{R}(0) = \int_0^\infty \Omega^2\ G(\Omega)\, d\Omega = (2\pi)^2 \int_0^\infty f^2\ G(f)\, df = \dot{\ell}_{rms}^2 \tag{5.75}$$

$$M_4 = -R^{(4)}(0) = \int_0^\infty \Omega^4\ G(\Omega)\, d\Omega = (2\pi)^4 \int_0^\infty f^4\ G(f)\, df = \ddot{\ell}_{rms}^2 \tag{5.76}$$

More generally, the n^{th} moment can be defined as [CHA 72] [CHA 85] [DEE 71] [PAR 64] [SHE 83] [SWA 63] [VAN 72] [VAN 75] [VAN 79]:

$$M_n = \int_0^\infty \Omega^n \, G(\Omega) \, d\Omega = R^{(n)}(0) \qquad [5.77]$$

or

$$M_n = (2\pi)^n \int_0^\infty f^n \, G(f) \, df$$

(n integer) where

$$R^{(2n)}(0) = (-1)^n \int_0^\infty \Omega^{2n} \, G(\Omega) \, d\Omega \qquad [5.78]$$

M_n are the moments of the PSD $G(\Omega)$ with respect to the vertical axis $f = 0$.

Application

One deduces from the preceding relations [CRA 68] [CHA 72] [LEY 65] [PAP 65] [SHE 83]:

$$\boxed{n_0^+ = \frac{1}{2\pi} \left(\frac{M_2}{M_0} \right)^{\frac{1}{2}}} \qquad [5.79]$$

$$\boxed{n_a^+ = n_0^+ \, e^{-\frac{a^2}{2 M_0}}} \qquad [5.80]$$

NOTES.

Some authors [CHA 85] [FUL 61] [KOW 69] [VAN 79] [WIR 73] [WIR 83] define M_n *by:*

$$M_n = \int_0^\infty f^n \, G(f) \, df \qquad [5.81]$$

which leads to [BEN 58] [CHA 85]:

$$n_0^+ = \left(\frac{M_2}{M_0} \right)^{\frac{1}{2}} \qquad [5.82]$$

(sometimes noted Ω_ℓ*) [VAN 79].*

5.4. Average frequency of PSD defined by straight line segments

5.4.1. *Linear-linear scales*

$$n_0^+ = \frac{1}{2\pi}\sqrt{\frac{M_2}{M_0}}$$

with

$$M_0 = \int_{f_1}^{f_2} G(f)\, df$$

where

$$G(f) = a\, f + b$$

$$a = \frac{G_2 - G_1}{f_2 - f_1} \text{ and } b = \frac{f_2\, G_1 - f_1\, G_2}{f_2 - f_1}$$

$$M_0 = \left[\frac{a}{2}\left(f_2^2 - f_1^2\right) + b\left(f_2 - f_1\right)\right] \qquad [5.83]$$

$$M_2 = (2\pi)^2 \int_{f_1}^{f_2} f^2\, G(f)\, df$$

$$M_2 = (2\pi)^2\left[\frac{a}{4}\left(f_2^4 - f_1^4\right) + \frac{b}{3}\left(f_2^3 - f_1^3\right)\right] \qquad [5.84]$$

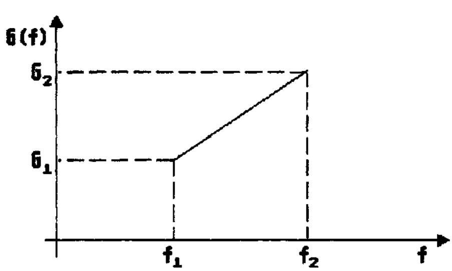


Figure 5.17. *PSD defined by a straight line segment on linear axes*

This yields, after having replaced a and b by their value according to f_1, f_2, G_1 and G_2 [BEN 62],

$$n_0^{+2} = \frac{\left(G_2 - G_1\right)\dfrac{f_2^4 - f_1^4}{4} + \dfrac{\left(f_2 G_1 - f_1 G_2\right)}{3}\left(f_2^3 - f_1^3\right)}{\dfrac{G_2 - G_1}{2}\left(f_2^2 - f_1^2\right) + \left(f_2 G_1 - f_1 G_2\right)\left(f_2 - f_1\right)} \qquad [5.85]$$

Particular cases

$G_1 = G_2 = G_0 = \text{constant}$

$$n_0^{+2} = \frac{f_2^3 - f_1^3}{3\left(f_2 - f_1\right)} = \frac{f_1^2 + f_1\, f_2 + f_2^2}{3} \qquad [5.86]$$

If $f_1 = 0$ and if $G = G_0$ until f_2, there results:

$$n_0^{+2} = \frac{f_2^2}{3} \qquad [5.87]$$

i.e.:

$$n_0^+ = 0.577\ f_2 \qquad [5.88]$$

If the PSD is a narrow band noise centered around f_0, we can set $f_1 = f_0 - \dfrac{\varepsilon}{2}$ and $f_1 = f_0 + \dfrac{\varepsilon}{2}$ [BEN 62], yielding:

$$n_0^{+2} = f_0 + \frac{\varepsilon^2}{12} \qquad [5.89]$$

If $\varepsilon \to 0$,

$$n_0^+ \to f_0$$

5.4.2. *Linear-logarithmic scales*

In this case, the PSD is represented by:

$$\ln G = a f + b \qquad [5.90]$$

$$M_0 = \int_{f_1}^{f_2} e^{a f+b} \, df = \frac{1}{a} \left(e^{a f+b} \right)_{f_1}^{f_2} \qquad [5.91]$$

$$M_2 = (2\pi)^2 \int_{f_1}^{f_2} f^2 \, e^{a f+b} \, df$$

After integration by parts, it becomes, if $a \neq 0$,

$$M_2 = (2\pi)^2 \frac{e^{a f+b}}{a^3} \left(a^2 f^2 - 2 a f + 2 \right) \qquad [5.92]$$

yielding

$$n_0^{+2} = \frac{e^{a f_2+b}\left(a^2 f_2^2 - 2 a f_2 + 2\right) - e^{a f_1+b}\left(a^2 f_1^2 - 2 a f_1 + 2\right)}{a^2 \left(e^{a f_2+b} - e^{a f_1+b}\right)} \qquad [5.93]$$

the constants a and b being calculated starting from the co-ordinates of the points f_1, G_1, f_2 and G_2.

Particular case

$$G_2 = G_1,$$

$(a = 0)$

$$M_2 = 2 \pi \frac{G}{3} \left(f_2^3 - f_1^3 \right) \qquad [5.94]$$

$$M_0 = G \left(f_2 - f_1 \right) \qquad [5.95]$$

and

$$n_0^{+2} = \frac{f_2^3 - f_1^3}{3 \left(f_2 - f_1 \right)} = \frac{f_1^2 + f_1 f_2 + f_2^2}{3} \qquad [5.96]$$

5.4.3. *Logarithmic-linear scales*

$$G(f) = a \ln f + b \quad [5.97]$$

$$M_0 = \int_{f_1}^{f_2} \left(a \ln f + b\right) df$$

$$M_0 = a \left(f \ln f\right)_{f_1}^{f_2} + \left(f_2 - f_1\right)(b - a) \quad [5.98]$$

$$M_2 = (2\pi)^2 \int_{f_1}^{f_2} f^2 \left(a \ln f + b\right) df$$

$$M_2 = (2\pi)^2 \left\{ \frac{f_2^3}{3}\left[a\left(\ln f_2 - \frac{1}{3}\right) + b\right] - \frac{f_1^3}{3}\left[a\left(\ln f_1 - \frac{1}{3}\right) + b\right] \right\} \quad [5.99]$$

$$n_0^{+2} = \frac{f_2^3\left[a\left(\ln f_2 - \frac{1}{3}\right) + b\right] - f_1^3\left[a\left(\ln f_1 - \frac{1}{3}\right) + b\right]}{a\left(f_2 \ln f_2 - f_1 \ln f_1\right) + \left(f_2 - f_1\right)(b - a)} \quad [5.100]$$

Particular case

$$G_1 = G_2 = G_0 = \text{constant}$$

In this case, $a = 0$ and $b = G_0$, yielding:

$$M_2 = (2\pi)^2 \frac{G_0}{3}\left(f_2^3 - f_1^3\right)$$

$$M_0 = G_0\left(f_2 - f_1\right)$$

and

$$\boxed{n_0^{+2} = \frac{f_2^3 - f_1^3}{3\left(f_2 - f_1\right)} = \frac{f_1^2 + f_1 f_2 + f_2^2}{3}} \quad [5.101]$$

5.4.4. *Logarithmic-logarithmic scales*

$$G(f) = G_1 \left(\frac{f}{f_1}\right)^b \qquad [5.102]$$

the constant b being such that $b = \dfrac{\ln G_2/G_1}{\ln f_2/f_1}$.

$$M_0 = \int_{f_1}^{f_2} G_1 \left(\frac{f}{f_1}\right)^b df = \frac{G_1}{f_1^b} \frac{1}{b+1} \left(f^{b+1}\right)_{f_1}^{f_2}$$

(if $b \neq -1$):

$$M_2 = (2\pi)^2 \int_{f_1}^{f_2} f^2\, G(f)\, df = (2\pi)^2 \frac{G_1}{f_1^b} \left(\frac{f^{b+3}}{b+3}\right)_{f_1}^{f_2}$$

(if $b \neq -3$). It yields:

$$n_0^{+2} = \frac{b+1}{b+3} \frac{f_2^{b+3} - f_1^{b+3}}{f_2^{b+1} - f_1^{b+1}} \qquad [5.103]$$

If b = -1:

$$M_0 = G_1\, f_1 \ln \frac{f_2}{f_1}$$

and

$$M_2 = (2\pi)^2\, G_1\, f_1 \frac{f_2^2 - f_1^2}{2}$$

$$n_0^{+2} = \frac{f_2^2 - f_1^2}{2 \ln f_2/f_1} \qquad [5.104]$$

If $b = -3$:

$$M_0 = \frac{G_1 f_1^3}{2}\left(\frac{1}{f_1^2} - \frac{1}{f_2^2}\right)$$

$$M_2 = (2\pi)^2 G_1 f_1^3 \ln \frac{f_2}{f_1}$$

$$n_0^{+2} = 2 f_1^2 f_2^2 \frac{2 \ln f_2/f_1}{f_2^2 - f_1^2} \qquad [5.105]$$

NOTE.

If the PSD is made up of n *straight line segments, the average frequency* n_0^+ *is obtained from:*

$$n_0^{+2} = \frac{1}{(2\pi)^2} \frac{\sum_{i=1}^{n} M_{2_i}}{\sum_{i=1}^{n} M_{0_i}} \qquad [5.106]$$

5.5. Fourth moment of PSD defined by straight line segments

The interest of this parameter lies in its participation, with M_0 and M_2 already studied, in the calculation of n_p^+ and r.

5.5.1. *Linear-linear scales*

By definition,

$$M_4 = (2\pi)^4 \int_0^\infty f^4 \, G(f) \, df$$

$$G(f) = a f + b$$

yielding:

$$\boxed{M_4 = (2\pi)^4 \left[\frac{a}{6}\left(f_2^6 - f_1^6\right) + \frac{b}{5}\left(f_2^5 - f_1^5\right)\right]} \qquad [5.107]$$

where $a = \dfrac{G_2 - G_1}{f_2 - f_1}$ and $b = \dfrac{f_2 G_1 - f_1 G_2}{f_2 - f_1}$.

Particular cases

1. $G_1 = G_2 = G_0 =$ constant, i.e. $a = 0$ and $b = G_0$:

$$M_4 = (2\pi)^4 \left[\frac{G_0}{5}\left(f_2^5 - f_1^5\right)\right] \qquad [5.108]$$

2. $f_1 = 0$

$$M_4 = (2\pi)^4 \left[\frac{a}{6} f_2^6 + \frac{b}{5} f_2^5\right] \qquad [5.109]$$

3. $f_1 = 0$ and $G_0 =$ constant

$$M_4 = (2\pi)^4 \frac{G_0}{5} f_2^5 \qquad [5.110]$$

5.5.2. *Linear-logarithmic scales*

$$G(f) = e^{af+b}$$

$$a = \frac{\ln G_2/G_1}{f_2 - f_1}$$

$$b = \frac{f_2 \ln G_1 - f_1 \ln G_2}{f_2 - f_1}$$

$$M_4 = (2\pi)^4 \int_{f_1}^{f_2} f^4\, G(f)\, df = (2\pi)^4 \int_{f_1}^{f_2} f^4\, e^{a f + b}\, df \qquad [5.111]$$

After several integrations by parts, we obtain, if $a \neq 0$,

$$M_4 = \frac{(2\pi)^4}{a} \left\{ e^{a f_2 + b} \left(f_2^4 - \frac{4 f_2^3}{a} + \frac{12 f_2^2}{a^2} - \frac{24 f_2}{a^3} + \frac{24}{a^4} \right) \right.$$

$$\left. - e^{a f_1 + b} \left(f_1^4 - \frac{4 f_1^3}{a} + \frac{12 f_1^2}{a^2} - \frac{24 f_1}{a^3} + \frac{24}{a^4} \right) \right\} \qquad [5.112]$$

Particular cases

1. $G_1 = G_2 = G_0 =$ constant. Then, $a = 0$ and $b = \ln G_0$

$$M_4 = (2\pi)^2 \, G_0 \, \frac{f_2^5 - f_1^5}{5} \qquad [5.113]$$

2. $f_1 = 0$ and $a = 0$

$$M_4 = \frac{(2\pi)^4}{a} \left[e^{a f_2 + b} \left(f_2^4 - \frac{4}{a} f_2^3 + \frac{12}{a^2} f_2^3 - \frac{24}{a^3} f_2 + \frac{24}{a^4} \right) - \frac{24}{a^4} e^b \right] \qquad [5.114]$$

and, if $G_1 = G_2 = G_0$

$$M_4 = (2\pi)^4 \, G_0 \, \frac{f_2^5}{5} \qquad [5.115]$$

5.5.3. *Logarithmic-linear scales*

$$G = a \ln f + b$$

$$M_4 = (2\pi)^4 \int_{f_1}^{f_2} f^4 \left(a \ln f + b \right) df$$

$$M_4 = (2\pi)^4 \left\{ \frac{f_2^5}{5} \left[a \left(\ln f_2 - \frac{1}{5} \right) + b \right] - \frac{f_1^5}{5} \left[a \left(\ln f_1 - \frac{1}{5} \right) + b \right] \right\} \qquad [5.116]$$

where

$$a = \frac{G_2 - G_1}{\ln f_2/f_1} \text{ and } b = \frac{G_2 \ln f_1 - G_1 \ln f_2}{\ln f_1 - \ln f_2}$$

Particular cases

$G_1 = G_2 = G_0 = \text{constant}$, i.e. $a = 0$ and $b = G_0$:

$$M_4 = (2\pi)^4 \frac{G_0}{5} \left(f_2^5 - f_1^5\right) \qquad [5.117]$$

If $f_1 = 0$:

$$M_4 = (2\pi)^4 \frac{G_0}{5} f_2^5$$

5.5.4. *Logarithmic-logarithmic scales*

$$G = G_1 \left(\frac{f}{f_1}\right)^b$$

yielding, if $b \neq -5$

$$M_4 = (2\pi)^4 \frac{G_1}{f_1^b} \frac{f_2^{b+5} - f_1^{b+5}}{b+5}$$

or

$$M_4 = \frac{(2\pi)^4}{b+5} \left(G_2\, f_2^5 - G_1\, f_1^5\right) \qquad [5.118]$$

If $b = -5$:

$$M_4 = (2\pi)^4 \int_{f_1}^{f_2} G_1\, f^4\, f_1^5 \frac{df}{f^5}$$

$$M_4 = (2\pi)^4\, f_1^5\, G_1 \ln \frac{f_2}{f_1} \qquad [5.119]$$

Particular case

If $G_1 = G_2 = G_0 = \text{constant}$ and if $b \neq -5$

$$M_4 = \frac{(2\pi)^4}{b+5} G_0 \left(f_2^5 - f_1^5\right) \qquad [5.120]$$

NOTE.

If the PSD is made up of n *horizontal segments, the value of* M_4 *is obtained by calculating the sum:*

$$M_4 = \sum_{i=1}^{n} M_{4_i} \qquad [5.121]$$

5.6. Generalization; moment of order n

In a more general way, the moment M_n is given, depending on the case, by the following relations.

5.6.1. *Linear-linear scales*

The order n being positive or zero,

$$M_n = (2\pi)^n \left[\frac{a}{n+2}\left(f_2^{n+2} - f_1^{n+2}\right) + \frac{b}{n+1}\left(f_2^{n+1} - f_1^{n+1}\right)\right] \qquad [5.122]$$

5.6.2. *Linear-logarithmic scales*

$$M_n = (2\pi)^n \left\{ e^{af_2+b}\left[f_2^n - \frac{n}{a} f_2^{n-1} + \frac{n(n-1)}{a^2} f_2^{n-2} \cdots + \frac{n!}{a^n}\right] \right.$$

$$\left. -e^{af_1+b}\left[f_1^n - \frac{n}{a} f_1^{n-1} + \frac{n(n-1)}{a^2} f_1^{n-2} \cdots + \frac{n!}{a^n}\right]\right\} \qquad [5.123]$$

5.6.3. *Logarithmic-linear scales*

$$M_n = (2\pi)^n \left\{ \frac{f_2^{n+1}}{n+1}\left[a\left(\ln f_2 - \frac{1}{n+1}\right) + b\right] - \frac{f_1^{n+1}}{n+1}\left[a\left(\ln f_1 - \frac{1}{n+1}\right) + b\right]\right\} \qquad [5.124]$$

$(n \ge 0)$

5.6.4. *Logarithmic-logarithmic scales*

If $b \neq -(n+1)$:

$$M_n = (2\pi)^n \frac{G_2 f_2^{n+1} - G_1 f_1^{n+1}}{b+n+1} = (2\pi)^n \frac{G_1}{f_1^b} \frac{f_2^{b+n+1} - G_1 f_1^{b+n+1}}{b+n+1} \quad [5.125]$$

If $b = -(n+1)$:

$$M_n = (2\pi)^n f_1^{n+1} G_1 \ln \frac{f_2}{f_1} \quad [5.126]$$

Chapter 6

Probability distribution of maxima of random vibration

6.1. Probability density of maxima

It can be useful, in particular for calculations of damage by fatigue, to know a vibration's average number of peaks per unit time, occurring between two close levels a and a + da as well as the average total number of peaks per unit time.

NOTE.

One is interested here in the maxima of the curve which can be positive or negative (Figure 6.1).

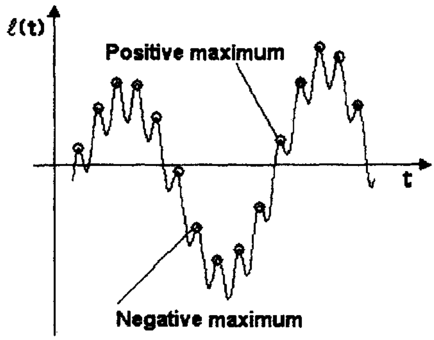


Figure 6.1. *Positive and negative peaks of a random signal*

For a fatigue analysis, it would of course be necessary to also count the minima. One can acknowledge that the average number of minima per unit time of a Gaussian random signal is equal to the average number the maxima per unit time, the distributions of the minima and maxima being symmetrical [CAR 68].

A maximum occurs when the velocity (derivative of the signal) cancels out with negative acceleration (second derivative of signal).

This remark leads one to think that the joint probability density between the processes $\ell(t)$, $\dot{\ell}(t)$ and $\ddot{\ell}(t)$ can be used to describe the maxima of $\ell(t)$. This supposes that $\ell(t)$ is derivable twice.

S.O. Rice [RIC 39] [RIC 44] showed that, if $p(a, b, c)$ is the probability density so that $\ell(t)$, $\dot{\ell}(t)$ and $\ddot{\ell}(t)$ respectively lie between a and $a + da$, b and $b + db$, c and $c + dc$, a maximum being defined by a zero derivative and a negative curvature, the average number the maxima located between levels a and $a + da$ in the time interval t, $t + dt$ (window a, $a + da$, t, $t + dt$) is:

$$\nu_a = -dt\ da \int_{-\infty}^{0} c\ \ p(a, 0, c)\ \ dc \qquad [6.1]$$

where, for a Gaussian signal as well as for its first and second derivatives [CRA 67] [KOW 63]:

$$p(a, 0, c) = (2\pi)^{-3/2}\ |M|^{-1/2}\ \exp\left[-\frac{\mu_{11}\ a^2 + \mu_{33}\ c^2 + 2\ \mu_{13}\ a\ c}{2\ |M|}\right] \qquad [6.2]$$

with

$$\|M\| = \begin{Vmatrix} \ell_{rms}^2 & 0 & -\dot{\ell}_{rms}^2 \\ 0 & \dot{\ell}_{rms}^2 & 0 \\ -\dot{\ell}_{rms}^2 & 0 & \ddot{\ell}_{rms}^2 \end{Vmatrix} \qquad [6.3]$$

Let us recall that:

$$\ell_{rms}^2 = R(0) = M_0$$

$$\dot{\ell}_{rms}^2 = -\ddot{R}(0) = M_2$$

$$\ddot{\ell}_{rms}^2 = R^{(4)}(0) = M_4$$

The determinant $|M|$ is written:

$$|M| = \dot{\ell}^2_{rms} \left(\ell^2_{rms}\, \ddot{\ell}^2_{rms} - \dot{\ell}^4_{rms} \right) = \ell^2_{rms}\, \dot{\ell}^2_{rms}\, \ddot{\ell}^2_{rms} \left(1 - r^2\right) \qquad [6.4]$$

$$|M| = M_0\, M_2\, M_4 \left(1 - r^2\right) \qquad [6.5]$$

if

$$\boxed{r = \frac{\dot{\ell}^2_{rms}}{\ell_{rms}\, \ddot{\ell}_{rms}} = \frac{M_2}{\sqrt{M_0\, M_4}} = \frac{R^{(2)}(0)}{\sqrt{R(0)\, R^{(4)}(0)}}} \qquad [6.6]$$

R is an important parameter named *irregularity factor*. $|M|$ is always positive. The cofactors μ_{ij} are equal, respectively to:

$$\mu_{11} = \dot{\ell}^2_{rms}\, \ddot{\ell}^2_{rms} = M_2\, M_4 \qquad [6.7]$$

$$\mu_{13} = \dot{\ell}^4_{rms} = M_2^2 \qquad [6.8]$$

$$\mu_{33} = \dot{\ell}^2_{rms}\, \ell^2_{rms} = M_0\, M_2 \qquad [6.9]$$

yielding

$$\nu_a = -da\, dt \int_{-\infty}^{0} \frac{c\,(2\pi)^{-3/2}}{\sqrt{M_0\, M_2\, M_4 \left(1 - r^2\right)}}$$

$$\exp\left[-\frac{M_2\, M_4\, a^2 + M_0\, M_2\, c^2 + 2\, M_2^2\, a\, c}{2\, M_0\, M_2\, M_4 \left(1 - r^2\right)} \right] dc \qquad [6.10]$$

$$\nu_a = -\frac{da\, dt\,(2\pi)^{-3/2}}{\sqrt{M_0\, M_2\, M_4 \left(1 - r^2\right)}}\, e^{-\frac{a^2}{2\, M_0 \left(1 - r^2\right)}}$$

$$\int_{-\infty}^{0} c\, \exp\left[-\frac{1}{2\, M_4 \left(1 - r^2\right)} \left(c^2 + \frac{2\, M_2\, a\, c}{M_0} \right) \right] dc$$

$$\nu_a = -\frac{da\ dt\ (2\pi)^{-3/2}}{\sqrt{M_0\ M_2\ M_4\ \left(1-r^2\right)}}\ e^{-\frac{a^2}{2\ M_0\ \left(1-r^2\right)}}$$

$$\int_{-\infty}^{0} c\ \exp\left\{-\frac{1}{2\ M_4\ \left(1-r^2\right)}\left[\left(c+\frac{M_2}{M_0}a\right)^2-\left(\frac{M_2}{M_0}a\right)^a\right]\right\} dc$$

$$\nu_a = -\frac{da\ dt\ (2\pi)^{-3/2}}{\sqrt{M_0\ M_2\ M_4\ \left(1-r^2\right)}}\ e^{-\frac{a^2}{2\ M_0\ \left(1-r^2\right)}}\ e^{\frac{a^2\ r^2}{2\ M_0\ \left(1-r^2\right)}}$$

$$\left\{\int_{-\infty}^{0}\left(c+\frac{M_2}{M_0}a\right)\exp\left[-\frac{\left(c+\frac{M_2}{M_0}a\right)^2}{2\ M_4\ \left(1-r^2\right)}\right]dc-\frac{M_2\ a}{M_0}\int_{-\infty}^{0}\exp\left[-\frac{\left(c+\frac{M_2}{M_0}a\right)^2}{2\ M_4\ \left(1-r^2\right)}\right]dc\right\}$$

Let us set $v = \dfrac{\left(c+\frac{M_2}{M_0}a\right)^2}{2\ M_4\ \left(1-r^2\right)}$ and $w = \sqrt{v}$. It becomes:

$$\nu_a = -\frac{da\ dt\ (2\pi)^{-3/2}}{\sqrt{M_0\ M_2\ M_4\ \left(1-r^2\right)}}\ e^{-\frac{a^2}{2\ M_0}}\left\{M_4\ \left(1-r^2\right)\int_{-\infty}^{a^2\ r^2/2\ M_0\ (1-r^2)} e^{-v}\ dv\right.$$

$$\left.-\frac{M_2\ a}{M_0}\int_{-\infty}^{M_2\ a/M_0\ \sqrt{2\ M_4\left(1-r^2\right)}}\sqrt{2M_4\left(1-r^2\right)}\ e^{-w^2}\ dw\right\}$$

After integration [BEN 58] [RIC 64],

$$\nu_a = \frac{(2\pi)^{-3/2}\ da\ dt}{M_0\ M_2}\left\{\sqrt{M_0\ M_2\ M_4\ \left(1-r^2\right)}\ e^{-\frac{a^2}{2\ M_0\ \left(1-r^2\right)}}\right.$$

$$+\sqrt{\frac{\pi}{2}}\, r \frac{M_2^{3/2}}{\sqrt{M_0}} e^{-\frac{a^2}{2 M_0}} \left[1 + \mathrm{Erf}\left(\frac{a\, r}{\sqrt{2\, M_0 \left(1 - r^2\right)}}\right)\right]\Bigg\} \qquad [6.11]$$

i.e.

$$\nu_a = n_p^+ \, q(a) \, da \, dt \qquad [6.12]$$

where

$$\boxed{n_p^+ = \frac{1}{2\pi} \sqrt{\frac{M_4}{M_2}}} \qquad [6.13]$$

(average number of maxima per second). n_p^+ can be also written:

$$n_p^+ = \frac{1}{2\,\pi} \sqrt{-\frac{R^{(4)}(0)}{R^{(2)}(0)}} \qquad [6.14]$$

NOTE.

ν_a can be written in the form [RIC 64]:

$$\nu_a = \frac{-R^{(2)}(0)}{2\, R(0) \sqrt{R^{(4)}(0)\, R(0)}} \exp\left[-\frac{a^2}{2\, R(0)}\right] \left\{ a \left[1 + \mathrm{Erf}\left(-\frac{a\, R^{(2)}(0)}{\sqrt{2}\, k\, R(0)}\right)\right] \right.$$

$$\left. -\frac{\sqrt{2\, k\, R(0)}}{\sqrt{\pi}\, R^{(2)}(0)} \exp\left[-\frac{\left(R^{(2)}(0)\, a\right)^2}{2\, k\, R(0)}\right] \right\} \qquad [6.15]$$

where

$$k = R(0)\, R^{(4)}(0) - \left[R^{(2)}(0)\right]^2 \qquad [6.16]$$

The probability density of maxima per unit time of a Gaussian signal whose amplitude lies between a and a + da is thus [BRO 63] [CAR 56] [LEL 73] [LIN 72]:

$$q(a) = \frac{\sqrt{1-r^2}}{\ell_{rms}\sqrt{2\pi}} e^{-\frac{a^2}{2\ell_{rms}^2(1-r^2)}} + \frac{r\,a}{2\,\ell_{rms}^2}\left[1 + \mathrm{Erf}\left(\frac{a\,r}{\ell_{rms}\sqrt{2(1-r^2)}}\right)\right] \quad [6.17]$$

where $\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\lambda^2}\, d\lambda$ (Appendix A4.1). The probability so that a maximum taken randomly is, per unit time, contained in the interval a, a + da is $q(a)da$. If we set $u = \frac{a}{\ell_{rms}}$, it becomes:

$$\frac{v_a}{dt} = q(a)\ da = q(u)\ du = q\left(\frac{a}{\ell_{rms}}\right)\frac{da}{\ell_{rms}} \quad [6.18]$$

yielding [BER 77] [CHA 85] [COU 70] [KOW 63] [LEL 73] [LIN 67] [RAV 70] [SCH 63]:

$$q(u) = \frac{\sqrt{1-r^2}}{\sqrt{2\pi}} e^{-\frac{u^2}{2(1-r^2)}} + \frac{r}{2}\, u\, e^{-\frac{u^2}{2}}\left[1 + \mathrm{Erf}\left(\frac{r\,u}{\sqrt{2(1-r^2)}}\right)\right] \quad [6.19]$$

The statistical distribution of the minima follows the same law. The probabilty density $q(u)$ is thus the weighted sum of a Gaussian law and Rayleigh's law, with coefficients function of parameter r. This expression can be written in various more or less practical forms according to its application. Since:

$$\int_{-\infty}^{\infty} e^{-\lambda^2} d\lambda = \sqrt{\pi} = 2 \int_0^x e^{-\lambda^2} d\lambda + 2 \int_x^{\infty} e^{-\lambda^2} d\lambda$$

where

$$\mathrm{Erf}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-\lambda^2}\ d\lambda \quad [6.20]$$

it becomes:

$$q(u) = \frac{\sqrt{1-r^2}}{\sqrt{2\pi}} e^{-\frac{u^2}{2(1-r^2)}} + r\,u\, e^{-\frac{u^2}{2}}\left[1 - \frac{1}{\sqrt{\pi}} \int_{\frac{r\,u}{\sqrt{2(1-r^2)}}}^{\infty} e^{-\lambda^2}\ d\lambda\right] \quad [6.21]$$

Setting $\lambda = \frac{t}{\sqrt{2}}$ in this relation, we obtain [BEN 61b] [BEN 64] [HIL 70] [HUS 56] [PER 74]:

$$q(u) = \frac{\sqrt{1-r^2}}{\sqrt{2\pi}}\, e^{-\frac{u^2}{2(1-r^2)}} + r\, u\, e^{-\frac{u^2}{2}} \left[1 - \frac{1}{\sqrt{2\,\pi}} \int_{\frac{r\,u}{\sqrt{1-r^2}}}^{\infty} e^{-\frac{t^2}{2}}\, dt \right] \qquad [6.22]$$

One also finds the equivalent expression [BAR 78] [CAR 56] [CLO 75] [CRA 68] [DAV 64] [KAC 76] [KOW 69] [KRE 83] [UDW 73]:

$$q(u) = \frac{\sqrt{1-r^2}}{\sqrt{2\pi}}\, e^{-\frac{u^2}{2(1-r^2)}} + r\, u\, e^{-\frac{u^2}{2}}\, \Phi(v) \qquad [6.23]$$

where

$$\Phi(v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{v} e^{-\frac{t^2}{2}}\, dt$$

and

$$v = \frac{r\,u}{\sqrt{1-r^2}}$$

$$q(u) = \sqrt{1-r^2}\; e^{-\frac{u^2}{2}} \left[\frac{e^{-\frac{r^2\,u^2}{2\sqrt{1-r^2}}}}{\sqrt{2\pi}} + \frac{r\,u}{\sqrt{1-r^2}}\, \Phi(v) \right]$$

$$\boxed{q(u) = \sqrt{1-r^2}\; e^{-\frac{u^2}{2}} \left[\frac{e^{-\frac{v^2}{2}}}{\sqrt{2\pi}} + v\, \Phi(v) \right]} \qquad [6.24]$$

or

$$q(u)=\sqrt{1-r^2}\ e^{-\frac{u^2}{2}}\left[\frac{d\Phi(v)}{dv}+v\ \Phi(v)\right]$$

$$q(u)=\sqrt{1-r^2}\ e^{-\frac{u^2}{2}}\ \frac{d\left[v\ \Phi(v)\right]}{dv} \qquad [6.25]$$

Particular cases

1. Let us suppose that the parameter r is equal to 1, $q(u)$ then becomes, starting from [6.19], knowing that $\text{Erf}(\infty) = 1$,

$$q(u)=u\ e^{-\frac{u^2}{2}} \qquad [6.26]$$

which is the probability density of Rayleigh's law of standard deviation equal to 1. Since $u=\dfrac{a}{\ell_{rms}}$ and:

$$q(a)\ da = q(u)\ du = q\left(\frac{a}{\ell_{rms}}\right)\frac{da}{\ell_{rms}} \qquad [6.27]$$

it becomes

$$q(a)=\frac{q(u)}{\ell_{rms}}=\frac{a}{\ell_{rms}^2}\ e^{-\frac{a^2}{2\ell_{rms}^2}} \qquad [6.28]$$

2. If $r = 0$,

$$q(u)=\frac{1}{\sqrt{2\pi}}\ e^{-\frac{u^2}{2}} \qquad [6.29]$$

(probability density of a normal ie Gaussian law). There exists in this (theoretical) case an infinite number of local maxima between two zero crossings with positive slope.

We will reconsider these particular cases.

6.2. Expected number of maxima per unit time

It was seen that the average number of maxima per second (frequency of maxima) can be written [6.13]:

$$n_p^+ = \frac{1}{2\pi}\sqrt{\frac{M_4}{M_2}}$$

Taking into account the preceding definitions, the expected maxima frequency is also equal to [CRA 67] [HUS 56] [LIN 67] [PAP 65] [PRE 56a] [RIC 64] [SJÖ 61]:

$$n_p^+ = \frac{1}{2\pi}\sqrt{-\frac{R^{(4)}(0)}{R^{(2)}(0)}} = \frac{1}{2\pi}\frac{\ddot{\ell}_{rms}}{\dot{\ell}_{rms}} \qquad [6.30]$$

$$n_p^+ = \frac{1}{2\pi}\left[\frac{\int_{-\infty}^{+\infty}\Omega^4\ G(\Omega)\ d\Omega}{\int_{-\infty}^{+\infty}\Omega^2\ G(\Omega)\ d\Omega}\right]^{\frac{1}{2}} = \left[\frac{\int_0^{+\infty} f^4\ G(f)\ df}{\int_0^{+\infty} f^2\ G(f)\ df}\right]^{\frac{1}{2}} \qquad [6.31]$$

In the case of a narrow band noise such as that of Figure 5.6, we have:

$$n_p^+ = \frac{1}{2\pi}\frac{\ddot{\ell}_{rms}}{\dot{\ell}_{rms}} = \frac{1}{2\pi}\sqrt{\frac{G_0\ \omega_0^4\ \Delta\Omega}{G_0\ \omega_0^2\ \Delta\Omega}} \qquad [6.32]$$

i.e.

$$n_p^+ = \frac{\omega_0}{2\pi} \qquad [6.33]$$

n_p^+ is thus approximately equal to n_0^+: there is approximately 1 peak per zero crossing; the signal resembles a sinusoid with modulated amplitude.

