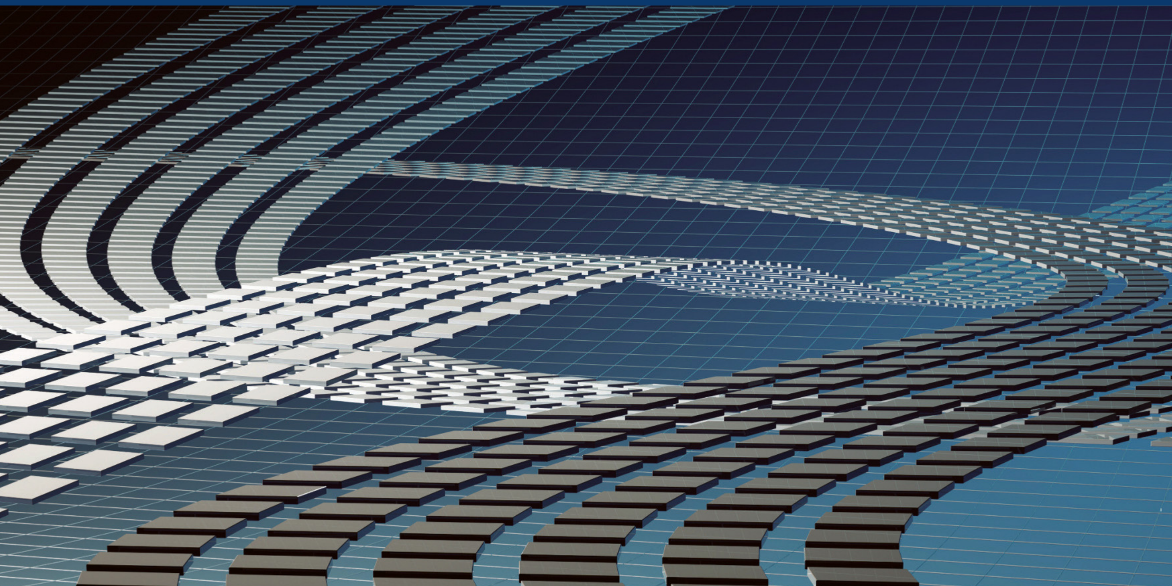
MECHANICAL ENGINEERING AND SOLID MECHANICS SERIES



Sinusoidal Vibration

Mechanical Vibration and Shock Analysis
Revised and Updated 3rd Edition
Volume 1

Christian Lalanne

iSTE

WILEY

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Third edition – Volume 1

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First edition published 2002 by Hermes Penton Ltd © Hermes Penton Ltd 2002
Second edition published 2009 in Great Britain and the United States by ISTE Ltd and John Wiley & Sons, Inc. © ISTE Ltd 2009
Third edition published 2014 in Great Britain and the United States by ISTE Ltd and John Wiley & Sons, Inc.

ISTE Ltd
27-37 St George's Road
London SW19 4EU
UK

www.iste.co.uk

John Wiley & Sons, Inc.
111 River Street
Hoboken, NJ 07030
USA

www.wiley.com

Library of Congress Control Number: 2014930266

British Library Cataloguing-in-Publication Data
A CIP record for this book is available from the British Library
ISBN 978-1-84821-643-3 (Set of 5 volumes)
ISBN 978-1-84821-644-0 (Volume 1)

MIX
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FSC® C013604
www.fsc.org

Printed and bound in Great Britain by CPI Group (UK) Ltd., Croydon, Surrey CR0 4YY

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Foreword to Series

In the course of their lifetime simple items in everyday use such as mobile telephones, wristwatches, electronic components in cars or more specific items such as satellite equipment or flight systems in aircraft, can be subjected to various conditions of temperature and humidity, and more particularly to mechanical shock and vibrations, which form the subject of this work. They must therefore be designed in such a way that they can withstand the effects of the environmental conditions to which they are exposed without being damaged. Their design must be verified using a prototype or by calculations and/or significant laboratory testing.

Sizing, and later, testing are performed on the basis of specifications taken from national or international standards. The initial standards, drawn up in the 1940s, were blanket specifications, often extremely stringent, consisting of a sinusoidal vibration, the frequency of which was set to the resonance of the equipment. They were essentially designed to demonstrate a certain standard resistance of the equipment, with the implicit hypothesis that if the equipment survived the particular environment it would withstand, undamaged, the vibrations to which it would be subjected in service. Sometimes with a delay due to a certain conservatism, the evolution of these standards followed that of the testing facilities: the possibility of producing swept sine tests, the production of narrowband random vibrations swept over a wide range and finally the generation of wideband random vibrations. At the end of the 1970s, it was felt that there was a basic need to reduce the weight and cost of on-board equipment and to produce specifications closer to the real conditions of use. This evolution was taken into account between 1980 and 1985 concerning American standards (MIL-STD 810), French standards (GAM EG 13) or international standards (NATO), which all recommended the *tailoring of tests*. Current preference is to talk of the *tailoring of the product to its environment* in order to assert more clearly that the environment must be taken into account from the very start of the project, rather than to check the behavior of the material *a* *posteriori*. These concepts, originating with the military, are currently being increasingly echoed in the civil field.

Tailoring is based on an analysis of the life profile of the equipment, on the measurement of the environmental conditions associated with each condition of use and on the synthesis of all the data into a simple specification, which should be of the same severity as the actual environment.

This approach presupposes a proper understanding of the mechanical systems subjected to dynamic loads and knowledge of the most frequent failure modes.

Generally speaking, a good assessment of the stresses in a system subjected to vibration is possible only on the basis of a finite element model and relatively complex calculations. Such calculations can only be undertaken at a relatively advanced stage of the project once the structure has been sufficiently defined for such a model to be established.

Considerable work on the environment must be performed independently of the equipment concerned either at the very beginning of the project, at a time where there are no drawings available, or at the qualification stage, in order to define the test conditions.

In the absence of a precise and validated model of the structure, the simplest possible mechanical system is frequently used consisting of mass, stiffness and damping (a linear system with one degree of freedom), especially for:

– the comparison of the severity of several shocks (shock response spectrum) or of several vibrations (extreme response and fatigue damage spectra);

– the drafting of specifications: determining a vibration which produces the same effects on the model as the real environment, with the underlying hypothesis that the equivalent value will remain valid on the real, more complex structure;

– the calculations for pre-sizing at the start of the project;

– the establishment of rules for analysis of the vibrations (choice of the number of calculation points of a power spectral density) or for the definition of the tests (choice of the sweep rate of a swept sine test).

This explains the importance given to this simple model in this work of five volumes on “Mechanical Vibration and Shock Analysis”.

Volume 1 of this series is devoted to *sinusoidal vibration*. After several reminders about the main vibratory environments which can affect materials during their working life and also about the methods used to take them into account, following several fundamental mechanical concepts, the responses (relative and absolute) of a mechanical one-degree-of-freedom system to an arbitrary excitation are considered, and its transfer function in various forms are defined. By placing the properties of sinusoidal vibrations in the contexts of the real environment and of laboratory tests, the transitory and steady state response of a single-degree-of-freedom system with viscous and then with non-linear damping is evolved. The various sinusoidal modes of sweeping with their properties are described, and then, starting from the response of a one-degree-of-freedom system, the consequences of an unsuitable choice of sweep rate are shown and a rule for choice of this rate is deduced from it.

Volume 2 deals with *mechanical shock*. This volume presents the shock response spectrum (SRS) with its different definitions, its properties and the precautions to be taken in calculating it. The shock shapes most widely used with the usual test facilities are presented with their characteristics, with indications how to establish test specifications of the same severity as the real, measured environment. A demonstration is then given on how these specifications can be made with classic laboratory equipment: shock machines, electrodynamic exciters driven by a time signal or by a response spectrum, indicating the limits, advantages and disadvantages of each solution.

Volume 3 examines the analysis of *random vibration* which encompasses the vast majority of the vibrations encountered in the real environment. This volume describes the properties of the process, enabling simplification of the analysis, before presenting the analysis of the signal in the frequency domain. The definition of the power spectral density is reviewed, as well as the precautions to be taken in calculating it, together with the processes used to improve results (windowing, overlapping). A complementary third approach consists of analyzing the statistical properties of the time signal. In particular, this study makes it possible to determine the distribution law of the maxima of a random Gaussian signal and to simplify the calculations of fatigue damage by avoiding direct counting of the peaks (Volumes 4 and 5). The relationships that provide the response of a one-degree-of-freedom linear system to a random vibration are established.

Volume 4 is devoted to the calculation of *damage fatigue*. It presents the hypotheses adopted to describe the behavior of a material subjected to fatigue, the laws of damage accumulation and the methods for counting the peaks of the response (used to establish a histogram when it is impossible to use the probability density of the peaks obtained with a Gaussian signal). The expressions of mean damage and its standard deviation are established. A few cases are then examined using other hypotheses (mean not equal to zero, taking account of the fatigue limit, non-linear accumulation law, etc.). The main laws governing low cycle fatigue and fracture mechanics are also presented.

Volume 5 is dedicated to presenting the method of *specification development* according to the principle of tailoring. The extreme response and fatigue damage spectra are defined for each type of stress (sinusoidal vibrations, swept sine, shocks, random vibrations, etc.). The process for establishing a specification as from the lifecycle profile of the equipment is then detailed taking into account the uncertainty factor (uncertainties related to the dispersion of the real environment and of the mechanical strength) and the test factor (function of the number of tests performed to demonstrate the resistance of the equipment).

First and foremost, this work is intended for engineers and technicians working in design teams responsible for sizing equipment, for project teams given the task of writing the various sizing and testing specifications (validation, qualification, certification, etc.) and for laboratories in charge of defining the tests and their performance following the choice of the most suitable simulation means.

Introduction

Materials which are transported by or loaded onto land vehicles, aircraft or marine vehicles, or which are installed close to turning machines, are subject to different vibrations and mechanical shocks. These materials must be able to endure such shocks and vibrations without being damaged. To achieve this goal, the first step consists of noting the values of these environments in the specifications of the material to be developed, so that the research departments can take them into account during dimensioning. The following step is the qualification of the designed material, starting from these specifications, to experimentally demonstrate its behavior under its future conditions of use.

The specifications used for dimensioning and testing today are elaborate, starting from measurements of the real environment which the equipment will undergo (*test tailoring*). It is thus necessary to correctly measure the vibrations and shocks before analyzing them and to synthesize them to obtain specifications leading to reasonable qualification tests of a reasonable duration.

Taking into account vibrations and shocks thus requires us:

– to identify the future conditions of use;

– to carry out, if possible, significant measurements;

– to digitize the measured signals;

– to identify each type of vibration in order to characterize them by analysis in the frequency domain, adapted to proceed to severity comparisons between the collected measurements under various conditions, or between real environments and values provided in normative documents, or with specifications established in another context;

– to finally transform measurements into specifications.

The object of this series of five volumes is thus to describe all the mathematical tools that are currently used in the analysis of vibrations and shocks, while starting with the sinusoidal vibrations.

Sinusoidal vibrations were first used in laboratory tests to verify the ability of equipment to withstand their future vibratory environment in service without damage. Following the evolution of standards and testing facilities, these vibrations, generally speaking, are currently studied only to simulate vibratory conditions of the same nature as encountered, for example, in equipment situated close to revolving machinery (motors, transmission shafts, etc.). Nevertheless, their value lies in their simplicity, enabling the behavior of a mechanical system subjected to dynamic stress to be demonstrated, and the introduction of basic definitions.

Given that, generally speaking, the real environment is more or less random in nature, with a continuous frequency spectrum in a relatively wide range, in order to overcome the inadequacies of the initial testing facilities, testing rapidly moved to the "swept sine" type. Here the vibration applied is a sinusoid, the frequency of which varies over time according to a sinusoidal or exponential law. Despite the relatively rapid evolution of electrodynamic exciters and electrohydraulic vibration exciters, capable of generating wideband random vibrations, these swept sine standards have lasted, and are in fact still used, for example in aerospace applications. They are also widely used for measuring the dynamic characteristics of structures.

After an introductory chapter (Chapter 1) to this series, pointing out the characteristics of some important vibratory environments and the various steps necessary to arrive at the qualification of a material, we follow-up with a few brief reminders of basic mechanics (Chapter 2). Chapter 3 examines the relative and absolute response of a mechanical system with one degree of freedom subjected to a given excitation, and defines the transfer function in different forms. Chapter 4 is devoted more particularly to the response of such a system to a unit impulse or to a unit step.

The properties of sinusoidal vibrations are then presented in the context of the environment and in laboratory tests (Chapter 5). The transitory and steady state response of a system with one degree of freedom to viscous damping (Chapter 6) and to non-linear damping (Chapter 7) is then examined.

Chapter 8 defines the various sinusoidal sweeping modes, with their properties and eventual justification. Chapter 9 is devoted to the response of a system with one degree of freedom subjected to linear and exponential sweeping vibrations, to illustrate the consequences of an unsuitable choice of sweep rate, resulting in the presentation of a rule for the choice of a rate.

The major properties of the Laplace transform are reviewed in the Appendix. This provides a powerful tool for the analytical calculation of the response of a system with one degree of freedom to a given excitation. Inverse transforms particularly suitable for this application are given in a table.

List of Symbols

The list below gives the most frequent definition of the main symbols used in this book. Some of the symbols can have another meaning locally which will be defined in the text to avoid confusion.

$A(t)$	Indicial admittance or step response
$A(p)$	Laplace transform of $A(t)$
c	Viscous damping constant
c_{eq}	Equivalent viscous damping constant
$C(\theta)$	Part of the response relating to non-zero initial conditions
d	Lever arm
D	Damping capacity
e	Neper number
E	Young's modulus
E_a	Damping energy
E_d	Dynamic modulus of elasticity
E_c	Kinetic energy
E_P	Potential energy
E( )	Function characteristic of sweep mode
f	Frequency of excitation
f_m	Expected frequency
f_{samp}	Sampling frequency
$\dot{f}$	Sweep rate
f_0	Natural frequency
F_i	Inertial force
F_r	Restoring force
$F(t)$	External force applied to a system
F_c	Peak factor (or crest factor)
F_d	Damping force
F_f	Form factor
F_m	Maximum value of $F(t)$
g	Acceleration due to gravity
G	Coulomb modulus
$G(\eta)$	Attenuation related to sweep rate
h	Interval (f/f_0)
H_{AD}	Transmissibility

H_{RD}	Dynamic amplification factor
H_{RV}	Relative transmissibility
$h(t)$	Impulse response
$H(\ )$	Transfer function
i	$\sqrt{-1}$
I	Moment of inertia
J	Damping constant
k	Stiffness or uncertainty coefficient
ℓ_{rms}	rms value of $\ell(t)$
ℓ_m	Maximum value of $\ell(t)$
$\ell(t)$	Generalized excitation (displacement)
$\dot{\ell}(t)$	First derivative of $\ell(t)$
$\ddot{\ell}(t)$	Second derivative of $\ell(t)$
$L(\ )$	Lagrange function
$L(p)$	Laplace transform of $\ell(t)$
$L(\Omega)$	Fourier transform of $\ell(t)$
m	Mass
M	Moment of a force
n	Number of cycles
n_d	Number of decades
N	Normal force
N_s	Number of cycles performed during swept sine test
p	Laplace variable
P	Reduced pseudo-pulsation
$\vec{P}$	Impulse vector
q_i	Generalized coordinate
q_m	Maximum value of $q(\theta)$
q_0	Value of $q(\theta)$ for $\theta = 0$
$q(\theta)$	Reduced response
$\dot{q}_0$	Value of $\dot{q}(\theta)$ for $\theta = 0$
$\dot{q}(\theta)$	First derivative of $q(\theta)$
$\ddot{q}(\theta)$	Second derivative of $q(\theta)$
Q	Q factor (quality factor)
$Q(p)$	Laplace transform of $q(\theta)$
$\vec{r}$	Position vector
R_m	Ultimate tensile strength
R_{om}	Number of octaves per minute
R_{os}	Number of octaves per second
s	Number of degrees of freedom
S	Action
t	Time
t_s	Sweep duration
T	Duration of application of vibration
T_0	Natural period
T_1	Time-constant of logarithmic swept sine
$u(t)$	Generalized response
U_s	Maximum elastic strain energy stored during one cycle
U_{ts}	Elastic strain energy per unit volume
$U(p)$	Laplace transform of $u(t)$
$U(\Omega)$	Fourier transform of $u(t)$
$\vec{v}$	Velocity vector
x_m	Maximum value of $x(t)$
$x(t)$	Absolute displacement of the base of a one-degree-of-freedom system

$\dot{x}(t)$	Absolute velocity of the base of a one-degree-of-freedom system
$\ddot{x}_m$	Maximum value of $\ddot{x}(t)$
$\ddot{x}(t)$	Absolute acceleration of the base of a one-degree-of-freedom system
$\ddot{X}(\Omega)$	Fourier transform of $\ddot{x}(t)$
$y(t)$	Absolute displacement response of the mass of a one-degree-of-freedom system
$\dot{y}(t)$	Absolute velocity response of the mass of a one-degree-of-freedom system
$\ddot{y}(t)$	Absolute acceleration response of the mass of a one-degree-of-freedom system
z_m	Maximum value of $z(t)$
z_s	Maximum static relative displacement
$z(t)$	Relative displacement response of the mass of a one-degree-of-freedom system with respect to its base
α	Rotation angle
δ	Logarithmic decrement
$\delta_g(\,)$	Dirac delta function
Δ	Energy dissipated per unit time
ΔE_d	Energy dissipated by damping in one cycle
Δf	Interval of frequency between half-power points
ΔN	Number of cycles between half-power points
ε	Relative deformation
$\dot{\varepsilon}$	Velocity of relative deformation
η	Coefficient of dissipation (or of loss) or reduced sweep rate
$\dot{z}(t)$	Relative velocity response
$\ddot{z}(t)$	Relative acceleration response
$Z(p)$	Generalized impedance
φ	Phase
$\lambda(\theta)$	Reduced excitation
$\Lambda(p)$	Laplace transform of $\lambda(\theta)$
μ	Coefficient of friction
π	3.14159265 ...
ρ	Radius of gyration
θ	Reduced time $\dot{q}(\theta)$
θ_b	Reduced sweep rate
Θ	Reduced pseudo-period
σ	Stress
σ_m	Mean stress
ω_0	Natural pulsation ($2\pi f_0$)
Ω	Pulsation of excitation ($2\pi f$)
ξ	Damping factor
ξ_{eq}	Equivalent viscous damping factor
ψ	Phase

Chapter 1

The Need

1.1. The need to carry out studies into vibrations and mechanical shocks

During their service life, many materials are subjected to vibratory environments, during their transport [OST 65], [OST 67], because they are intended to equip themselves with means of transport (airplanes, road vehicles, etc.) or because they are placed beside vibratory sources (engines, wind mills, roads, etc.). These vibratory environments (vibrations and shocks) create dynamic strains and stresses in the structures which can, for example, produce intermittent or permanent breakdowns in electrical equipment, plastic deformations or fractures by up-crossing an ultimate stress of the material (yield limit, rupture limit), optical misalignments of systems or may contribute to the fatigue and the wear of the machine elements.

It is therefore necessary to take all of these points into consideration during the design phase of structures and of mechanical equipment. The approach is normally made up of several steps:

– measuring the vibration phenomena;

– analyzing the results of the measurements, bearing in mind that this analysis will be used for different objectives, including:

- the characterization of the frequency contents of the vibration (the search for predominant frequencies, amplitudes, etc.), for example, to compare the natural frequencies of the structures,

- comparing the relative severity of several different vibratory environments (transport on various vehicles) or comparing the severity of such vibration environments with a standard,

- confirming *a posteriori* the validity of a dimensioning or test specification which is established starting from *fallback level* values, from data collected at the time of a preceding project or starting from values resulting from normative documents;

– the transformation of measurements into dimensioning specifications for research departments; these are presented in the simplest possible form requiring a synthesis of all the measured data;

– during and at the end of the design phase, at the time of the qualification, realization of tests intended to validate the behavior of the materials developed from these environments.

The vibrations most frequently encountered in the real environment are of a random nature. Along with shocks, they constitute the main part of mechanical excitations. These two environments can be severe, shocks by their amplitude and random vibrations by their duration.

In certain situations, however (near turning machines), it is possible to observe sinusoidal vibrations which are often polluted by noise. This is especially the case for vibrations which are produced by propeller airplanes and helicopters. In these cases, the random noise which is produced is significantly important compared to the sinusoidal lines (fundamental and harmonics).