NOTE.

Using the definition of expression [5.81], n_p^+ *would be written [BEN 58] [CHA 85]:*

$$n_p^+ = \sqrt{\frac{M_4}{M_2}}$$

Starting from the number of maxima ν_a lying between a and a + da in the time interval t, t + dt, one can calculate, by integration between t_1 and t_2 for time, and between $-\infty$ and $+\infty$ for the levels, the average total number of maxima between t_1 and t_2:

$$\nu_a = \frac{1}{2\pi}\sqrt{\frac{M_4}{M_2}}\; q(a)\, da\, dt \qquad [6.34]$$

Per second,

$$n_p^+ = \frac{1}{2\pi}\sqrt{\frac{M_4}{M_2}}\int_{-\infty}^{+\infty} q(a)\, da \qquad \left(= \frac{N_p^+}{dt}\right)$$

$$n_p^+ = \frac{1}{2\pi}\sqrt{\frac{M_4}{M_2}}$$

and, between t_1 and t_2,

$$N_p^+ = \frac{1}{2\pi}\sqrt{\frac{M_4}{M_2}}\int_{t_1}^{t_2} dt$$

$$\boxed{N_p^+ = \frac{1}{2\pi}\sqrt{\frac{M_4}{M_2}}\left(t_2 - t_1\right) = n_p^+\left(t_2 - t_1\right)} \qquad [6.35]$$

Application to the case of a noise with constant PSD between two frequencies

Let us consider a vibratory signal $\ell(t)$ whose PSD is constant and equal to G_0 between two frequencies f_1 and f_2 (and zero elsewhere) [COU 70]. We have:

$$M_4 = (2\pi)^4\,\frac{G_0}{5}\left(f_2^5 - f_1^5\right)$$

$$M_2 = (2\pi)^2\,\frac{G_0}{3}\left(f_2^3 - f_1^3\right)$$

This yields

$$\boxed{n_p^+ = \left[\frac{3}{5}\,\frac{f_2^5 - f_1^5}{f_2^3 - f_1^3}\right]^{\frac{1}{2}}} \qquad [6.36]$$

If $f_1 \to 0$,

$$n_p^+ \to \sqrt{\frac{3}{5}}\; f_2 = 0.775\; f_2 \qquad [6.37]$$

If $f_1 = f_0 - \dfrac{\Delta f}{2}$ and $f_2 = f_0 + \dfrac{\Delta f}{2}$ (narrow band noise Δf small).

$$n_p^{+2} = \frac{3}{5}\,\frac{5\,f_0^4 + 10\,f_0^2\left(\dfrac{\Delta f}{2}\right)^2 + \left(\dfrac{\Delta f}{2}\right)^4}{3\,f_0^2 + \left(\dfrac{\Delta f}{2}\right)^2}$$

$$n_p^{+2} = f_0^2\,\frac{1 + 2\left(\dfrac{\Delta f}{2\,f_0}\right)^2 + \dfrac{1}{5}\left(\dfrac{\Delta f}{2\,f_0}\right)^4}{1 + \dfrac{1}{3}\left(\dfrac{\Delta f}{2\,f_0}\right)^2} \qquad [6.38]$$

If $\Delta f \to 0$,

$$n_p^+ \to f_0$$

Figure 6.2 shows the variations of $\dfrac{n_p^+}{f_0}$ versus $\dfrac{\Delta f}{f_0}$.

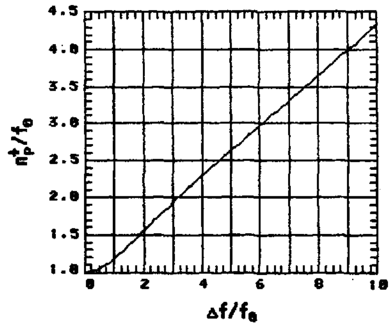


Figure 6.2. *Average number of maxima per second of a narrow band noise versus its width*

6.3. Average time interval between two successive maxima

This average time is calculated directly starting from n_p^+ [COU 70]:

$$\tau_m = \frac{1}{n_p^+} \qquad [6.39]$$

In the case of a narrow band noise, centered on frequency f_0:

$$\boxed{\tau_m = \frac{1}{f_0}\left[\frac{1+\dfrac{1}{3}\left(\dfrac{\Delta f}{2\,f_0}\right)^2}{1+2\left(\dfrac{\Delta f}{2\,f_0}\right)^2+\dfrac{1}{5}\left(\dfrac{\Delta f}{2\,f_0}\right)^4}\right]^{\frac{1}{2}}} \qquad [6.40]$$

$\tau_m \to \dfrac{1}{f_0}$ when $\Delta f \to 0$.

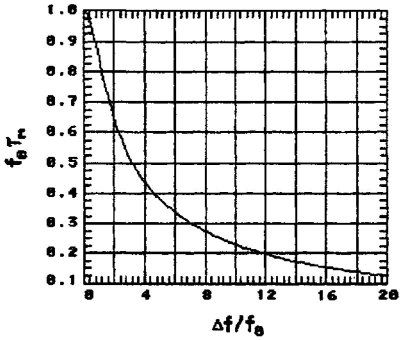


Figure 6.3. *Average time interval between two successive maxima of a narrow band noise versus its width*

6.4. Average correlation between two successive maxima

This correlation coefficient [$\rho(\tau_m)$] is obtained by replacing τ by τ_m in equation [2.70] previously established [COU 70]. If we set:

$$\delta = \frac{\Delta f}{2\, f_0}$$

it becomes:

$$\rho = \frac{1}{2\pi\delta}\left[\frac{1+2\delta^2+\frac{\delta^4}{5}}{1+\frac{\delta^2}{3}}\right]^{1/2} \cos\left(\frac{1+\frac{\delta^2}{3}}{1+2\delta^2+\frac{\delta^4}{5}}\right)^{1/2} \sin\left(\frac{1+\frac{\delta^2}{3}}{1+2\delta^2+\frac{\delta^4}{5}}\right)^{1/2} \qquad [6.41]$$

Figure 6.4 shows the variations of $|\rho|$ versus δ.

The correlation coefficient does not exceed 0.2 when δ is greater than 0.4.

We can thus consider the amplitudes of two successive maxima of a wide-band process as independent random variables [COU 70].

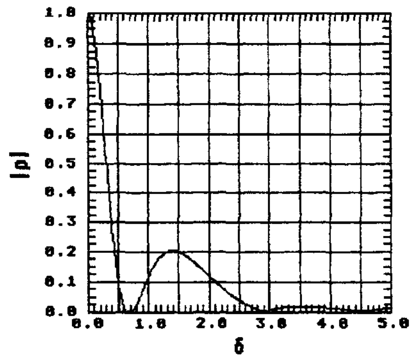


Figure 6.4. *Average correlation between two successive maxima of a narrow band noise versus its bandwidth*

6.5. Properties of the irregularity factor

6.5.1. *Variation interval*

The irregularity factor:

$$r = \frac{M_2}{\sqrt{M_0\, M_4}} = \frac{\dot{\ell}^2_{rms}}{\ell_{rms}\ \ddot{\ell}_{rms}} = \frac{-R^{(2)}(0)}{\sqrt{R(0)\, R^{(4)}(0)}}$$

can vary in the interval [0, 1]. We have indeed [PRE 56b]:

$$r = \frac{M_2}{\sqrt{M_0\, M_4}} = \frac{\int_0^\infty \Omega^2\, G(\Omega)\, d\Omega}{\sqrt{\int_0^\infty G(\Omega)\, d\Omega\ \int_0^\infty \Omega^4\, G(\Omega)\, d\Omega}} \qquad [6.42]$$

According to Schwarz's inequality,

$$\int_0^\infty \Omega^2\, G(\Omega)\, d\Omega \le \sqrt{\int_0^\infty G(\Omega)\, d\Omega}\ \sqrt{\int_0^\infty \Omega^4\, G(\Omega)\, d\Omega}$$

i.e.

$$M_2 \le \sqrt{M_0\, M_4} \qquad [6.43]$$

Since $M_2 \ge 0$, it becomes:

$$0 \le \frac{M_2}{\sqrt{M_0 M_4}} \le 1 \qquad [6.44]$$

Another definition

The irregularity factor r can also be defined like the ratio of the average number of zero crossings per unit time with positive slope to the average number of positive and negative maxima (or minima) per unit time. Indeed,

$$r = \frac{M_2}{\sqrt{M_0\, M_4}} = \frac{1}{2\,\pi}\sqrt{\frac{M_2}{M_0}}\;2\,\pi\,\sqrt{\frac{M_2}{M_4}} = \frac{n_0^+}{n_p^+} = \frac{n_0}{2\,n_p^+} \qquad [6.45]$$

Example

Let us consider the sample of acceleration signal as a function of time represented in Figure 6.5 (with few peaks to facilitate calculations).

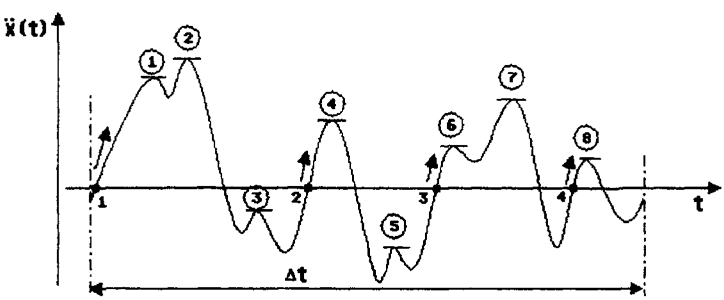


Figure 6.5. *Example of peaks of a random signal*

The number of maxima in the considered time interval Δt is equal to 8, the number of zero-crossing with positive slope to 4 yielding:

$$r = \frac{4}{8} = 0.5$$

The parameter r is a measure of the width of the noise:

– for a broad band process, the number of maxima is much higher than the number of zeros. This case corresponds to the limiting case where $r = 0$. The maxima occur above or below the zero line with an equal probability [CAR 68]. We saw that the probability density of the peaks then tends towards that of a Gaussian law [6.29]:

$$q(u) = \frac{1}{\sqrt{2\,\pi}}\, e^{-\frac{u^2}{2}}$$

– when the number of passages through zero is equal to the number of peaks, r is equal to 1 and the signal appears as a sinusoidal wave, of about constant frequency and slowly modulated amplitude passing successively through a zero, one peak (positive or negative), a zero, and so on. We are dealing with what is called a *narrow band signal,* obtained in response to a narrow rectangular filter or in response of a one-degree-of-freedom system of rather high Q factor (higher than 10 for example).

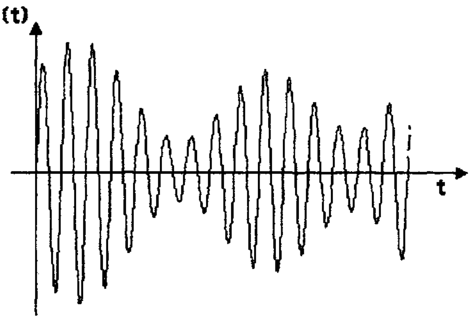


Figure 6.6. *Narrow band signal*

All the maxima are positive and the minima negative. For this value of r, $q(u)$ tends towards Rayleigh's law [6.26]:

$$q(u) = u\, e^{-\frac{u^2}{2}}$$

The value of the parameter r depends on the PSD of the noise via n_0 and n_p (or the moments M_0, M_2 and M_4). Figure 6.7 shows the variations of $q(u)$ for r varying from 0 to 1 per step $\Delta r = 0.15$.

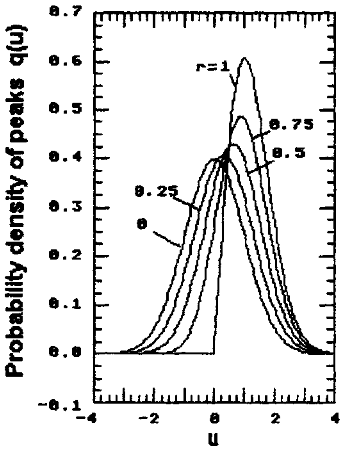


Figure 6.7. *Probability density function of peaks for various values of* r

Example

The probability that $u_0 \le u \le u_0 + \Delta u$ is defined by:

$$\frac{n_u}{n_R} = \int_{u_0}^{u_0+\Delta u} q(u)\, du$$

where n_R is the total number of occurrences.

For example, the probability that a peak exceeds the rms value is approximately 60.65%. the probability of exceeding 3 times the rms value is only approximately 1.11% [CLY 64].

NOTES.

1. *Some authors prefer to use the parameter* $k = 1/r$ *[SCH 63] instead of* r. *Others, more numerous, prefer the quantity [CAR 56] [KRE 83]:*

$$q = \sqrt{1 - r^2} \tag{6.46}$$

(sometimes noted ε*) whose properties are similar:*

– *since* r *varies between 0 and 1,* q *lies between 0 and 1,*

– q *is close to 0 for a band narrow process and close to 1 for a wide-band process,*

– $q = 0$ *for a pure sinusoid with random phase [UDW 73].*

One should not confuse this parameter q *with the quantity*

$$q = \frac{r.m.s.\ \text{value of the slope of the envelope of the process}}{r.m.s.\ \text{value of the slope of the process}}$$ *, often noted using the same letter; this spectral parameter also varies between 0 and 1 (according to the Schwartz inequality) and is function of the form of the PSD [VAN 70] [VAN 72] [VAN 75] [VAN 79]. It is shown that it is equal to the ratio of the rms value of the envelope of the signal to that of the slope of the signal itself. To avoid any confusion, it will hereafter be noted* q_E *(Volume 5).*

2. *The parameter* r *depends on the form of the PSD and there is only one probability density of maxima for a given* r. *But PSD of different forms can have the same* r.

3. *A measuring instrument for the parameter* r *("R meter") has been developed by the Brüel and Kjaer Company[CAR 68].*

6.5.2. *Calculation of irregularity factor for band-limited white noise*

The following definition can be used:

$$r^2 = \frac{M_2^2}{M_0\ M_4}$$

$$M_0 = G\left(f_2 - f_1\right) \qquad [6.47]$$

$$M_2 = (2\ \pi)^2 \int_{f_1}^{f_2} G\ f^2\ df = (2\ \pi)^2\ G\ \frac{f_2^3 - f_1^3}{3} \qquad [6.48]$$

$$M_4 = (2\ \pi)^4 \int_{f_1}^{f_2} G\ f^4\ df = (2\ \pi)^4\ G\ \frac{f_2^5 - f_1^5}{5} \qquad [6.49]$$

yielding

$$r^2 = \frac{5}{9}\ \frac{\left(f_2^3 - f_1^3\right)^2}{\left(f_2 - f_1\right)\left(f_2^5 - f_1^5\right)} \qquad [6.50]$$

i.e., if $h = \frac{f_2}{f_1}$,

$$\boxed{r^2 = \frac{5}{9} \frac{\left(h^3 - 1\right)^2}{(h-1)\left(h^5 - 1\right)}} \qquad [6.51]$$

$$r^2 = \frac{5}{9} \frac{\left(h^2 + h + 1\right)^2}{h^4 + h^3 + h^2 + h + 1}$$

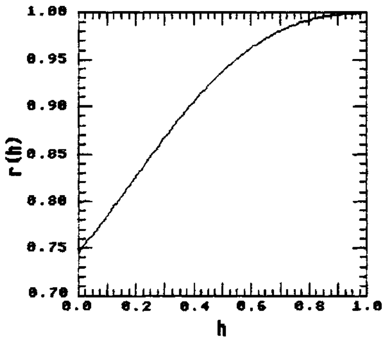


Figure 6.8. *Irregularity factor of band-limited white noise with respect to* h

If $f_2 \to f_1$, $h \to 1$ and $r \to 1$. If $f_2 \to \infty$, $h \to \infty$ and $r \to \frac{\sqrt{5}}{3}$.

When the bandwidth tends towards the infinite, the parameter r tends towards $\frac{\sqrt{5}}{3} = 0.7454$. This is also true if $f_1 \to 0$ whatever value f_2 [PRE 56b].

The limiting case r = 0 can be obtained only if the number of peaks between two zero-crossings is very large, infinite at the limit. That is for example the case for a composite signal made up of the sum of a harmonic process of low frequency f_2 and of a band-limited process at very high frequency and of low amplitude compared with the harmonic movement.

L.P. Pook [POO 76] uses as an analogy the rectangular filter – a one-degree-of-freedom mechanical filter in which $\Delta f = \frac{f_0}{Q} = 2\,\xi\,f_0$ to demonstrate, by considering that the band-limited PSD is the response of the system (f_0, Q) to a white noise, that:

$$r^2 = \frac{4}{9}\,\xi^2\,\frac{15+\xi^2}{5+10\xi^2+\xi^4}$$

$$r = \frac{1+\frac{\xi^2}{3}}{\sqrt{1+2\,\xi^2+\frac{\xi^4}{5}}} \qquad [6.52]$$

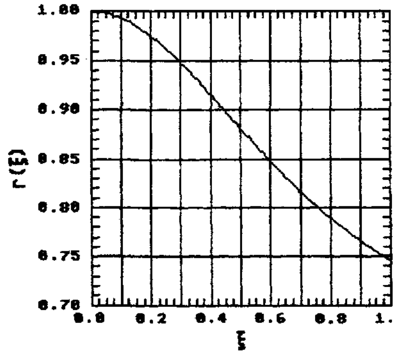


Figure 6.9. *Irregularity factor of band-limited white noise versus damping factor*

It is noted that $r \to 1$ if $\xi \to 0$.

NOTE.

The parameter r *of a narrow band noise centered on frequency* f_0*, whose PSD has a width* Δf*, is written, from the above expressions [COU 70] [RUD 75]:*

$$r = \frac{n_0^+}{n_p^+} = \frac{1 + \frac{1}{3}\left(\frac{\Delta f}{2\, f_0}\right)^2}{\sqrt{1 + 2\left(\frac{\Delta f}{2\, f_0}\right)^2 + \frac{1}{5}\left(\frac{\Delta f}{2\, f_0}\right)^4}} \qquad [6.53]$$

6.5.3. *Calculation of irregularity factor for noise of form* G = Const. f^b

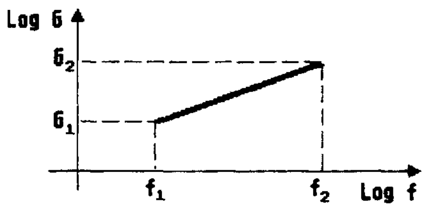


Figure 6.10. *PSD of a noise defined by a straight line segment in logarithmic scales*

$$r = \frac{M_2}{\sqrt{M_0\, M_4}}$$

The moments are expressed

$$\left.\begin{array}{ll} M_0 = \dfrac{G_1}{f_1^b}\, \dfrac{f_2^{b+1} - f_1^{b+1}}{b+1} & \text{if } b \neq -1 \\ M_0 = f_1\, G_1 \ln\dfrac{f_2}{f_1} & \text{if } b = -1 \end{array}\right\} \qquad [6.54]$$

$$\left.\begin{array}{ll} M_2 = (2\,\pi)^2\, \dfrac{G_1}{f_1^b\,(b+3)} \left(f_2^{b+3} - f_1^{b+3}\right) & \text{if } b \neq -3 \\ M_2 = (2\,\pi)^2\, f_1\, G_1 \ln\dfrac{f_2}{f_1} & \text{if } b = -3 \end{array}\right\} \qquad [6.55]$$

$$\left.\begin{aligned} M_4 &= (2\pi)^4 \frac{G_1}{f_1^b (b+5)} \left(f_2^{b+5} - f_1^{b+5}\right) && \text{if } b \neq -5 \\ M_4 &= (2\pi)^4 f_1^5 G_1 \ln\frac{f_2}{f_1} && \text{if } b = -5 \end{aligned}\right\} \qquad [6.56]$$

Case: $b \neq -1$, $b \neq -3$, $b \neq -5$

Let us set $h = \frac{f_2}{f_1}$. Then:

$$r^2 = \frac{(b+1)(b+5)}{(b+3)^2} \frac{\left(h^{b+3}-1\right)^2}{\left(h^{b+1}-1\right)\left(h^{b+5}-1\right)} \qquad [6.57]$$

The curves of Figures 6.11 and 6.12 show the variations of $r(h)$ for various values of b ($b \leq 0$ and $b \geq 0$).

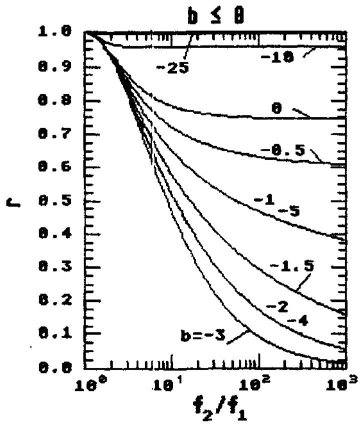


Figure 6.11. *Irregularity factor versus* h, *for various values of the negative exponent* b

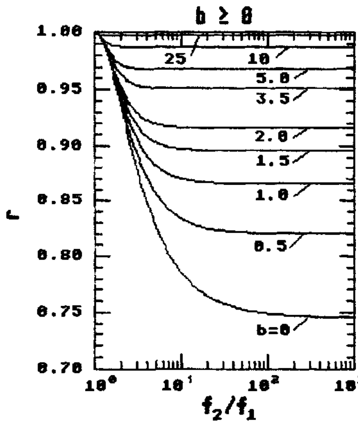


Figure 6.12. *Irregularity factor versus* h, *for various values of the positive exponent* b

For $b < 0$, we note (Figure 6.11) that, when b varies from 0 to -25, the curve, always issuing from the point $r = 1$ for $h = 1$ goes down to $b = -3$, then rises; the curves for $b = -2$ and $b = -4$ are thus superimposed, just as those for $b = -1$ and $b = -5$. This behaviour can be highlighted in a more detailed way while plotting, for given h, the variations of r with respect to b (Figure 6.13) [BRO 63].

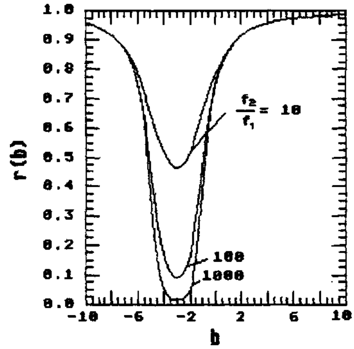


Figure 6.13. *Irregularity factor versus the exponent* b

We observe, moreover, that, for $b = 0$, the curve $r(h)$ tends, for large h, towards $r_0 = \frac{\sqrt{5}}{3} = 0.7454$. All occurs then as if f_1 were zero (signal filtered by a low-pass filter).

Case: $b = -1$

$$M_0 = f_1 \, G_1 \ln \frac{f_2}{f_1}$$

$$M_2 = (2\pi)^2 \frac{f_1 \, G_1}{2} \left(f_2^2 - f_1^2\right)$$

$$M_4 = (2\pi)^4 \frac{f_1 \, G_1}{4} \left(f_2^4 - f_1^4\right)$$

yielding

$$r = \frac{h^2 - 1}{\sqrt{\left(h^4 - 1\right) \ln h}} \qquad [6.58]$$

Case: b = −3

$$M_0 = -\frac{f_1^3\, G_1}{2}\left(\frac{1}{f_2^2} - \frac{1}{f_1^2}\right)$$

$$M_2 = (2\,\pi)^2\, f_1^3\, G_1\, \ln\frac{f_2}{f_1}$$

$$M_4 = (2\,\pi)^4\, \frac{f_1^3\, G_1}{2}\left(f_2^2 - f_1^2\right)$$

$$r = \frac{2\,h\,\ln h}{\left|h^2 - 1\right|} \qquad [6.59]$$

This curve gives, for given h, the lowest value of r.

Case: b = −5

$$M_0 = \frac{f_1^5\, G_1}{4}\left(\frac{1}{f_1^4} - \frac{1}{f_2^4}\right)$$

$$M_2 = -(2\,\pi)^2\, \frac{f_1^5\, G_1}{2}\left(\frac{1}{f_2^2} - \frac{1}{f_1^2}\right)$$

$$M_4 = (2\,\pi)^4\, f_1^5\, G_1\, \ln\frac{f_2}{f_1}$$

$$r = \frac{\left|h^2 - 1\right|}{\sqrt{\left(h^4 - 1\right)\ln h}} \qquad [6.60]$$

6.5.4. *Study: variations of irregularity factor for two narrow band signals*

Let us set $\Delta f = f_2 - f_1$ in the case of a single narrow band noise. The expressions [6.47], [6.48] and [6.49] can be approximated by supposing that Δf being small, the frequencies f_1 and f_2 are close to the central frequency of the band $f_0 = \frac{f_1 + f_2}{2}$. We have then:

$$M_0 = G\ \Delta f$$

$$M_2 \approx (2\ \pi)^2\ G\ \Delta f\ f_0^2$$

and

$$M_4 \approx (2\ \pi)^4\ G\ \Delta f\ f_0^4$$

Now let us apply the same process to a two narrow bands noise whose central frequencies and widths are respectively equal to f_0, Δf_0 and f_1, Δf_1.

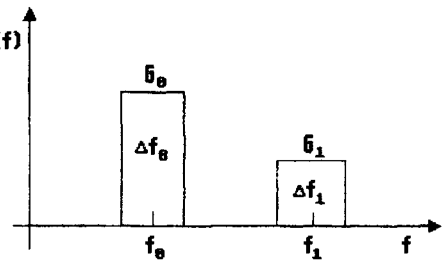


Figure 6.14. *Random noise composed of two narrow bands*

With the same procedure, the factor r obtained is roughly given by [BRO 63]:

$$r^2 = \frac{(2\ \pi)^4\ \left(\Delta f_0\ f_0^2\ G_0 + \Delta f_1\ f_1^2\ G_1\right)^2}{\left(\Delta f_0\ G_0 + \Delta f_1\ G_1\right)\ (2\ \pi)^4\ \left(\Delta f_0\ f_0^4\ G_0 + \Delta f_1\ f_1^4\ G_1\right)}$$

$$r = \frac{1 + \frac{\Delta f_1}{\Delta f_0}\ \frac{f_1^2\ G_1}{f_0^2\ G_0}}{\sqrt{\left(1 + \frac{\Delta f_1}{\Delta f_0}\ \frac{G_1}{G_0}\right)\left(1 + \frac{\Delta f_1}{\Delta f_0}\ \frac{f_1^4\ G_1}{f_0^4\ G_0}\right)}} \qquad [6.61]$$

Figures 6.15 and 6.16 show the variations of r with $\frac{f_1}{f_0}$ and of $\frac{\Delta f_1 \; G_1}{\Delta f_0 \; G_0}$. It is observed that if $\frac{f_1}{f_0} = 1$, r is equal to 1 for whatever $\frac{\Delta f_1 \; G_1}{\Delta f_0 \; G_0}$.

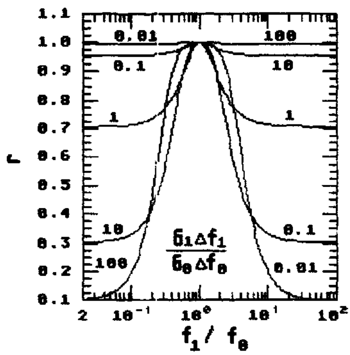


Figure 6.15. *Irregularity factor of a two narrow band noise*

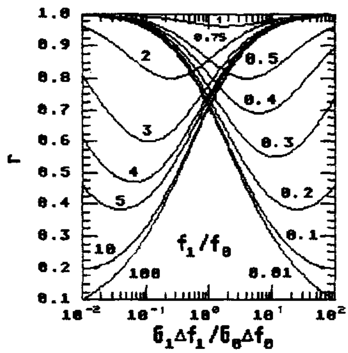


Figure 6.16. *Irregularity factor of a two narrow band noise*

These results can be useful to interpret the response of a two-degrees-of-freedom linear system to a white noise, each of the two peaks of the PSD response being able to be compared to a rectangle of amplitude equal to Q_i^2 times the PSD of the excitation, and of width $\Delta f_i = \frac{\pi}{2} \frac{f_0}{Q}$ [BRO 63].

6.6. Error related to the use of Rayleigh's law instead of complete probability density function

This error can be evaluated by plotting, for various values of r, variations of the ratio [BRO 63]:

$$\frac{q(u)}{p_r(u)}$$

where $q(u)$ is given by [6.19] and where $p_r(u)$ is the probability density from Rayleigh's law (Figure 6.17):

$$p_r(u) = u\, e^{-\frac{u^2}{2}}$$

When u becomes large, these curves tend towards a limit equal to r. This result can be easily shown from the above ratio, which can be written:

$$\frac{q(u)}{p_r(u)} = \frac{\sqrt{1-r^2}}{\sqrt{2\,\pi}}\,\frac{e^{-\frac{r^2 u^2}{2(1-r^2)}}}{u} + \frac{r}{2}\left\{1 + \operatorname{Erf}\left[\frac{r\,u}{\sqrt{2\left(1-r^2\right)}}\right]\right\} \qquad [6.62]$$

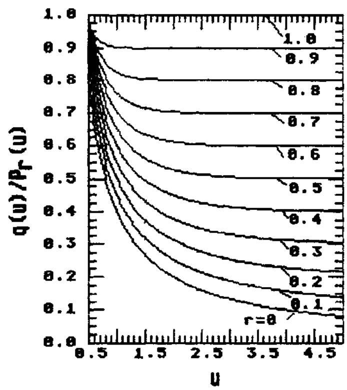


Figure 6.17. *Error related to the approximation of the peak distribution by Rayleigh's law*

It is verified that, when u becomes large, $\frac{q(u)}{p_r(u)} \to r$. One notes in addition from these curves that:

– this ratio is closer to 1 the larger is r,

– the greatest maxima tend to obey a law close to Rayleigh's, the difference being related to the value of r (which characterizes the number of maxima which occur in alternating between two zero-crossings).

6.7. Peak distribution function

6.7.1. *General case*

From the probability density $q(u)$, one can calculate by integration the probability that a peak (maximum) randomly selected among all the maxima of a random process be higher than a given value (per unit time) [CAR 56] [LEY 65]:

$$Q_p(u) = \int_u^{\infty} q(u)\, du = P\left(\frac{u}{\sqrt{1-r^2}}\right) + r\, e^{-\frac{u^2}{2}} \left[1 - P\left(\frac{r\, u}{\sqrt{1-r^2}}\right)\right] \qquad [6.63]$$

where

$$P(x_0) = \frac{1}{\sqrt{2\,\pi}} \int_{x_0}^{\infty} e^{-\frac{\lambda^2}{2}}\, d\lambda$$

$P(x_0)$ is the probability that the normal random variable x exceeds a given threshold x_0. If $u \to \infty$, $P(x_0) \to 1$ and $Q_p(u) \to 0$. Figure 6.18 shows the variations of $Q_P(u)$ for r = 0; 0.25; 0.5; 0.75 and 1.

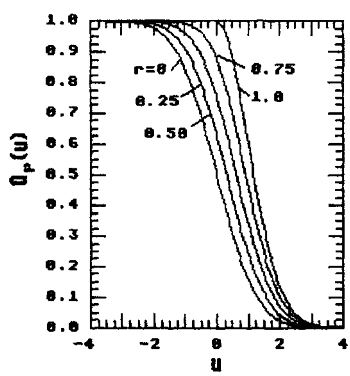


Figure 6.18. *Probability that a peak is higher than a given value* u

NOTES.

1. *The distribution function of the peaks is obtained by calculating* $1 - Q_P(u)$.
2. *The function* $Q_p(u)$ *can also be written in several forms.*

Knowing that:

$$A = \frac{1}{\sqrt{2\pi}} \int_{x_0}^{\infty} e^{-\frac{\lambda^2}{2}} \, d\lambda = \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_0^{x_0} e^{-\frac{\lambda^2}{2}} \, d\lambda$$

$$A = \frac{1}{2} - \frac{1}{\sqrt{\pi}} \int_0^{x_0/\sqrt{2}} e^{-\lambda^2} \, d\lambda = \frac{1}{2}\left[1 - \text{Erf}\left(\frac{x_0}{\sqrt{2}}\right)\right]$$

there results [HEA 56] [KOW 63]:

$$Q_p(u) = \frac{1}{2}\left\{1 - \text{Erf}\left[\frac{u}{\sqrt{2\left(1-r^2\right)}}\right]\right\} + \frac{r}{2} e^{-\frac{u^2}{2}} \left\{1 + \text{Erf}\left[\frac{r\,u}{\sqrt{2\left(1-r^2\right)}}\right]\right\} \quad [6.64]$$

or [HEA 56]:

$$Q_p(u) = \frac{1}{2}\left\{\text{Erfc}\left[\frac{u}{\sqrt{2\left(1-r^2\right)}}\right]\right\} + \frac{r}{2} e^{-\frac{u^2}{2}} \left\{1 + \text{Erf}\left[\frac{r\,u}{\sqrt{2\left(1-r^2\right)}}\right]\right\} \quad [6.65]$$

This form is most convenient to use, the error function Erf *being able to be approximated by a series expansion with very high precision (cf. Appendix A4.1). One also sometimes encounters the following expression:*

$$Q_p(u) = 1 - \frac{\sqrt{1-r^2}}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-\frac{\lambda^2}{2\left(1-r^2\right)}} \, d\lambda$$

$$- \frac{r^2}{\sqrt{2\pi}} \int_{-\infty}^{u/\sqrt{1-r^2}} e^{-\frac{\lambda^2}{2}} \, d\lambda + \frac{r}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \int_{-\infty}^{r\,u/\sqrt{1-r^2}} e^{-\frac{\lambda^2}{2}} \, d\lambda \quad [6.66]$$

3. *For large* u *[HEA 56],*

$$Q_p(u) \approx r\, e^{-\frac{u^2}{2}}.$$

yielding the average amplitude of the maximum (or minimum):

$$\overline{u_{max}} = r \sqrt{\frac{\pi}{2}} \qquad [6.67]$$

6.7.2. *Particular case of narrow band Gaussian process*

For a narrow band Gaussian process (r = 1), we saw that [6.28]:

$$q(a) = \frac{a}{\ell_{rms}^2} \, e^{-\frac{a^2}{2\,\ell_{rms}^2}}$$

The probability so that a maximum is greater than a given threshold a is then:

$$Q_p(a) = e^{-\frac{a^2}{2\,\ell_{rms}^2}} \qquad [6.68]$$

It is observed that, in this case [5.38],

$$Q_p(a) = \frac{n_a^+}{n_0^+} = \frac{p(a)}{p(0)}$$

yielding

$$q(a) = -\frac{d[p(a)]/da}{p(0)} \qquad [6.69]$$

These two last relationships suppose that the functions $\ell(t)$ and $\dot{\ell}(t)$ are independent. If this is not the case, in particular if $p(\ell)$ is not Gaussian, J.S. Bendat [BEN 64] notes that these relationships nonetheless give acceptable results in the majority of practical cases.