Whenever such rotating machines are switched on and off, their frequency varies, in a continuous way, generating a vibration similar to a swept sine. This type of environment is primarily used in laboratory tests in order to carry out research into the resonance frequency of structures.

The mechanical excitations which are then analyzed, resulting from measurements of the environment or test laboratory, belong to one of the following groups:

– sinusoidal vibrations;

– swept sine vibrations;

– random vibrations;

– mechanical shocks;

or a combination of these vibrations:

– sine on random (one or several lines);

– a swept sine on random (with a sweeping on one or several frequency bands);

– a narrowband random vibration swept on a wideband noise, etc.

The vibrations which are produced in the real world have quite different frequency domains:

– between approximately 1 and 500 Hz for road vehicles;

– between approximately 10 and 2,000 Hz for airplanes and spacecraft;

– between approximately 1 and 35 Hz for earthquakes;

– more than 10,000 Hz for shocks which are created by metal–metal impacts, several tens of thousands of Hz for shocks which are created by pyrotechnic devices.

Vibrations are often classed into three different categories, depending on their frequency. The different categories are as follows:

– very low frequency for frequency values between 0 and 2 Hz;

– medium frequency for frequency values between 2 and 20 Hz;

– high frequency for frequency values between 20 and 2,000 Hz.

These values in conventional matter are given only as an indication and do not have any theoretical legitimacy. The low frequency concept can in fact be definite only according to the natural frequency of the system which undergoes the vibration. The frequency of a vibration will be low for a mechanical system if it induces any dynamic response (no attenuation and no amplification).

1.2. Some real environments

1.2.1. *Sea transport*

The sources of vibrations on board ships have various origins and natures. They are primarily due to:

– the propeller (periodic vibrations);

– the propelling unit and the auxiliary groups (periodic vibrations);

– the equipment used on board (for example, winches);

– the effects of the sea (random vibrations).

The measured levels are in general the lowest amongst all the means of surface transport.

1.2.1.1. *Vibrations produced by the ship's propeller*

The rotation of the propeller can excite the modes of the ship's frame in different ways:

– the accelerations transmitted to the hull via the line shafts;

– forces exerted on the ship's rudder;

– hydroelastic coupling between the propeller and the shafts' line;

– fluctuations in pressure distributed on all parts of the back hull, having as an origin the wake in which the propeller works. These fluctuations in pressure are dependent on:

- the variations of propeller's push. When the propeller provides a push, the back of each blade is subjected to a "negative pressure" (suction) compared to the environmental pressure, and the front face is subjected to an overpressure,

- the number, area and thickness of the blades. The fluctuations in pressure are a linear function of the average thickness of the blades and decrease very quickly when the number of blades increases,

- the presence of a variable vapor pocket on the surface of the blade and in its slipstream, as a consequence of cavitation.

Around the propeller is formed a cavity filled with vapor within the liquid, due to a local pressure lower than the saturating steam pressure. When the vapor bubbles reach higher pressure zones, they condense brutally. This phenomenon, known as cavitation, involves very strong mechanical actions (vibrations, noises, etc.).

Cavitation is the source of the majority of vibration problems encountered on ships. It is equivalent to an increase of the thickness of blades and, as a result, increases the pressure fluctuations. The variation of the volume of the cavitation pocket over time is a second source of pressure fluctuation. The fundamental frequency is around 20 Hz for fixed blade propellers from 5 to 6 m in diameter and 10 Hz for propellers from 8 to 10 m in diameter. The natural frequencies of the blades decrease when the diameter increases.

1.2.1.2. *Vibrations produced by the ship's engine*

The vibrations which are produced by a ship's engine are caused by the alternate movements of the piston, connecting rod and crankshaft systems.

They can excite the modes of the ship's frame, especially for medium-sized ships. Their vibratory frequency generally lies between 3 and 30 Hz.

1.2.1.3. *Vibrations produced by the state of the sea*

Vibrations due to the swell

The swell heave leads to the creation of vibrations of a long duration and of very low frequency (less than 2 Hz) in both the longitudinal (pitching) and transverse direction (rolling). These random oscillations are always of a seismic nature.

Their frequency varies between 0.01 Hz (when the sea is very calm) and 1.5 Hz (during bad weather). Their associated accelerations range from approximately 0.1 m/s^2 to 9 m/s^2.

Vibrations of the whole of the ship due to the state of the sea

In general, two types of vibrations are considered:

– hydrodynamic shocks applied to the front of the ship lead to the vibration of the whole of the ship, which works like a beam. This phenomenon occurs whenever the ship navigates the sea with its front first, with relative movements of the stem sufficiently significant to create impacts. These impacts can be distinguished as follows:

- shocks which are produced on the flat part at the bottom of the ship, when the ship makes contact with the sea, after it emerges from the water,

- shocks on planking of the stem, without emergence, without the ship resurfacing from the sea,

- areas of seawater;

– excitations which are caused by the swell's variable hydrodynamic forces, which lead to a steady state free vibration of the entire ship.

These vibrations generally have low or very low frequencies and, to a lesser extent, some can have high frequencies [VIB 06]. The frequencies range from 0.01 Hz to 80 Hz, with a maximum value of between 3 Hz and 30 Hz. The vibrations are periodic or random.

1.2.2. *Earthquakes*

The rapid release of the deformation energy which is accumulated in the Earth's crust or mantle (the underlying layer) is felt as a vibration on the Earth's surface: an earthquake. The vibration (the tremor) lasts in general for a few tens of a second. Their amplitude on the ground level can reach several m/s^2.

The shock response spectrum was created in the 1930s in order to group together the different effects that earthquakes of different amplitudes have on buildings. The amplitudes are taken from actual acceleration signals which were measured from real earthquakes (see Volume 2).

1.2.3. *Road vibratory environment*

The road transport vibratory environment is complex. It can be described as a mixture of permanent vibrations and discrete superimposed vibrations. The permanent part is comprised of variable proportions of the following types of vibrations:

– wideband noise, with a distribution of the instantaneous values which is generally Gaussian;

– very narrowband excitation with amplitude distribution very close to a Gauss law (for example, in response to a suspension);

– excitation of only one frequency and of constant amplitude (a poorly balanced rotor).

The discrete components can be recurring (i.e. with a periodicity), for example at the time of the passage of joints of a road made up of concreted plates, or intermittent (only one or some occurrences), for example during the crossing of a railway crossing.

Four main sources of vibrations can be distinguished: the suspension system, tires, the driving system and parts of the vehicle's framework [FOL 72]. The spectrum's characteristics depend on the state of the road or the type of terrain on which the vehicle is being used, the speed at which the vehicle is traveling and the vehicle's suspension.

The vehicle suspension generates vibrations at quite high amplitudes with frequencies between 3 and 6 Hz. The tires produce recurring components between 15 and 25 Hz. The engine and the driving train produce continuous excitation with frequencies between 60 and 80 Hz. The structural responses can range from 100 Hz to 120 Hz [FOL 72]. Other frequency domains can reach frequencies of up to 1,000 Hz according to the type of vehicle that is being used, due for example to the operation of electrical brakes.

The road vibratory environment is mainly made up of the following components:

– longitudinal movements which are linked to the acceleration and slowing down of a vehicle;

– lateral movements which correspond to driving around bends;

– vibrations which occur along the vertical axis, related to rolling on the road;

– longitudinal and lateral movements which are both associated with vertical non-symmetric excitation.

The first two environments are relatively weak and quasi-static. The last two are dependent on the state of the road. The frequencies of the spectrum can reach up to approximately 30 Hz, with low frequencies being able to produce large displacements. Frequencies larger than 30 Hz can also exist, being able to excite local resonances of structures [HAG 63]. The vibrations according to the vertical axis are generally dominant.

The rms acceleration of these vibrations ranges between 2 and 7 m/s^2 approximately [RIS 08].

The spectrum measured on the tracked vehicles is comprised of a random broadband noise and other higher energy bands of random vibrations which are created by the interaction of the caterpillar with the track and the toothed wheels. It is preferable to simulate these types of vibrations by using a swept sine on a wideband noise.

1.2.4. *Rail vibratory environment*

The permanent excitation measured during the rail transport is of a slightly smaller amplitude than that measured on the road [VIB 06].

The origin of the vibrations is primarily related to defects which exist on railway lines, for example, gaps between the rails, distance between the rails, switch point areas, etc. These examples are only a few of those that exist.

The vertical axis is in general the most excited, but the vibrations according to the transverse axes can also be severe, at least for particular frequency bands. The highest levels correspond to the frequency of the suspension (between 1 and 10 Hz), to the frequency of the train's framework (between 10 and 100 Hz) and to the areas where there are joints which hold the rails together (between 10 and 30 Hz). The switch point areas produce the strongest excitations [FOL 72] like the shocks between coaches during the process of putting the train together – attaching the wagons of the trains (the most severe levels of all types of surface transport).

1.2.5. *Propeller airplanes*

The vibrations measured on the propeller planes have a spectrum that is made up of a wideband noise and of several sinusoidal or narrowband lines. Wideband noise comes from the flow of air that occurs around the airplane and also from the multiple periodic components which are produced by all the elements in rotation in the propeller.

The peaks come from the flow of air that exists between the blades of the propellers, creating periodic aerodynamic pressure fields on the structure of the plane. The narrowbands are centered on a frequency which corresponds to the number of propeller blades multiplied by the engine’s rotation speed and on its harmonics.

The most visible lines are generally the fundamental frequency as well as the first two or three harmonics. The amplitude of these rays depends on the stage of the flight, i.e. take-off, ascent, cruise, landing, etc., and also depends on the point at which the measurement is taken.

The same spectrum can also be observed around the airplane’s engine. The majority of engines have an almost constant rotation speed. This rotation speed can be modified by supplying fuel to the engine, or by changing the angle of the propeller’s blades. The frequency of the peaks is also quite stable. Their width is linked to the small change in rotation speed and to the fact that the vibrations which are generated are not purely sinusoidal vibrations.

Other engines function with a more variable rotation speed. In this case, simulation in a laboratory is instead carried out by specifying a test defined by a swept sine on a wideband random vibration.

1.2.6. *Vibrations caused by jet propulsion airplanes*

1.2.6.1. *During take-off and ascent*

The strongest vibrations occur along the vertical axis of an airplane during its take-off and ascent. The weakest vibrations occur along the airplane’s horizontal axis.

Depending on the type of airplane, the typical frequency has a value of between 60 and 90 Hz, with a root mean square of about 5 m/s^2.

1.2.6.2. *The cruising phase*

The amplitudes of the vibrations are much lower during the cruising phase of the plane than is the case during the take-off and ascent phases. Nevertheless, the amplitudes remain stronger along the vertical axis. These values are much lower along the other two axes. There is also a constant frequency of between 60 and 90 Hz.

1.2.7. *Vibrations caused by turbofan aircraft*

We observe here a tendency towards a continuous rise of the levels of amplitude between 20 and 1,000 Hz, then a decrease of the amplitudes.

Once again, the strongest vibrations occur along the vertical axis and the weakest vibrations along the longitudinal axis. The vibration signal tends to be made up of a sine wave which is superimposed onto a wideband Gaussian noise.

This type of vibration occurs on the fighter airplane and is produced by many sources, including:

– the engine's noise which is then transmitted by the airplane's bodywork;

– aerodynamic flow;

– dynamic responses due to operations (airbrakes, missile launches, etc.).

In addition to these vibrations, shocks (which are sometimes severe) also occur, related to landing, taking-off, catapult-launchings, etc.

1.2.8. *Helicopters*

The vibrations which are produced by helicopters are made up of a random wideband noise and sinusoidal lines which are produced by the helicopter's main rotor, tail rotor and engine. The frequency of the sinusoidal lines does not vary much, the rotation speed of all of these components remaining relatively constant (variation of approximately 5%). The fundamental frequency which can be found in the sinusoidal lines corresponds to the rotation speed of the rotor and to its harmonic frequencies.

The amplitude of the lines is a function of the type of the helicopter and the point of measurement (proximity of the source).

The helicopter produces the most severe environment among all the means of air transport, producing high amplitudes at low frequency. The permanent random wideband component is very complex and has an extremely large amplitude.

The dynamic environment of the helicopters is different from that created by fixed wing airplanes. There is little difference here between take-off and cruising, and the amplitudes are generally larger.

The rotation speed does not vary much during flight for helicopters, except during hovering flight. Random vibrations (approximately Gaussian) are superimposed on sine lines, with a significant component at very low frequency. These lines are difficult to identify (frequency and amplitude) and extract. The amplitude of the rays varies depending on whether the vibration was recorded close to the rotors and engine, or not.

The vertical axis is in general the most severe. The fundamental frequency of the vibration depends on the rotation speed of the blades and also on the number of blades present.

The first component, between 15 and 25 Hz for the main rotor, is easily identifiable on the three axes and is more important according to the longitudinal and transverse axes [FOL 72]. The back rotor produces higher frequencies in general, between 20 and 100 Hz approximately, according to the type of apparatus and the number of blades.

The tail rotor tends to produce frequencies of a higher value, i.e. between 20 Hz and 100 Hz. These values depend on the type of helicopter and on the number of blades on the helicopter's propeller.

Example 1.1.

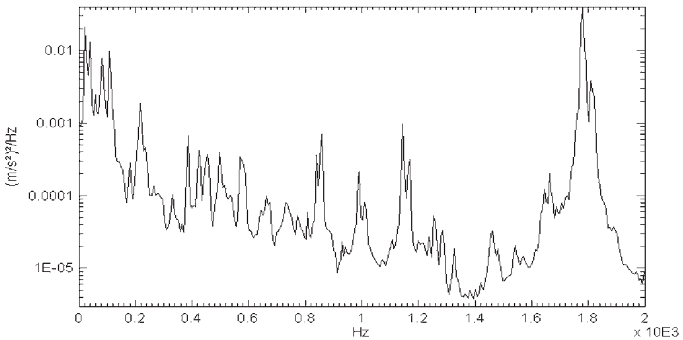


Figure 1.1. *Frequency contents of a vibration measured on a helicopter (power spectral density)*

1.3. Measuring vibrations and shocks

Different physical parameters can be *a priori* used for characterizing a vibration: an acceleration, a velocity, a displacement, a force or even a stress directly. All these parameters are measurable, but the most frequently used is undoubtedly acceleration. The main reason for this is due to the diversity of the different sensors which are available, their different acceleration and frequency ranges and their different sizes.

A sensor is an energy converter. Accelerometers are composed of a seismic mass suspended by an elastic element. Measuring the force F at which the mass m is subjected allows the acceleration G to be derived. The dynamic mass may carry compressive, bending or shearing forces. The different types of accelerometers differ in the force measurement principle.

Accelerometers are mechanically one-degree-of-freedom systems (see Figure 1.2). The system's mass response, which is subject to a certain level of acceleration applied at its base, will be studied in later chapters.

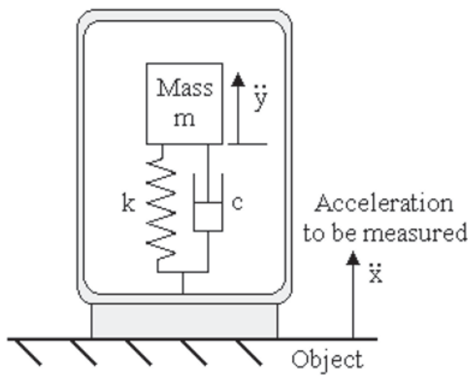


Figure 1.2. *Mechanical principle of an accelerometer*

Several physical principles are used to convert movement into an electric signal. These principles are as follows [ERE 99], [WAL 07]:

– the piezoelectric effect: a crystal which has a dynamic stress applied to it produced, in response to the acceleration which is to be measured, electrical charges which are converted into tension;

– a variation of capacity between two very near microstructures. This variation in capacity is also transformed into a variation of tension;

– the piezoresistive effect (change in resistance with acceleration);

– etc.

Figure 1.3. *Example of piezoelectric accelerometer (PCB 357B81, 2000 g, 20 pC/g, 9kHz shear ceramic) (courtesy of PCB Piezotronics)*

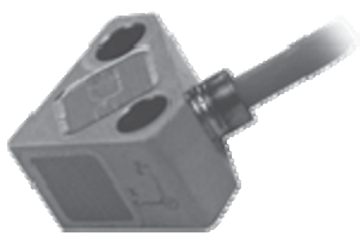


Figure 1.4. *Example of piezoresistive accelerometer (MEMS, 20000 g – 0 to 10 kHz – 2.83 g, -54 to 121°C, shock measurements) (courtesy of PCB Piezotronics)*

The resulting signal can be analog (continuous tension proportional to acceleration) or digital.

These sensors are characterized by their bandwidth (frequency domain, which is a function of the sensor's resonance frequency), by their effective range, by their sensitivity (V/g) and their size (or masses). Some make it possible to measure acceleration according to three axes.

Accelerometer	Advantages	Disadvantages	Field of Use
Piezoelectric	- Usable at high temperature (up to 700°C) - Generally low costs - Large measurement scale (from 10–5 to 105 g) - Sensitive to weak amplitude vibrations - Low volume - Response in a wide frequency, from 0.5 Hz to 40 kHz	- Does not filter DC components - Badly adapted to pyroshocks beyond 100,000 g	- Impulse or non-impulse vibratory phenomena - Characterization of structure and equipment behavior - Measurements at high temperatures - Seismic measurements - Shock measurements - Low frequency vibratory phenomena (vibratory comfort analysis)
Piezoresistive	- Does not filter DC components - Low volume - Adapted to the measurement of amplitude shocks (greater than 100,000 g)	- Temperatures lower than 130°C - More expensive than piezoelectric - Less sensitive to weak levels than piezoelectric	- Low amplitude and low frequency vibration acceleration measurements (up to several thousand Hz) - Shock measurements - Characterization of structures and equipment (quasi-static measurement): vehicle behavior, suspension during road tests, crash-tests, etc.
Capacitive	- Does not filter DC components - Very high resolution (up to 10^{-6} g) - High output signal	- Cost - Fragility - Volume - Temperatures lower than 150°C	- Low amplitude and low frequency inertial phenomena measurements - Examples: trajectory correction, stabilization of platforms, etc.

Table 1.1. *Advantages and disadvantages of the different types of accelerometers*

Advantages	Disadvantages	Applications
Large dynamic field Large frequency range Resistance to high level shocks Low supply costs Less sensitive to the electromagnetic environment Easy to use High impedance output Large length of cable possible without noise Output parameters fixed by construction	Field limited by temperatures of use Maximum: 170°C Integrated electronics subjected to same environment as sensor Low frequency response determined by construction	Modal analysis Motors In-flight tests Drop tests Earthquake behavior tests HALT/HASS Cold environment tests

Table 1.2. *Advantages and disadvantages of piezoelectric accelerometers with integrated electronics*

MEMS are Micro Electro Mechanical Systems which use small silicon surfaces (the material used for CMOS technology). They are measured in micrometers.

Theoretically, MEMS accelerometers do not have a zero derivative. One drawback with MEMS accelerometers in shock measurement is their considerable amplification at resonance (for example, 1000:1). This can lead to a rupture in response to high frequency inputs (for example, metal–metal impacts, pyroshocks, etc.). This defect can be improved by incorporating a small damping film.

Signal conditioners

Conditioners are used to carry out a load/voltage or voltage/voltage conversion, with an amplification and attenuation gain. Some conditioners also make it possible to integrate the signal in order to obtain at the output velocity or displacement signals. Signal pre-filtering functions often enable us to optimize the signal before saving and/or analysis.

Measurements must be carried out under real conditions of use if possible, for example, the same vehicle (if the material is embarked), the same interfaces, etc.

Some simple rules must be respected:

– the vibration should be measured at the input of material, the sensor being placed onto its support near an area which is very close to the material's fixation point, preferably on the most rigid surface available [STA 62]. It would be best to avoid placing the sensor on a sheet of metal or on the hood, etc.;

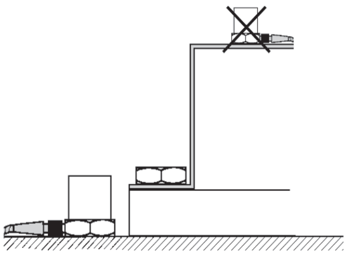


Figure 1.5. *Position of the sensor for measuring vibrations experienced by equipment*

– a sufficient number of sensors should be used so that a better understanding of how the material works is obtained. However, caution is required, as we do not want to have too many sensors present in order to avoid modifying its mechanical behavior.