NOTE.

The relationship [6.28] can also be established as follows [CRA 63] [FUL 61] [POW 58]. We showed that the number of threshold level crossings with positive slope, per unit time, n_a^+ *is, for a Gaussian stationary noise [5.47]:*

$$n_a^+ = n_0^+ \, e^{-\frac{a^2}{2\,\ell_{rms}^2}}$$

where

$$n_0^+ = \frac{1}{2\pi}\,\frac{\dot{\ell}_{rms}}{\ell_{rms}}$$

The average number of maxima per unit time between two neighbouring levels a *and* a + da *must be equal, for a narrow band process, to:*

$$n_a^+ - n_{a+da}^+ = -\frac{dn_a^+}{da}\,da$$

yielding, by definition of $q(a)$,

$$n_p^+\,q(a)\,da = -\frac{dn_a^+}{da}\,da$$

The signal being assumed narrow band, $n_p^+ = n_0^+$. *This yields*

$$q(a) = -\frac{1}{n_0^+}\,\frac{dn_a^+}{da}$$

and

$$q(a) = \frac{a}{\ell_{rms}^2}\,e^{-\frac{a^2}{2\,\ell_{rms}^2}}$$

It is shown that the calculation of the number of peaks from the number of threshold crossings using the difference $n_a^+ - n_{a+da}^+$ *is correct only for one perfectly narrow band process [LAL 92]. In the general case, this method can lead to errors.*

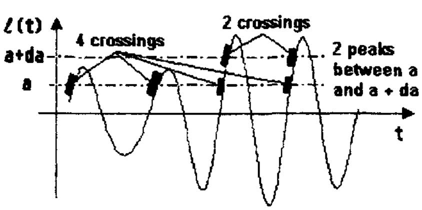


Figure 6.19. *Threshold crossings of a narrow band noise*

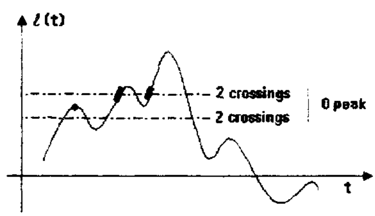


Figure 6.20. *Threshold crossings of a wide-band noise*

Particular case where $f_1 \to 0$

We saw that, for a band-limited noise, $r \to \frac{\sqrt{5}}{3}$ when $f_1 \to 0$. Figures 6.21 and 6.22 respectively show the variations of the density $q(u)$ and of $P(a < u\ \ell_{rms}) = 1 - Q_p(u)$ versus u, for $r = \frac{\sqrt{5}}{3}$.

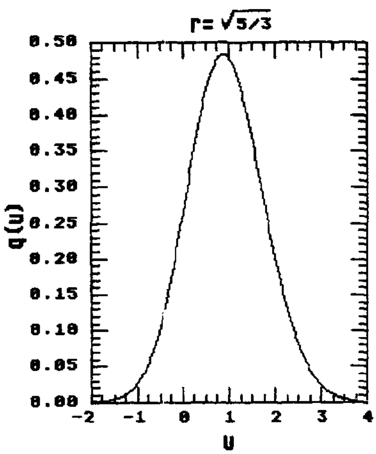


Figure 6.21. *Peak probability density of a band-limited noise with zero initial frequency*

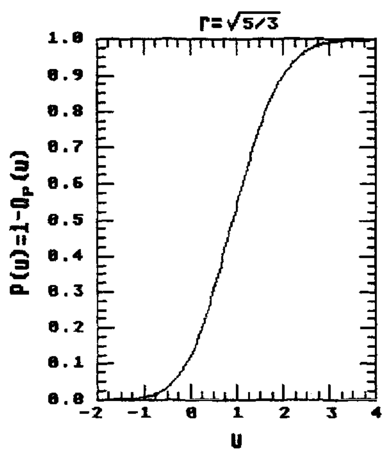


Figure 6.22. *Peaks distribution function of a band-limited noise with zero initial frequency*

6.8. Mean number of maxima greater than given threshold (by unit time)

The mean number of maxima which, per unit time, exceeds a given level $a = u\ \ell_{rms}$ is equal to:

$$M_a = n_p^+\ Q_p(u) \qquad [6.70]$$

If a is large and positive, the functions $P\left(\frac{u}{\sqrt{1-r^2}}\right)$ and $P\left(\frac{u\ r}{\sqrt{1-r^2}}\right)$ tend towards zero; yielding:

$$Q_p \approx r\ e^{-u^2/2} \qquad [6.71]$$

and

$$M_a \approx n_p^+ \, r \, e^{-\frac{u^2}{2}} \quad [6.72]$$

i.e. [RAC 69], since $r = \dfrac{n_0^+}{n_p^+}$,

$$M_a \approx n_0^+ \, e^{-\frac{u^2}{2}} \quad [6.73]$$

This expression gives acceptable results for $u \geq 2$ [PRE 56b]. For $u < 2$, it results in underestimating the number of maxima. To evaluate this error, we have plotted in Figure 6.23 variations of the ratio $\dfrac{\text{exact value}}{\text{approximate value}}$ of M_a:

$$\frac{n_p^+ \; Q_p(u)}{n_0^+ \; e^{-u^2/2}} = \frac{Q_p(u) \; e^{u^2/2}}{r}$$

with respect to u, for various values of r. This ratio is equal to 1 when r = 1 (narrow band process).

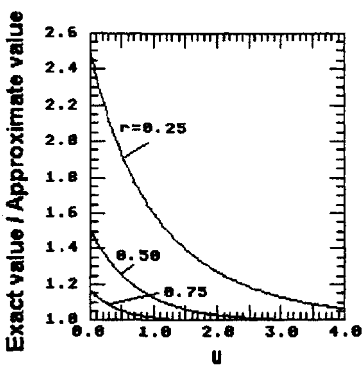


Figure 6.23. *Error related to the use of the approximate expression of the average number of maxima greater than a given threshold*

This yields $Q_p(u) \approx r \, e^{-u^2/2}$ and $M_a \approx n_0^+ \, e^{-u^2/2}$ (same result as for large a). In these two particular cases, the average number per second of the maxima located above a threshold a is thus equal to the average number of times per second which

$\ell(t)$ crosses the threshold a with a positive slope; this equivalent to saying that there is only one maximum between two successive threshold crossings (with positive slope). For a narrow band noise, one thus has:

$$M_a = n_p^+ \; Q_p(a)$$

$$M_a = \frac{1}{2\pi}\sqrt{\frac{M_4}{M_2}}\, e^{-\frac{a^2}{2\,\ell_{rms}^2}} = \frac{1}{2\pi}\sqrt{\frac{M_2}{M_0}}\, e^{-\frac{a^2}{2\,M_0}} \qquad [6.74]$$

NOTE.

The expression [5.47] ($n_a^+ = n_0^+ \, e^{-\frac{a^2}{2\,\ell_{rms}^2}}$) *is an asymptotic expression for large* a *[PRE 56b]. The average frequency* $n_0^+ = \frac{1}{2\pi}\left(\frac{\int_0^\infty \Omega^2 \, G(\Omega)\, d\Omega}{\int_0^\infty G(\Omega)\, d\Omega}\right)^{1/2}$ *is independent of noise intensity and depends only on the form of the PSD. In logarithmic scales, [5.47] becomes:*

$$\ln n_a^+ = \ln n_0^+ - \frac{a^2}{2\,\ell_{rms}^2}$$

$\ln n_a^+$ *is thus a linear function of* a^2*, the corresponding straight line having a slope* $-\frac{1}{2\,\ell_{rms}^2}$*. One often observes this property in practice. Sometimes however, the curve* $\left(\ln n_a^+, a^2\right)$ *resembles that in Figure 6.24. It is in particular the case for turbulence phenomena. One then carries out a combination of Gaussian processes [PRE 56b] when calculating:*

$$M(a) = \sum_{i=1}^{k} P_i \; n_{a_i}^+(a) \qquad [6.75]$$

where P_i *is a coefficient characterizing the contribution brought by the* i^{th} *component and* $n_{a_i}^+$ *is the number of crossings per second for this* i^{th} *component. If it is supposed that the shape of the atmospheric turbulence spectrum is invariant and that only the intensity varies,* n_0^+ *is constant. A few components then often suffice to representing the curve correctly.*

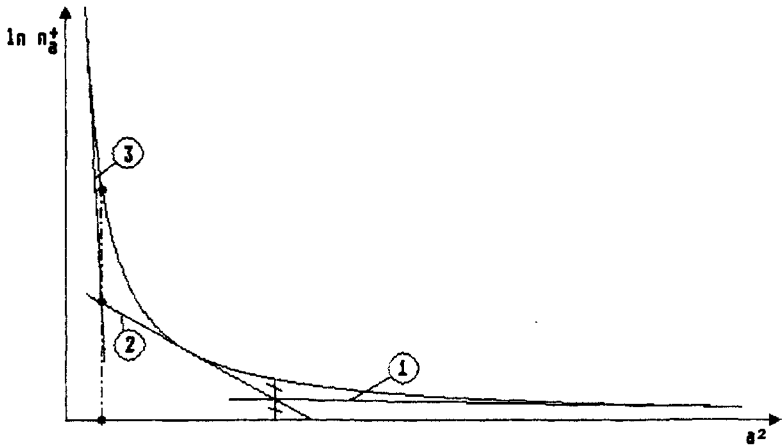


Figure 6.24. *Decomposition of the number of threshold crossings into Gaussian components*

One can for example proceed according to the following (arbitrary) steps:

– plot the tangent at the tail of the observed distribution ①;

– plot the straight line ② starting from the point of the straight line 1 which underestimates the distribution observed by a factor 2, and tangent to the higher part of the distribution;

– plot straight line ③ from ② in the same way.

The sum of these three lines gives a good enough approximation of the initial curve. The slopes of these lines allow the calculation of the squares of the rms values of each component. The coefficients P_i *are obtained from:*

$$M_i(a) = P_i\, n_0^+\, e^{-\frac{a^2}{2\,\ell_{rms}^2}} \qquad [6.76]$$

for each component. Each term M_i *can be evaluated directly by reading the ordinate at the beginning of each line (for* $a = 0$*), yielding*

$$P_i = \frac{M_i}{n_0^+\, e^{-\frac{a^2}{2\,\ell_{rms}^2}}} \qquad [6.77]$$

6.9. Mean number of maxima above given threshold between two times

If a is the threshold, and t_1 and t_2 the two times, this number is given by [CRA 67] [PAP 65]:

$$E(a) = N_a^+ = \frac{1}{2\pi}(t_2 - t_1)\sqrt{\frac{M_4}{M_2}}\, e^{-\frac{a^2}{2\,\ell_{rms}^2}} \qquad [6.78]$$

6.10. Mean time interval between two successive maxima

Let T be the duration of the sample. The average number of positive maxima which exceeds the level a in time T is:

$$M_a T = n_p^+\, Q(a)\, T \qquad [6.79]$$

and the average time between positive peaks above a is:

$$\boxed{T_a = \frac{1}{M_a} = \frac{1}{n_p^+\, Q(a)}} \qquad [6.80]$$

For a narrow band noise,

$$T_a = \frac{1}{M_a} = \frac{1}{n_p^+\, Q_p(a)} = \frac{1}{n_0^+\, Q_p(a)}$$

$$T_a = \frac{2\pi\, e^{\frac{a^2}{2\,\ell_{rms}^2}}}{\sqrt{\frac{M_4}{M_2}}} = 2\pi\,\frac{M_4}{M_2}\, e^{\frac{a^2}{2\,\ell_{rms}^2}} \qquad [6.81]$$

or

$$\boxed{T_a = 2\pi\,\frac{M_0}{M_2}\, e^{\frac{a^2}{2\,M_0}}} \qquad [6.82]$$

6.11. Mean number of maxima above given level reached by signal excursion above this threshold

The parameter $r = \dfrac{n_0}{2\, n_p^+}$ makes it possible to compare the number of zero-crossings and the number of peaks of the signal. Another interesting parameter can be the ratio N_m of the mean number, per unit time, of maxima which occur above a level a_0 to the mean number, per unit time, of crossings of the same level a_0 with positive slope [CRA 68].

The mean number, per unit time, of maxima which occur above a level a_0 is equal to:

$$M_{a_0} = n_p \int_{u_0}^{\infty} q(u)\, du \qquad [6.83]$$

where $u_0 = \dfrac{a_0}{\ell_{rms}}$ and $q(u)$ is given by [6.19]. The mean number, per unit time, of crossings of the level a_0 with positive slope is [5.47]:

$$n_a^+ = n_0^+ \, e^{-\frac{u_0^2}{2}}$$

This yields

$$N_m = \frac{M_{a_0}}{n_a^+} \qquad [6.84]$$

$$\boxed{N_m = \frac{1}{r}\, Q(u_0)\, e^{\frac{u_0^2}{2}}} \qquad [6.85]$$

Figure 6.25 shows the variations of N_m versus u_0, for various values of r.

It is noted that N_m is large for small u_0 and r: there are several peaks of amplitude greater than u_0 for only one crossing of this u_0 threshold.

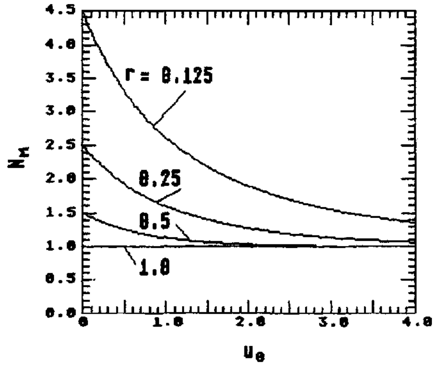


Figure 6.25. *Average number of maxima above a given level through excursion of the signal above this threshold*

For large u_0, N_m decreases quickly and tends towards unity for whatever r. In this case, there is on average only one peak per level crossing. During a time interval $t_1 - t_0$, the average number of maxima which exceed level a is:

$$M_a\left(t_1 - t_0\right) = n_0^+ \left(t_1 - t_0\right) Q_p\left(\frac{a}{\ell_{rms}}\right) \qquad [6.86]$$

Let us replace the rms value ℓ_{rms} by ℓ_{rms_1} and seek the rms value ℓ_{rms2} of another random vibration which has the same number n_p^+ of peaks so that, over time $t_3 - t_2 = t_1 - t_0$, we have [BEN 61b] [BEN 64]:

$$M_a\left(t_3 - t_2\right) = n_p^+ \left(t_1 - t_0\right) Q_p\left(\frac{a}{\ell_{rms_1}}\right) = n_p^+ \left(t_3 - t_2\right) Q\left(\frac{a}{\ell_{rms_2}}\right) \qquad [6.87]$$

It is thus necessary that:

$$\frac{t_3 - t_2}{t_1 - t_0} = \frac{Q_p\left(\dfrac{a}{\ell_{rms_1}}\right)}{Q_p\left(\dfrac{a}{\ell_{rms_2}}\right)} \qquad [6.88]$$

If the two vibrations follow each other, applied successively over $t_1 - t_0$ and $t_2 - t_1$, the *equivalent* stationary noise of rms value $\ell_{rms_{eq}}$ applied over:

$$T = \left(t_1 - t_0\right) + \left(t_2 - t_1\right)$$

which has the same number of maxima n_p^+ exceeding the threshold a as the two vibrations ℓ_{rms_1} and ℓ_{rms_2}, is such that:

$$M_a\ T = M_a\left(t_1 - t_0\right) + M_a\left(t_2 - t_1\right)$$

$$M_a\ T = n_p^+\left(t_1 - t_0\right) Q_p\left(\frac{a}{\ell_{rms_1}}\right) + n_p^+\left(t_2 - t_1\right) Q_p\left(\frac{a}{\ell_{rms_2}}\right) \qquad [6.89]$$

and

$$M_a\ T = n_p^+\ T\ Q_p\left(\frac{a}{\ell_{rms_{eq}}}\right) \qquad [6.90]$$

This yields

$$T = \left[\frac{Q_p\left(\frac{a}{\ell_{rms_1}}\right)}{Q_p\left(\frac{a}{\ell_{rms_{eq}}}\right)}\right]\left(t_1 - t_0\right) + \left[\frac{Q_p\left(\frac{a}{\ell_{rms_2}}\right)}{Q_p\left(\frac{a}{\ell_{rms_{eq}}}\right)}\right]\left(t_2 - t_1\right) \qquad [6.91]$$

and

$$Q_p\left(\frac{a}{\ell_{rms_{eq}}}\right) = \frac{t_1 - t_0}{T} Q_p\left(\frac{a}{\ell_{rms_1}}\right) + \frac{t_2 - t_1}{T} Q_p\left(\frac{a}{\ell_{rms_2}}\right) \qquad [6.92]$$

This expression makes it possible to calculate the value of $\ell_{rms_{eq}}$ (for $a \neq 0$).

6.12. Time during which the signal is above a given value

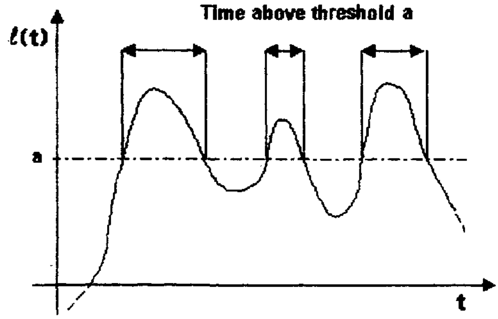


Figure 6.26. *Time during which signal is above a given value*

Let a be the selected threshold; the time during which $\ell(t)$ is greater than a is a random variable [RAC 69]. The problem of research of the statistical distribution of this time is not yet solved.

One can however consider the average value of this time for a stationary random process. The average time during which one has $a \le \ell(t) \le b$ is equal to:

$$\overline{T}_{ab} = \int_a^b \frac{1}{\ell_{rms}\sqrt{2\pi}} e^{-\frac{\ell^2}{2\ell_{rms}^2}} d\ell \qquad [6.93]$$

and, if $b \to \infty$, the time for which $\ell(t) \ge a$ is given by:

$$\overline{T}_{a\infty} = \int_a^\infty \frac{1}{\ell_{rms}\sqrt{2\pi}} e^{-\frac{\ell^2}{2\ell_{rms}^2}} d\ell \qquad [6.94]$$

(ℓ_{rms} = rms value of $\ell(t)$). This result is a consequence of the theorem of ergodicity. It should be noted that this average time does not describe in any way how time is spent above the selected threshold. For high frequency vibrations, the response of the structure can have many excursions above the threshold with a relatively small average time between two excursions. For low frequency vibrations, having the same probability density p as for the preceding high frequencies, there would be fewer excursions above the threshold, but longer, with the excursions being more spaced.

Proportion of time during which $\ell(t) > a$

Given a process $\ell(t)$ defined in [0, T] and a threshold a, let us set [CRA 67]:

$$\left.\begin{array}{ll} \eta(t) = 1 & \text{if } \ell(t) > a \\ \eta(t) = 0 & \text{elsewhere} \end{array}\right\} \qquad [6.95]$$

and

$$Z_0(t) = \frac{1}{T} \int_0^T \eta(t)\, dt = m_\eta \qquad [6.96]$$

the proportion of time during which $\ell(t) > a$, the average of Z_0 is:

$$m_Z = \frac{1}{T} \int_0^T m_\eta\, dt = m_\eta$$

$$m_Z = P[\ell(t) > a]$$

$$m_Z = 1 - \phi\left(\frac{a}{\ell_{rms}}\right) \qquad [6.97]$$

where $\ell_{rms}^2 = M_0 = R(0)$ and $\phi(\ )$ is the Gaussian law. The variance of Z_0 is of the form $\frac{A}{\pi}\, e^{-\frac{a^2}{\ell_{rm}^2}}\, \frac{\ln T}{T}$ when $T \to \infty$.

6.13. Probability that a maximum is positive or negative

These probabilities, respectively q_{max}^+ and q_{max}^-, are obtained directly from the expression of $Q_p(u)$. If we set $u = 0$, it becomes [CAR 56] [COU 70] [KRE 83]:

$$q_{max}^+ = \frac{1+r}{2} \qquad [6.98]$$

$q_{max}^- = 1 - q_{max}^+$ since, for u equal to $-\infty$, $Q_p(u) = 1$ [POO 76] [POO 78], yielding

$$q_{max}^- = \frac{1-r}{2} \qquad [6.99]$$

q_{max}^{+} is the percentage of positive maxima (number of positive maxima divided by the total number of maxima), q_{max}^{-} the percentage of negative maxima [CAR 56]. These relations can be used to estimate r by simply counting the number of positive and negative maxima over a rather long time.

For a wide-band process, $r = 0$ and $q_{max}^{+} = q_{max}^{-} = \frac{1}{2}$.

For a narrow band process, $r = 1$ and $q_{max}^{+} = 1$, $q_{max}^{-} = 0$.

6.14. Probability density of positive maxima

This density has the expression [BAR 78] [COU 70]:

$$q^{+}(u) = \frac{2}{1+r}\, q(u) \qquad [6.100]$$

6.15. Probability that positive maxima is lower than given threshold

Let u be this threshold. This probability is given by [COU 70]:

$$P(u) = 1 - \frac{2}{1+r}\, Q_p(u) \qquad [6.101]$$

yielding

$$P(u) = \frac{1}{1+r}\,\mathrm{Erf}\left(\frac{u}{\sqrt{2\,(1-r^2)}}\right) + \frac{r}{1+r}\left\{1 - e^{-\frac{u^2}{2}}\left[1 + \mathrm{Erf}\left(\frac{u\, r}{\sqrt{2\,(1-r^2)}}\right)\right]\right\} \qquad [6.102]$$

6.16. Average number of positive maxima per unit time

The average number of maxima per unit time is equal to [BAR 78]:

$$n_p^{+} = \int_{-\infty}^{0} \int_{-\infty}^{+\infty} \left|\ddot{\ell}\right|\, p\left(\ell, 0, \ddot{\ell}\right)\, d\ell\, d\ddot{\ell} \qquad [6.103]$$

i.e. [6.13]

$$n_p^+ = \frac{1}{2\pi}\sqrt{\frac{M_4}{M_2}}$$

(the notation + means that it is a maximum, which is not necessarily positive). The average number of positive maxima per unit time is written:

$$n_{p>0}^+ = \int_{-\infty}^{0}\int_{0}^{+\infty} |\ddot{\ell}|\, p(\ell,0,\ddot{\ell})\, d\ell\, d\ddot{\ell}$$

$$n_{p>0}^+ = \frac{1}{4\pi}\left(\sqrt{\frac{M_2}{M_0}} + \sqrt{\frac{M_4}{M_2}}\right) \qquad [6.104]$$

6.17. Average amplitude jump between two successive extrema

Being given a random signal $\ell(t)$, the total height swept in a time interval $(-T, T)$ is [RIC 64]:

$$\int_{-T}^{T} \left|\frac{d\ell(t)}{dt}\right| dt$$

Let $dn(t)$ be the random function which has the value 1 when an extremum occurs and 0 at all the other times. The number of extrema in $(-T, T)$ is $\int_{-T}^{T} dn(t)$.

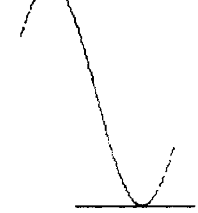


Figure 6.27. *Amplitude jump between two successive extrema*

The average height $\bar{h}_T$ between two successive extrema (maximum - minimum) in $(-T, T)$ is the total distance divided by the number of extrema:

$$\overline{h}_T = \frac{\int_{-T}^{T} \left|\frac{d\ell}{dt}\right| dt}{\int_{-T}^{T} dn(t)} = \frac{\frac{1}{2T} \int_{-T}^{T} \left|\frac{d\ell}{dt}\right| dt}{\frac{1}{2T} \int_{-T}^{T} dn(t)} \tag{6.105}$$

If the temporal averages are identical to the ensemble averages, the average height $\overline{h}$ is:

$$\overline{h} = \lim_{T\to\infty} \overline{h_T} = \frac{\lim\limits_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left|\frac{d\ell}{dt}\right| dt}{\lim\limits_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dn(t)} = \frac{E\left(\left|\frac{d\ell}{dt}\right|\right)}{n_p} \tag{6.106}$$

where n_p is the number of extrema per unit time.

For a Gaussian process, the average height $\overline{h}$ of the rises or falls is equal to [KOW 69] [LEL 73] [RIC 65] [SWA 68]:

$$E(h) = \overline{h} = \sqrt{2\pi}\, \frac{n_0^+}{n_p^+}\, \ell_{rms} = \sqrt{2\pi}\, r\, \ell_{rms} \tag{6.107}$$

or

$$\overline{h} = \sqrt{2\pi}\, \frac{\dot{\ell}_{rms}}{\ddot{\ell}_{rms}} = -R^{(2)}(0) \sqrt{\frac{2\pi}{R^{(4)}(0)}} = M_2 \sqrt{\frac{2\pi}{M_4}} \tag{6.108}$$

For a narrow band process, r = 1 and:

$$\overline{h} = \ell_{rms} \sqrt{2\pi} \tag{6.109}$$

This value constitutes an upper limit when r varies [RIC 64].

NOTE.

Calculation of $\overline{h}$ can be also carried out starting from the average number of crossings per second of the threshold [KOW 69]. For a Gaussian signal, this number is equal to [5.47]:

$$n_a = n_0 \exp\left(-\frac{a^2}{2\,\ell_{rms}^2}\right)$$

The total rise or fall (per second) is written:

$$R = \int_{-\infty}^{+\infty} n_a \, da = n_0 \int_{-\infty}^{+\infty} e^{-\frac{a^2}{2 \ell_{rms}^2}} \, da = \sqrt{2\pi} \; n_0 \; \ell_{rms} \qquad [6.110]$$

This yields the average rise or fall [PAR 62]:

$$\bar{h} = \frac{R}{n_p} = \frac{\sqrt{2\pi} \, n_0}{n_p} \ell_{rms} = \sqrt{2\pi} \, r \, \ell_{rms} \qquad [6.111]$$

Example

Let us consider a stationary random process defined by [RIC 65]:

$$\left.\begin{aligned} &G(\Omega) = \frac{\ell_{rms}^2}{(1-\beta)\,\omega_0} && \text{for } \beta\,\omega_0 < \Omega < \omega_0 \\ &(0 \le \beta \le 1) && \\ &G(\Omega) = 0 && \text{elsewhere} \end{aligned}\right\} \qquad [6.112]$$

J.R. Rice and F.P. Beer [RIC 65] show that:

$$\frac{\bar{h}}{\ell_{rms}} = \sqrt{\frac{10\,\pi}{(1-\beta)\left(1-\beta^5\right)} \frac{\left(1-\beta^3\right)}{3}} \qquad [6.113]$$

For $\beta = 0$ (perfect low-pass filter),

$$\frac{\bar{h}}{\ell_{rms}} = \frac{\sqrt{10\,\pi}}{3} \qquad [6.114]$$

If $\beta \to 1$ (narrow band process),

$$\frac{\bar{h}}{\ell_{rms}} \to \sqrt{2\pi} \qquad [6.115]$$

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Chapter 7

Statistics of extreme values

7.1. Probability density of maxima greater than given value

Let us consider a signal $\ell(t)$ having a distribution of instantaneous values of probability density $p(\ell)$ and distribution function $P(\ell)$.

$$\text{Prob}\left[\ell < \ell_{\text{peak}} < \ell + d\ell\right] = p(\ell)\, d\ell$$

$$P(\ell) = \text{Prob}\left[\ell_{\text{peak}} < \ell\right] = \int_{-\infty}^{\ell} p(\ell)\, d\ell$$

Let λ_N be a new random variable such that $\lambda_N = \max_{i=1,n} \ell_{\text{peak}_i}$. λ_N is the largest peak obtained among the N_p peaks of the signal $\ell(t)$ over a given duration. The distribution function of λ_N is equal to:

$$P(\lambda_N < \ell) = P_N(\ell) = [P(\ell)]^{N_p} \qquad [7.1]$$

and the probability density function to:

$$p_N(\ell) = \frac{dP_N}{d\ell}$$

$$p_N(\ell) = N_p\, [P(\ell)]^{N_p - 1}\, p(\ell) \qquad [7.2]$$

If the probability Q that a maximum is higher than a given value is used,

$$Q = 1 - P$$

we have

$$\boxed{p_N(\ell) = N_p\ [1 - Q(\ell)]^{N_p - 1}\ p(\ell)} \qquad [7.3]$$

where $N_p\ [1 - Q(\ell)]^{N_p - 1}$ is the probability of having $(N_p - 1)$ peaks less than a value ℓ among the N_p peaks.

7.2. Return period

The *return period* $T(X)$ is the number of peaks necessary such that, on average, there is a peak equal to or higher than X. $T(X)$ is a monotonous increasing function of X.

$$T(X) = \frac{1}{1 - P(X)} \qquad [7.4]$$

where $P(X)$ is related to the distribution of ℓ. It becomes:

$$T(X)\ [1 - P(X)] = T(X)\ \text{Prob}(x > X) = 1 \qquad [7.5]$$

7.3. Peak ℓ_p expected among N_p peaks

ℓ_p is the value exceeded once on average in a sample containing N_p peaks. We have:

$$P(\ell_p) = 1 - \frac{1}{N_p} \qquad [7.6]$$

and

$$N_p\left[1 - P(\ell_p)\right] = N_p\ \text{Prob}\left(\ell > \ell_p\right)$$

The return period of ℓ_p is equal to:

$$T(\ell_p) = N_p \qquad [7.7]$$

7.4. Logarithmic rise

The logarithmic rise α_N characterizes the increase in the expected maximum ℓ_p in accordance with the Napierian logarithm of the sample size:

$$\frac{1}{\alpha_N} = \frac{d\ell_p}{d\left(\ln N_p\right)} \qquad [7.8]$$

From [7.6], we have

$$\frac{dP\left(\ell_p\right)}{\ell_p} d\ell_p = p\left(\ell_p\right) d\ell_p = \frac{dN_p}{N_p^2}$$

yielding

$$N_p\, p\left(\ell_p\right) d\ell_p = \frac{dN_p}{N_p} = d\left(\ln N_p\right)$$

and

$$N_p\, p\left(\ell_p\right) = \frac{d\left(\ln N_p\right)}{d\ell_p} = \alpha_N$$

i.e.

$$\alpha_N = N_p\, p\left(\ell_p\right) \qquad [7.9]$$

7.5. Average maximum of N_p peaks

$$\overline{\ell_N} = \int_{-\infty}^{+\infty} \ell\, p_N(\ell)\, d\ell \qquad [7.10]$$

7.6. Variance of maximum

$$s_n^2 = \int_{-\infty}^{+\infty} \left(x - \overline{\ell_N}\right) p_N(\ell)\, d\ell \qquad [7.11]$$

7.7. Mode (most probable maximum value)

Let us set ℓ_M such that $p_N(\ell_M)$ is maximum. The calculation of $\dfrac{dp_N(\ell)}{d\ell} = 0$ gives:

$$\left(N_p - 1\right) \frac{p(\ell_M)}{P(\ell_M)} + \frac{p'(\ell_M)}{p(\ell_M)} = 0 \qquad [7.12]$$

7.8. Maximum value exceeded with risk α

This value, noted ℓ_{N_α}, is defined by:

$$P_N\left(\ell_{N_\alpha}\right) = 1 - \alpha \qquad [7.13]$$

α is the probability of recording a maximum value higher than ℓ_{N_α} among N_p peaks.

7.9. Application to case of centred narrow band normal process

7.9.1. *Distribution function of largest peaks over duration* T

If it be considered that the maxima are distributed according to a Rayleigh density law

$$p(\ell) = \frac{\ell}{s_\ell^2} \exp\left(-\frac{\ell^2}{2\, s_\ell^2}\right)$$

and if it be supposed that the peaks of the narrow band random signal are themselves randomly distributed (a broad assumption in a strict sense, because such a signal may have a correlation between consecutive peaks), the probability that an arbitrary peak ℓ_{peak} is lower than a given value ℓ is equal to:

$$P\left(\ell_{peak} \le \ell\right) = \int_0^\ell \frac{\ell}{s_\ell^2} \exp\left(-\frac{\ell^2}{2\, s_\ell^2}\right) d\ell$$

i.e.

$$P(\ell) = 1 - \exp\left(-\frac{\ell^2}{2\, s_\ell^2}\right)$$

We obtain, from the above relationships, the distribution function of the largest peaks

$$P_N = P\left(\ell_{\text{peak}_i} \le \ell\right) = \left[1 - \exp\left(-\frac{\ell^2}{2\, s_\ell^2}\right)\right]^{N_p} \qquad [7.14]$$

$\left(1 \le i \le N_p\right)$. P_N is the probability that each of the N_p peaks is lower than ℓ, if the peaks are independent [KOW 69]. Figure 7.1 shows this probability for some values of $n_0^+ T$ (equal to N_p since, for a narrow band noise, $n_p^+ = n_0^+$), plotted versus $u = \dfrac{\ell}{\ell_{\text{rms}}}$.

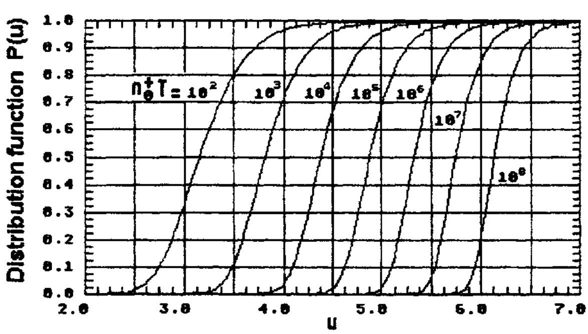


Figure 7.1. *Distribution function of largest peaks of a narrow band noise*

Figure 7.2 presents the variations of the function $Q_N = 1 - P_N$, Q_N being the probability so that the largest peak is higher than a given value U during a length of time T.

NOTES.