It is important to evaluate the representativeness of measurement compared to the physical phenomenon. Is one measurement enough? Does the variability of the results require the realization of several recordings, statistical processing, etc.?

1.4. Filtering

1.4.1. *Definitions*

Filters are used to remove components of undesired frequencies in a measured signal, shock or random vibration. They can also be used to extract the useful components of a signal in a given frequency domain. The filter transfer function (the ratio of the response divided by the input to each frequency) should have a value of 1, or as close to 1 as possible for the frequencies which are to be kept. For all of the other frequencies this value should be zero. The transition zone needs to be as small as possible.

There are two types of filters that exist:

– analog filters. These filters use electronic circuits. The original signal is analog (current, tension), such as the filter's response signal, or filtered signal as it is otherwise known. Examples of such filters include the Butterworth filter, the Tchebycheff filter and the Bessel filter;

– digital filters. Using these filters makes it possible to process signals which have already been digitized and which rely on the use of data processing calculations.

1.4.1.1. *Low-pass filter*

A low-pass filter is a filter which lets low frequencies pass through the filter without making any modification to them. This type of filter then rejects frequencies which have a value of more than f_c. This frequency is known as the cutoff frequency.

An ideal low-pass filter has a constant gain of 1 in its frequency range, and a zero gain in its stop band. For the frequency values of between zero and f_c, the shape of this filter is rectangular. In practice, the transition from a value of 1 to a value of zero is done with a more or less important slope according to the quality of the filter.

The most simple analog low-pass filter (order 1 filter) has the characteristic

$$H(j\,f) = \frac{1}{1 + \dfrac{j\,f}{f_c}} \qquad [1.1]$$

The gain equals

$$\left|H(j\,f)\right| = \frac{1}{\sqrt{1 + \left(\dfrac{f}{f_c}\right)^2}} \qquad [1.2]$$

where f_c is the *cut-off frequency*, the frequency for which the gain has decreased by 3 dB.

We use the larger n-order filters instead, in which the gain is given by (Butterworth filter):

$$\left|H(j\,f)\right| = \frac{1}{\sqrt{1+\left(\frac{f}{f_c}\right)^{2n}}} \qquad [1.3]$$

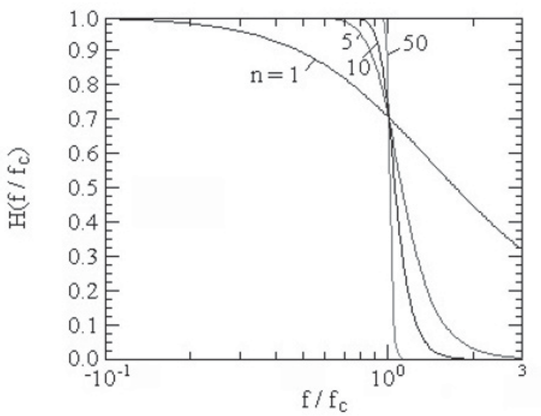


Figure 1.6. *Low-pass filter – gain versus* $f\ /\ f_c$ *, for different values of the filter order n*

The larger the order of the filter, the quicker the return to zero (Figure 1.6). It is easy to show than the decrease slope is about equal to –6 n dB / octave. A 20 order filter is therefore necessary to obtain a decrease of –120 dB / octave.

1.4.1.2. *High-pass filter*

A high-pass filter is a filter which lets high frequencies pass through the filter, and rejects the low-value frequencies which have a value that is less than the cutoff frequency. An ideal high-pass filter has a constant gain of 1 for frequencies which are greater than f_c and a zero gain for frequencies which are lower than f_c .

The n-order filter gain is as follows

$$\left|H(j\,f)\right| = \frac{\left(\dfrac{f}{f_c}\right)^n}{\sqrt{1+\left(\dfrac{f}{f_c}\right)^{2n}}} \qquad [1.4]$$

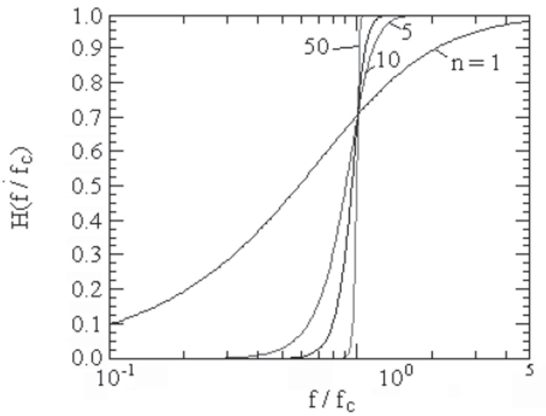


Figure 1.7. *High-pass filter - Gain versus* $f\ /\ f_c$*, for different values of the filter order n*

1.4.1.3. *Band-pass filter*

A band-pass filter is a filter which only lets frequencies within a certain range pass through the filter. This range includes frequencies which are greater than the low cutoff frequency and which are lower than the high cutoff frequency. The ideal filter gain is zero for all frequencies except for the frequencies which can be found in this particular range. Here the value of the filter gain is 1.

1.4.1.4. *Band-stop filter*

A band-stop filter is a filter which prevents some frequencies, which can be found in a certain interval, from passing through the filter.

The band-stop filter is made up of a band-pass filter and a high-pass filter, and whose cutoff frequency is greater than the cutoff frequency of the low-pass frequency. The band-stop filter can be used to remove any parasite frequencies.

1.4.2. *Digital filters*

The digital filters can be grouped into two different categories:

– Finite impulse response (FIR) filters. These filters are said to be finite because their impulse response is stabilized ultimately to zero. The response which is provided by these filters depends entirely on the entry signal. There is no counter-reaction. FIR filters are said to be non-recursive. Each point of the filtered signal is calculated from the entry signal at the same time and also from preceding points of the signal. These filters are always stable.

The method used consists of numerically carrying out filtering by means of a convolution product, which makes it possible to produce any filter, but requires longer calculations.

Its specifications must specify:

- the ripple ratio in the passing band,
- the all-off rate in the rejected band,
- the width of the transition band.

– Infinite impulse response (IIR) filters. These filters use analog filtering techniques. Their impulse response does not settle. This type of filter is said to be recursive: the response which is provided by this type of filter depends on both the input signal and the output signal because of the existence of a feedback loop. Each point from the filtered signal is calculated from the original signal at the same time, from the amplitudes of the preceding points of the original signal and from the preceding values of the filtered signal. These filters require fewer calculations to be carried out in comparison to their FIR equivalents.

The response of a digital filter can be written as follows:

$$y(n) = \sum_{j=0}^{N} a_j \; x(n-j) - \sum_{k=0}^{M} b_k \; y(n-k) \qquad [1.5]$$

where a_j and b_k are coefficients, x is the current point of the original signal (the input signal) and y is the current point of the filtered signal (the output signal).

The b_k coefficients have a value of zero for the FIR filters.

The order of a non-recursive filter is the largest number of values of the original signal that are necessary to calculate one point of the filter's response.

The order of a recursive filter is equal to the largest number of values from the original signal from the response which is taken into account in this calculation. In general, the number of values considered in the original signal and the response is the same. Thus, each point of index n of the response of the second order filter is calculated starting from the last two points of the original signal (i.e. indices n–1 and n) and of the two preceding points of the response (indices n–2 and n–1).

The slope of the filter at its cut-off frequency is dependent on the order of the filter:

$$\text{Slope in dB/oct} = 6 \times \text{Order} \tag{1.6}$$

If no particular precaution is taken, it is possible that the filters might introduce a type of phase difference (or delay) when compared to the original signal. It is possible to remove this dephasing during the calculation of the response.

Advantages and disadvantages of digital filters

Advantages	Disadvantages
Not sensitive to environmental conditions (temperature, humidity, etc.)	Filtering limited to 100 MHz
Can process low frequency signals with precision	Analog to digital conversion necessary
Designed and tested directly on a computer	Requires an analog anti-aliasing filter for sampling and restitution
As they are programmable, their characteristics can be changed easily without changing the hardware	Performance of the filter directly proportional to the power of the calculation unit (processor or DSP)
No problem with deriving their components	
Some filters can only be realized digitally (FIR)	
Known and controlled precision	
Reproducible without fine-tuning	

*Advantages and disadvantages of FIR (**F**inite **I**mpulse **R**esponse) filters*

These non-recursive filters have no feedback.

Advantages	Disadvantages
Always stable Linear phase coefficient symmetry No phase distortion Possible to create all sorts of filters (through calculation of the inverse Fourier transform from a gauge in the frequency range)	Larger calculations with respect to an equivalent IIR filter Delay of the filter can be significant

Advantages and disadvantages of IIR (Infinite Impulse Response) filters

These recursive filters have feedback.

Advantages	Disadvantages
Much less calculation with respect to an FIR	Need to check stability Nonlinear phase (phase distortion)

1.5. Digitizing the signal

In order to be processed by a computer, the measured signals must be digitized and represented as a time–amplitude couple. How is it possible to choose the number of points per second that need to be digitized, i.e. to choose the sampling frequency?

Digitization consists of:

– sampling, which consists of representing an analog signal using a series of n values quantified at integer multiple instants of a time interval δt, the sampling period;

– and quantization, which consists of approaching each value of the signal using an integer multiple of a basic quantity Δ, called the quantization step.

1.5.1. *Signal sampling frequency*

In 1920, H. Nyquist, from Bell Laboratories, was the first person to demonstrate, without any practical application, that “if a function does not contain any frequency which is larger than f_{max} Hz, then it is completely determined by sampling it with a frequency equal to 2 f_{max}” [SHA 49].

This theory is often associated with Claude Shannon, who worked in the same laboratory. It was Shannon who in 1948 used this theory once again, but this time on applications which were part of the world's first computers.

If we want to analyze any signal with a frequency value of up to f_{max}, it is therefore necessary to make sure that there are no frequencies which have a value that is greater than the value of f_{max} before it is finally digitized at a value of 2 f_{max}. These frequencies can sometimes resemble a real physical object or can simply be a noise. In Volume 3 we will see that these frequencies lead to a phenomenon known as spectrum folding (or aliasing). As far as this phenomenon is concerned, the signal is filtered with the help of a low-pass analog filter, whose cutoff frequency value is f_{max}.

NOTE.– *The Nyquist frequency can be shown as* $f_{Nyquist} = f_{samp.} / 2$.

Thus, it should be considered that the true contents of the filtered signal extend to the frequency corresponding to this attenuation (f_{-40}), which is calculated as follows.

In practice, however, the low-pass filters are not perfect as they do not always reject the frequencies which are above the requested cutoff level. Let us take the example of a low-pass filter which decreases by 120 dB per octave once the cutoff frequency has been passed It is estimated that the signal is sufficiently attenuated with –40 dB. Thus, it should be considered that the true contents of the filtered signal extend to the frequency corresponding to this attenuation (f_{-40}), which is calculated as follows [1.7].

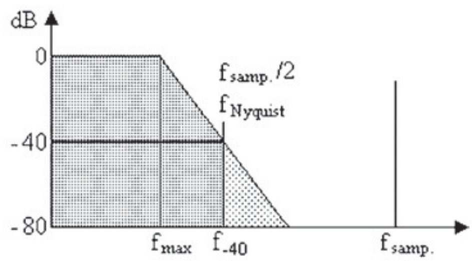


Figure 1.8. *Taking into account of the real characteristics of the low-pass filter for the determination of the sampling rate*

A reduction of 120 dB per octave means that:

$$-120 = \frac{10 \log \frac{A_1}{A_0}}{\log \frac{f_1}{f_0}} \log 2 \qquad [1.7]$$

where A_0 and A_1 are the amplitudes of the non-reduced signal (with a frequency of f_{max}) and the reduced signal to −40 dB (with a frequency of f_{-40}) respectively.

This yields:

$$-120 = \frac{10\,\frac{-40}{10}}{\log \frac{f_{-40}}{f_{max}}} \log 2 \quad [1.8]$$

and:

$$\frac{f_{-40}}{f_{max}} = 10^{\frac{\log 2}{3}} \approx 1.26 \quad [1.9]$$

If f_{-40} is the largest frequency signal, then according to Shannon's theorem we obtain the following equation: $f_{samp.} = 2\,f_{-40}$, i.e.:

$$\frac{f_{samp.}}{f_{max}} \approx 2.52 \quad [1.10]$$

f_{-40} is the Nyquist frequency and is written as $f_{Nyquist}$.

A number like 2.5 times would be adequate, but in order to comply with the computer world, 2.56 is usually the number employed (sometimes 2.6) [BRA 11], [SHR 95]. This result has sometimes led us to state that Shannon's theorem imposes a sampling rate equal to 2.6 times the largest frequency of the signal to be analyzed.

Using this theorem makes it possible to determine the minimum sampling frequency that is required, so that a signal keeps its full frequency contents.

According to this theorem, the sampled signal possesses all of the characteristics of the original signal without any loss of information. This means that it is possible to reconstruct the original signal from the sampled signal (see section 1.5). However, the sampled signal tends not to have the same effects on a mechanical system when it is compared to the original signal.

Example 1.2.

Let us consider the sinusoid from Figure 1.9. The sinusoid has a frequency of 100 Hz and is sampled with a sufficiently large frequency to represent the signal correctly. Figure 1.10 shows the same sampled sinusoid which is sampled at a frequency of 200 Hz (two times the frequency of the sinusoid). The signal's frequency can be read without ambiguity, but the signal is very deformed. It is easily understood that it will not have the same effects on any mechanical system on which it will be applied.

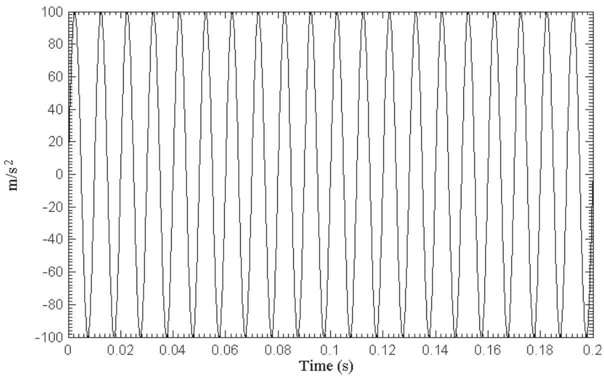


Figure 1.9. *Sampled sinusoid (100 Hz) with a frequency of 8,000 Hz (1,600 points over 200 ms)*

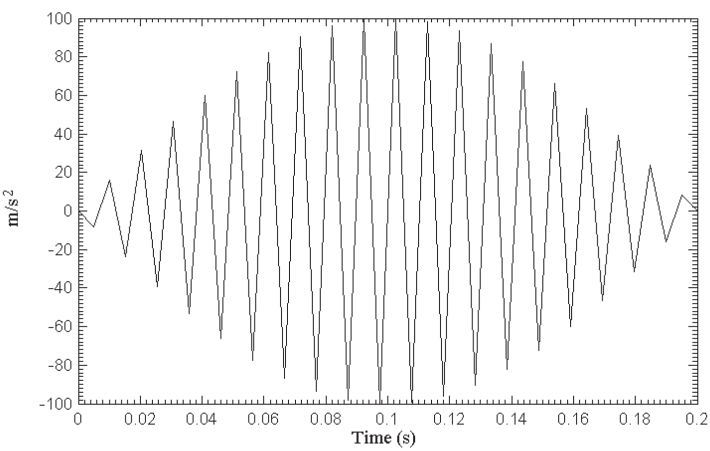


Figure 1.10. *Sampled sinusoid (100 Hz) with a frequency of 200 Hz (40 points over 200 ms)*

A.G. Marshall and F. R. Verdun [MAR 90] have shown that it is necessary to sample a signal with a frequency equal to 20 times its maximum frequency to be able to correctly reproduce its initial form. T. E. Rosenberger and J. DeSpirito [ROS 93] proposed to use a factor of 5 as a set standard.

The best practice today for each signal which is used to digitally calculate the responses produced by a mechanical system is to sample it with a frequency that is:

– ten times larger than the mechanical system's natural frequency for shocks (Volume 2);

– seven times higher than the signal maximum frequency if it is a vibration (Volume 5).

In Volume 3 we will see that Shannon's sampling frequency is sufficient for the calculation of power spectral densities.

1.5.2. *Quantization error*

The variation field of the signal $\left[-X_m, X_m\right]$ is divided into intervals of width Δ.

A signal x(t) is quantified correctly (without clipping) with a converter on n bits if its amplitude x_m is in the interval $\left[-X_m, X_m\right]$ where $X_m = 2^{n-1}\,\Delta$. In the opposite case, the signal will be clipped [HAY 99].

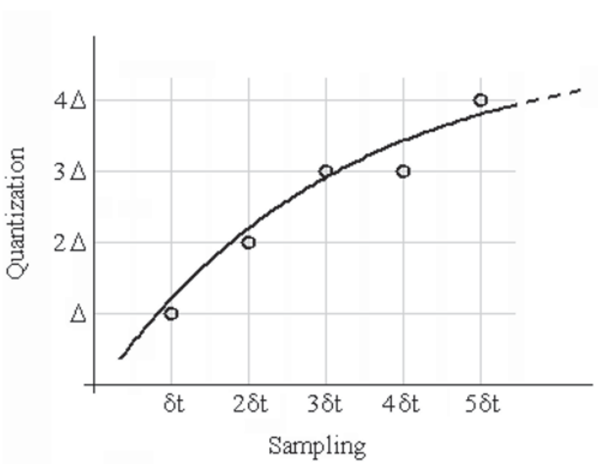


Figure 1.11. *Sampling and quantization*

Δ is called the *quantization step size* or the *resolution of the quantizer*, and the quantizer is said to be a *uniform* or a *linear* quantizer.

We have

$$\Delta = 2\,X_m\,10^{\frac{\text{resolution (dB)}}{20}} \qquad [1.11]$$

Each signal value can thus be written as

$$x = \sum_{i=0}^{n-1} a_i\,2^i \qquad [1.12]$$

where a_i is equal to 0 or 1.

This operation cannot be carried out without error. The difference between the actual analog value and quantized digital value is called the quantization error. This error is either due to rounding or truncation.

Assume that each error is independent of the rest, and that the error amplitude is evenly distributed in the range $-\Delta/2$ to $\Delta/2$, where Δ is the step in the analog-to-digital converter (ADC) process, its probability density p(x) being equal to $1/\Delta$.

We may then calculate the mean square value of the error:

$$\sigma^2 = \int_{-\infty}^{\infty} x^2\,p(x)\,dx = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} x^2\,\frac{1}{\Delta}\,dx = \frac{\Delta^2}{12} \qquad [1.13]$$

The noise standard deviation (rms quantization error) is equal to $\sigma = \dfrac{\Delta}{2\sqrt{3}} \approx 0.29\,\Delta$, i.e.

$$\sigma = \frac{2\,X_m}{2^n\,\sqrt{12}} \qquad [1.14]$$

Figure 1.12 shows the variations of this error as a function of the number of bits n for different values of X_m.

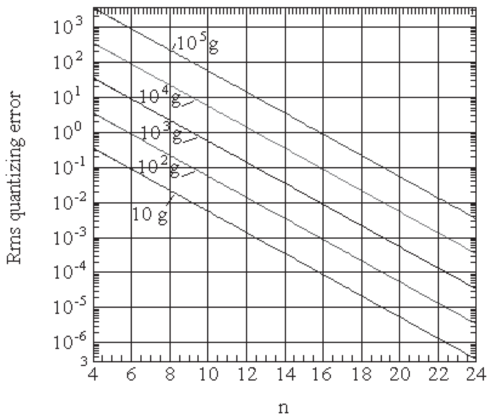


Figure 1.12. *Rms quantizing error versus bits number n*

Number of bits	8	10	12	16	20	24
Number of levels	256	1024	4096	65536	$1.05\ 10^{6}$	$1.68\ 10^{7}$
Absolute error (mV)	40	10	2.5	0.15	0.01	0.0006
Relative error (%)	0.4	0.1	0.025	$1.5\ 10^{-3}$	$1\ 10^{-4}$	$6\ 10^{-6}$

Table 1.3. *Quantization error for an input range of the analog-digital converter of 0 V to 10 V*

The quantization effects can be reduced by a low-pass filtering of the digital signal [BAC 87], with a cut-off frequency a little larger than that of the filter used before the digitization (anti-aliasing filter).