1. *For* N_p *large (i.e., in practice, for* $s_\ell / \ell \le 0.2$*) [KOW 69], we have*

$$P_N \approx e^{-N_p \exp\left(-\ell^2/2\, s_\ell^2\right)} \qquad [7.15]$$

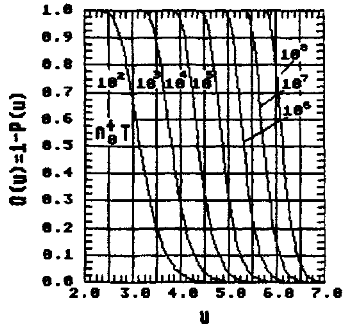


Figure 7.2. *Probability that the largest peak is higher than a given value*

2. *This relation can be written in the form:*

$$\frac{\ell}{s_\ell} = \sqrt{-2 \ln\left\{1 - \exp\left[\left(\ln P_N\right) / N_p\right]\right\}} \qquad [7.16]$$

Figure 7.3 shows the variations of ℓ / s_ℓ *versus* N_p*, for various values of* P_N.

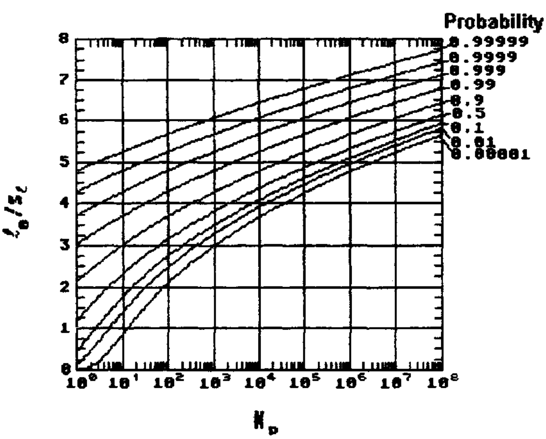


Figure 7.3. *Amplitude of the largest peak against number of peaks, for a given probability*

7.9.2. *Probability that one peak at least exceeds a given threshold*

The probability that one peak at least exceeds the threshold ℓ is equal to:

$$P\left(\ell_i \geq \ell\right) = 1 - \left(1 - e^{-\frac{\ell^2}{2\,s_\ell^2}}\right)^{N_p} \qquad [7.17]$$

where $\left(1 \leq i \leq N_p\right)$, yielding the probability so that a maximum ℓ_{peak_i} lies between ℓ and $\ell + d\ell$:

$$P\left(\ell \leq \ell_{peak_i} \leq \ell + d\ell\right) = P\left(\ell_{peak_i} \geq \ell\right) - P\left(\ell_{peakc_i} \geq \ell + d\ell\right)$$

i.e.

$$P\left(\ell \leq \ell_{peak_i} \leq \ell + d\ell\right) = -d\left[1 - \left(1 - e^{\frac{\ell^2}{2\,s_\ell^2}}\right)^{N_p}\right] \qquad [7.18]$$

7.9.3. *Probability density of the largest maxima over duration* T

The probability density of the largest maxima is thus

$$p_N(\ell) = N_p \left[1 - \exp\left(-\frac{\ell^2}{2\,s_\ell^2}\right)\right]^{N_p - 1} \frac{\ell}{s_\ell^2} \exp\left(-\frac{\ell^2}{2\,s_\ell^2}\right) \qquad [7.19]$$

or, while noting $v = \left(\dfrac{\ell}{\sqrt{2}\,s_\ell}\right)^2$:

$$p_N(v) = N_p \left(1 - e^{-v}\right)^{N_p - 1} v^{1/2}\, e^{-v} \qquad [7.20]$$

Over time T, the number of maxima higher than $u = \dfrac{\ell}{\ell_{rms}}$ is

$$\nu = Q(u)\, N_p \qquad [7.21]$$

where ν is such that $0 \le \nu \le N_p$.

$$p_N(u)\ du = -\left\{N_p\ [1 - Q(u)]\right\}^{N_p - 1}\ dQ$$

$$p_N(u)\ du = \left\{N_p\ [1 - Q(u)]\right\}^{N_p - 1}\ d[1 - Q(u)]$$

$$p_N(u)\ du = d\left[\left(1 - \frac{\nu}{N_p}\right)^{N_p}\right]$$

$$p_N(u)\ du = d\left(e^{-\nu}\right) = -e^{-\nu}\ d\nu \qquad [7.22]$$

For large u, we can accept that $Q(u)$ can be approximated by [CAR 56]:

$$Q(u) \approx r\ e^{-\frac{u^2}{2}} \qquad [7.23]$$

(Rayleigh's law). For large N_p, we have, on average, for a given duration T,

$$N_p = n_p^+\ T$$

In addition, we still have $\nu = N_p\ Q(u)$, yielding, since $r = 1$,

$$\nu \approx n_0^+\ T\ e^{-\frac{u^2}{2}} \qquad [7.24]$$

and

$$p_N(u)\ du = d\left[\exp\left(-n_0^+\ T\ e^{-\frac{u^2}{2}}\right)\right] \qquad [7.25]$$

From the relationship [7.25], we can express, by integration, this density in the form:

$$\boxed{p_N(u) = n_0^+\ T\ u\ \exp\left\{-\left[\frac{u^2}{2} + n_0^+\ T\ \exp\left(-\frac{u^2}{2}\right)\right]\right\}} \qquad [7.26]$$

Figure 7.4 shows the variations of $p_N(u)$ for various values of $n_0^+ T$ between 10^2 and 10^8.

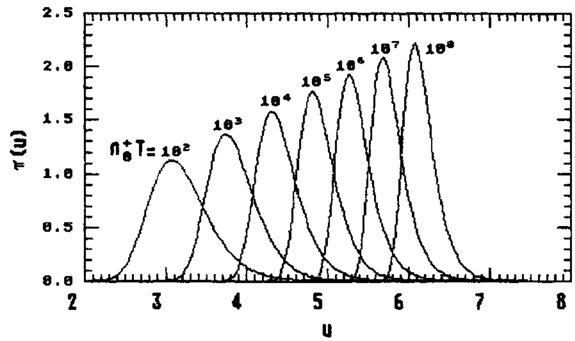


Figure 7.4. *Probability density of the largest maximum over duration* T

Each one of these curves gives the distribution law of the largest maximum over duration T of n signal samples to be studied (Figure 7.5).

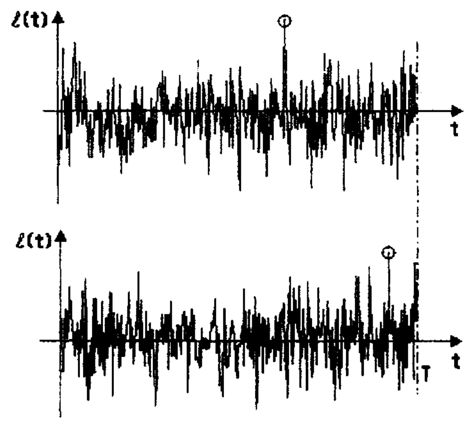


Figure 7.5. *Largest peak of a sample of given duration*

Figure 7.6 shows this same probability density for $n_0^+ T = 3.6\,10^4$ to $3.6\,10^6$, superimposed over the probability density curve of the instantaneous values of the random signal (Gauss's law).

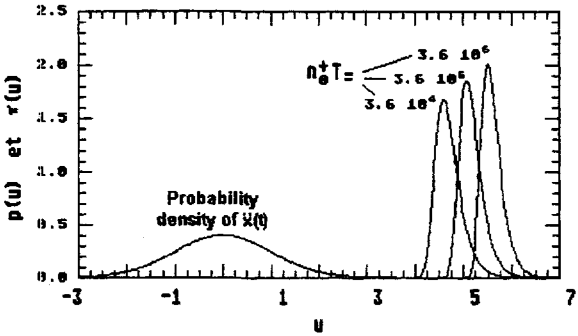


Figure 7.6. *Probability densities of peaks and highest maxima*

7.9.4. *Average of highest peaks*

$$\overline{u_0} = \int_{-\infty}^{+\infty} u \; p_N(u) \; du = \int_{-\infty}^{+\infty} u \; e^{-v} \; dv$$

The relation [7.24] makes it possible to express u according to v:

$$u = 2 \ln\left(n_0^+ \; T\right) - 2 \ln(v)$$

On the assumption that $\ln\left(n_0^+ \; T\right)$ is large compared to $\ln(v)$, A.G. Davenport [DAV 64] deduces the average value of ℓ_0:

$$E(\ell) = -\int_0^{\infty} \ell \; d\left[1 - \left(1 - e^{-\frac{\ell^2}{2 s_\ell^2}}\right)^{N_p}\right] \qquad [7.27]$$

i.e., after a MacLaurin series development and an integration by parts [KOW 69] [LON 52]:

$$\overline{u}_0 = \frac{E(\ell)}{s_\ell} = \sqrt{\frac{\pi}{2}} \left[\frac{N_p}{1! \sqrt{1}} - \frac{N_p \left(N_p - 1\right)}{2! \sqrt{2}} \right.$$

$$+\frac{N_p\left(N_p-1\right)\left(N_p-2\right)}{3!\sqrt{3}}-\cdots+(-1)^{N_p+1}\frac{1}{\sqrt{N_p}}\Bigg] \qquad [7.28]$$

For large values of N_p, M.S. Longuet-Higgins [KRE 83] [LON 52] shows that one can use the asymptotic expression

$$\overline{u_0} \approx \sqrt{2\ln\left(n_0^+ T\right)}+\frac{\varepsilon}{\sqrt{2\ln\left(n_0^+ T\right)}} \qquad [7.29]$$

where ε is the Euler's constant equal to 0.577 215 664 90 ... (cf. Appendix A4.3), the difference with the whole expression being about $\left(\ln N_p\right)^{-3/2}$ [UDW 73].

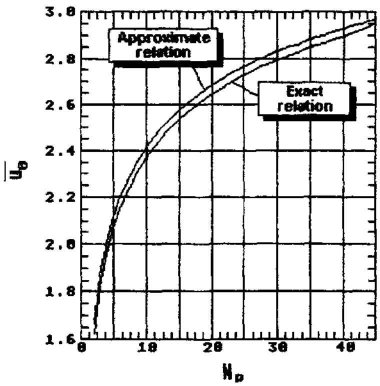


Figure 7.7. *Comparison of the approximate average value of the distribution of the highest peaks to the exact value*

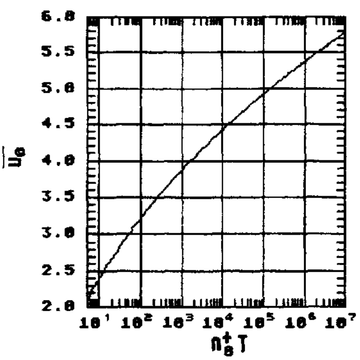


Figure 7.8. *Average of the highest peaks*

The approximation is very good [CAR 56] , even for small N_p (error less than 3% for all $N_p \geq 2$ and less than 1% if $N_p > 50$).

NOTE.

On the assumption $\ln\left(n_0^+ T\right) >> \ln(\nu)$.

The ratio

$$\frac{\ln \nu}{\ln n_0^+ T} = \frac{-\frac{u^2}{2} + \ln n_0^+ T}{\ln n_0^+ T} = -\frac{u^2}{2 \ln n_0^+ T} + 1 \qquad [7.30]$$

is small with regard to the unit if $u^2 \approx 2 \ln n_0^+ T$. *The approximation [7.29] is very acceptable for a narrow band process, i.e. for* r *close to 1 [CAR 56] [POO 76].*

7.9.5. *Standard deviation of highest peaks*

On the same assumptions, the standard deviation of the largest peak distribution is calculated from

$$s_{u_0} = \sqrt{\overline{u_0^2} - \left(\overline{u_0}\right)^2}$$

$$\boxed{s_{u_0} \approx \frac{\pi}{\sqrt{6}} \frac{1}{\sqrt{2 \ln\left(n_0^+ T\right)}}} \qquad [7.31]$$

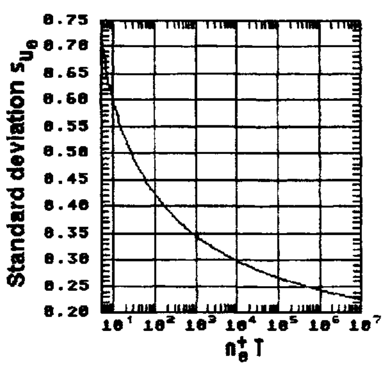


Figure 7.9. *Standard deviation of the distribution law of the highest peaks*

Figures 7.8 and 7.9 respectively show the average $\overline{u_0}$ and the standard deviation s_u as a function of $n_0^+ T$. We note on these curves that, when $n_0^+ T$ increases, the average increases and the standard deviation decreases very quickly.

We notice in Figure 7.10 that the slope of the curve $P_N(u)$ increases with $n_0^+ T$, result in conformity with the decrease of s_u.

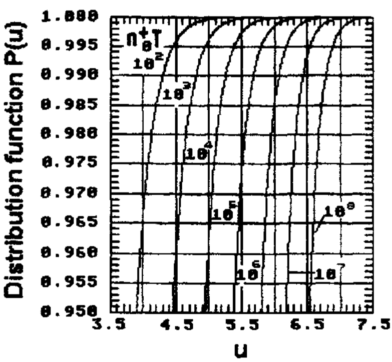


Figure 7.10. *Probability density of the largest peaks close to unity*

7.9.6. *Most probable value*

The most probable value of ℓ corresponds to the peak of the probability density curve defined by [7.19], i.e. to the mode ℓ_m (or to the reduced mode $m = \frac{\ell_m}{s_\ell}$). If we let $v = \left(\frac{\ell}{\sqrt{2}\, s_\ell}\right)^2$, it occurs when

$$\frac{d}{dv}\left[\left(1 - e^{-v}\right)^{N_p - 1} v^{1/2}\, e^{-v}\right] = 0$$

i.e. when [PRA 70] [UDW 73]

$$v = \ln N_p - \ln\left[1 - \frac{1}{2\,v}\left(1 - e^{-v}\right)\right] \qquad [7.32]$$

If N_p is large

$$v \approx \ln N_p \qquad [7.33]$$

yielding the most probable value

$$m = \frac{\ell_m}{s_\ell} = \sqrt{2}\,\sqrt{v} \approx \sqrt{2 \ln N_p} \qquad [7.34]$$

$$m = \sqrt{2 \ln\left(n_0^+ \, T\right)} \qquad [7.35]$$

7.9.7. *Value of density at mode*

$$p_{N_m} = \frac{1}{e} \sqrt{2 \ln\left(n_0^+ \, T\right)} \qquad [7.36]$$

A typical example of the use of the preceding relations relates to the study of the distribution of the wave heights, starting from an empirical relationship of the acceleration spectral density [PIE 63].

7.9.8. *Expected maximum*

The expected maximum ℓ_p is such that

$$P\left(\ell_p\right) = 1 - \frac{1}{N_p} = 1 - \exp\left(-\frac{\ell_p^2}{2\, s_\ell^2}\right) \qquad [7.37]$$

$$\ell_p = 2\, s_\ell \ln N_p \qquad [7.38]$$

7.9.9. *Average maximum*

$$\overline{\ell_N} = \int_0^\infty \ell\, N_p \left[1 - \exp\left(-\frac{\ell^2}{2\, s_\ell^2}\right)\right]^{N_p - 1} \frac{\ell}{s_\ell^2} \exp\left(-\frac{\ell^2}{2\, s_\ell^2}\right) d\ell \qquad [7.39]$$

7.9.10. *Maximum exceeded with given risk* α

$$P_N\left(\ell_{N\alpha}\right) = 1 - \alpha = \left[1 - \exp\left(-\frac{\ell_{N\alpha}^2}{2\, s_\ell^2}\right)\right]^{N_p} \qquad [7.40]$$

$$\ell_{N\alpha} = \sqrt{2 \ln \frac{1}{\left[1 - (1 - \alpha)^{1/N_p}\right]}}\; s_\ell \qquad [7.41]$$

i.e., for $\alpha << 1$,

$$\ell_{N\alpha} \approx s_\ell \sqrt{2 \ln \frac{N_p}{\alpha}} \qquad [7.42]$$

One finds in Table 7.1 the value of the parameters above defined from the relationships [7.29] [7.31] [7.35] [7.36] for some values of $n_0^+ T$.

Table 7.1. *Examples of values of parameters from the highest peaks distribution law*

$n_0^+ T$	$\overline{u_0}$	s_{u_0}	$s_{u_0} / \overline{u_0}$	m	p_{N_m}
3.6 10^2	3.5993	0.3738	10.386 10^{-2}	3.4311	1.2622
3.6 10^3	4.1895	0.3169	7.565 10^{-2}	4.0469	1.4888
3.6 10^4	4.7067	0.2800	5.949 10^{-2}	4.5807	1.6851
3.6 10^5	5.1725	0.2535	4.901 10^{-2}	5.0584	1.8609
3.6 10^6	5.5999	0.2334	4.168 10^{-2}	5.4948	2.0214

It is noted that $s_{u_0} / \overline{u_0}$ is always very small and tends to decrease when $n_0^+ T$ increases. Table 7.2 gives, with respect to $n_0^+ T$, the values of $Q = 1 - P$ for $u = \overline{u_0}$ and $u = m$.

Table 7.2. *Examples of values of parameters from the highest peaks distribution law*

$n_0^+ T$	$\overline{u_0}$	$Q(\overline{u_0})$	m	Q(m)
10^2	3.2250	0.4239	3.0349	0.6321
10^3	3.8722	0.4258	3.7169	0.6321
10^4	4.4264	0.4267	4.2919	0.6321
$3.6\ 10^4$	4.7067	0.4271	4.5807	0.6321
10^5	4.9188	0.4273	4.7985	0.6321
$3.6\ 10^5$	5.1725	0.4275	5.0584	0.6321
10^6	5.3663	0.4277	5.2565	0.6321
$3.6\ 10^6$	5.5999	0.4279	5.4948	0.6321
10^7	5.7794	0.4280	5.6777	0.6321
10^8	6.1648	0.4282	6.0697	0.6321

It is noticed that, for whatever n_0^+ T, $Q(\overline{u_0})$ and $Q(m)$ are practically constant.

In many problems, one can suppose that with slight error the highest value is equal to the average value $\overline{u_0}$. It is also noted that the average is higher than the mode, but the deviation decreases when n_0^+ T increases.

Over one hour of vibrations and for an average frequency n_0^+ of the signal varying between 10 Hz and 1000 Hz, one notes that the average $\overline{u_0}$ varies between 4.7 and 5.6 times the rms value ℓ_{rms} (Figure 7.8 and Table 7.2). The amplitude of the largest peak therefore remains lower than 5.6 ℓ_{rms}.

The amplitude of the probability density to the mode increases with respect to n_0^+ T (Figure 7.11).

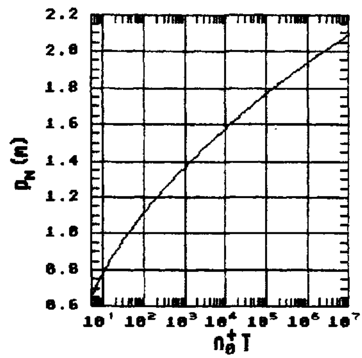


Figure 7.11. *Value of density of largest peak at the mode*

7.10. Wide-band centered normal process

7.10.1. *Average of largest peaks*

The preceding calculations were carried out on the assumption of a narrow band noise $(r \approx 1)$. For a wide-band noise ($r \neq 1$), D.E. Cartwright and M.S. Longuet-Higgins [CAR 56] show that the average value of the largest peak in a sample of N_p peaks is equal to:

$$\overline{u}_0 \approx \sqrt{2 \ln\left(r\ N_p\right)} + \frac{\varepsilon}{\sqrt{2 \ln\left(r\ N_p\right)}} \qquad [7.43]$$

($\varepsilon = 0.57721566490... =$ Euler's constant). One obtains the relationship [7.29] for $r = 1$, N_p being then equal to $n_0^+\ T$. Let us set $\sqrt{m_2}$ as the rms value of the peak distribution, where

$$m_2 = 1 + r^2 \qquad [7.44]$$

Figure 7.12 shows the variations of $\dfrac{\overline{u_0}}{\sqrt{m_2}}$ with respect to r, for various values of N_p.

For large N_p, $\dfrac{\overline{u_0}}{\sqrt{m_2}}$ is a decreasing function of r. When the spectrum widens, the average value of the highest peak decreases. When $r \to 0$ (Gaussian case), the expression [7.43] cannot be used any more, the quantity $r\, N_p$ becoming small compared to unity. The general expression is complicated and without much interest. R.A. Fisher and L.H.C. Tippett [CAR 56] [FIS 28] [TIP 25] propose an asymptotic expression of the form,

$$\overline{u_0} = m + \frac{\varepsilon\, m}{1 + m^2} \qquad [7.45]$$

where m is the mode of the distribution of maxima, given in this case by

$$m\, e^{\frac{m^2}{2}} = \frac{N_p}{\sqrt{2\,\pi}} \qquad [7.46]$$

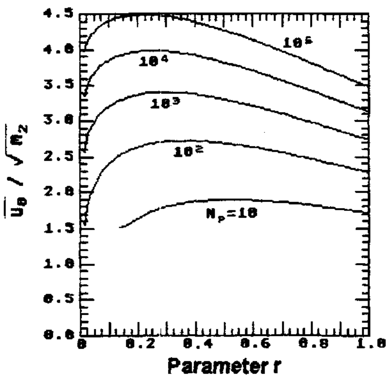


Figure 7.12. *Average value of the highest peak of a wide-band process versus the irregularity factor*

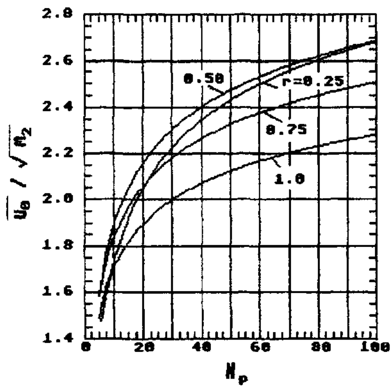


Figure 7.13. *Average value of the highest peak of a wide-band process versus the number of peaks*

The distribution [7.19] is thus centered around this mode for large N_p.

From [7.46], it becomes:

$$m^2 = \ln\left(\frac{N_p^2}{2\,\pi}\right) - \ln\left(m^2\right)$$

yielding

$$m \approx \left\{ \ln\left(\frac{N_p^2}{2\,\pi}\right) - \ln\left[\ln\left(\frac{N_p^2}{2\,\pi}\right)\right] \right\}^{1/2}$$

and

$$\boxed{\overline{u_0} \approx \sqrt{2\,\ln\left(\frac{N_p}{\sqrt{2\,\pi}}\right)}} \qquad [7.47]$$

One can show that $\overline{u_0}$ converges only very slowly towards this limit.

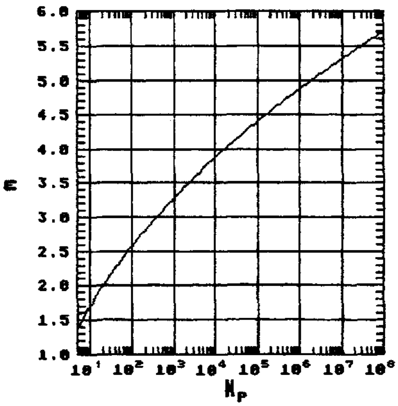


Figure 7.14. *Mode of the distribution law of the highest peaks of a wide-band noise*

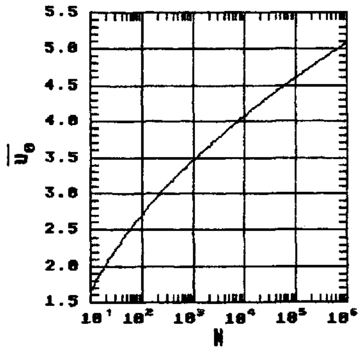


Figure 7.15. *Average value of the highest peaks of a wide-band noise over duration* T

7.10.2. *Variance of the largest peaks*

The variance is given by [FIS 28]

$$\boxed{s^2 = \frac{\pi^2}{6}\,\frac{m^2}{\left(m^2+1\right)^2}} \qquad [7.48]$$

The standard deviation is plotted against N_p in Figure 7.16.

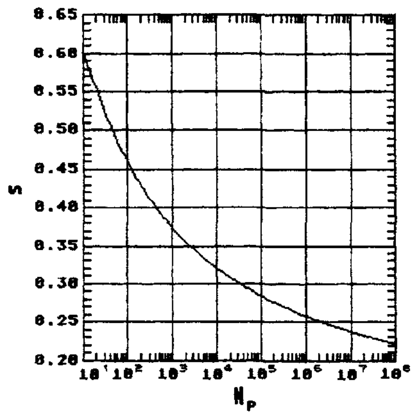


Figure 7.16. *Standard deviation of the distribution law of the highest peaks of a wide-band noise*

Table 7.3 makes it possible to compare the values of $\frac{E(\ell)}{s_\ell}$ calculated from [7.28] with those given exactly by L.H.C. Tippet for some values of N_p [TIP 25].

Table 7.3. *Comparison of exact and approximate values of* $\frac{E(\ell)}{s_\ell}$

N_p		10	20	100	200	500	1000
M		1.43165	1.74393	2.37533	2.61467	2.90799	3.11528
$\frac{E(\ell)}{s_\ell}$	Relation [7.28]	1.70263	1.99302	2.58173	2.80726	3.08549	3.28326
	L.H.C. Tippett	1.53875	1.86747	2.50758	2.74604	3.03670	3.24138

7.11. Asymptotic laws

The use of exact laws of probability for extreme values, established from the initial distribution law of the instantaneous values or from the distribution law of the maxima, leads to calculations which quickly become very complicated.

They can be simplified by treating only the tail of the initial law, but with many precautions because, as one well imagines, several asymptotic laws can be used in this domain. Moreover, the values contained therein, of weak probability, appear only occasionally and the real law is not well known.

7.11.1. *Gumbel asymptote*

This approximation is used for the distribution functions of the exponential type, which tend towards 1 at least as quickly as exponential for the great values of the variable [GUM 54]. This asymptotic law applies in particular to the normal and lognormal laws. Let us consider a distribution function which, for x large, is of the form

$$P(x) = 1 - a \exp(-b\ x) \qquad [7.49]$$

The constants a and b are selected according to the law being simulated. If, for example, we want to respect the values of the expected maximum x_p and the logarithmic increase α_N:

$$P(x_p) = 1 - \frac{1}{N_p} \qquad [7.50]$$

$$\alpha_N = N_p\ p(x_p) \qquad [7.51]$$

In comparing these expressions with those derived from the $P(x)$ law, it becomes:

$$\frac{1}{N_p} = a\ e^{-b\,x_p} \qquad [7.52]$$

$$\alpha_N = N_p\ a\ b\ e^{-b\,x_p} \qquad [7.53]$$

yielding

$$\begin{cases} b = \alpha_N \\ a = \dfrac{1}{N_p}\ e^{\alpha_N\,x_p} \end{cases}$$

The adjusted distribution function around x_p is thus

$$P(x) = 1 - \frac{1}{N_p} \exp\left[-\alpha_N \left(x - x_p\right)\right] \qquad [7.54]$$

For large N_p, one obtains an approximate value of the distribution function of the extreme values making use of the relationship

$$\left(1 - \frac{x}{N_p}\right)^{N_p} \approx e^{-x}$$

which yields

$$P_N(x) \approx \exp\left\{-\exp\left[-\alpha_N \left(x - x_p\right)\right]\right\} \qquad [7.55]$$

7.11.2. *Case: Rayleigh peak distribution*

We have

$$x_p = s_x \sqrt{2 \ln N_p}$$

$$\alpha_N = \frac{1}{s_x} \sqrt{2 \ln N_p}$$

If we set $x = u\ s_x$ and if the reduced variable $\eta = \alpha_N \left(x - x_p\right)$ is considered, we have

$$\eta = \sqrt{2 \ln N_p}\left(u - \sqrt{2 \ln N_p}\right) \qquad [7.56]$$

The distribution function is expressed as

$$P_N(x) = \exp[-\exp(-\eta)] \qquad [7.57]$$

while the probability density is written:

$$\boxed{p(\eta) = \exp[-\eta - \exp(-\eta)]} \qquad [7.58]$$

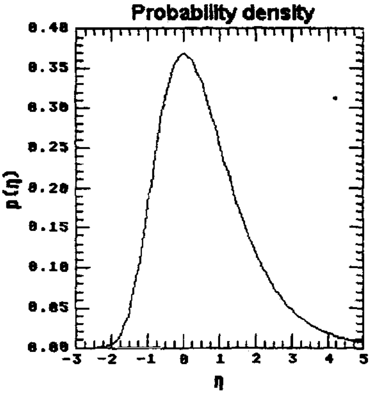


Figure 7.17. *Probability density of extreme values for a Rayleigh peak distribution*

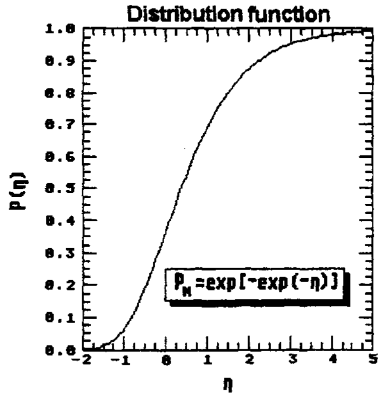


Figure 7.18. *Distribution function of extreme values for a Rayleigh peak distribution*

7.11.3. *Expressions for large values of* N_p

Average maximum

$$\overline{x_{N_p}} = x_p + \frac{\varepsilon}{\alpha_N} \qquad [7.59]$$

where ε = 0.57722 … (Euler's constant).

Standard deviation of maxima

$$s_N = \frac{\pi}{\sqrt{6}} \frac{1}{\alpha_N} \qquad [7.60]$$

Probability of an extreme value less than x_p

$$P_N(x_p) \approx \frac{1}{e} \approx 0.36788 \qquad [7.61]$$

7.12. Choice of type of analysis

The prime objective is to simplify the analysis by reducing the number and duration of the signals studied. The starting datum is in general composed of one or more records of an acceleration time history. If there are several records, the first step is to carry out check of the stationarity of the process and, if it is the case, its ergodicity. If one has only one record, one checks the autostationarity of the signal and its ergodicity. These properties make it possible to reduce the analysis of the whole of the process to that of only one signal sample of short duration (a few tens of seconds for example).

This procedure is not always followed and one often prefers to plot the rms value of the record with respect to time (sliding average on a few tens of points). In a complementary way, one can add the time variations of skewness and kurtosis. This work makes it possible to identify the various events characteristic of the phenomenon, to isolate the shocks, the transitional phases and the time intervals when, the rms value varying little, the signal can be analysed from a sample of short duration. It also makes it possible to make sure that the signal is Gaussian.

The rms value ℓ_{rms} of the signal gives an overall picture of the excitation intensity. It can be useful to calculate the average $E(\ell) = m$. If it differs from zero, one can either centre the signal, if it is estimated that the physical phenomenon has really zero average and that the DC component is due to an imperfection of measurement, or calculate the rms value of the total signal and the standard deviation $s = \sqrt{\ell_{rms}^2 - m^2}$.

In order to have a precise idea of the frequential content of the vibration, it is also important to calculate *the power spectral density* of the signal in a sufficiently broad range not to truncate its frequency contents. If one has measurements carried out at several points of a structure, the PSDs can be used to calculate the transfer functions between these various points. The PSDs are in addition very often used as source data for other more specific analyses.

The test facilities are controlled starting from the PSD and it is still from the PSD that one can most easily evaluate the test feasibility on a given facility: calculation of the rms value of acceleration (on all the whole frequency band or a given band), of the velocity and displacement, average frequency etc.

The autocorrelation function is a little more specific mode of analysis . We saw that this function is the inverse Fourier transform of the PSD. Strictly, there is no more information in the autocorrelation than in the PSD. These two functions however underline different properties of the signal. The autocorrelation makes it possible in particular to identify more easily the periodic signals which can be superimposed on the random vibration (measurement of the periods of the periodic components, measurement of coherence time etc) [VIN 72].

The identification of the nature of *the probability density law of the instantaneous values* of the signal is seldom carried out, for two essential reasons [BEN 61b]:

– this analysis is very long if one wants points representative of the density around 3 to 4 times that of the rms value ℓ_{rms} (a recording lasting 18.5 minutes is necessary to estimate the probability density to $4\ \ell_{rms}$ of a normal law with an error of 30%);

– the tendency is generally, and sometimes wrongly, to consider *a priori* that the signal studied is Gaussian. Skewness and kurtosis are however simple indicators to use.

Peak value distribution

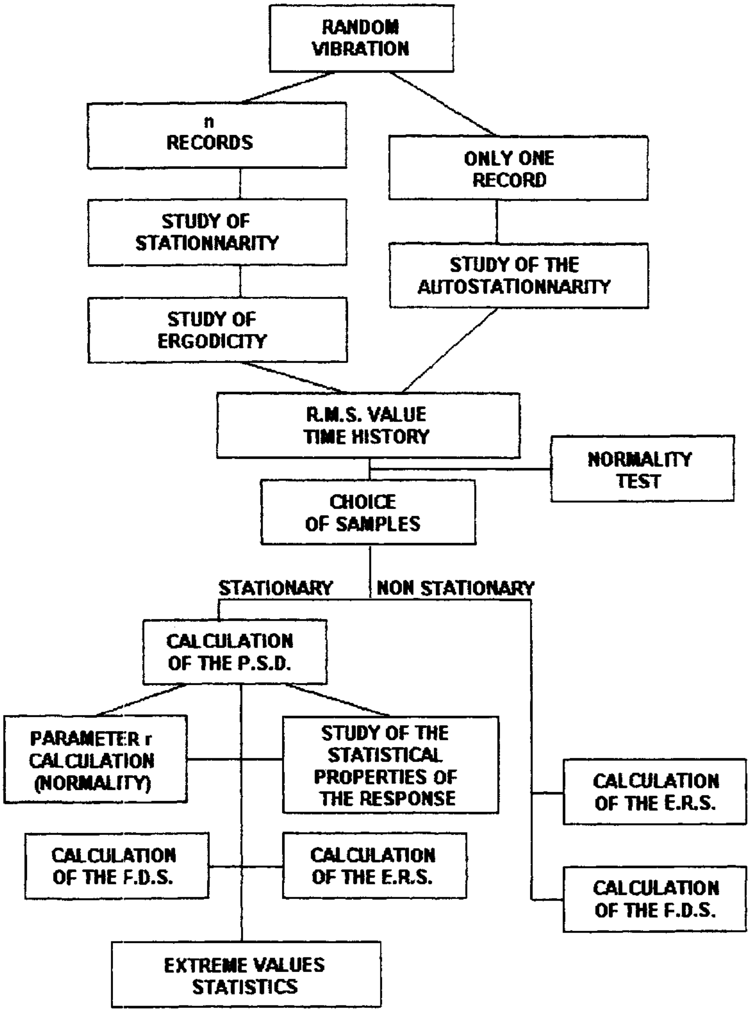
The distribution of the peak values is especially useful to know when one wishes to make a study of the fatigue damage. The parameter as function of time to study must be, in this case, not acceleration at the input or in a point of the specimen, but rather the relative displacement between two given points (or better, directly strains or stresses in the part). The maxima of this displacement are proportional to the maximum stresses in the part on the assumption of linearity. We saw that if the signal is Gaussian, the probability density of the distribution of the peak values follows a law made up of the sum of a Gaussian law and Rayleigh law.