If n is the number of binary bits in the converter, the dynamic range is given by

$$D_R = \frac{\text{rms of largest sine}}{\text{rms of quantisation noise}} = \frac{\dfrac{\Delta}{\sqrt{2}}}{\dfrac{\Delta}{\sqrt{12}}} = \sqrt{6}\,\frac{\Delta}{\dfrac{2\,\Delta}{2^n}} = \sqrt{1.5}\;2^n \qquad [1.15]$$

i.e. in decibels:

$$D_R = 20 \log_{10}(\sqrt{1.5}\ 2^n) \approx 1.76 + 6.02\ n \qquad [1.16]$$

Today, current ADC have 24 bits.

n	11	12	14	16	18	20	22	24
D_R	68	74	86	98	110	122	134	146

Table 1.4. *Dynamic range versus bit number*

Example 1.3.

Let us consider a pyroshock measured with a sensor ± 100 000 g. With an ADC 11 bits (+ sign bit), the quantization step is equal to 200 000 / 2^{11} = 97.6g.

Influence on the calculation of a PSD

The error related to the quantization appears as a white noise having a PSD of amplitude [BAC 87]:

$$e_{PSD} = \frac{X_m^2}{3\ f_{samp}\ 2^{2n}} \qquad [1.17]$$

where f_{samp} is the sample rate of the signal.

1.6. Reconstructing the sampled signal

Sampling a signal transforms a continuous analog curve into a series of points. Shannon’s theorem states that the sampling frequency must be equal to twice the largest signal’s frequency. This sampling leads to the creation of high frequencies.

It is possible to reconstruct the signal by removing these high frequencies by applying a rectangular window into the frequency domain (a low-pass filter), and at the same time by increasing the number of points of the signal [LAL 04], [SMA 00], [WES 10]. This can be carried out using the following remarks.

The inverse Fourier transform of a rectangular window becomes a function in the form sin x/x in the time domain.

Let us suppose that the functions mentioned below are continuous. Consider a function defined in the frequency interval – f_{max}, f_{max} (after a low-pass filtering if the studied signal refers to a measurement) by n points with a sampling rate of $f_{samp.} \geq 2\, f_{max}$.

If we only consider the physical case in which frequencies only have positive values then this function can be expressed in the form of a Fourier integral:

$$\ddot{x}(t) = \frac{1}{2\pi} \int_0^{\Omega_{max}} \ddot{X}(\Omega)\, e^{i\Omega t}\, d\Omega \qquad [1.18]$$

where $\Omega = 2\pi f$ and $\Omega_{max} = 2\pi f_{max}$.

In this frequency band, the function $\ddot{X}(\Omega)$ can be developed into a Fourier series:

$$\ddot{X}(\Omega) = \sum_{n=0}^{\infty} a_n\, e^{-\frac{i n \Omega}{\Omega_{max}}} \qquad [1.19]$$

yielding:

$$\ddot{x}(t) = \sum_{n=0}^{\infty} \frac{a_n}{2\pi} \int_0^{\Omega_{max}} e^{i\Omega(t - t_n)}\, d\Omega \qquad [1.20]$$

where $t_n = \dfrac{n\pi}{\Omega_{max}}$.

After integration:

$$\ddot{x}(t) = \sum_{n=0}^{\infty} \frac{a_n}{\pi} \frac{\sin \Omega_{max}(t - t_n)}{t - t_n} \qquad [1.21]$$

Since:

$$\lim_{t \to t_j} \ddot{x}(t) = \frac{a_j \, \Omega_{max}}{\pi} \qquad [1.22]$$

it becomes:

$$\ddot{x}(t) = \sum_{n=0}^{\infty} \ddot{x}(t_n) \frac{\sin \Omega_{max} (t - t_n)}{\Omega_{max} (t - t_n)} \qquad [1.23]$$

Knowing that $f_{max} = \frac{f_{samp.}}{2}$ and that the signal's temporal step is equal to $\delta t = \frac{1}{f_{samp.}}$, this expression can be written as:

$$\ddot{x}(t) = \sum_{n=0}^{\infty} \ddot{x}(n \, \delta t) \frac{\sin\left[\frac{\pi}{\delta t}(t - n \, \delta t)\right]}{\frac{\pi}{\delta t}(t - n \, \delta t)} \qquad [1.24]$$

In order to reconstruct the signal at a given time t, the procedure thus consists of centering a function of the form sinc = sin x/x on each point of the signal and adding all the sinc functions thus defined [BRA 11].

Theoretically, in order to perfectly reconstruct a signal, it is necessary for the signal to have an infinite number of points. In practice, the number of sampling points is necessarily limited and its sum of all of these functions is truncated. Due to this fact, the reconstructed signal can differ slightly from the original signal. This is, however, only a small error which can be ignored whenever the initial sampling frequency is multiplied by 10.

Example 1.4.

Consider a sinusoid which has an amplitude of 100 m/s^2 and a frequency of 100 Hz. The sinusoid is sampled with a frequency of 250 points/s (50 points over 0.2 s).

The signal is reconstructed using equation [2.7]. The number of points of the new curve is multiplied by 20 (i.e. 1,000 points over 0.2 s). The reconstructed signal is compared with the signal sampled with 50 points in Figure 1.13 and, just like a reference, the reconstructed signal is also compared with the original sinusoid which has a very large sampling frequency (5,000 points/s).

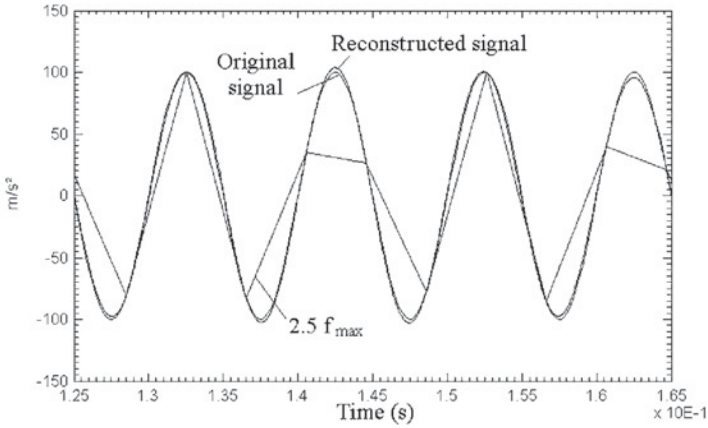


Figure 1.13. *Sampled sinusoid with a frequency of 250 Hz, which is superimposed on a reconstructed signal and on the original signal*

The reconstructed signal is very similar to the ideal signal.

1.7. Characterization in the frequency domain

The recorded signal is generally made up of several types of successive signals, such as random stationary vibrations, shocks, non-stationary vibrations, etc. It is necessary to split the signal so that, with the appropriate mathematical tools, it becomes possible to study the individual components of the signal.

The mechanical shocks are generally characterized by the effects they have on a one-degree-of-freedom linear system according to its natural frequency, i.e. the *shock response spectrum* (see Volume 2).

The frequency content of the random vibrations is studied, when they are stationary, by using a spectrum called *power spectral density* obtained by taking the average of all the Fourier transforms of several samples of the signal (see Volume 3).

Vibrations, just like shocks, can be analyzed by using another spectrum, the *extreme response spectrum*, giving the largest response of a linear one-degree-of-freedom system over the studied duration (see Volume 5).

If we take the duration of the vibrations (which can be quite long) into consideration, they are capable of damaging the mechanical parts of a system by the fatigue which is created by the repetition of stress cycles (see Volume 4). To take this mode of failure into account, a second spectrum is defined, the *fatigue damage spectrum*, which gives the fatigue damage experienced by this same one-degree-of-freedom system according to its natural frequency when it is subjected to the vibration for a given duration of time. These two spectra can be calculated for any type of vibration, for random stationary and non-stationary vibrations in particular or for a large number of repeated shocks (Volume 5).

1.8. Elaboration of the specifications

The dimensioning of a material and the realization of a qualification test with this material require environmental specifications which can result from normative documents or are developed from measurements of the real environment. The MIL STD 810 standards in the USA, GAM EG 13 in France and the international NATO standard recommend this last method, called "*test tailoring*". This approach involves:

– analyzing the conditions in which a material is used (life profile);

– linking environment measurements with each of the different conditions in which the material is used;

– synthesizing all the data thus joined together; and

– for tests, establishing the test program in the most representative and least expensive way.

Each of these operations which make up the four step approach is extremely important, but the most technical is the synopsis which will lead us, for the vibrations, to define a test of the same severity as all vibrations and shocks of the life profile. This test must be able to produce the same failures in the material that would also be created if the material were to be used in a real environment.

Two different synopsis methods exist nowadays. One of these methods involves using envelopes from power spectral densities, whilst the other method aims to reproduce the largest instantaneous stress which is produced by the vibrations, as well as the fatigue damage which is caused by the accumulation of all the different stress cycles. Volume 5 will deal with the second of these methods, which is based on the behavioral laws of fatigue of materials described in Volume 4. As the structure is generally not known at the time of the writing of specifications, the search for a specification respecting these two criteria is carried out by studying the response of a simple mechanical system, the one-degree-of-freedom linear system already used to characterize shocks. This choice highlights the advantage of standardizing the methods that are used to analyze vibrations and shocks.

1.9. Vibration test facilities

1.9.1. *Electro-dynamic exciters*

1.9.1.1. *Principle*

An electro-dynamic exciter converts electrical energy into mechanical energy. The force which is generated on the table supporting the specimen to be tested is produced by the presence of a constant magnetic field which acts upon a conductor coil. The conductor coil is linked to the table and has a variable current that runs through it. The conductor being placed perpendicularly to the magnetic field is subjected to a force perpendicular to the flow and the current.

The constant magnetic field in the air-gap where the coil moves is created by a DC current circulating in two fixed coils.

1.9.1.2. *Main components*

An electro-dynamic exciter is made up of:

– a table supporting the specimen to test, made from an aluminum alloy. This table is connected to an armature by suspension, which makes it possible for the table to move in the axial direction, minimizing the movements in the other directions;

– a mobile coil which is firmly attached to the table and which is placed inside the magnetic circuit's air-gap. This coil is guided using hydrostatic bearings;

– an armature, which forms the polar parts of the magnetic circuit;

– field coils;

– a fixed frame to which the exciter is connected by using two pivots allowing its rotation (for the large exciters).