Extreme values analyis

This type of analysis can also be interesting, either for studies of fatigue damage, or for studies of damage due to crossing a threshold stress, while working under the same conditions as above.

It can also be useful to determine these values directly on the acceleration signal to anticipate possible disjunctions of the test facility as a result of going beyond its possibilities.

Table 7.4. *Possibilities of analysis of random vibrations*



Threshold level crossings

The study of threshold crossings of a random signal can have some interest in certain cases:

– to reduce the test duration by preserving the shape of the PSD and that of the threshold level crossings curve (by rotation of this last curve) [HOR 75] [LAL 81]. This method is little used;

– to predict collisions between parts of a structure or to choose the dimension of the clearance between parts (the signal being a relative displacement);

– to anticipate disjunctions of the test facility.

7.13. Study of envelope of narrow band process

7.13.1. *Probability density of maxima of envelope*

It was previously shown how one can estimate the maxima distribution of a random vibration.

Another method of analysing the properties of the maxima can consist in studying the smoothed curve connecting all the peaks of the signal [BEN 58] [BEN 64] [CRA 63] [CRA 67] [RIC 44].

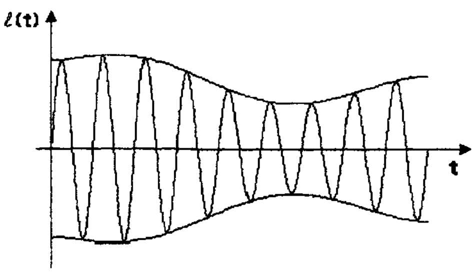


Figure 7.19. *Narrow band vibration and its envelope*

Given a random vibration $\ell(t)$, one can use a diagram giving $\dot{\ell}(t)$ with respect to $\ell(t)$. For a sinusoidal movement, one would have:

$$\ell(t) = A \sin \omega_0 t \qquad [7.62]$$

$$\dot{\ell}(t) = A\, \omega_0 \cos \omega_0 t \qquad [7.63]$$

and the diagram $\dfrac{\dot{\ell}(t)}{\omega_0}$ according to $\ell(t)$ would be a circle of radius A, since:

$$\ell^2(t) + \frac{\dot{\ell}^2(t)}{\omega_0^2} = A^2 \qquad [7.64]$$

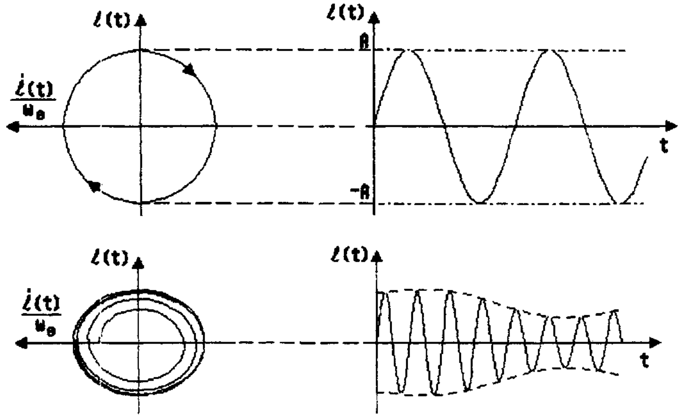


Figure 7.20. *Study of the envelope of a sinusoidal signal and of a narrow band signal*

The envelope of this sinusoid is made up of two straight lines: $\pm A$. In the case of a narrow band random signal, envelope A is a time function and can be regarded as the amplitude of a function of the form [DEE 71]:

$$u(t) = A(t)\ \sin[\omega_0\ t + \theta(t)] \qquad [7.65]$$

in which $A(t)$ and the phase $\theta(t)$ are random functions that are supposed to be slowly variable with ω_0. There are in reality two symmetrical curves with respect to the time axis which are envelopes of the curve $\ell(t)$.

By analogy with the case of a pure sinusoid, $A(t)$ can be considered the radius of the image point in the diagram $\ell(t)$, $\dot{\ell}(t)$:

$$A^2(t) = \ell^2(t) + \frac{\dot{\ell}^2(t)}{\omega_0^2}$$

$(A \geq 0)$, where

$$\ell(t) = A(t)\ \sin[\theta(t)]$$

$$\dot{\ell}(t) = A(t)\ \omega_0\ \cos[\theta(t)]$$

The probability that the envelope lies between A and A + dA is equal to the joint probability that the curves ℓ and $\frac{\dot{\ell}}{\omega_0}$ are located in the hatched field ranging between the two circles of radius A and A + dA (Figure 7.21).

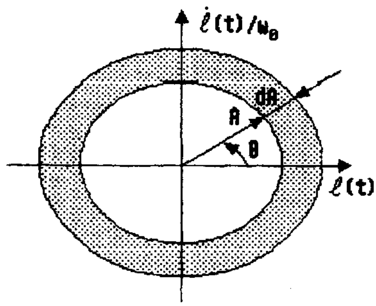


Figure 7.21. *Probability that the envelope lies between* A *and* A + dA

Consider the corresponding two dimensional probability density $p\left(\ell, \frac{\dot{\ell}}{\omega_0}\right)$. We have:

$$p\left(\ell, \frac{\dot{\ell}}{\omega_0}\right) dx \, d\left(\frac{\dot{\ell}}{\omega_0}\right) = \omega_0 \; p\left(\ell, \dot{\ell}\right) \, d\ell \, d\left(\frac{\dot{\ell}}{\omega_0}\right)$$

$$p\left(\ell, \frac{\dot{\ell}}{\omega_0}\right) d\ell \, d\left(\frac{\dot{\ell}}{\omega_0}\right) = p\left(A \sin\theta, A \, \omega_0 \cos\theta\right) A \, dA \, d\theta$$

$$p\left(\ell, \frac{\dot{\ell}}{\omega_0}\right) d\ell \, d\left(\frac{\dot{\ell}}{\omega_0}\right) = q\left(A, \theta\right) dA \, d\theta \qquad [7.66]$$

where

$$q\left(A, \theta\right) = A \;\; p\left(A \sin\theta, \, A \, \omega_0 \cos\theta\right) \qquad [7.67]$$

The probability density function $q(A)$ of the envelope $A(t)$ is obtained by making the sum of all the angles θ:

$$q(\theta) = \int_0^{2\pi} q(A,\theta)\, d\theta \qquad [7.68]$$

Let us suppose now that the random vibration $\ell(t)$ and its derivative $\dot{\ell}(t)$ are statistically independent, with zero averages and equal variances $s_\ell^2 = s_{\dot{\ell}}^2$ $(= \ell_{rms}^2)$, according to a two dimensional Gaussian law:

$$p\left(\ell, \frac{\dot{\ell}}{\omega_0}\right) = \frac{1}{2\pi s_\ell^2} \exp\left[-\frac{\ell^2 + \dfrac{\dot{\ell}^2}{\omega_0^2}}{2 s_\ell^2}\right] = \frac{1}{2\pi s_\ell^2} \exp\left[-\frac{A^2}{2 s_\ell^2}\right] \qquad [7.69]$$

$$q(A, \theta) = \frac{A}{2\pi s_\ell^2} \exp\left[-\frac{A^2}{2 s_\ell^2}\right] \qquad [7.70]$$

and

$$q(A) = \frac{A}{s_\ell^2} \exp\left[-\frac{A^2}{2 s_\ell^2}\right] \qquad [7.71]$$

$(A \geq 0)$. The probability density of the envelope $A(t)$ follows Rayleigh's law.

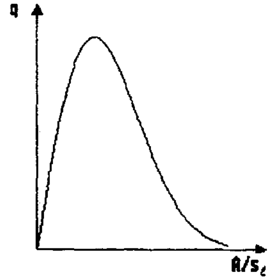


Figure 7.22. *Probability density of envelope* A(t)

NOTES.

1. *The probability density* $q(A)$, *calculated at a given time* t, *is independent of* t, *the process being supposed stationary.*

One could calculate this density from an arbitrary signal ${}^{i}\ell(t)$. *The result would be independent of the sample chosen in* ${}^{i}\ell(t)$ *if the process is ergodic [CRA 63].*

2. *The density* $q(A)$ *has the same form as the probability density* $q(a)$ *of maxima [CRA 63]. It is a consequence of the assumption of a Gaussian law for* $\ell(t)$ *and* $\dot{\ell}(t)$. *In the case of a narrow band noise for which this assumption would not be observed, or if the system were nonlinear, the densities* $q(A)$ *and* $q(a)$ *would have different forms [BEN 64] [CRA 61].*

When the process has only one maximum per cycle, the maxima have the same distribution as its envelope (this remark is strictly true when $r = 1$).

When the number of maxima per second increases and tends towards infinity, it has been seen that the distribution of maxima becomes identical to that of the instantaneous values of the signal (Gaussian law) [CRA 68].

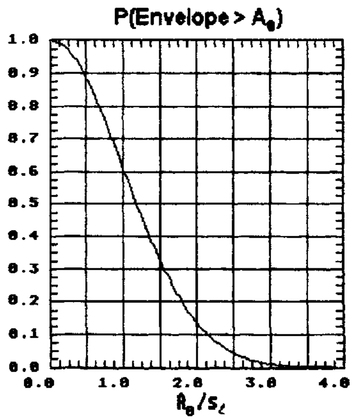


Figure 7.23. *Probability that the envelope exceeds a given threshold* A_0

The probability that the envelope exceeds a certain given value A_0 is obtained by integrating $q(A)$ between A_0 and infinity.

$$P(\text{Envelope} > A_0) = \int_{A_0}^{\infty} q(A)\, dA$$

$$P(\text{Envelope} > A_0) = \exp\left[-\frac{A_0^2}{2\, s_\ell^2}\right]$$

Table 7.5. *Examples of probabilities of threshold crossings*

$\frac{A_0}{s_\ell}$	P
0.5	0.8825
1	0.6065
2	0.1353
3	0.0111

7.13.2. *Distribution of maxima of envelope*

S.O. Rice [RIC 44] showed that the average number of maxima (per second) of the envelope of a white noise between two frequencies f_a and f_b is:

$$N \approx 0.64110\,(f_b - f_a) \qquad [7.72]$$

Let us set $v = \frac{A_{max}}{s_\ell}$. If v is large (superior to 2.5), the probability density $q(v)$ can be approximated by:

$$\boxed{q(v) \approx \frac{\sqrt{\frac{\pi}{6}}}{0.64110}\left(v^2 - 1\right) e^{-\frac{v^2}{2}}} \qquad [7.73]$$

and the corresponding distribution function by:

$$Q_A = Q\left(A_{max} < v\, s_\ell\right) \approx 1 - \frac{\sqrt{\frac{\pi}{6}}}{0.64110}\, v\, e^{-\frac{v^2}{2}} \qquad [7.74]$$

Q_A is the probability that a maximum of the envelope chosen randomly is lower than a given value $A = v\, s_\ell$. The functions $q(v)$ and Q_A are respectively plotted in the general case (arbitrary v) on Figures 7.24 and 7.25.

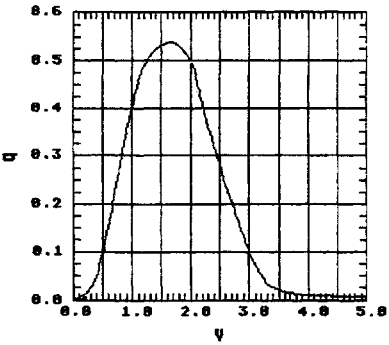


Figure 7.24. *Probability density of envelope maxima*

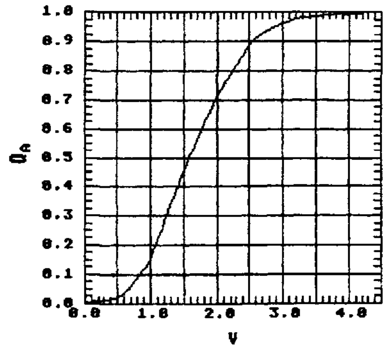


Figure 7.25. *Distribution function of envelope maxima*

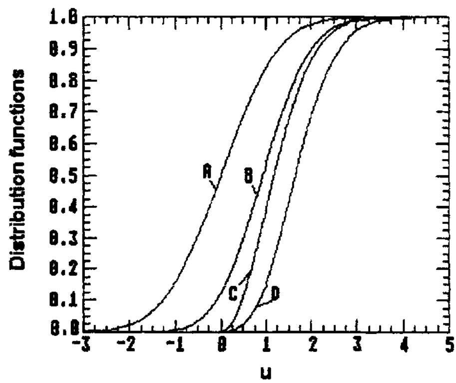


Figure 7.26. *Comparison of distribution functions*

Figure 7.26 shows, by way of comparison, the distribution functions of:

– the instantaneous values of the signal (Gaussian law) (A):
$$P = \frac{1}{2}\left[1 + \text{Erf}\left(\frac{u}{\sqrt{2}}\right)\right],$$

– the maxima of the signal [6.64] (B),

– the instantaneous values of the envelope (Rayleigh's law) (C): $P = 1 - e^{-\frac{u^2}{2}}$,

– the maxima of the envelope (curve given by S.O. Rice [RIC 44]) (D).

7.13.3. *Average frequency of envelope of narrow band noise*

It is shown that [BOL 84]:

$$\varphi_M^2 = \frac{\int_0^\infty (\Omega - \theta)^2 \, G(\Omega) \, d\Omega}{\ell_{rms}} \qquad [7.75]$$

where

ℓ_{rms} = rms value of the noise $\ell(t)$

θ = average pulsation of the noise $(2 \pi f_0)$

f_0 = average frequency of $\ell(t)$

For a signal $\ell(t)$ whose PSD $G(f)$ is constant between frequencies f_1 and f_2 and centered on f_0, this relationship leads to:

$$\varphi_M^2 = \frac{\int_{f_1}^{f_2} (f - f_0)^2 \, df}{f_2 - f_1}$$

i.e. to:

$$\boxed{\varphi_M = \left[\frac{f_1^2 + f_1 \, f_2 + f_2^2}{3} - f_0 \left(f_1 + f_2\right) + f_0^2\right]^{1/2}} \qquad [7.76]$$

Summary tables of the main results

Table 7.6 (a). *Main results*

Parameter	Relation	Expression
Number of crossings of a threshold a with positive slope per unit time	[5.47]	$n_a = n_0 \, e^{-\frac{a^2}{2 \ell_{rms}^2}}$
Average frequency	[5.53] [5.79]	$n_0^+ = \left[\frac{\int_0^\infty f^2 \, G(f) \, df}{\int_0^\infty G(f) \, df} \right]^{\frac{1}{2}} = \frac{1}{2\pi} \sqrt{\frac{M_2}{M_0}}$
Moments	[5.77]	$M_n = (2\pi)^n \int_0^\infty f^n \, G(f) \, df$
Irregularity factor	[6.6]	$r = \frac{\dot{\ell}_{rms}^2}{\ell_{rms}^2 \, \ddot{\ell}_{rms}^2} = \frac{M_2}{\sqrt{M_0 \, M_4}} = \frac{R^{(2)}(0)}{\sqrt{R(0) \, R^{(4)}(0)}}$
Probability density of the maxima	[6.19]	$q(u) = \frac{\sqrt{1-r^2}}{\sqrt{2\pi}} \, e^{-\frac{u^2}{2(1-r^2)}} + \frac{r}{2} \, u \, e^{-\frac{u^2}{2}} \left[1 + \text{Erf}\left(\frac{r \, u}{\sqrt{2\left(1-r^2\right)}} \right) \right]$
Average number of maxima per second	[6.13] [6.31]	$n_p^+ = \left[\frac{\int_0^{+\infty} f^4 \, G(f) \, df}{\int_0^{+\infty} f^2 \, G(f) \, df} \right]^{\frac{1}{2}} = \frac{1}{2\pi} \sqrt{\frac{M_4}{M_2}}$
Average time between two successive maxima (narrow band noise)	[6.40]	$\tau_m = \frac{1}{f_0} \left[\frac{1 + \frac{1}{3} \left(\frac{\Delta f}{2 \, f_0} \right)^2}{1 + 2 \left(\frac{\Delta f}{2 \, f_0} \right)^2 + \frac{1}{5} \left(\frac{\Delta f}{2 \, f_0} \right)^4} \right]^{\frac{1}{2}}$

Average correlation between two successive maxima	[6.41]	$\rho = \frac{1}{2\pi\delta}\left[\frac{1+2\delta^2+\frac{\delta^4}{5}}{1+\frac{\delta^2}{3}}\right]^{1/2} \cos\left(\frac{1+\frac{\delta^2}{3}}{1+2\delta^2+\frac{\delta^4}{5}}\right)^{1/2} \sin\left(\frac{1+\frac{\delta^2}{3}}{1+2\delta^2+\frac{\delta^4}{5}}\right)^{1/2}$

Table 7.6 (b). *Main results*

Parameter	Relation	Expression
Distribution function of the peaks	[6.64]	$Q_p(u) = \frac{1}{2}\left\{1 - \mathrm{Erf}\left[\frac{u}{\sqrt{2\left(1-r^2\right)}}\right]\right\} + \frac{r}{2}\, e^{-\frac{u^2}{2}} \left\{1 + \mathrm{Erf}\left[\frac{r\,u}{\sqrt{2\left(1-r^2\right)}}\right]\right\}$
Average number of maxima greater than a threshold a per unit time	[6.74]	$M_a = \frac{1}{2\pi}\sqrt{\frac{M_4}{M_2}}\, e^{-\frac{a^2}{2\ell_{rms}^2}} = \frac{1}{2\pi}\sqrt{\frac{M_2}{M_0}}\, e^{-\frac{a^2}{2 M_0}}$
Average number of positive maxima per second	[6.104]	$n_p^+ = \frac{1}{4\pi}\left(\sqrt{\frac{M_2}{M_0}} + \sqrt{\frac{M_4}{M_2}}\right)$
Average time interval between the maxima	[6.80]	$T_a = \frac{1}{M_a} = \frac{1}{n_p^+\, Q(a)}$
Average time interval between the maxima (narrow band noise)	[6.82]	$T_a = 2\,\pi\,\frac{M_0}{M_2}\, e^{\frac{a^2}{2 M_0}}$
Probability so that a maximum is positive or negative	[6.98] [6.99]	$q_{max}^+ = \frac{1+r}{2} \qquad q_{max}^- = \frac{1-r}{2}$
Time during which the signal is above a given value	[6.93]	$\bar{T}_{ab} = \int_a^b \frac{1}{\ell_{rms}\sqrt{2\,\pi}}\, e^{-\frac{\ell^2}{2\,\ell_{rms}^2}}\, d\ell$
Average amplitude jump between two successive maxima	[6.107]	$\bar{h} = \sqrt{2\,\pi}\, r\, \ell_{rms}$

Probability density of the largest peaks	[7.19]	$p_N(\ell) = N_p \left[1 - \exp\left(-\frac{\ell^2}{2\,s_\ell^2}\right)\right]^{N_p-1} \frac{\ell}{s_\ell^2} \exp\left(-\frac{\ell^2}{2\,s_\ell^2}\right)$

Table 7.6 (c). *Main results*

Parameter	Relation	Expression
Probability density of the largest maximum over duration T	[7.26]	$p_N(u) = n_0^+\,T\,u\,\exp\left\{-\left[\frac{u^2}{2} + n_0^+\,T\,\exp\left(-\frac{u^2}{2}\right)\right]\right\}$
Average for the large values of the number of peaks (narrow band noise)	[7.29]	$\overline{u_0} \approx \sqrt{2\,\ln\left(n_0^+\,T\right)} + \frac{\varepsilon}{\sqrt{2\,\ln\left(n_0^+\,T\right)}}$
Standard deviation (narrow band noise)	[7.31]	$s_{u_0} \approx \frac{\pi}{\sqrt{6}}\,\frac{1}{\sqrt{2\,\ln\left(n_0^+\,T\right)}}$
Most probable value (mode)	[7.35]	$m = \sqrt{2\,\ln\left(n_0^+\,T\right)}$
Maximum exceeded with a risk α	[7.41]	$\ell_{N\alpha} = \sqrt{2\,\ln\frac{1}{\left[1-(1-\alpha)\right]^{1/N_p}}}\;s_\ell$
Average for the great values of the number of peaks (wide-band noise)	[7.43]	$\overline{u}_0 \approx \sqrt{2\,\ln\left(r\,N_p\right)} + \frac{\varepsilon}{\sqrt{2\,\ln\left(r\,N_p\right)}}$
Standard deviation (wide-band noise)	[7.48]	$s^2 = \frac{\pi^2}{6}\,\frac{m^2}{\left(m^2+1\right)^2}$
Probability density of the envelope of a narrow band Gaussian process	[7.71]	$q(A) = \frac{A}{s_\ell^2}\,\exp\left[-\frac{A^2}{2\,s_\ell^2}\right]$

Distribution of maxima of the envelope of a narrow band process	[7.73]	$q(v) \approx \dfrac{\sqrt{\dfrac{\pi}{6}}}{0.64110}\left(v^2 - 1\right) e^{-\frac{v^2}{2}}$
Average frequency of the envelope of a narrow band noise of constant PSD	[7.76]	$\varphi_M = \left[\dfrac{f_1^2 + f_1\, f_2 + f_2^2}{3} - f_0\left(f_1 + f_2\right) + f_0^2\right]^{1/2}$

Appendices

A1. Laws of probability

A1.1. *Gauss's law*

This law is also called the *Laplace-Gauss* or *normal law*.

Probability density	$p(x) = \dfrac{1}{s\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-m}{s}\right)^2}$ m = mean, s = standard deviation The law is referred to as reduced centered normal if m = 0 and s = 1.
Distribution function	$F(X) = P(x < X) = \dfrac{1}{s\sqrt{2\pi}} \int_{-\infty}^{X} e^{-\frac{1}{2}\left(\frac{x-m}{s}\right)^2} dx$ Reduced variable: $t = \dfrac{x-m}{s}$
	If E_1 is the error function $F(T) = \dfrac{1}{2}\left[1 + E_1\left(\dfrac{T}{\sqrt{2}}\right)\right]$ where $T = \dfrac{X-m}{s}$
Mean	M
Variance (central moment of order 2)	s^2

Central moments	$\mu_k = \frac{\sigma^k}{\sqrt{2\,\pi}} \int_{-\infty}^{+\infty} t^k \, e^{-\frac{t^2}{2}} \, dt$ k even (k = 2 r): $\mu_{2\,r} = \frac{2\,r!}{2^r\,r!}\,\sigma^{2\,r}$ k odd (k = 2 r + 1) $\mu_{2\,r+1} = 0$
Moment of order 3 and skewness	0
Kurtosis	$\mu'_4 = \frac{\mu_4}{s^4} = 3$
Median and mode	0

A1.2. *Log-normal law*

Probability density	$p(x) = \frac{1}{x\,s_y\,\sqrt{2\,\pi}}\,e^{-\frac{1}{2}\left(\frac{\ln x - m_y}{s_y}\right)^2}$ m_y = mean s_y = standard deviation of the normal random variable $y = \ln x$ $p(y)\,dy = \frac{1}{s_y\,\sqrt{2\,\pi}}\,e^{-\frac{1}{2}\left(\frac{y - m_y}{s_y}\right)^2}\,dy$ The log-normal law is thus obtained from the normal law by the change of variable $x = e^y$.
Distribution function	$F(X) = P(x < X) = \int_0^X \frac{1}{x\,s_y\,\sqrt{2\,\pi}}\,e^{-\frac{1}{2}\left(\frac{\ln x - m_y}{s_y}\right)^2}\,dx$
Mean	$m = E(x) = e^{m_y + \frac{s_y^2}{2}}$

<table>
<tr><td>Variance (central moment of order 2)</td><td>$s^2 = e^{2\,m_y + s_y^2}\left(e^{s_y^2} - 1\right) = m^2\left(e^{s_y^2} - 1\right)$
$s^2 = \left(e^{s_y^2} - 1\right)\left[e^{m_y + \frac{s_y^2}{2}}\right]^2$</td><td>$s^2 = [E(x)]^2\ v^2$
$v = \sqrt{e^{s_y^2} - 1}$
v = variation coefficient</td></tr>
<tr><td rowspan="3">Expressions for m_y and s_y^2 with respect to $E(x)$ and s^2</td><td>$m_y = \ln[E(x)] - \frac{1}{2}\ln\left[1 + \frac{s^2}{E^2(x)}\right]$</td><td>$s_y^2 = \ln\left[1 + \frac{s^2}{E^2(x)}\right]$</td></tr>
<tr><td>$m_y = \ln\left[E^2(x)\right] - \frac{1}{2}\ln\left[s^2 + E^2(x)\right]$</td><td>$s_y = -\ln\left[E^2(x)\right] + \ln\left[s^2 + E^2(x)\right]$</td></tr>
<tr><td colspan="2">It is noted that the transformation $x = e^y$ applies neither to $E(x)$, nor to s^2.</td></tr>
<tr><td rowspan="3">Variation coefficient</td><td>$m_y = \ln\tilde{x} = \ln m - \frac{1}{2}\ln\left(1 + v^2\right)$
$s_y^2 = \ln\left(1 + v^2\right)$</td><td>$v = \sqrt{e^{s_y^2} - 1}$</td></tr>
<tr><td colspan="2">If v is the variation coefficient of x, we have also $m = \tilde{x}\sqrt{1 + v^2}$</td></tr>
<tr><td colspan="2">If two log-normal distributions have the same variation coefficient, they have equal values of s_y (and conversely).</td></tr>
<tr><td>Moment of order j</td><td colspan="2">$\lambda'_j = e^{j\,m_j + \frac{1}{2} j^2\, s_y^2}$</td></tr>
<tr><td>Central moment of order 3</td><td colspan="2">$\lambda_3 = \left(e^{s_y^2} - 1\right)^{3/2} e^{\frac{3}{2}\left(2\,m + s_y^2\right)}\left(e^{s_y^2} + 2\right)^2 = m^3\left(v^6 + 3\,v^4\right)$</td></tr>
<tr><td>Skewness</td><td colspan="2">$\lambda'_3 = \frac{\lambda_3}{s^3} = \left(e^{s_y^2} - 1\right)\left(e^{s_y^2} + 2\right)^2 = v^3 + 3\,v$</td></tr>
<tr><td>Central moment of order 4</td><td colspan="2">$\lambda_4 = m^4\left(v^{12} + 6\,v^{10} + 15\,v^8 + 16\,v^6 + 3\,v^4\right)$</td></tr>
<tr><td>Kurtosis</td><td colspan="2">$\frac{\lambda_4}{s^4} = v^8 + 6\,v^6 + 15\,v^4 + 16\,v^2 + 3$</td></tr>
<tr><td>Median</td><td colspan="2">$\tilde{x} = e^{m_y}$</td></tr>
</table>

Mode	$M = e^{m_y - s_y^2}$ For this value, the probability density has a maximum equal to $\frac{1}{s_y \sqrt{2\pi}} e^{-m_y + \frac{s_y^2}{2}}$

[AIT 81] [CAL 69] [KOZ 64] [PAR 59] [WIR 81] [WIR 83].

NOTES.

1. *Another definition can be: a random variable* x *follows a log-normal law if and only if* y = ln x *is normally distributed, with average* m_y *and variance* s_y^2.

2. *This law has several names: the Galton, Mc Alister, Kapteyn, Gibrat law or the logarithmic-normal or logarithmo-normal law.*

3. *The definition of the log-normal law can be given starting from base 10 logarithms* ($y = \log_{10} x$)*:*

$$p(x) = \frac{1}{x\, s_y \sqrt{2\pi}\, \ln 10} e^{-\frac{1}{2}\left(\frac{\log_{10} x - m_y}{s_y}\right)^2} \qquad [A1.1]$$

With this definition for base 10 logarithms, we have:

$$m_y = \log_{10} \tilde{x} \qquad [A1.2]$$

$$m = 10^{\left[m_y + \frac{1}{2}\log_{10}\left(s_y^2/0.434\right)\right]} \qquad [A1.3]$$

$$v = \sqrt{10^{s_y^2/0.434} - 1} \qquad [A1.4]$$

$$m_y = \log_{10} x - \frac{1}{2}\log_{10}\left(1 + v^2\right) \qquad [A1.5]$$

$$s_y^2 = 0.434 \log_{10}\left(1 + v^2\right) \qquad [A1.6]$$

Hereafter, we will consider only the definition based on Napierian logarithms.

4. *Some authors make the variable change defined by* $y = 20 \log x$, y *being expressed in decibels. We then have:*

$$v = \sqrt{e^{s_y^2/75.44} - 1} \qquad [A1.7]$$

since $\left(\frac{20}{\ln 10}\right)^2 \approx 75.44$.

5. *Depending on the values of the parameters* m_y *and* s_y, *it can sometimes be difficult to imagine a priori which is the law which is best adjusted to an enseble of experimental values. A method allowing for choosing between the normal law and the lognormal law consists in calculating:*

– *the variation coefficient* $v = \frac{s}{m}$,

– *the skewness* $\frac{\lambda_3}{s^3}$,

– *the kurtosis* $\frac{\lambda_4}{s^4}$,

knowing that

$$E(x) = m = \frac{\sum_i x_i}{n} \qquad [A1.8]$$

$$s^2 = \frac{\sum_i (x_i - m)^2}{n} \qquad [A1.9]$$

$$\lambda_3 = \frac{\sum_i (x_i - m)^3}{n} \qquad [A1.10]$$

and

$$\lambda_4 = \frac{\sum_i (x_i - m)^4}{n} \qquad [A1.11]$$

If skewness is close to zero and kurtosis close to 3, the normal law is that which is best adjusted. If $v < 0.2$ *and* $\frac{\lambda_3'}{v} \approx 3$*, the log-normal law is preferable.*

A1.3. *Exponential law*

This law is often used with reliability where it expresses the time expired up to failure (or the time interval between two consecutive failures).

Probability density	$p(x) = \lambda\, e^{-\lambda x}$
Distribution function	$F(X) = P(x < X) = 1 - e^{-\lambda X}$
Mean	$m_1 = E(x) = \frac{1}{\lambda}$
Moments	$m_n = \frac{n\,!}{\lambda^n} = \frac{n}{\lambda}\, m_{n-1}$
Variance (central moment of order 2)	$s^2 = \frac{1}{\lambda^2}$
Central moments	$\mu_n = 1 + \frac{n}{\lambda}\, \mu_{n-1}$
Variation coefficient	$v = 1$
Moment of order 3 (skewness)	$\mu_3' = \frac{\mu_3}{s^3} = \lambda^3 + 3$
Kurtosis	$\mu_4' = \lambda^4 + 4\,\lambda^3 + 12$

A1.4. *Poisson's law*

It is said that a random variable X is a Poisson variable if its possible values are countable to infinity $x_0, x_1, x_2 \ldots, x_k \ldots$, the probability that $X = x_k$ being given by:

$$p_k = P(X = x_k) = e^{-\lambda}\, \frac{\lambda^k}{k\,!} \qquad [A1.12]$$

where λ is an arbitrary positive number.

One can also define it in a similar way as a variable able to take all on the integer values, a countable infinity, 0, 1, 2, 3..., k..., the value k having the probability:

$$p_k \equiv P(X = k) = e^{-\lambda} \frac{\lambda^k}{k\,!} \qquad \text{[A1.13]}$$

The random variable is here a number of events (we saw that, with an exponential law, the variable is the time interval between two events).

The set of possible values n and their probability p_k constitutes the Poisson law for parameter λ. This law is a discrete law. It is shown that the Poisson law is the limit of the binomial distribution when the probability p of this last law is equal to $\frac{\lambda}{k}$ and when k tends towards infinity.

Distribution function	$F(X) = P(0 \le x < X) = \sum_{k=0}^{n} e^{-\lambda} \frac{\lambda^k}{k\,!} \quad (n < X \le n+1)$
Mean	$m_1 = E(x) = \sum_{k=0}^{\infty} k\, e^{-\lambda} \frac{\lambda^k}{k\,!} = \lambda$
Moment of order 2	$m_2 = \lambda\,(\lambda + 1)$
Variance (central moment of order 2)	$s^2 = \lambda$
Central moments	$\mu_3 = \lambda$ $\mu_4 = \lambda + 3\,\lambda^2$
Variation coefficient	$v = \frac{1}{\sqrt{\lambda}}$
Skewness	$\mu'_3 = \frac{1}{\sqrt{\lambda}}$
Kurtosis	$\mu'_4 = \frac{1 + 3\,\lambda}{\lambda}$

NOTES.

1. *Skewness* μ'_3 *being always positive, the Poisson distribution is dissymmetrical, more spread out on the right.*

2. *If λ tends towards infinity, μ'_3 tends towards zero and μ'_4 tends towards 3. There is convergence from Poisson's law towards the Gaussian law. When λ is large, the Poisson distribution is very close to a normal distribution.*

A1.5. *Chi-square law*

Given ν random variables $u_1, u_2 \ldots, u_\nu$, supposed independent reduced normal, i.e. such that:

$$f(u_i) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u_i^2}{2}} du_i \qquad [A1.14]$$

one calls chi-square law with ν degrees of freedom (ν independent variables) the probability law of the variable χ_ν^2 defined by:

$$\chi_\nu^2 = u_1^2 + u_2^2 + \cdots + u_\nu^2 = \sum_{i=1}^{\nu} u_i^2 \qquad [A1.15]$$

The variables u_i being continuous, the variable χ^2 is continuous in $(0, \infty)$.

NOTES.

1. *The variable χ^2 can also be defined starting from ν independent non reduced normal random variables x_i whose averages are respectively equal to $m_i = E(x_i)$ and the standard deviations s_i, while referring back to the preceding definition with the reduced variables $u_i = \dfrac{x_i - m_i}{s_i}$ and the sum $\chi_\nu^2 = \sum_{i=1}^{\nu} u_i^2$.*

2. *The sum of the squares of independent non reduced normal random variables does not follow a chi-square law.*

Probability density	$p(\chi_\nu^2) = \dfrac{1}{2^{\nu/2}\,\Gamma(\nu/2)} (\chi_\nu^2)^{\frac{\nu}{2}-1} e^{-\frac{\chi_\nu^2}{2}}$ ν= number of degrees of freedom Γ= Euler function-second kind (gamma function)

Mean	$E\left(\chi_\nu^2\right) = \sum_{i=1}^{\nu} E\left(u_i^2\right) = \nu$
Moment of order 2	$m_2 = \nu\,(2+\nu)$
Standard deviation	$s = \sqrt{2\,\nu}$
Central moments	$\mu_2 = 2\,\nu \quad \mu_3 = 8\,\nu \quad \mu_4 = 12\,\nu\,(\nu+4)$
Variation coefficient	$V = \sqrt{\frac{2}{\nu}}$
Skewness	$\mu_3' = 2\sqrt{\frac{2}{\nu}}$
Kurtosis	$\mu_4' = 3\,\frac{\nu+4}{\nu}$
Mode	$M = \nu - 2$

This law is comparable to a normal law when ν is greater than 30 approximately.

A1.6. *Rayleigh's law*

Probability density	$p(x) = \frac{x}{\nu^2}\, e^{-\frac{x^2}{2\,\nu^2}}$ ν is a constant
Distribution function	$F(X) = P(x < X) = 1 - e^{-\frac{X^2}{2\,\nu^2}}$
Mean	$m = \sqrt{\frac{\pi}{2}}\,\nu$
Median	$X = \nu\sqrt{2\,\ln 2} \approx 1.1774\,\nu$
Rms value	$\sqrt{E\left(x^2\right)} = \nu\sqrt{2}$

Variance	$s^2 = \left(2 - \frac{\pi}{2}\right) v^2 = \left(\frac{4}{\pi} - 1\right) m^2$
Moment of order k	If k is odd (k = 2r − 1) $m_{2r-1} = \frac{(2r)!}{2^r\, r!} \sqrt{\frac{\pi}{2}}\, v^{2r-1}$
	If k is even (k = 2 r) $m_{2r} = 2^r\, r!\, v^{2r}$
Central moments	$\mu_0 = 1$ $\mu_1 = 0$ $\mu_2 = \left(2 - \frac{\pi}{2}\right) v^2$ $\mu_3 = \sqrt{\frac{\pi}{2}}\, (\pi - 3)\, v^3$
Variation coefficient	$v = \sqrt{\frac{4}{\pi} - 1}$
Skewness	$a = \sqrt{\frac{\pi}{2}}\, \frac{\pi - 3}{\left(2 - \frac{\pi}{2}\right)^{3/2}} \approx 0.6311$ $\mu_4 = \left(8 - \frac{3\pi^2}{4}\right) v^4$
Kurtosis	$b = \frac{32 - 3\pi^2}{(4 - \pi)^2} \approx 3.2451$
Mode	$M = v$

Reduced law

If we set $u = \frac{x}{s}$, it becomes, knowing that $p(x) = \frac{x}{s^2}\, e^{-\frac{x^2}{2\sigma^2}}$,

$$p(x) = \frac{1}{s}\, \frac{x}{s}\, e^{-\frac{x^2}{2 s^2}} = \frac{1}{s}\, u\, e^{-\frac{u^2}{2}} = \frac{1}{s}\, p(u) \qquad \text{[A1.16]}$$

$$p(u)\ du = p(u)\ d\left(\frac{x}{s}\right) = s\ p(x)\ \frac{dx}{s} = p(x)\ dx \qquad [A1.17]$$

Table A1.1. *Particular values of the Rayleigh distribution*

$\frac{X}{\nu}$	$\text{Prob}\left(\frac{x}{\nu} > \frac{X}{\nu}\right)$
1	0.60653
1.5	0.32465
2	0.13534
2.5	$4.3937\ 10^{-2}$
3	$1.1109\ 10^{-2}$
3.5	$2.1875\ 10^{-3}$
4	$3.355\ 10^{-4}$
4.5	$4.01\ 10^{-5}$
5	$3.7\ 10^{-6}$

A1.7. *Weibull distribution*

Probability density	$p(x) = \begin{cases} \frac{\alpha}{\nu-\alpha}\left(\frac{x-\varepsilon}{\nu-\varepsilon}\right)^{\alpha-1} \exp\left[-\left(\frac{x-\varepsilon}{\nu-\varepsilon}\right)^{\alpha}\right] & x > \varepsilon \\ 0 & x < \varepsilon \end{cases}$ α and ν= positive constants
Distribution function	$F(X) = \begin{cases} 1 - \exp\left[-\left(\frac{X-\varepsilon}{\nu-\varepsilon}\right)^{\alpha}\right] & X > \varepsilon \\ 0 & X < \varepsilon \end{cases}$
Mean	$m = \varepsilon + (\nu - \varepsilon)\ \Gamma\left(1 + \frac{1}{\alpha}\right)$
Median	$X = \varepsilon + (\nu - \varepsilon)\ (\ln 2)^{1/\alpha}$

Variance	$s^2 = (\nu - \varepsilon)^2 \left[\Gamma\left(1 + \frac{2}{\alpha}\right) - \Gamma^2\left(1 + \frac{1}{\alpha}\right)\right]$
Mode	$M = \varepsilon + (\nu - \varepsilon)\left(1 - \frac{1}{\alpha}\right)^{1/\alpha}$

[KOZ 64] [PAR 59].

NOTE.

One sometimes uses the constant $\eta = \nu - \varepsilon$ *in the above expressions.*

A1.8. *Normal Laplace-Gauss law with n variables*

Let us set $x_1, x_2 ..., x_n$ n random variables with zero average. The normal law with n variable x_i is defined by its probability density:

$$p(x_1, x_2, \cdots, x_n) = (2\pi)^{-n/2} |M|^{-1/2} \exp\left[-\frac{1}{2|M|} \sum_{i,j}^{n} M_{ij}\, x_i\, x_j\right] \quad \text{[A1.18]}$$

where $|M|$ is the determinant of the square matrix:

$$\|M\| = \begin{Vmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1n} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n1} & \mu_{n2} & \cdots & \mu_{nn} \end{Vmatrix} \quad \text{[A1.19]}$$

$\mu_{ij} = E(x_i, x_j)$ = moments of second order of the random variables

M_{ij} = cofactor of μ_{ij} in $|M|$.

Examples

1. n = 1

$$p(x_1) = (2\pi)^{-1/2} |M|^{-1/2} \exp\left(-\frac{1}{2} \frac{M_{11}}{|M|} x_1^2\right)$$

with

$$\|M\| = \|\mu_{11}\|$$
$$|M| = \mu_{11}$$
$$M_{11} = 1$$
$$\mu_{11} = E\left(x_1^2\right) = s^2$$

yielding

$$p(x_1) = \frac{1}{s\sqrt{2\pi}}\, e^{-\frac{x_1^2}{2s^2}} \qquad [A1.20]$$

which is the probability density of a one dimensional normal law as defined previously.

2. n = 2

$$p(x_1, x_2) = (2\pi)^{-1} |M|^{-1/2} \exp\left[-\frac{1}{2|M|}\left(M_{11}\, x_1^2 + M_{12}\, x_1\, x_2 + M_{21}\, x_2\, x_1 + M_{22}\, x_2^2\right)\right] \qquad [A1.21]$$

with

$$\|M\| = \begin{Vmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{Vmatrix} \qquad |M| = \mu_{11}\,\mu_{22} - \mu_{12}\,\mu_{21}$$

$$\mu_{11} = E\left(x_1^2\right) = s_1^2 \qquad M_{11} = \mu_{22}$$

$$\mu_{12} = E(x_1\, x_2) = E(x_2\, x_1) = \mu_{21} = \rho\, s_1\, s_2 \qquad M_{12} = -\mu_{21} = -\mu_{12} = M_{21}$$

$$\mu_{22} = E\left(x_2^2\right) = s_2^2 \qquad M_{22} = \mu_{11}$$

ρ is the coefficient of linear correlation between the variables x_1 and x_2, defined by:

$$\rho = \frac{\mathrm{cov}(x_1, x_2)}{s(x_1)\, s(x_2)} \qquad [A1.22]$$

where $\mathrm{cov}(x_1, x_2)$ is the covariance between the two variables x_1 and x_2:

$$\text{cov}(X_1, X_2) = \int\int_{-\infty}^{+\infty} [x_1 - E(X_1)][x_2 - E(X_2)]\, p(x_1, x_2)\, dx_1\, dx_2 \quad \text{[A1.23]}$$

Covariance can be negative, zero or positive. Covariance is zero when x_1 and x_2 are completely independent variables. Conversely, a zero covariance is not a sufficient condition that x_1 and x_2 be independent.

It is shown that ρ is included in the interval $[-1, 1]$. $\rho = 1$ is a necessary and sufficient condition of linear dependence between x_1 and x_2.

Yielding

$$p(x_1, x_2) = \frac{1}{2\,\pi\, s_1\, s_2 \sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[\left(\frac{x_1}{s_1}\right)^2 - 2\,\rho\,\frac{x_1\, x_2}{s_1\, s_2} + \left(\frac{x_2}{s_2}\right)^2\right]\right\}$$

[A1.24]

NOTE.

If the averages were not zero, we would have

$$p(x_1, x_2) = \frac{1}{2\,\pi\, s_1\, s_2 \sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[\left(\frac{x_1 - E(x_1)}{s_1}\right)^2 \right.\right.$$

$$\left.\left. -2\,\rho\,\frac{x_1 - E(x_1)}{s_1}\,\frac{x_2 - E(x_2)}{s_2} + \left(\frac{x_2 - E(x_2)}{s_2}\right)^2\right]\right\} \quad \text{[A1.25]}$$

If $\rho = 0$, we can write $p(x_1, x_2) = p(x_1)\,p(x_2)$ where $p(x_1) = \frac{1}{s_1\sqrt{2\,\pi}}\, e^{-\frac{x_1^2}{2\, s_1^2}}$

and $p(x_2) = \frac{1}{s_2\sqrt{2\,\pi}}\, e^{-\frac{x_2^2}{2\, s_2^2}}$. x_1 and x_2 are independent random variables.

It is easily shown, by using the reduced centered variables $t_1 = \frac{x_1 - E(x_1)}{s_1}$ and $t_2 = \frac{x_2 - E(x_2)}{s_2}$, that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p(x_1\ x_2)\ dx_1\ dx_2 = 1 \qquad [A1.26]$$

Indeed, with these variables,

$$p(t_1, t_2) = \frac{1}{2\ \pi\ s_1\ s_2\ \sqrt{1-\rho^2}} \exp\left[-\frac{1}{2\left(1-\rho^2\right)}\left(t_1^2 - 2\ \rho\ t_1\ t_2 + t_2^2\right)\right] \qquad [A1.27]$$

and

$$t_1^2 - 2\ \rho\ t_1\ t_2 + t_2^2 = \left(t_1 - \rho\ t_2\right)^2 + \left(1-\rho^2\right) t_2^2 \qquad [A1.28]$$

Let us set $u = \frac{t_1 - \rho\ t_2}{\sqrt{1-\rho^2}}$ and calculate

$$\frac{1}{2\ \pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-u^2/2}\ e^{-t_2^2/2}\ dt_2\ du \qquad [A1.29]$$

i.e.

$$\frac{1}{\sqrt{2\ \pi}}\int_{-\infty}^{+\infty} e^{-u^2/2}\ du \cdot \frac{1}{\sqrt{2\ \pi}}\int_{-\infty}^{+\infty} e^{-t_2^2/2}\ dt_2 \qquad [A1.30]$$

We thus have

$$\int_{-\infty}^{+\infty} p(u)\ du \int_{-\infty}^{+\infty} p(t_2)\ dt_2 = 1 \qquad [A1.31]$$

NOTE.

It is shown that, if the terms μ_{ij} *are zero when* $i \neq j$, *i.e. if all the correlation coefficients of the variables* x_i *and* x_j *are zero* ($i \neq j$), *we have:*

$$\|M\| = \begin{Vmatrix} \mu_{11} & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ 0 & \cdots & \ddots & 0 \\ 0 & 0 & 0 & \mu_{nn} \end{Vmatrix} \qquad [A1.32]$$

$$|M| = \prod_{i=1}^{n} \mu_{ii}$$

$$M_{ij} = 0 \quad \text{if } i \neq j$$

$$M_{ij} = \frac{|M|}{\mu_{ii}} \quad \text{if } i = j$$

and

$$p(x_1, x_2, \cdots, x_n) = (2\,\pi)^{-n/2}\, |M|^{-1/2} \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{x_i^2}{\mu_{ii}}\right) \qquad [A1.33]$$

$$p(x_1, x_2, \cdots x_n) = \prod_{i=1}^{n} p(x_i) \qquad [A1.34]$$

For normally distributed random variables, it is sufficient that the cross-correlation functions are zero for these variables to be independent.

A1.9. *Student law*

The Student law with n degrees of freedom of the random variable x whose probable value would be zero for probability density:

$$p(x) = \frac{1}{\sqrt{2\,\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}} \qquad [A1.35]$$

A2. 1/nth octave analysis

Some analysers make it possible to express the power spectral densities calculated in dB from an analysis into 1/3 octave. We propose here to give the relations which make it possible to go from such a representation to the traditional representation. We will place ourselves in the more general case of a distribution of the points in 1/nth octave.

A2.1. *Center frequencies*

A2.1.1. *Calculation of the limits in 1/nth octave intervals*

By definition, an octave is the interval between two frequencies f_1 and f_2 such that $\frac{f_2}{f_1} = 2$. In 1/nth octave, we have

$$\frac{f_2}{f_1} = 2^{1/n} \qquad \text{[A2.1]}$$

i.e.

$$\log f_2 = \log f_1 + \frac{\log 2}{n} \qquad \text{[A2.2]}$$

Example

Analysis in 1/3 octave between $f_1 = 5$ Hz and $f_2 = 10$ Hz.

$$\log f_a = \log 5 + \frac{\log 2}{3} = 0.7993$$

$$f_a = 6.3 \text{ Hz}$$

$$\log f_b = \log f_a + \frac{\log 2}{3} = 0.8997$$

$$f_b = 7.937 \text{ Hz}$$

$$\log f_c = \log f_b + \frac{\log 2}{3} = 1$$

$$f_c = f_2 = 10 \text{ Hz}$$

A2.1.2. *Width of the interval* Δf *centered around* f

The width of this interval is equal to

Δf = upper limit − lower limit

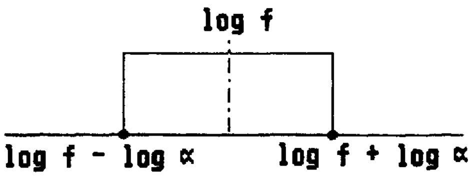


Figure A2.1. *Frequency interval*

Let α be a constant characteristic of width Δf (Figure A2.1) such that:

$$\left(\log f + \log \alpha\right) - \left(\log f - \log \alpha\right) = \frac{\log 2}{n} \qquad \text{[A2.3]}$$

yielding

$$\boxed{\alpha = 2^{1/2n}} \qquad [A2.4]$$

One deduces

$$\Delta f = \alpha\, f - \frac{f}{\alpha} \qquad [A2.5]$$

$$\boxed{\Delta f = \left(\alpha - \frac{1}{\alpha}\right) f} \qquad [A2.6]$$

This value of Δf is particularly useful for the calculation of the rms value of a vibration defined by a PSD expressed in dB.

Example

For $n = 3$, it becomes $\alpha \approx 1.122462\ldots$ and $\Delta f \approx 0.231563\ldots f$. At 5 Hz, we that have $\Delta f = 1.15$ Hz.

A2.2. *Ordinates*

We propose here to convert the decibels into unit of amplitude $[(m/s^2)^2/Hz]$. We have, if $\ddot{x}_{rms}$ is the rms value of the signal filtered by the filter $(f, \Delta f)$ defined above:

$$N(dB) = 20 \log \frac{\ddot{x}_{rms}}{\ddot{x}_{ref}} \qquad [A2.7]$$

$\ddot{x}_{ref}$ is a reference value. If the parameter studied is an acceleration, the reference value is by convention equal to 1 $\mu m/s^2 = 10^{-6}$ m/s^2 (one finds sometimes 10^{-5} m/s^2 in certain publications).

Table A2.1 lists the reference values quoted by Standard ISO/DIS 1683.2.

Table A2.1. *Values of reference (Standard ISO 1683 [ISO 94])*

Parameter	Formulate (dB)	Reference level
Sound pressure level	$20\log\left(p/p_0\right)$	20 µPa in air 1 µPa in other media
Acceleration level	$20\log\left(\ddot{x}/\ddot{x}_0\right)$	1 µm/s^2
Velocity	$20\log\left(v/v_0\right)$	1 nm/s
Force level	$20\log\left(F/F_0\right)$	1 µN
Power level	$10\ \log\left(P/P_0\right)$	1 pW
Intensity level	$10\ \log\left(I/I_0\right)$	1 pW/m^2
Energy density level	$10\ \log\left(W/W_0\right)$	1 pJ/m^3
Energy	$10\ \log\left(E/E_0\right)$	1 pJ

Yielding

$$\ddot{x}_{rms} = \ddot{x}_{ref}\ 10^{\frac{N}{20}} \qquad [A2.8]$$

The amplitude of the corresponding PSD is equal to

$$G = \frac{\ddot{x}_{rms}^2}{\Delta f} = \frac{\ddot{x}_{ref}^2\ 10^{\frac{N}{10}}}{\Delta f} \qquad [A2.9]$$

$$\boxed{G = \frac{2^{1/2n}}{2^{1/n} - 1}\ \frac{\ddot{x}_{ref}^2\ 10^{\frac{N}{10}}}{f}} \qquad [A2.10]$$

or

$$N = 10\left[\log \frac{2^{1/n} - 1}{2^{1/2n}}\ \frac{f\ G}{\ddot{x}_{ref}}\right] \qquad [A2.11]$$

Example

If $\ddot{x}_{ref} = 10^{-5}$ m/s^2

$$G = \frac{10^{\frac{N}{10}-10}}{\Delta f}$$

and if n = 3

$$G = \frac{2^{1/6}}{2^{1/3}-1}\,\frac{10^{\frac{N}{10}-10}}{f}$$

$$G \approx \frac{10^{\frac{N}{10}-10}}{0.23\,f}$$

If, at 5 Hz, the spectrum gives N = 50 dB,

$$G = \frac{2^{1/6}}{2^{1/3}-1}\,\frac{10^{\frac{50}{10}-10}}{5}$$

$$G \approx 8.6369\ 10^{-6}\ (\text{m/s}^2)^2/\text{Hz}$$

A3. Conversion of an acoustic spectrum into a power spectral density

A3.1. *Need*

When the real environment is an acoustic noise, it is possible to evaluate the vibratory levels induced by this noise in a structure and the stresses which result from it using finite element computation software.

At the stage of writing of specifications, one does not normally have such a model of the structure. It is nevertheless very necessary to obtain an evaluation of the vibratory levels for the dimensioning of the material.

To carry out this estimate, F Spann and P. Patt [SPA 84] propose an approximate method based once again on calculation of the response of a one-degree-of-freedom system (Figure A3.1).

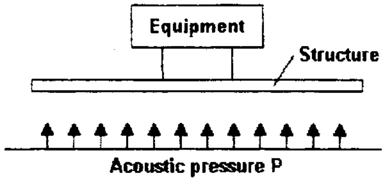


Figure A3.1. *Model for the evaluation of the effects of acoustic pressure*

Let us set:

P = acoustic pressure

G_P = power spectral density of the pressure

A = area exposed to the pressure

β = effectiveness vibroacoustic factor

M = mass of the specimen and support unit.

The method consists of:

– transforming the spectrum of the pressure expressed into dB into a PSD G_P expressed in $(N/m^2)^2/Hz$,

– calculating, in each frequency interval (in general in the third octave), the response of an equivalent one-degree-of-freedom system from the value of the PSD pressure, the area A exposed to the pressure P and the effective mass M,

– smoothing the spectrum obtained.

A3.2. *Calculation of the pressure spectral density*

By definition, the number N of dB is given by

$$N = 20 \log_{10} \frac{P}{P_0} \qquad [A3.1]$$

P_0 = reference pressure = $2\ 10^{-5}\ N/m^2$

$$P = \text{rms pressure} = \sqrt{G_P\ \Delta f}$$

For a 1/n^th octave filter centered on the frequency f_c, we have

$$\Delta f = \left(2^{1/2n} - \frac{1}{2^{1/2n}}\right) f_c \qquad [A3.2]$$

yielding

$$G_P = \frac{\left(P_0\ 10^{N/20}\right)^2}{\Delta f} \qquad [A3.3]$$

In the particular case of an analysis in third octave, we would have

$$\Delta f = \left(2^{1/6} - \frac{1}{2^{1/6}}\right) f_c \approx 0.23\ f_c \approx \frac{f_c}{4.32} \qquad [A3.4]$$

and

$$G_P = \frac{\left(P_0\ 10^{N/20}\right)^2}{\left(2^{1/6} - \frac{1}{2^{1/6}}\right) f_c} \qquad [A3.5]$$

A3.3. *Response of an equivalent one-degree-of-freedom system*

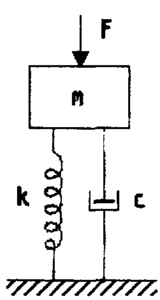


Figure A3.2. *One-degree-of-freedom system subjected to a force*

Let us consider the one-degree-of-freedom linear system in the Figure A3.2, excited by a force F applied to mass m. The transfer function of this system is equal to:

$$H(f) = \frac{\ddot{z}}{F} = \frac{\ddot{y}}{F} = \frac{h^2}{m\left[\left(1-h^2\right)^2 + h^2/Q^2\right]^{1/2}} \qquad [A3.6]$$

y and z being respectively the absolute response and the relative response of the mass m, and

$$h = \frac{f}{f_0}$$

At resonance, h = 1 and

$$H = \frac{Q}{m} \qquad [A3.7]$$

The power spectral density G_F of the transmitted force is given by:

$$G_F = \left(\beta\ A\right)^2\ G_P \qquad [A3.8]$$

$(F = \beta\ A\ P)$ and the PSD of the response $\ddot{y}$ to the force F applied to the one-degree-of-freedom system is equal, at resonance, to:

$$G_{\ddot{y}} = H^2\ G_F \qquad [A3.9]$$

$$G_{\ddot{y}} = \frac{Q^2}{m^2}\left(\beta\ A\right)^2\ G_P \qquad [A3.10]$$

$$G_{\ddot{y}} = \beta^2 \left(\frac{A}{m}\right)^2 Q^2 \frac{\left(P_0\ 10^{N/20}\right)^2}{\left(2^{1/2n} - \frac{1}{2^{1/2n}}\right) f_c} \qquad [A3.11]$$

In the case of third octave analysis,

$$G_{\ddot{y}} = \beta^2 \left(\frac{A}{m}\right)^2 Q^2 \frac{\left(P_0\ 10^{N/20}\right)^2}{\left(2^{1/6} - \frac{1}{2^{1/6}}\right) f_c} \qquad [A3.12]$$

F. Spann and P. Patt set $Q = 4.5$ and $\beta = 2.5$; yielding

$$G_{\ddot{y}} = 126.6 \left(\frac{A}{m}\right)^2 G_P \qquad \text{[A3.13]}$$

A4. Mathematical functions

The object of this appendix is to provide tools facilitating the evaluation of some mathematical expressions, primarily integrals, intervening very frequently in calculations related to the analysis of random vibrations and their effect on a one-degree-of-freedom mechanical system.

A4.1. *Error function*

This function, also named *probability integral*, is the subject of two definitions.

A4.1.1. *First definition*

The error function is expressed:

$$\boxed{E_1(x) = \frac{2}{\sqrt{\pi}} \int_0^X e^{-t^2} dt} \qquad \text{[A4.1]}$$

If $x \to \infty$, $E_1(x)$ tends towards E_{1_∞} which is equal to

$$E_{1\,\infty} = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} dt = 1 \qquad \text{[A4.2]}$$

and if $x = 0$, $E_1(0) = 0$. If we set $t = \dfrac{u}{\sqrt{2}}$, it becomes

$$E_1\left(\frac{x}{\sqrt{2}}\right) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\frac{u^2}{2}} \frac{du}{\sqrt{2}}$$

$$\boxed{E_1\left(\frac{x}{\sqrt{2}}\right) = \sqrt{\frac{2}{\pi}} \int_0^x e^{-\frac{u^2}{2}} du} \qquad \text{[A4.3]}$$

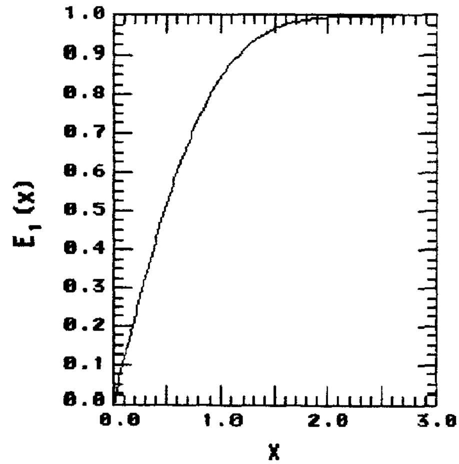


Figure A4.1. *Error function* $E_1(x)$

One can express a series development of $E_1(x)$ by integrating the series development of e^{-t^2} between 0 and x:

$$E_1(x) = \frac{2}{\sqrt{\pi}} \left[x - \frac{x^3}{1!\,3} + \frac{x^5}{2!\,5} - \cdots + (-1)^n \frac{x^{2n+1}}{n!\,(2n+1)} + \cdots \right] \qquad \text{[A4.4]}$$

This series converges for any x. For large x, one can obtain the asymptotic development according to [ANG 61] [CRA 63]:

$$E_1(x) \approx 1 - \frac{e^{-x^2}}{x\sqrt{\pi}} \left[1 - \frac{1}{2\,x^2} + \frac{1.3}{2^2\,x^4} - \frac{1.3.5}{2^3\,x^6} + \cdots + (-1)^{n-1} \frac{1.3.5 \cdots (2n-3)}{2^{n-1}\,x^{2n-2}} + \cdots \right] \qquad \text{[A4.5]}$$

For sufficiently large x, we have

$$E_1(x) \approx 1 - \frac{e^{-x^2}}{x\,\sqrt{\pi}} \qquad \text{[A4.6]}$$

If $x = 1.6$, $E_1(x) = 0.976$, whilst the value approximated by the expression above is

$E_1(x) \approx 0.973$.

For $x = 1.8$, $E_1(x) \approx 0.9877$ instead of 0.890.

The ratio of two successive terms, equal to $\frac{2n-1}{x^2}$, is close to 1 when n is close to x^2. This remark makes it possible to limit the calculation by minimizing the error on $E_1(x)$.

NOTE.

$E(x)$ *is the error function and* $[1 - E(x)]$, *noted* $ERFC(x)$, *is the ('complementary error function').*

$$ERFC(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \qquad [A4.7]$$

Function $E_1(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$

Table A4.1. *Error function* $E_1(x)$ *(continuation)*

x	$E_1(x)$	ΔE_1	X	$E_1(x)$	ΔE_1	x	$E_1(x)$	ΔE_1
0.025	0.02820	0.02820	0.425	0.45219	0.2380	0.825	0.75668	0.01458
0.030	0.05637	0.02817	0.450	0.47548	0.02329	0.850	0.77067	0.01399
0.075	0.08447	0.02810	0.475	0.49826	0.02278	0.875	0.78408	0.01341
0.100	0.11246	0.02799	0.500	0.520500	0.02224	0.900	0.79691	0.01283
0.125	0.14032	0.02786	0.525	0.54219	0.02169	0.925	0.80918	0.01227
0.150	0.16800	0.02768	0.550	0.56332	0.02113	0.950	0.82089	0.01171
0.175	0.19547	0.02747	0.575	0.58388	0.02056	0.975	0.83206	0.01117
0.200	0.22270	0.02723	0.600	0.60386	0.01998	1.000	0.84270	0.01064
0.225	0.24967	0.02697	0.625	0.62324	0.01938	1.025	0.85282	0.01012
0.250	0.27633	0.02666	0.650	0.64203	0.01879	1.050	0.86244	0.00962
0.275	0.30266	0.02633	0.675	0.66022	0.01819	1.075	0.87156	0.00912
0.300	0.32863	0.02597	0.700	0.67780	0.01758	1.100	0.88021	0.00865
0.325	0.35421	0.02558	0.725	0.69478	0.01698	1.125	0.88839	0.00818
0.350	0.37938	0.02517	0.750	0.711156	0.01638	1.150	0.89612	0.00773
0.375	0.40412	0.02474	0.775	0.72693	0.01577	1.175	0.90343	0.00731
0.400	0.42839	0.02427	0.800	0.74210	0.01517	1.200	0.91031	0.00688

1.225	0.91680	0.00649	1.650	0.98038	0.00194	2.075	0.99666	0.00040
1.250	0.92290	0.00610	1.675	0.98215	0.00177	2.100	0.99702	0.00036
1.275	0.929863	0.00573	1.700	0.98379	0.00164	2.125	0.99735	0.00033
1.300	0.93401	0.00538	1.725	0.98529	0.00150	2.150	0.99764	0.00029
1.325	0.93905	0.00504	1.750	0.98667	0.00138	2.175	0.99790	0.00026
1.350	0.094376	0.00472	1.775	0.98793	0.00126	2.200	0.99814	0.00024
1.375	0.94817	0.00441	1.800	0.989090	0.00116	2.225	0.99835	0.00021
1.400	0.95229	0.00412	1.825	0.99015	0.00106	2.250	0.99854	0.00019
1.425	0.95612	0.00383	1.850	0.99111	0.00096	2.275	0.99871	0.00017
1.450	0.95970	0.00356	1.875	0.99199	0.00088	2.300	0.99886	0.00015
1.475	0.96302	0.00332	1.900	0.99279	0.00080	2.325	0.99899	0.00013
1.500	0.96611	0.00309	1.925	0.99352	0.00073	2.350	0.99911	0.00012
1.525	0.96897	0.00286	1.950	0.99418	0.00066	2.375	0.99922	0.00011
1.550	0.97162	0.00265	1.975	0.99478	0.00060	2.400	0.99931	0.00009
1.575	0.97408	0.00246	2.000	0.99532	0.00054	2.425	0.99940	0.00009
1.600	0.97635	0.00227	2.025	0.99781	0.00049	2.450	0.99947	0.00007
1.625	0.97844	0.00209	2.050	0.99626	0.00045	2.475	0.99954	0.00007
						2.500	0.99959	0.00005

Approximate calculation of E_1

The error function can be estimated using the following approximate relationships [ABR 70] [HAS 55]:

$$E_1(x) = 1 - (a_1\, t + a_2\, t^2 + a_3\, t^3 + a_4\, t^4 + a_5\, t^5)\, e^{-x^2} + e(x) \qquad [A4.8]$$

where

$$t = \frac{1}{1 + px} \quad (0 \le x < \infty)$$

$$|\varepsilon(x)| \le 1.5\ 10^{-7}$$

$p = 0.327\ 591\ 1$ $a_3 = 1.421\ 413\ 741$
$a_1 = 0.254\ 829\ 592$ $a_4 = -1.453\ 152\ 027$
$a_2 = -0.284\ 496\ 736$ $a_5 = 1.061\ 495\ 429$

$$E_1(x) = 1 - (a_1\ t + a_2\ t^2 + a_3\ t^3)\ e^{-x^2} + e(x) \qquad [A4.9]$$

$$t = \frac{1}{1 + px} \qquad |\varepsilon(x)| \leq 2.5\ 10^{-5}$$

$p = 0.470\ 47$ $a_2 = -0.095\ 879$
$a_1 = 0.348\ 024\ 2$ $a_3 = 0.747\ 855\ 6$

Other approximate relationships of this type have been proposed [HAS 55] [SPA 87], with developments of 3rd, 4th and 5th order. C. Hastings also suggests the expression

$$E_1(x) = 1 - \frac{1}{(1 + a_1\ x + a_2\ x^2 + a_3\ x^3 + a_4\ x^4 + a_5\ x^5 + a_6\ x^6)^{16}} \qquad [A4.10]$$

$a_1 = 0.070\ 523\ 078\ 4$ $a_4 = 0.000\ 152\ 014\ 3$
$a_2 = 0.042\ 282\ 012\ 3$ $a_5 = 0.000\ 276\ 567\ 2$
$a_3 = 0.009\ 270\ 527\ 2$ $a_6 = 0.000\ 043\ 063\ 8$

$(0 \leq x \leq \infty)$

Derivatives

$$\frac{d\ E_1(x)}{dx} = \frac{2}{\sqrt{\pi}} e^{-x^2} \qquad [A4.11]$$

$$\frac{d^2 E_1(x)}{dx^2} = -\frac{4}{\sqrt{\pi}} x\ e^{-x^2} \qquad [A4.12]$$

Approximate formula

The approximate relationship [DEV 62]

$$\boxed{E_1(x) = \sqrt{1 - \exp\left(-\frac{4\,x^2}{\pi}\right)}} \qquad [A4.13]$$

gives results of a sufficient precision for many applications (error lower than some thousandths for whatever x).

A4.1.2. *Second definition*

The error function is often defined by [PAP 65] [PIE 70]:

$$E_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{t^2}{2}}\, dt \qquad [A4.14]$$

With this definition

$$E_2(x) = \frac{E_1\left(\frac{x}{\sqrt{2}}\right)}{2} \qquad [A4.15]$$

yielding

$$E_1(x) = 2\ E_2(x\sqrt{2}) \qquad [A4.16]$$

Applications

$$\frac{1}{\alpha\sqrt{2\pi}} \int_{x_1}^{x_2} e^{-\frac{(x-\beta)^2}{2\,\alpha^2}}\, dx = E_2\left(\frac{x_2-\beta}{\alpha}\right) - E_2\left(\frac{x_1-\beta}{\alpha}\right) \qquad [A4.17]$$

where α and β are two arbitrary constants [PIE 70] and

$$\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\, dt = \sqrt{2\pi} \qquad [A4.18]$$

Properties of $E_2(x)$

$E_2(x)$ tends towards 0.5 when $x \to \infty$:

$$E_2 = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{t^2}{2}} \, dt = 0.5 \qquad [A4.19]$$

$$E_2(0) = 0$$

$$E_2(-x) = -E_2(x)$$

Function $E_2(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_0^{x} e^{-\frac{t^2}{2}} \, dt$

Table A4.2. *Error function* $E_2(x)$

x	$E_2(x)$	ΔE_2	X	$E_2(x)$	ΔE_2	x	$E_2(x)$	ΔE_2
0.05	0.01994	0.01994	0.85	0.30234	0.01420	1.65	0.45053	0.00533
0.10	0.03983	0.01989	0.90	0.31594	0.01360	1.70	0.45543	0.00490
0.15	0.05962	0.01979	0.95	0.32894	0.01300	1.75	0.45994	0.00451
0.20	0.07926	0.01964	1.00	0.34134	0.01240	1.80	0.46407	0.00413
0.25	0.09871	0.01945	1.05	0.35314	0.01180	1.85	0.46784	0.00377
0.30	0.11791	0.01920	1.10	0.36433	0.01119	1.90	0.47128	0.00344
0.35	0.13683	0.01892	1.15	0.37493	0.01060	1.95	0.47441	0.00313
0.40	0.15542	0.01859	1.20	0.38493	0.01000	2.00	0.47725	0.00284
0.45	0.17364	0.01822	1.25	0.39435	0.00942	2.05	0.47982	0.00257
0.50	0.19146	0.01782	1.30	0.40320	0.00885	2.10	0.48214	0.00232
0.55	0.20884	0.01738	1.35	0.41149	0.00829	2.15	0.48422	0.00208
0.60	0.22575	0.01691	1.40	0.41924	0.00775	2.20	0.48610	0.00188
0.65	0.24215	0.01640	1.45	0.42647	0.00723	2.25	0.48778	0.00168
0.70	0.25804	0.01589	1.50	0.43319	0.00672	2.30	0.48928	0.00150
0.75	0.27337	0.01533	1.55	0.43943	0.00624	2.35	0.49061	0.00133
0.80	0.28814	0.01477	1.60	0.44520	0.00577	2.40	0.49180	0.00119

2.45	0.49286	0.00106	3.00	0.49865	0.00024	3.55	0.49841	0.00004
2.50	0.49379	0.00093	3.05	0.49886	0.00021	3.60	0.49984	0.00003
2.55	0.49461	0.00082	3.10	0.49903	0.00017	3.65	0.49987	0.00003
2.60	0.49534	0.00072	3.15	0.49918	0.00015	3.70	0.49989	0.00002
2.65	0.49598	0.00064	3.20	0.49931	0.00013	3.75	0.49991	0.00002
2.70	0.49653	0.00055	3.25	0.49942	0.00011	3.80	0.49993	0.00002
2.75	0.49702	0.00049	3.30	0.49952	0.00010	3.85	0.49994	0.00001
2.80	0.49744	0.00042	3.35	0.49960	0.00008	3.90	0.49995	0.00001
2.85	0.49781	0.00037	3.40	0.49966	0.00006	3.95	0.049996	0.00001
2.90	0.49813	0.00032	3.45	0.49972	0.00006	4.00	0.49997	0.00001
2.95	0.49981	0.00028	3.50	0.49977	0.00005			

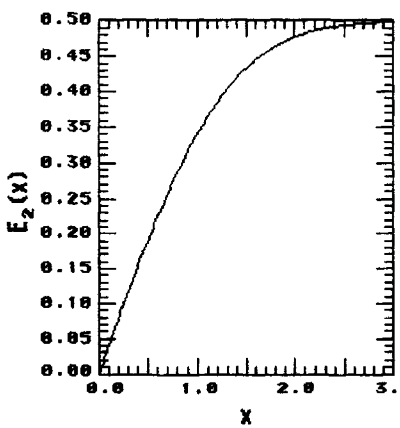


Figure A4.2. *Error function* $E_2(x)$

Approximate calculation of $E_2(x)$

The function $E_2(x)$ can be approximated, for $x > 0$, by the expression defined as follows [LAM 76] [PAP 65]:

$$E_2(x) \approx \frac{1}{2}\left[1-(a\,t+b\,t^2+c\,t^3+d\,t^4+e\,t^5)\,e^{-\frac{x^2}{2}}\right] \qquad [A4.20]$$

where

$$t=\frac{1}{1+0.2316418\;x} \qquad a=0.254\,829\,592 \qquad b=-0.284\,496\,736$$

$$c=1.421\,413\,741 \qquad d=-1.453\,152\,027 \qquad e=1.061\,405\,429$$

The approximation is very good (at least 5 decimal points).

NOTE.

With these notations, the function $E_2(x)$ *is none other than the integral of the Gauss function:*

$$G(x)=\frac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}}$$

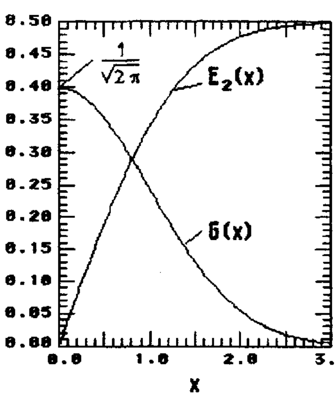


Figure A4.3. *Comparison of the error function* $E_2(x)$ *and of* $G(x)$

Figure A4.3 shows the variations of $G(x)$ *and of* $E_2(x)$ *for* $0 \le x \le 3$. *We thus have:*

$$\int_0^x \exp\left(-\frac{u^2}{2\sigma^2}\right) du = \sqrt{\frac{\pi}{2}}\ \sigma\ E_1\left(\frac{x}{\sigma\sqrt{2}}\right) \qquad [A4.21]$$

Calculation of x *for* $E_2(x) = E_0$

The method below applies if x is positive and where $0 < E_0 < 0.5$ [LAM 80]. One calculates successively:

$$z = \sqrt{-2\ \ln(1 - 2\ E_0)}$$

and

$$x = g_0 + g_1\ z + g_2\ z^2 + \cdots + g_{10}\ z^{10} \qquad [A4.22]$$

where

$g_0 = 6.55864\ 10^{-4}$

$g_1 = -0.02069$

$g_2 = 0.737563$

$g_3 = -0.207071$

$g_4 = -2.06851\ 10^{-2}$

$g_5 = 0.03444$

$g_6 = -1.17213\ 10^{-2}$

$g_7 = 2.10941\ 10^{-3}$

$g_8 = -2.18541\ 10^{-4}$

$g_9 = 1.23163\ 10^{-5}$

$g_{10} = -2.93138\ 10^{-7}$

For negative values, one will use the property

$$E_2(-x) = -\ E_2(x)$$

NOTE.

To calculate x *from given* E_1 *set* $E_2 = \frac{E_1}{2}$, *calculate x, and then* $\frac{x}{\sqrt{2}}$.

A4.2. *Calculation of the integral* $\int e^{a x}/x^n \, dx$

One has [DWI 66]:

$$\int \frac{e^{a x}}{x} dx = \ln |x| + \frac{a x}{1!} + \frac{a^2}{2} \frac{x^2}{2!} + \frac{a^3}{3} \frac{x^3}{3!} + \cdots + \frac{a^n}{n} \frac{x^n}{n!} + \cdots \qquad [A4.23]$$

yielding, since

$$\int \frac{e^{a x}}{x^n} dx = -\frac{e^{a x}}{(n-1)\, x^{n-1}} + \frac{a}{n-1} \int \frac{e^{a x}}{x^{n-1}} dx \qquad [A4.24]$$

$$\boxed{\int \frac{e^{ax}}{x^n} dx = -\frac{e^{ax}}{(n-1)\, x^{n-1}} - \frac{a\, e^{ax}}{(n-1)(n-2)\, x^{n-2}} - \cdots - \frac{a^{n-2}\, e^{ax}}{(n-1)!\, x} + \frac{a^{n-1}}{(n-1)!} \int \frac{e^{ax}}{x} dx} \qquad [A4.25]$$

A4.3. *Euler's constant*

Definition

$$\varepsilon = \lim_{n \to \infty} \left[1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n \right] \qquad [A4.26]$$

$$\varepsilon \approx 0.577\ 215\ 664\ 90\ldots$$

An approximate value is given by [ANG 61]:

$$\varepsilon \approx \frac{1}{2} \left(\sqrt[3]{10} - 1 \right)$$

i.e.

$$\varepsilon \approx 0.577\ 217\ 3\ldots$$

Applications

It is shown that [DAV 64]:

$$\int_0^\infty \ln \lambda \, e^{-\lambda} \, d\lambda = -\varepsilon \qquad [A4.27]$$

and that

$$\int_0^{\infty} (\ln \lambda)^2 \, e^{-\lambda} \, d\lambda = \frac{\pi}{6} + \varepsilon^2 \qquad [A4.28]$$

A5. Complements to the transfer functions

A5.1. *Error related to digitalization of transfer function*

The transfer function is defined by a certain number of points. According to this number, the peak of this function can be more or less truncated and the measurement of the resonance frequency and Q factor distorted [NEU 70].

Any complex system with separate modes is comparable in the vicinity of a resonance frequency to a one-degree-of-freedom system of quality factor Q.

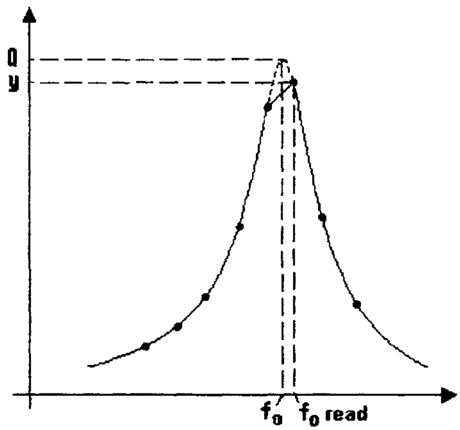


Figure A5.1. *Transfer function of a one dof system close to resonance*

Let us set y as the value of the quality factor read on the curve, Q being the true value.

Let us set $\beta = \frac{y}{Q}$ and $\alpha = \frac{f}{f_0} = \frac{\text{read resonance frequency}}{\text{true resonance frequency}}$. When α is different from unity, one can set $\alpha = 1 - \delta$, if δ is the relative deviation on the value of the resonance frequency. For $\delta = 0$, one has $\alpha = 1$ and $\beta = 1$. For $\delta \neq 0$, β is less than one. The resolution error is equal to $\varepsilon_R = 1 - \beta$. The amplitude of the transfer function away from resonance is given by y such that:

$$y^2 = \frac{1 - \dfrac{\alpha^2}{Q^2}}{\left(1-\alpha^2\right)^2 + \dfrac{\alpha^2}{Q^2}} = \frac{Q^2 + \alpha^2}{Q^2 \left(1-\alpha^2\right)^2 + \alpha^2} \qquad \text{[A5.1]}$$

$$\beta^2 = \frac{y^2}{Q^2} = \frac{1 + \dfrac{\alpha^2}{Q^2}}{Q^2 \left(1-\alpha^2\right)^2 + \alpha^2} = \frac{1 + \dfrac{(1-\alpha)^2}{Q^2}}{Q^2\,\delta^2 + (1-\delta)^2} \qquad \text{[A5.2]}$$

For large Q, we have

$$\beta^2 \approx \frac{1}{Q^2 \left(1-\alpha^2\right)^2 + \alpha^2} \qquad \text{[A5.3]}$$

i.e., replacing α by $1 - \delta$ and supposing Q^2 large compared to 1,

$$\beta^2 \approx \frac{1}{1 - 2\,\delta + 4\,Q^2\,\delta^2} \qquad \text{[A5.4]}$$

and

$$\varepsilon_R = 1 - \beta \approx 1 - \frac{1}{\sqrt{1 - 2\,\delta + 4\,Q^2\,\delta^2}} \qquad \text{[A5.5]}$$

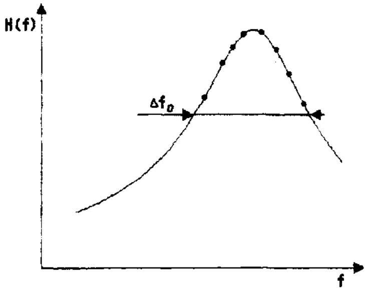


Figure A5.2. *Digitalization of n points of the transfer function between the half-power points*

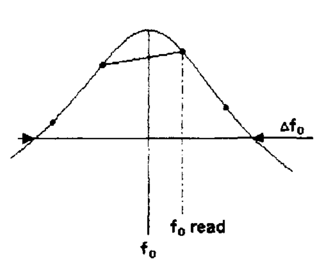


Figure A5.3. *Effect of a too low a sampling rate*

If $4\,Q^2\,\delta^2 >> 2\,\delta$, i.e. if $Q^2 >> \frac{1}{2\,\delta}$,

$$\varepsilon_R \approx 1-\left(1+4\,Q^2\,\delta^2\right)^{-1/2} \qquad [A5.6]$$

$$\boxed{\varepsilon_R \approx 2\,Q^2\,\delta^2} \qquad [A5.7]$$

Let us suppose that there are n points in the interval Δf_0 between the half-power points, i.e. n − 1 intervals. We have:

$$\delta = 1-\alpha = 1-\frac{f}{f_0} \qquad [A5.8]$$

$$f_0 - f = \frac{\Delta f_0}{2\,(n-1)} \qquad [A5.9]$$

yielding

$$\delta = \frac{\Delta f_0}{2\,(n-1)\,f_0} = \frac{1}{2\,(n-1)\,Q} \qquad [A5.10]$$

i.e., since $\varepsilon_R \approx 2\,\alpha^2\,\delta^2$,

$$\varepsilon_R \approx \frac{1}{2\,(n-1)^2} \qquad [A5.11]$$

Figure A5.4 shows variations of the error ε_R versus the number of points n in Δf_0. To measure the Q factor with an error less than 2%, it is necessary for n to be greater than 6 points.

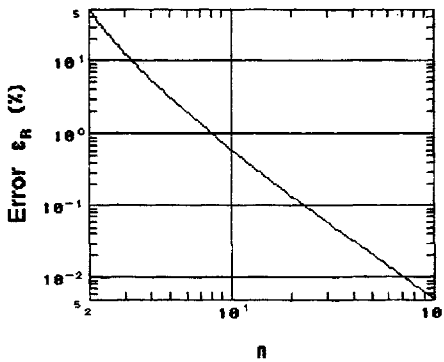


Figure A5.4. *Error of resolution versus number of points in* Δf

NOTE.

In the case of random vibrations, the frequency increment Δf *is related to the sampling frequency* f_s *by the relationship*

$$\Delta f = \frac{f_s}{2\,M} \qquad [A5.12]$$

where M *is the total number of points representing the spectrum. Ideally, the increment* Δf *should be a very small fraction of the bandwidth* Δf_0 *around resonance. The number of points* M *is limited by the memory size of the calculator and the frequency* f_s *should be at least twice as large as the highest frequency of the analysed signal, to avoid aliasing errors (Shannon's theorem). A too large* Δf *leads to a small value of* n *and therefore to an error to the* Q *factor measurement. Decreasing* f_s *to reduce* Δf *(with* M *constant) can lead to poor representation of the temporal signal and thus to an inaccuracy in the amplitude of the spectrum at high frequencies. It is recommended to choose a sampling frequency greater than 6 times the largest frequency to be analysed [TAY 75].*

A5.2. *Use of a fast swept sine for the measurement of transfer functions*

The measurement of a transfer function starting from a traditional swept sine test leads to a test of relatively long duration and requires in addition material having a great measurement dynamics.

Transfer functions can also be measured from random vibration tests or by using shocks, the test duration being obviously in this latter case very short. On this assumption, the choice of the form of shock to use is important, because the transfer function being calculated from the ratio of the Fourier transforms of the response (in a point of the structure) and excitation, it is necessary that this latter transform does not present one zero or too small an amplitude in a certain range of frequency. In the presence of noise, the low levels in the denominator lead to uncertainties in the transfer function [WHI 69].

The interest of the fast linear swept sine resides in two points:

– the Fourier transform of a linear swept sine has a roughly constant amplitude in the swept frequency range. W.H. Reed, A.W. Hall, L.E. Barker [REE 60], then R.G. White [WHI 72] and R.J. White and R.J. Pinnington [WHI 82] showed that the average module of the Fourier transform of a linear swept sine is equal to:

$$\overline{\left|\ddot{X}(\omega)\right|} = \frac{\left|\ddot{x}_m\right|}{2\sqrt{b}} \qquad [A5.13]$$

where $\ddot{x}_m$ = amplitude of acceleration defining the swept sine

$$b = \frac{f_2 - f_1}{t_b} = \text{sweep rate}$$

and that, more generally,

$$\overline{\left|\ddot{X}\right|} = \frac{\left|\ddot{x}_m\right|}{2\sqrt{\dot{f}}} \qquad [A5.14]$$

where $\dot{f}$ is the sweep rate for an arbitrary law.

Example

Linear sweep: 10 Hz to 200 Hz
Durations: 1 s – 0.5 s – 0.1 s and 10 ms
$\ddot{x}_m = 10 \text{ ms}^{-2}$

Depending on the case, the relationship [A5.14] gives 0.3627 – 0.256 – 0.1147 or 0.03627 (m/s).

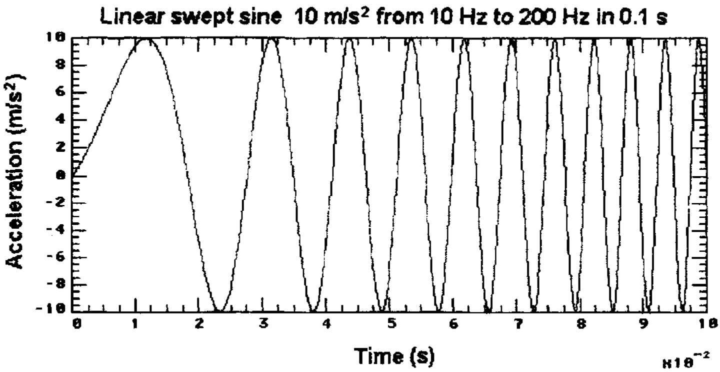


Figure A5.5. *Example of fast swept sine*

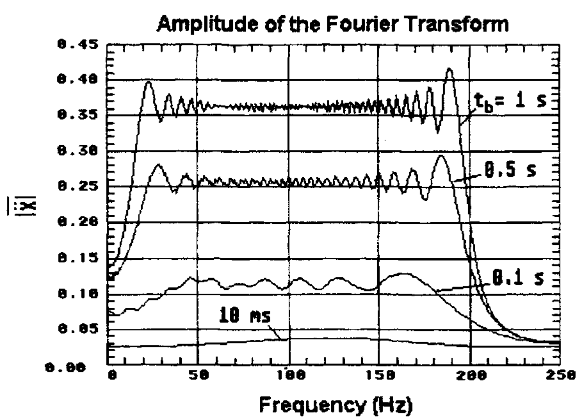


Figure A5.6. *Examples of fast swept sine Fourier transforms*

– sweeping being fast (a few seconds or a fraction of a second, depending on the studied frequency band), the mechanical system responds as to a shock and does not have time to reach the response which it would have in steady state operation or with a slow sweep (Q time the excitation). Accordingly, the dynamics of the necessary instrumentation is less constraining and measurement is taken in a domain where the non linearities of the structure are less important.

The Fourier transform of the response must be calculated over the whole duration of the response, including the residual signal after the end of sweep.

A5.3. *Error of measurement of transfer function using a shock related to signal truncation*

With a transient excitation, of shock type or fast swept sine, the transfer function is calculated from the ratio of the Fourier transforms of response and excitation:

$$H(i\,\Omega) = \frac{Y(i\,\Omega)}{X(i\,\Omega)} \qquad [A5.15]$$

where

$$X(i\,\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-i\,\Omega\,t}\, dt \qquad [A5.16]$$

$$Y(i\,\Omega) = \int_{-\infty}^{+\infty} y(t)\, e^{-i\,\Omega\,t}\, dt \qquad [A5.17]$$

If $x(t)$ is an impulse unit applied to the time $t = 0$, we have $X(i\,\Omega) = 1$ for whatever Ω and (Volume 1, expression [3.115]):

$$H(i\,\Omega) = \int_0^{\infty} h(t)\, e^{-i\,\Omega\,t}\, dt \qquad [A5.18]$$

where $h(t)$ is the impulse response. For a single-degree-of-freedom system of natural frequency f_0 (Volume 1, relationship [3.114]),

$$h(t) = \frac{\omega_0}{\sqrt{1-\xi^2}}\, e^{-\xi\,\omega_0\,t}\, \sin\sqrt{1-\xi^2}\;\omega_0\, t \qquad [A5.19]$$

yielding the complex transfer function

$$H(i\,\Omega) = \frac{1}{\left(1 - \frac{\omega^2}{\omega_0^2}\right) + i\,2\,\xi\,\frac{\omega}{\omega_0}} \qquad [A5.20]$$

The relationship [A5.18] could be used in theory to determine $H(i\,\Omega)$ from the response to an impulse, but, in practice, a truncation of the response is difficult to avoid, either because the decreasing signal becomes non measurable, or because the time of analysis is limited to a value τ_m [WHI 69]. The effects of truncation have been analysed by B.L. Clarkson and A.C. Mercer [CLA 65] who showed:

– that the resonance frequency can still be identified from the diagram vector as the frequency to which the rate of variation in the length of arc with frequency, $\frac{ds}{df}$, is maximum,

– but that the damping measured from such a diagram (established with a truncated signal) is larger than the true value.

These authors established by theoretical analysis that the error (in %) introduced by truncation is equal to:

$$e\,(\%) = 100\left\{1 - \left[\frac{1 - e^{-\xi\,\omega_0\,\tau_m}\left(1 + \xi\,\omega_0\,\tau_m + \frac{1}{2}\,\xi^2\,\omega_0^2\,\tau_m^2\right)}{1 - e^{-\xi\,\omega_0\,\tau_m}\left(1 + \xi\,\omega_0\,\tau_m\right)}\right]\right\} \qquad [A5.21]$$

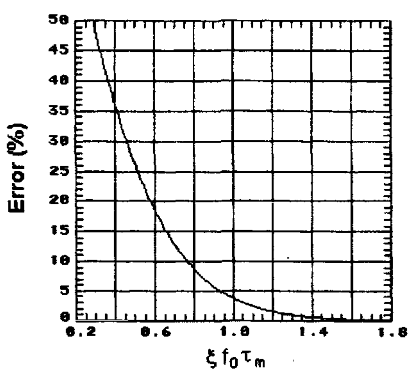


Figure A5.7. *Error of measured value of ξ due to truncation of the signal*

It is seen that if $\xi\ \omega_0\ \tau_m > 1$, the error of the measured value of ξ is lower than 5%.

It is noted that one can obtain a very good precision without needing to analyse extremely long records. For example, for $f_0 = 100$ Hz and $\xi = 0.005$, a duration of 2 s led to an error less than 5% ($\xi\ \omega_0\ \tau_m = 1$).

A5.4. *Error made during measurement of transfer functions in random vibration*

The function of coherence γ^2 is a measurement of the precision of the calculated value of the transfer function $H(f)$ and is equal to [2.97]:

$$\gamma^2 = \frac{\left[G_{xy}(f)\right]^2}{G_x(f)\ G_y(f)} \qquad [A5.22]$$

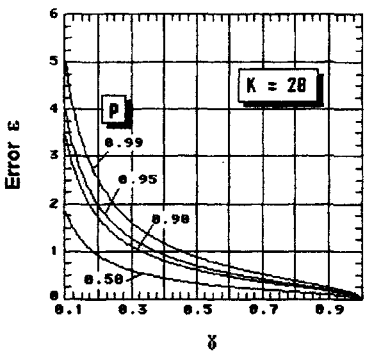


Figure A5.8. *Error of measurement of the transfer function in random excitation versus* γ*, for* K = 20

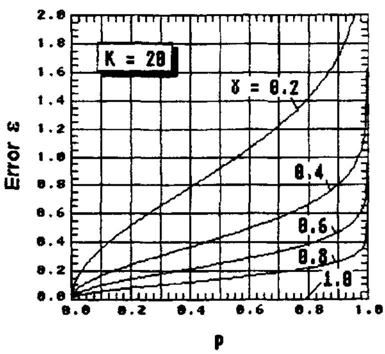


Figure A5.9. *Error of measurement of the transfer function in random excitation versus the probability, for* K = 20

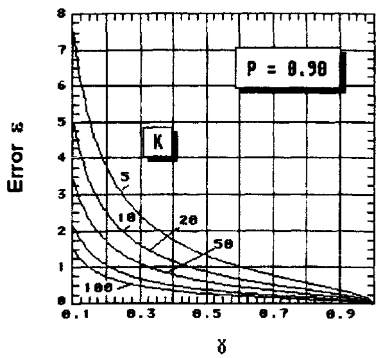


Figure A5.10. *Error of measurement of the transfer function in random excitation versus* γ, *for* P = 0.90

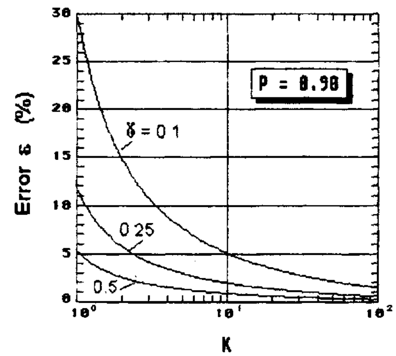


Figure A5.11. *Error of measurement of the transfer function in random excitation versus* K, *for* P = 0.90

If the system is linear and if there is no interference, $\gamma^2 = 1$ and the calculated value of $H(f)$ is correct. If $\gamma^2 < 1$, the error in the estimate of $H(f)$ is provided with a probability P by:

$$\varepsilon = \frac{\Delta\left|H(f)\right|}{\left|H(f)\right|} \le \left\{\left[(1-P)^{-1/K} - 1\right]\left[\frac{1}{\gamma_{xy}^2} - 1\right]\right\}^{\frac{1}{2}} \qquad [A5.23]$$

where K is the number of spectra (blocks) used to calculate each PSD [WEL 70].

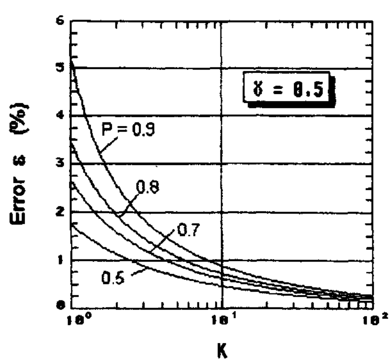


Figure A5.12. *Error of measurement of the transfer function in random excitation versus* K, *for* γ = 0.5

A5.5. *Derivative of expression of transfer function of a one-degree-of-freedom linear system*

Let us consider the transfer function:

$$H(\omega) = \frac{1}{\omega_0^2 \left[1 - h^2 + 2i\xi h\right]} \qquad [A5.24]$$

where

$$h = \frac{\omega}{\omega_0}$$

through multiplication of the denominator's conjugate quantity, we obtain

$$H(\omega) = \left(1 - h^2\right) A - 2\, i\, \xi\, h\, A$$

if we set

$$A = \frac{1}{\omega_0^2 \left[\left(1 - h^2\right)^2 + \left(2\, \xi\, h\right)^2\right]}$$

yielding

$$\frac{dH}{dh} = -2\, h\, A - 2\, i\, \xi\, A + \left(1 - h^2\right)\frac{dA}{dh} - 2\, \xi\, h\, i\, \frac{dA}{dh} \qquad [A5.25]$$

with

$$\frac{dA}{dh} = \frac{4\, h \left(1 - 2\, \xi^2 - h^2\right)}{\omega_0^2 \left[\left(1 - h^2\right)^2 + \left(2\, \xi\, h\right)^2\right]^2} \qquad [A5.26]$$

$$\frac{dH}{dh} = -2(h + i\, \xi)\, A + \left[1 - h^2 - 2\, \xi\, h\, i\right] \frac{dA}{dh} \qquad [A5.27]$$

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Bibliography

[ABR 70] ABRAMOWITZ M. and STEGUN I., *Handbook of mathematical functions*, Dover Publications, Inc., New York, 1970.

[AIT 81] AITCHISON J. and BROWN J.A.C., *The log-normal distribution with special reference to its uses in economics*, Cambridge University Press, Monograph 5, 1981.

[ALL 82] ALLIOUD J.P., 'Prévision des niveaux vibratoires au moyen de l'analyse statistique dynamique – Application à la maintenance prévisionnelle', *Mécanique-Matériaux-Electricité*, n° 386/387, 2–3, 1982, p. 84/95.

[ANG 61] ANGOT A., 'Compléments de mathématiques à l'usage des ingénieurs de l'électrotechnique et des télécommunications', *Editions de la Revue d'Optique*, 4[e] édition, Collection Scientifique et Technique du CNET, 1961.

[BAN 78] BANG B., PETERSEN B.B. and MORUP E., *Random vibration introduction*, Danish Research Centre for Applied Electronics, Elektronikcentralen, Horsholm, Denmark, September 1978.

[BAR 55] BARTLETT M.S. and MEDHI J., 'On the efficiency of procedures for smoothing periodograms from time series with continuous spectra', *Biometrika*, Vol. 42, p. 143/150, 1955.

[BAR 69] BARNOSKI R.L. and MAURER J.R., 'Mean-square response of simple mechanical systems to non-stationary random excitation', *Journal of Applied Mechanics*, 36, 1969, p. 221/227.

[BAR 78] BARNOUIN B. and THEBAULT J., 'Problèmes posés par l'estimation des distributions des contraintes pour le calcul en fatigue des structures tubulaires offshore', *Annales de l'Institut Technique de Batiment et des Travaux Publics*, n° 365, Série: Théories et Méthodes de Calcul, n° 221, October 1978, p. 54/72.

[BEA 72] BEAUCHAMP K.G., PITTEM S.E. and WILLIAMSON M.E., 'Analysing vibration and shock data – Part II: Processing and presentation of results', *Journal of the Society of Environmental Engineers*, Vol. 11-4, Issue 55, December 1972, p. 26/32.

[BEN 58] BENDAT J.S., *Principles and applications of random noise theory*, John Wiley and Sons, Inc., 1958.

[BEN 61a] BENDAT J.S., 'The establishment of test levels from field data – Panel Session', *The Shock and Vibration Bulletin*, n° 29, Part IV, 1961, p. 359/376.

[BEN 61b] BENDAT J.S., ENOCHSON L.D., KLEIN G.H. and PIERSOL A.G., 'The application of statistics to the flight vehicle vibration problem', *ASD Technical Report 61-123*, December 1961.

[BEN 62] BENDAT J.S., ENOCHSON L.D., KLEIN G.H. and PIERSOL A.G., 'Advanced concepts of stochastic processes and statistics for flight vehicle vibration estimation and measurement', *ASD-TDR-62-973*, December 1962.

[BEN 63] BENDAT J.S., ENOCHSON L.D. and PIERSOL A.G., *Analytical study of vibration data reduction methods*, The Technical Products Company, Contract NAS8-5093, September 1963.

[BEN 64] BENDAT J.S., *Probability functions for random responses: prediction of peaks, fatigue damage and catastrophic failures*, NASA CR-33, April 1964.

[BEN 71] BENDAT J.S. and PIERSOL A.G., *Random data: analysis and measurement procedures*, Wiley – Interscience, 1971.

[BEN 75] BENNET F.E., 'Spectral measurements made from finite duration observations', *Journal of the Society of Environmental Engineers*, Vol. 14-1, n° 64, March 1975, p. 11/16.

[BEN 78] BENDAT J.S., 'Statistical errors in measurement of coherence functions and input/output quantities', *Journal of Sound and Vibration*, 59, 3, 1978, p. 405/421.

[BEN 80] BENDAT J.S. and PIERSOL A.G., *Engineering applications of correlation and spectral analysis*, John Wiley and Sons, 1980.

[BER 77] BERNSTEIN M., 'Single mode stress required for equivalent damage to multimode stress in sonic feature', *AIAA Dynamics Specialist Conference*, San Diego, California, 24–25 March 1977, p. 191/197.

[BIN 67] BINGHAM C., GODFREY M.D. and TUKEY J.W., 'Modern techniques of power spectrum estimation', *IEEE Trans. Audio and Electroacoustics*, Vol. AU-15, n° 2, 1967, p. 56/66.

[BLA 58] BLACKMAN R.B. and TUKEY J.W., *The measurement of power spectra*, Dover Publications, Inc., New York, 1958, or *Bell System Technical Journal*, Vol. XXXVII, January and March 1958.

[BLA 91] BLANCHET G. and PRADO J., *Méthodes numériques pour le traitement du signal*, Masson, 1991.

[BOD 72] BODMER R., 'Qu'est ce que la corrélation?', *EMI*, 155/15-4, p. 45/48, 1972.

[BOL 84] BOLOTIN V.U., *Random vibrations of elastic systems*, Martinus Nishoff Publishers, The Hague, Vol. 8, 1984.

[BOO 56] BOOTH G.B., 'Random motion', *Product Engineering*, November 1956, p. 169/176.

[BRO 63] BROCH J.T., 'Effects of spectrum non-linearities upon the peak distribution of random signals', *Brüel and Kjaer Technical Review*, n° 3, 1963.

[CAL 69] CALOT G., *Cours de statistique descriptive*, Dunod, 1969.

[CAR 56] CARTWRIGHT D.E. and LONGUET-HIGGINS M.S., 'Statistical distribution of the maxima of a random function', *Proc. Roy. Soc. A.*, Vol. 237, 1956, p. 212/232.

[CAR 68] CARY H., 'Facts from figures – The values of amplitude distribution analysis', *IES Proceedings*, May 1968, p. 49/54.

[CAR 73] CARTER G.C., KNAPP C.H. and NUTTALL A.H., 'Estimation of the magnitude-squared coherence function via overlapped fast Fourier transform processing', *IEEE Trans. on Audio and Electroacoustics*, Vol. AU-21, n° 4, August 1973, p. 337/344.

[CAR 80] CARTER G.C. and NUTTALL A.H., 'On the weighted overlapped segment averaging method for power spectral estimation', *Proceedings of the IEEE*, Vol. 68, n° 10, p. 1352/1354, October 1980.

[CAU 61] CAUGHEY T.K. and STUMPF H.J., 'Transient response of a dynamic system under random excitation', *Transactions of the ASME Journal of Applied Mechanics*, 28, December 1961, p. 563/566.

[CHA 72] CHAKRAVORTY M., 'Transient spectral analysis of linear elastic structures and equipment under random excitation', Department of Civil Engineering, *MIT Research Report N° R72-18*, Structures Publication, n° 340, April 1972.

[CHA 85] CHAUDHURY G.K. and DOVER W.D., 'Fatigue analysis of offshore platforms subject to sea wave loadings', *International Journal of Fatigue*, Vol. 7, n° 1, January 1985, p. 13/19.

[CLA 65] CLARKSON B.L. and MERCER C.A., 'Use of cross-correlation in studying the response of lightly damped structures to random forces', *AIAA Journal*, Vol. 12, 1965, p. 2287/2291.

[CLO 75] CLOUGH R.W. and PENZIIEN, *Dynamics of structures*, McGraw-Hill Book Company Inc., New York, 1975.

[CLY 64] Mc CLYMONDS J.C. and GANOUNG J.K., 'Combined analytical and experimental approach for designing and evaluating structural systems for vibration environments', *The Shock and Vibration Bulletin*, n° 34, Part 2, December 1964, p. 159/175.

[CON 95] MCCONNEL K.G., *Vibration testing: theory and practice*, John Wiley & Sons Inc., 1995.

[COO 65] COOLEY J.W. and TUKEY J.W., 'An algorithm for the machine calculation of complex Fourier series', *Mathematics of Computation*, Vol. 19, n° 90, April 1965, p. 297/301.

[COU 66] COUPRY G., 'Une nouvelle méthode d'analyse des phénomènes aléatoires', ONERA, TP n° 330, 1966, *Communication présentée au IVème Symposium International sur l'Instrumentation Aérospatiale*, Cranfield, 21/24 March 1966.

[COU 70] COUPRY G., 'Distribution statistique des maximums absolus des processus localement stationnaires et gaussiens à large bande', *La Recherche Aérospatiale*, n° 1, January/February 1970, p. 43/48.

[CRA 61] CRANDALL S.H., 'Random vibration of a nonlinear system with a set-up spring', *ASME Journal of Applied Mechanics*, Paper n° 61-WA-152, 1961, p. 1/6.

[CRA 63] CRANDALL S.H. and MARK W.D., *Random vibration in mechanical systems*, Academic Press, 1963.

[CRA 67] CRAMER H. and LEADBETTER M.R., *Stationary and related stochastic processes*, John Wiley and Sons Inc., New York, 1967.

[CRA 68] CRANDALL S.H., 'Distribution of maxima in the response of an oscillator to random excitation', *MIT Acoust. and Vib. Lab. Rept n° DSR 78867-3*, 1968.

[CRA 83] CRANDALL S.H. and ZHU W.Q., 'Random vibration: a survey of recent developments', *Transactions of the ASME Journal of Applied Mechanics*, Vol. 50, n° 4b, December 1983, p. 953/962.

[CRE 56] CREDE C.E. and LUNNEY E.J., 'Establishment of vibration and shock tests for missile electronics as derived from the measured environment', *WADC Technical Report 56-503*, ASTIA Document n° AD118133, 1 December 1956.

[CUR 64] CURTIS A.J., 'Random and complex vibration theory', *IES Proceedings*, 1964, p. 35/52.

[CUR 71] CURTIS A.J., TINLING N.G., ABSTEIN H.T., *Selection and performance of vibration tests*, The Shock and Vibration Information Center, USDD-SVM8, 1971.

[CUR 87] CURTIS A.J., 'Applications of digital computers', *Shock and Vibration Handbook*, 27, third edition, C.M. Harris ed., McGraw-Hill Book Company, 1987.

[CUR 88] CURTIS A.J., 'Concepts in vibration data analysis', *Shock and Vibration Handbook*, p. 22-1/22-28, C.M. Harris ed., McGraw Hill Book Company, 1988.

[DAH 74] DAHLQUIST G. and BJÖRCK A., *Numerical methods*, Prentice-Hall, Series in Automatic Computation, 1974.

[DAR 72] DART M., *Traitement des signaux aléatoires*, INSA Lyon, CAST, Stage Etude des Vibrations Aléatoires, 1972.

[DAS 89] DA SILVA PASSOS A., *Méthodes mathématiques du traitement numérique du signal*, Eyrolles, 1989.

[DAV 58] DAVENPORT W.B. and ROOT W.L., *Random signals and noise*, McGraw Hill Book Company, 1958.

[DAV 64] DAVENPORT A.G., 'Note on the distribution of the largest value of a random function with application to gust loading', *Proceedings of Institute of Civil Engineers*, Vol. 28, 1964, p. 187/196.

[DEE 71] DE ESTRADA M.B., *Average response spectra for structures excited by filtered white noise*, Thesis, Department of Civil Engineering, Massachusetts Institute of Technology, June 1971.

[DER 80] DER KIUREGHIAN A., 'A response spectrum method for random vibrations', Earthquake Engineering Research Center, *Report n° UCB/EERC-80/15*, June 1980.

[DEV 62] DE VRIENDT A.B., *La transmission de la chaleur*, Editions Gaëtan Morin, Volume 1, Tome 2, 1962.

[DUB 59] DUBOIS W., 'Random vibration testing', *Shock, Vibration and Associated Environments Bulletin*, n° 27, Part 2, June 1959, p. 103/112.

[DUR 72] DURRANI T.S. and NIGHTINGALE J.M., 'Data windows for digital spectral analysis', *Proceedings IEE*, Vol. 119, n° 3, March 1972, p. 342/352.

[DWI 66] DWIGHT H.B.; *Tables of integrals and others mathematical data*, fourth edition, Macmillan, 1966.

[ELD 61] ELDRED K., ROBERTS W.M. and WHITE R., 'Structural vibrations in space vehicles', *WADD Technical Report 61-62*, December 1961.

[ENO 69] ENOCHSON L.D., 'Modern aspects of data processing applied to random vibrations and acoustics', *Proceedings of the Conference on Applications and Methods of Random Data Analysis*, ISCR, University of Southampton, July 1969.

[FAU 69] FAUQUE J.M., BERTHIER D. and MAX J., 'Analyse spectrale par corrélation', *Journées de l'Automatique dans la Recherche et l'Industrie Nucléaires*, Grenoble, 24–27 June 1969.

[FIS 28] FISHER R.A. and TIPPETT L.H.C., 'Limiting forms of the frequency distribution of the largest or smallest member of a sample', *Proceedings of Camb. Phil. Soc.*, 24, 180, 1928.

[FOR 64] FORLIFER W.R., 'The effects of filter bandwith in spectrum analysis of random vibration', *The Shock and Vibration Bulletin*, n° 33, Part II, p. 273/278, 1964.

[FUL 61] FULLER J.R., 'Cumulative fatigue damage due to variable-cycle loading', *Noise Control*, July/August 1961, or *The Shock and Vibration Bulletin*, n° 29, Part 4, 1961, p. 253/273.

[FUL 62] FULLER J.R., 'Research on techniques of establishing random type fatigue curves for broad band sonic loading', ASTIA, The Boeing Co., *Final Report N° ASD-TDR-62-501*, October 1962, or *National Aeronautical Meeting*, SAE Paper 671-C, Washington DC, 8–11 April, 1963.

[GAD 87] GADE S. and HERLUFSEN H., 'Use of weighting functions in DFT/FFT analysis', *Technical Review*, Brüel & Kjaer, n° 3 (Part I) and n° 4 (Part II), 1987.

[GIL 88] GILBERT R.J., *Vibrations des structures. Interactions avec les fluides. Sources d'excitation aléatoires*, Editions Eyrolles, 1988.

[GMU 68] GMURMAN V.E., *Fondamentals of probability theory and mathematical statistics*, ILIFFE Books LTD, London, 1968.

[GOL 53] GOLDMAN S., *Information theory*, Prentice-Hall Inc., New York, 1953.

[GOO 57] GOODMAN N.R., 'On the joint estimation of the spectra, cospectrum and quadrature spectrum of a two dimensional stationary gaussian process', *Scientific Paper*, n° 10, Engineering Statistics Laboratory, New York University, 1957.

[GRE 61] Mc GREGOR H.N. et al., 'Acoustic problems associated with underground launching of a large missile', *Shock, Vibration and Associated Environments Bulletin*, n° 29, Part IV, June 1961, p. 317/335.

[GRE 81] GREGOIRE R., 'La fatigue sous charge programmée', *CETIM*, Note Technique, n° 20, May 1981.

[GUE 80] GUESDON M., GARCIN G. and BOCQUET P., 'Surveillance des machines par analyse des vibrations', *Mécanique-Matériaux-Electricité*, n° 371–372, November/ December 1980, p. 395/403.

[GUM 54] GUMBEL E.J., 'Statistical theory of extreme values and some practical applications', *National Bureau of Standards Applied Stress Math*, Series 33, February 1954.

[HAM 68] HAMMOND J.K., 'On the response of single and multidegree of freedom systems to nonstationary random excitations', *Journal of Sound and Vibration* 7, 1968, p. 393/416.

[HAR 78] HARRIS F.J., 'On the use of windows for harmonic analysis with the discrete Fourier transform', *Proceedings IEEE*, Vol. 66, n° 1, January 1978, p. 51/83.

[HAS 55] HASTINGS C., Jr, *Approximations for digital computers*, Princeton University Press, 1955.

[HEA 56] HEAD A.K. and HOOKE F.H., 'Random noise fatigue testing', *International Conference on Fatigue of Metals*, Session 3, Paper 9, London, September 1956.

[HIL 70] HILLBERRY B.M., 'Fatigue life of 2024-T3 aluminum alloy under narrow- and broad-band random loading', *ASTM.STP*, 462, 1970, p. 167/183.

[HIM 59] HIMELBLAU H., 'Random vibration nomograph', *Noise Control*, Vol. 5, n° 3, p. 49/50, July 1959.

[HIM 64] HIMELBLAU H., 'Graphical method of calculating R.M.S. values for shaped random vibration spectra', *The Shock and Vibration Bulletin*, n° 34, Part II, p. 225/237, 1964.

[HOR 75] HORBETTE B., *Etablissement des spécifications d'essais*, Stage ADERA, Théorie des Vibrations et des Chocs, 1975.

[HUS 56] HUSTON W.B. and SKOPINSKI T.H., 'Probability and frequency characteristics of some flight buffet loads', *NACA – TN 3733*, August 1956.

[ISO 94] ISO 1683, 'Acoustics – Preferred Reference Quantities for Acoustic Levels', first edition, *BS EN 21683. CEN EN 21683*, 1994.

[JAM 47] JAMES H.M., NICHOLS N.B. and PHILLIPS R.S., *Theory of servomechanisms*, MIT Radiation Laboratory Series, Vol. 25, McGraw Hill Book Company, 1947.

[JEN 68] JENKINS G.M. and WATTS D.G., *Spectral analysis and its applications*, Holden-Day, 1968.

[KAC 76] KACENA W.J. and JONES P.J., 'Fatigue prediction for structures subjected to random vibration', *The Shock and Vibration Bulletin*, n° 46, Part III, August 1976, p. 87/96.

[KAT 65] KATZ H. and WAYMON G.R., 'Utilizing in-flight vibration data to specify design and test criteria for equipment mounted in jet aircraft', *The Shock and Vibration Bulletin*, n° 34, Part 4, February 1965, p. 137/146.

[KAY 81] KAY S.M. and MARPLE S.L,, 'Spectrum analysis – A modern perspective', *Proceedings of the IEEE*, Vol. 69, n° 11, November 1981, p. 1380/1419.

[KAZ 70] KAZMIERCZAK F.F., 'Objective criteria for comparison of random vibration environments', *The Shock and Vibration Bulletin*, n° 41, Part 3, December 1970, p. 43/54.

[KEL 61] KELLY R.D., 'A method for the analysis of short-duration non-stationary random vibration', *Shock, Vibration and Associated Environments Bulletin*, n° 29, Part IV, June 1961, p. 126/137.

[KEL 67] KELLER A.C., 'Considerations in the analysis of arbitrary waveforms', *Proceedings IES*, 1967, p. 659/671.

[KNU 98] KNUTH D.E., *The art of computer programming*, Addison-Wesley, third edition, Vol. 2, 1998.

[KOR 66] KORN G.A., *Random-process simulation and measurements*, McGraw-Hill Book Company, 1966.

[KOW 63] KOWALEWSKI J., *Über die Beziehungen zwischen der Lebensdauer von Bauteilen bei unregelmäßig schwankenden und bei geordneten Belastungsfolgen*, DVL-Bericht Nr 249, Porz-Wahn, September 1963.

[KOW 69] KOWALEWSKI J., *Beschreibung regerlloser Vorgänge*, Forstchr. Ber-VDI-Z, Reihe 5, Nr. 7, S7/28, VDI, Verlag GmbH, Düsseldorf, 1969.

[KOZ 64] KOZIN F., 'Final report on statistical models of cumulative damage', *Midwest Applied Science Corp. Report*, n° 64 - 17, N 6520969, December 1964.

[KRE 83] KREE P. and SOIZE C., *Mécanique aléatoire*, Dunod, 1983.

[LAL 81] LALANNE C., *Utilisation de l'environnement réel pour l'établissement des spécifications d'essais,* Synthèse sur les méthodes classiques et nouvelles, Cours ADERA, 1981.

[LAL 92] LALANNE C., 'Fatigue des matériaux', *Stage Intespace Environnements Vibratoires Réels - Analyse et Spécifications*, March 1992.

[LAL 94] LALANNE C., *Vibrations aléatoires - Dommage par fatigue subi par un système mécanique à un degré de liberté*, CESTA/DT/EX/EC, n° 1019/94, 1994.

[LAL 95] LALANNE C., *Analyse des vibrations aléatoires*, CESTA/DQS DO 60, 10 May 1995.

[LAM 76] LAMBERT R.G., 'Analysis of fatigue under random vibration', *The Shock and Vibration Bulletin*, n° 46, Part 3, 1976, p. 55/72.

[LAM 80] LAMBERT R.G., *Computation methods,* General Electric Company, Aerospace Electronic Systems Department, Utica, New York, March 1980.

[LEA 69] LEADBETTER M.R., 'Extreme value theory and stochastic processes', *Proceedings of the International Conference on Structural Reliability*, Washington DC, April 1969.

[LEL 73] LELEUX F., 'Fatigue sous charge aléatoire. Analyse des sollicitations et estimations des durées de vie', *CETIM*, Note Technique, n° 3, September 1973.

[LEY 65] LEYBOLD H.A., 'Techniques for examining statistical and power-spectral properties of random time histories', *NASA-TND-2714*, March 1965 (MS Dissertation, VPI, May 1963).

[LIN 67] LIN Y.K., *Probabilistic theory of structural dynamics*, McGraw-Hill Book Company, 1967.

[LIN 72] LINSLEY R.C. and HILLBERRY B.M., 'Random fatigue of 2024-T3 aluminium under two spectra with identical peak-probability density functions', *Probabilistic Aspects of Fatigue*, ASTM STP 511, 1972, p. 156/167.

[LON 52] LONGUET-HIGGINS M.S., 'On the statistical distribution of heights of sea waves', *Journal of Marine Research*, n° 3, p. 245/266, 1952.

[MAR 58] MARSHALL J.T. and HARMEN R.A., 'A proposed method for assessing the severity of the vibration environment', *The Shock and Vibration Bulletin*, n° 26, Part 2, December 1958, p. 259/277.

[MAR 70] MARK W.D., 'Spectral analysis of the convolution and filtering of non-stationary stochastic processes', *Journal of Sound and Vibration*, 11, 1970, p. 19/63.

[MAX 65] MAX J., *Les méthodes de corrélation dans le traitement de l'information*, CEA/CENG, Note technique EL/277, May 1965.

[MAX 69] MAX J., FAUQUE J.M. and BERTHIER D., 'Les corrélateurs dans le traitement des mesures et l'analyse spectrale', *Journées de l'Automatique dans la Recherche et l'Industrie Nucléaires*, Grenoble, 24–27 June 1969.

[MAX 81] MAX J., *Méthodes et techniques de traitement du signal et applications aux mesures physiques*, Tome II, Masson, 1981.

[MAX 86] MAX J., DIOT M. and BIGRET R., 'Les analyseurs de spectre à FFT et les analyseurs de spectre à corrélation', *Traitement du Signal*, Vol. 3, n° 4–5, 1986, p. 241/256.

[MAZ 54] MAZELSKY B., 'Extension of power spectral methods of generalized harmonic analysis to determine non-gaussian probability functions of random input disturbances and output responses of linear systems', *The Journal of the Aeronautical Sciences*, Vol. 21, n° 3, March 1954, p. 145/153.

[MIX 69] MIX D.F., *Random signal analysis*, Addison-Wesley Publishing Company, 1969.

[MOO 61] MOODY R.C., 'The principles involved in choosing analyzer bandwidth, averaging time, scanning rate and the length of the sample to be analyzed', *The Shock and Vibration Bulletin*, n° 29, Part IV, 1961, p. 183/190.

[MOR 55] MORROW C.T. and MUCHMORE R.B., 'Shortcomings of present methods of measuring and simulating vibration environments', *Journal of Applied Mechanics*, September 1955, p. 367/371.

[MOR 56] MORROW C.T., 'Shock and vibration environments', *Shock and Vibration Instrumentation, ASME*, June 1956, p. 75/86.

[MOR 58] MORROW C.T., 'Averaging time and data-reduction time for random vibration spectra', *The Journal of Acoustical Society of America*; I – Vol. 30, n° 5, p. 456/461, May 1958; II – Vol. 30, n° 6, p. 572/578, June 1958.

[MOR 63] MORROW C.T., *Shock and vibration engineering*, John Wiley and Sons, Inc., Vol. 1, 1963.

[MOR 75] MORROW C.T. and MUCHMORE R.B., 'Shortcomings of present methods of measuring and simulating vibration environments', *Journal of Applied Mechanics*, Vol. 22, n° 3, September 1975, p. 367/371.

[NEU 70] NEUVILLE J.C., 'Erreurs de principe sur les essais sinusoïdaux traités sur ordinateur', *CEA/CESTA/ Z-M EX – DO 449*, 25 March 1970.

[NEW 75] NEWLAND D.E., *An introduction to random vibrations and spectral analysis*, Longman, London, 1975.

[NUT 71] NUTTALL A.H., 'Spectral estimation by means of overlapped fast Fourier transform processing of windowed data', *NUSC Tech. Rep. 4169*, New London, CT, 13 October 1971.

[NUT 76] NUTTALL A.H., 'Probability distribution of spectral estimates obtained via overlapped FFT processing of windowed data', Naval Underwater Systems Center, *Tech. Rep. 5529*, New London, CT, 3 December, 1976.

[NUT 80] NUTTAL A.H. and CARTER G.C., 'A generalized framework for power spectral estimation', *IEEE Transactions on Acoustics, Speech, and Signal Processing*, Vol. ASSP-28, n° 3, June 1980, p. 334/335.

[NUT 81] NUTTALL A.H., 'Some windows with very good sidelobe behavior', *IEEE Trans. on Acoustics, Speech and Signal Processing*, Vol. ASSP-29, n° 1, February 1981, p. 84/91.

[OSG 69] OSGOOD C.C., 'Analysis of random responses for calculation of fatigue damage', *The Shock and Vibration Bulletin*, n° 40, Part II, December 1969, p. 1/8.

[OSG 82] OSGOOD C.C., *Fatigue design*, Pergamon Press, 1982.

[PAP 65] PAPOULIS A., *Probability random variables and strochastic processes*, McGraw-Hill Book Company, 1965.

[PAR 59] PARZEN E., 'On models for the probability of fatigue failure of a structure', *App. Math. and Stat. Lab.*, Stanford University, Tech. Report, n° 45, 17 April 1959.

[PAR 62] PARIS P.C., *The growth of cracks due to variations in loads*, PhD Dissertation, Lehigh University, Bethlehem, Pennsylvania, AD63-02629, 1962.

[PAR 64] PARIS P.C., 'The fracture mechanics approach to fatigue', *Proceedings 10th Sagamore Army Mater. Res. Conf.*, Syracuse University Press, Syracuse, New York, 1964, p. 107/132.

[PER 74] PERUCHET C. and VIMONT P., *Résistance à la fatigue des matériaux en contraintes aléatoires*, ENICA, 1974.

[PIE 63] PIERSOL W.J. Jr. and MOSKOWITZ L., 'A proposed spectral form for fully developed wind seas based on the similarity theory of S.Z. KITAIGORODSKII', NYU Dept. Meteorology and Oceanography Geophysical Sciences, *Lab. Report 63.12*, October 1963, or *J. Geophysical Res.*, 69, 5181, 1964.

[PIE 64] PIERSOL A.G., 'The measurement and interpretation of ordinary power spectra for vibration problems', *NASA-CR 90*, 1964.

[PIE 70] PIERSOL A.G. and MAURER J.R., 'Investigation on statistical techniques to select optimal test levels for spacecraft vibration tests', Digital Corporation, *Report 10909801-F*, October 1970.

[PIE 93] PIERSOL A.G., 'Optimum resolution bandwidth for spectral analysis of stationary random vibration data', *Shock and Vibration*, John Wiley and Sons, Vol. 1, n° 1, 1993, p. 33/43.

[POO 76] POOK L.P., 'Proposed standard load histories for fatigue testing relevant to offshore structures', National Engineering Laboratory, *NEL Report*, n° 624, 1976.

[POO 78] POOK L.P., 'An approach to practical load histories for fatigue testing relevant to offshore structures', *Journal of the Society of Environmental Engineers*, Vol. 17-1, Issue 76, March 1978, p. 22/35.

[POT 78] POTTER R. and LORTSCHER J., *What in the world is overlapped processing?*, UPDATE, Hewlett-Packard, Santa Clara DSA/Laser Division, n° 1, September 1978, p. 4/6.

[POW 58] POWELL A., 'On the fatigue failure of structures due to vibrations excited by random pressure fields', *The Journal of the Acoustical Society of America*, 30, Vol. 12, p. 1130/1135, December 1958.

[PRA 70] PRATT J.S., *Random accelerations: R.M.S. value*, Test Engineering, March 1970, p.10.

[PRE 54] PRESS H. and MAZELSKY B., 'A study of the application of power-spectral methods of generalized harmonic analysis to gust loads on airplanes', National Advisory Committee for Aeronautics, *NACA Report 1172*, 1954.

[PRE 56a] PRESS H. and TUKEY J.W., 'Power spectral methods of analysis and their application to problems in airplane dynamics', *Agard Flight Test Manual*, Vol. IV, Part IV-C, June 1956, p. 1/41, or *Bell Telephone System Monograph 26006*, 1957.

[PRE 56b] PRESS H., MEADOWS M.T. and HADLOCK I., 'A reevaluation of data on atmospheric turbulence and airplane gust loads for application in spectral calculations', *NACA Report 1272*, 1956.

[PRE 90] PREUMONT A., *Vibrations aléatoires et analyse spectrale*, Presses Polytechniques et Universitaires Romandes, 1990.

[PRI 67] PRIESTLEY M.B., 'Power spectral analysis of nonstationary random processes', *Journal of Sound and Vibration*, 6, p. 86/97, 1967.

[RAC 69] RACICOT R.L., *Random vibration analysis – Application to wind loaded structures*, Solid Mechanics, Structures and Mechanical Design Division, Report n° 30, Case Western Reserve University, 1969 (PhD Thesis).

[RAP 69] RAPIN P., 'Applications de la statistique mathématique aux problèmes vibratoires de la mécanique industrielle', *Mécanique*, n° 240, December 1969, p. 30/39.

[RAP 82] RAPIN P., 'L'utilisation pratique des machines tournantes – Les comportements anormaux', *Mécanique-Matériaux-Electricité*, n° 386/387, 1982, p. 51/57.

[RAV 70] RAVISHANKAR T.J., *Simulation of random load fatigue in laboratory testing*, Institute for Aerospace Studies, UTIAS Review, n° 29, University of Toronto, March 1970.

[REE 60] REED W.H., HALL A.W. and BARKER L.E., 'Analog techniques for measuring the frequency response of linear physical systems excited by frequency sweep inputs', *NASA TN D 508*, 1960.

[RIC 39] RICE S.O., 'The distribution of the maxima of a random curve', *Amer. J. Math.*, Vol. 61, April 1939, p. 409/416.

[RIC 44] RICE S.O., 'Mathematical analysis of random noise', *Bell System Technical Journal*, n° 23 (July 1944) and 24 (January 1945).

[RIC 64] RICE J.R., Theoretical prediction of some statistical characteristics of random loadings relevant to fatigue and fracture, Lehigh University, PhD, 1964.

[RIC 65] RICE J.R. and BEER F.P., 'On the distribution of rises and falls in a continuous random process', *Trans. of the ASME, Journal of Basic Engineering*, June 1965, p. 398/404.

[ROB 71] ROBERTS J.B., 'The covariance response of linear systems to non-stationary random excitation', *Journal of Sound and Vibration*, 14, p. 385/400, 1971.

[ROT 70] ROTH P.R., *Digital Fourier analysis*, Hewlett-Packard, 1970.

[RUB 64] RUBIN S., 'Introduction to dynamics', *Proceedings IES*, 1964, p. 3/7.

[RUD 75] RUDDER F.F. and PLUMBEE H.E., 'Sonic fatigue design guide for military aircraft', *Technical Report AFFDL-TR-74-112*, May 1975.

[SAN 63] SANDLER I.J., 'Techniques of analysis of random and combined random-sinusoïdal vibration', *Shock, Vibration and Associated Environments Bulletin*, n° 31, Part 3, 1963, p. 211/224.

[SAN 66] Sandia Corporation Standard Environmental Test Methods, *SC-4452 D (M)*, June 1966.

[SCH 63] SCHIJVE J., 'The analysis of random load-time histories with relation to fatigue tests and life calculations', *Fatigue of Aircraft Structures*, Pergamon, 1963, p. 115/149.

[SHA 49] SHANNON C.E., 'Communication in the presence of noise', *Proceedings of the IRE*, 37, January 1949, pp. 10/21.

[SHE 83] SHERRAT F., 'Vibration and fatigue: basic life estimation methods', *Journal of the Society of Environmental Engineers*, Vol. 22-4, n° 99, December 1983, p. 12/17.

[SHI 70a] SHINKLE R.K. and STRONG S.A., 'A sine of the times, the detection of discrete components in a complex spectrum', *Proceedings IES*, 1970, p. 282/292.

[SHI 70b] SHINOZUKA M., 'Random processes with evolutionary power', *Journal of the Engineering Mechanics Division, ASCE 96*, 1970, n° EM4, p. 543/545.

[SJÖ 61] SJÖSTRÖM S., 'On random load analysis', KTH Avh., Thesis n° 156, *Transactions of the Royal Institut of Technology*, Stockholm, Sweden, 1961.

[SKO 59] SKOOG J.A. and SETTERLUND G.G., 'Space requirements for equipment items subjected to random excitation', *The Journal of the Acoustical Society of America*, Vol. 31, n° 2, February 1959, p. 227/232.

[SPA 84] SPANN F. and PATT P., 'Component vibration environment predictor', *The Journal of Environmental Sciences*, Vol. XXVII, n° 5, September/October 1984, p. 19/24.

[SPA 87] SPANIER J. and OLDHAM K.B., *An atlas of functions*, Hemisphere Publishing Corporation, 1987.

[SPI 74] SPIEGEL M.R., *Formules et tables de mathématiques*, McGraw-Hill International Book Company, Série SCHAUM, 1974.

[STE 67] STERN J., De BARBEYRAC J. and POGGI R., *Méthodes pratiques d'étude des fonctions aléatoires*, Dunod, 1967.

[SVE 80] SVETLICKIJ V.A., *Vibrations aléatoires des systèmes mécaniques*, Technique et Documentation, 1980.

[SWA 63] SWANSON S.R., *An investigation of the fatigue of aluminum alloy due to random loading*, Institute of Aerophysics, University of Toronto, UTIA Report n° 84, AD 407071, February 1963.

[SWA 68] SWANSON S.R., 'Random load fatigue testing: a state of the art survey', *ASTM Materials Research and Standards*, Vol. 8, n° 4, April 1968, p. 10/44.

[TAY 75] TAYLOR H.R., *A study of instrumentation systems for the dynamic analysis of machine tools*, PhD Thesis, University of Manchester, 1975.

[THR 64] THRALL G.P., *Extreme values of random processes in seakeeping applications*, Measurement Analysis Corporation, MAC 307-03, AD 607912, August 1964.

[TIP 25] TIPPET L.H.C., 'On the extreme individuals and the range of samples taken from a normal population', *Biometrika*, 17, 1925, p. 364/387.

[TIP 77] TIPTON R., 'Noise testing – Part 1. An introduction to random vibration testing', *Test*, June/July 1977, p. 12/15.

[TUS 67] TUSTIN W., 'Vibration and shock tests do not duplicate service environment', *Test Engineering*, Vol. 18, n° 2, August 1967, p. 18/21.

[TUS 72] TUSTIN W., *Environmental vibration and shock*, Tustin Institute of Technology, 1972.

[TUS 73] TUSTIN W., 'Basic considerations for simulation of vibration environments', *Experimental Mechanics*, September 1973, p. 390/396.

[TUS 79] TUSTIN W., 'In wide-band random vibration, all frequencies are present', *Test*, Vol. 41, n° 2, April/May 1979, p. 12/13.

[UDW 73] UDWADIA F.E. and TRIFUNAC M.D., 'The Fourier transform, response spectra and their relationship through the statistics of oscillator response', Earthquake Engineering Research Laboratory, Calif. Institute of Technology, Pasadena, Calif., *Report NO-EERL 73-01*, April 1973.

[VAN 70] VANMARCKE E.H., 'Parameters of the spectral density function: their significance in the time and frequency domains', Department of Civil Engineering, *MIT Research Report n° R70-58*, 1970.

[VAN 72] VANMARCKE E.H., 'Properties of spectral moments with applications to random vibration', *Journal of Engineering Mechanics Division, ASCE*, Vol. 98, 1972, p. 425/446 [AMR 26, 1973, Rev. 259].

[VAN 75] VANMARCKE E.H., 'On the distribution of the first-passage time for normal stationary random processes', *Journal of Applied Mechanics*, Vol. 42, March 1975, p. 215/220 [AMR 29, 1976, Rev. 1184].

[VAN 79] VANMARCKE E.H., 'Some recent developments in random vibration', *Applied Mechanics Reviews*, Vol. 32, n° 10, October 1979, p. 1197/1202.

[VIN 72] VINH T. and LABOUREAU B., *Problème d'identification des corrélogrammes*, INSA, CAST, Stage Etude des Vibrations Aléatoires, 1972.

[WAL 81] WALKER A.W., 'The effects of bandwidth on the accuracy of transfert function measurements of single degree of freedom system response to random excitation', *Journal of Sound and Vibration*, 74 (2), 1981, p. 251/263.

[WAN 45] WANG M.C. and UHLENBECK G.E., 'On the theory of the Brownian motion II', *Reviews of Modern Physics*, Vol. 17, n° 2 and 3, April–July 1945, p. 323/342.

[WEL 67] WELCH P.D., 'The use of Fast Fourier Transform for the estimation of power spectra: a method based on time averaging over short, modified periodograms', *IEEE Trans. on Audio and Electroacoustics*, Vol. AU-15, n° 2, June 1967, p. 70/73.

[WEL 70] WELLSTEAD P.E., *Aspects of real time digital spectral analysis*, PhD Thesis, University of Warwick, 1970.

[WHI 69] WHITE R.G., 'Use of transient excitation in the dynamic analysis of structures', *The Aeronautical Journal of the Royal Aeronautical Society*, Vol. 73, December 1969, p. 1047/1050.

[WHI 72] WHITE R.G., 'Spectrally shaped transient forcing functions for frequency response testing', *Journal of Sound and Vibration*, Vol. 23, n° 3, 1972, p. 307/318.

[WHI 82] WHITE R.G. and PINNINGTON R.J., 'Practical application of the rapid frequency sweep technique for structural frequency response measurement', *Aeronautical Journal*, Paper n° 964, May 1982, pp 179/199.

[WIE 30] WIENER N., 'Generalized harmonic analysis', *Acta Mathematica*, 55, 1930, p. 117/258.

[WIR 73] WIRSCHING P.H. and HAUGEN E.B., 'Probabilistic design for random fatigue loads', *Journal of the Engineering Mechanics Division, ASCE*, 99, n° EM 6, December 1973.

[WIR 81] WIRSCHING P.H., 'The application of probability design theory to high temperature low cycle fatigue', *NASA - CR 165488, N82-14531/9*, November 1981.

[WIR 83] WIRSCHING P.H., 'Probability-based fatigue design criteria for offshore structures', *Final Project Report API-PRAC 81-15*, The American Petroleum Institute, Dallas, Texas, January 1983.

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Index

Synopsis of five volume series: Mechanical Vibration and Shock

This is the third volume in this five volume series.

Volume 1 is devoted to *sinusoidal vibration*. The responses, relative and absolute, of a mechanical one-degree-of-freedom system to an arbitrary excitation are considered, and its transfer function in various forms defined. By placing the properties of sinusoidal vibrations in the contexts of the environment and of laboratory tests, the transitory and steady state response of a single-degree-of-freedom system with viscous and then with non-linear damping is evolved. The various sinusoidal modes of sweeping with their properties are described, and then, starting from the response of a one-degree-of-freedom system, the consequences of an unsuitable choice of the sweep rate are shown and a rule for choice of this rate deduced from it.

Volume 2 deals with *mechanical shock*. This volume presents the shock response spectrum (SRS) with its different definitions, its properties and the precautions to be taken in calculating it. The shock shapes most widely used with the usual test facilities are presented with their characteristics, with indications how to establish test specifications of the same severity as the real, measured environment. A demonstration is then given on how these specifications can be made with classic laboratory equipment: shock machines, electrodynamic exciters driven by a time signal or by a response spectrum, indicating the limits, advantages and disadvantages of each solution.

Volume 3 examines the analysis of *random vibration*, which encompass the vast majority of the vibrations encountered in the real environment. This volume describes the properties of the process enabling simplification of the analysis, before presenting the analysis of the signal in the frequency domain. The definition of the power spectral density is reviewed as well as the precautions to be taken in calculating it, together with the processes used to improve results (windowing, overlapping). A complementary third approach consists of analyzing the statistical properties of the time signal. In particular, this study makes it possible to determine the distribution law of the maxima of a random Gaussian signal and to simplify the calculations of fatigue damage by avoiding direct counting of the peaks (Volumes 4 and 5).

Having established the relationships which provide the response of a linear system with one degree of freedom to a random vibration, Volume 4 is devoted to the calculation of *damage fatigue*. It presents the hypotheses adopted to describe the behaviour of a material subjected to fatigue, the laws of damage accumulation, together with the methods for counting the peaks of the response, used to establish a histogram when it is impossible to use the probability density of the peaks obtained with a Gaussian signal. The expressions of mean damage and of its standard deviation are established. A few cases are then examined using other hypotheses (mean not equal to zero, taking account of the fatigue limit, non linear accumulation law, etc.).

Volume 5 is more especially dedicated to presenting the method of *specification development* according to the principle of tailoring. The extreme response and fatigue damage spectra are defined for each type of stress (sinusoidal vibrations, swept sine, shocks, random vibrations, etc.). The process for establishing a specification as from the life cycle profile of the equipment is then detailed, taking account of an uncertainty factor, designed to cover the uncertainties related to the dispersion of the real environment and of the mechanical strength, and of another coefficient, the test factor, which takes into account the number of tests performed to demonstrate the resistance of the equipment.

This work is intended first and foremost for engineers and technicians working in design teams, which are responsible for sizing equipment, for project teams given the task of writing the various sizing and testing specifications (validation, qualification, certification, etc.) and for laboratories in charge of defining the tests and their performance, following the choice of the most suitable simulation means.