However, a certain number of extra components are required within the electro-exciter, such as water circulating pumps for the cooling process, different security devices, a control system, etc.

The exciter is installed in a seismic mass with the aim of protecting the room from the vibrations.

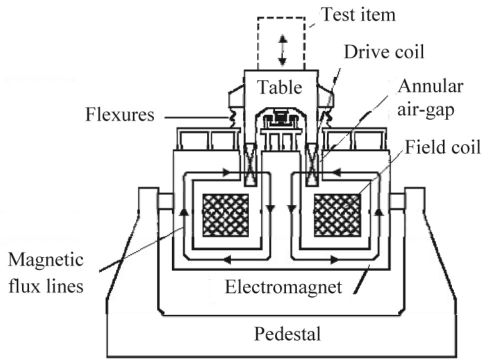


Figure 1.14. *Composition of an electro-dynamic exciter*

1.9.1.3. *Moving assembly*

The moving assembly includes:

– the specimen-holder table which is made out of a cast aluminum alloy. The mobile coil is firmly attached to the table. The table is connected by eight tighteners to a central tube guided by using two hydrostatic bearings;

– the mobile coil which is made up of two superimposed coils:

- the interior coil is made out of hollow aluminum and has a variable current running through it. This coil is cooled by water. It is this coil which transforms electrical energy into mechanical energy,

- the exterior coil is stuck onto the main coil. The exterior coil has a DC current running through it, intended to compensate for the axial loadings;

– the mounting fixture and the test specimen.

1.9.1.4. *Control system*

To obtain a given acceleration on the table at the specimen input, it is necessary to generate an electrical signal which takes account of the transfer function of the facility, the non-linearities of the shaker, the dynamic interactions, the fixture, etc.

The compensation for the transfer function is obtained from feedback making it possible to create the required level of acceleration on the table according to the frequency.

The acceleration signal which is to be generated is sinusoidal, random or a shock.

The control system, which was originally analog, is nowadays digital.

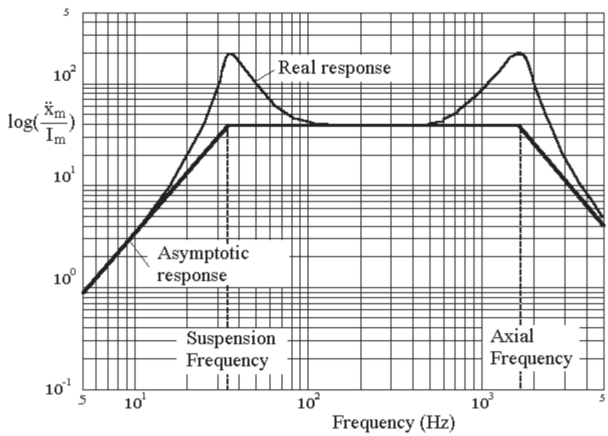


Figure 1.15. *The acceleration/current transfer function of an exciter*

Figure 1.16 shows a diagram which highlights the way in which an exciter provides feedback.

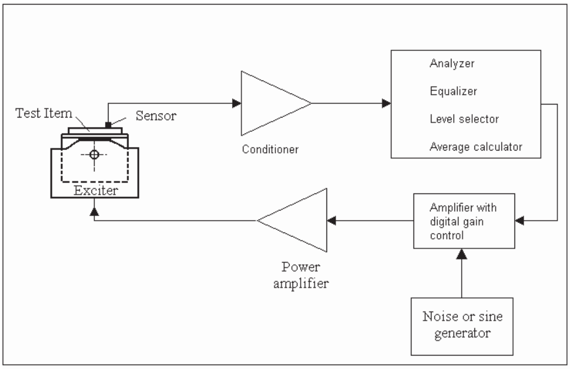


Figure 1.16. *Diagram showing the principle of the feedback process*

1.9.1.5. *Main characteristics*

The *maximum force* which is generated is generally defined by a peak value for sinusoids and by a root mean square for random materials. As far as the random materials are concerned, in order to ensure that the test is reliable, it is necessary to have a root mean square which is approximately 5.5 times smaller than the maximum force.

The *mass of the moving assembly* includes the masses of the table and of the coils, the masses of the mounting fixture and of the specimen. The moving assembly mass limits the maximum value of the specimen's acceleration. Other characteristics include:

– the *maximum mass* which can be dealt with without any external compensation;

– the *maximum couple* which can be applied to the moving assembly by a horizontal charge;

– the *maximum displacement* that can take place between mechanical stops;

– the *maximum velocity*;

– the maximum frequency range.

1.9.1.6. *The horizontal table*

The exciter's axis is generally the vertical axis. When a specimen needs to experience a vibration in any other direction, the exciter's axis is changed by turning the specimen over in order to vertically place the new test axis, using a square fixture to keep the table horizontal.

If the specimen is too heavy, it is best to keep its axis in a vertical position. The solution therefore involves turning the exciter around (using its pivots) so that it is possible to slide a horizontal table onto a thin layer of oil (see Figure 1.17).

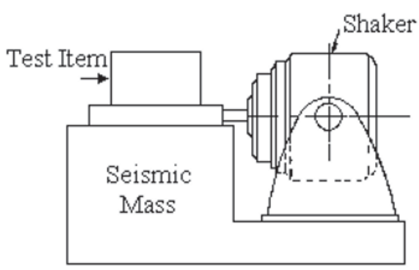


Figure 1.17. *Assembly with a horizontal table*

1.9.2. *Hydraulic actuators*

1.9.2.1. *Principle*

Electro-hydraulic vibration systems are remote power transmitter systems which use a low pressure fluid that is not very compressible.

These vibration systems are made up of three main parts:

– the generator of pressure (pump), which receives the energy of the external medium (electrical motor) and which communicates it to the fluid;

– the receiver (the actuator), which receives the energy of the fluid and restores it in the external medium;

– focal points, which exist between the pump and the receiver (tubes, valves, etc.).

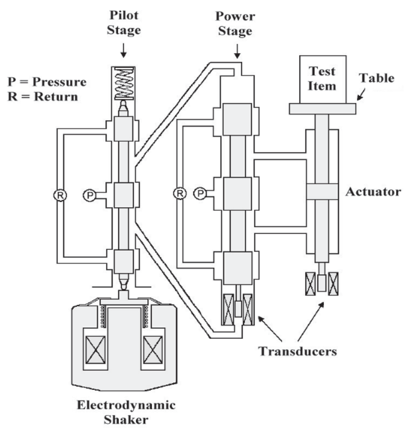


Figure 1.18. *How a hydraulic actuator works*

1.9.2.2. *Description*

The hydraulic actuator is made up of:

– a hydraulic power unit which supplies oil throughout the jack with the help of several pumps that are equipped with a cooling system and an oil reservoir (example: a flow of 600 l/min for 210 bars);

– an electro-hydraulic exciter which converts electrical energy into mechanical energy with the help of a hydraulic amplifier. The electro-hydraulic exciter receives its commands from a servo-valve;

– the servo-valve is responsible for supplying oil within the actuator. The servo-valve is made up of an electro-dynamic exciter attached to the servo-valve distributor;

– a double acting jack made up of a sliding piston in a cylinder, receiving oil on its two sides from the servo-valve distributor.

The piston is guided by hydrostatic bearings at the ends of the cylinder.

1.9.2.3. *How the hydraulic actuator functions*

The servo-valve distributor, which is connected to the mobile coil of the electro-dynamic exciter, moves in relation to the current which runs through the mobile coil.

The distributor's main high pressure supply is connected to one of the pipes that supplies one of the jack's chambers with oil. The other chamber is connected to the low pressure hydraulic return.

Due to the difference in pressure that exits on its two sides, the piston moves at a speed which is proportional to the opening of the pipes in the servo-valve's casing.

1.9.2.4. *Principal characteristics*

– The *maximum force* generated (for example, 120 kN). At higher frequencies, the performances in acceleration are limited. This limitation is due to the maximum dynamic effort which is allowed, and also due to the effects of the hydraulic natural frequency.

– The *mass of the moving assembly,* including the masses of the table and of the piston, the mass of the mounting fixture and of the test specimen. The moving assembly's total mass limits the maximum value of the specimen's acceleration.

– The *maximum displacement*, e.g. of 10 cm (limitation at low frequencies).

– The *maximum velocity*, e.g. of 1.56 m/s. In the medium frequency range, the velocity is limited by the maximum flow of oil throughout the system.

– The *frequency range* (for example, between 0.1 and 300 Hz).

1.9.3. *Test Fixtures*

It is generally impossible to fix a test object directly to the shaker table itself. The fixture acts as a transition piece between the two. They are generally used to enable us to carry out tests in the three directions.

The real vibratory environment is generally tri-axial. Tests are usually carried out axis by axis, basically due to the cost of tri-axial testing installations. To reduce parastic vibrations as far as possible on the two axes perpendicular to the axis under test, the rule is to place the center of gravity of the specimen and the assembly over that of the testing table.

In real service conditions, equipment is often fixed onto structures which may to a greater or lesser extent deform under the vibrations according to the mass of the specimen. Ideally, the assemblies should reproduce the real fixture conditions, such as stiffnesses, support masses (mechanical impedances). However, these characteristics are generally not specified are not even known.

The assemblies are thus designed instead to be as rigid as possible in order to transmit uniformly to the specimen the forces produced by the exciter at all its fixation points. They are designed so as to not deform the spectrum that will be applied to the specimen. It is thus necessary *a priori* that their resonance frequencies be larger than the maximum specification frequency. It is however difficult to completely suppress the resonance frequencies between 1000 and 2000 Hz. In order to reduce their effects, we can add materials that create a damping reducing the amplitude of the resonance peaks.

Amongst the rules for a good design, the following must be retained [LEV 07]:

– the contact surfaces between the specimen and the test table must be machined to be as flat as possible;

– the joints between the elements making up the assembly must be welded, in a continuous manner (no simple welded joints) and bolts should be avoided;

– the bolts used to fix the specimen to the table must be screwed over a length at least equal to twice their diameter.

The most commonly used materials are steel, aluminum and magnesium, sometimes titanium. The disadvantage of steel is its weight, and magnesium its cost [FIX 87].

The natural frequency is linked with respect to the Young's modulus E and the density ρ, which varies slightly according to the material and is therefore not a chosen criterion (Table 1.5).

	Steel	Magnesium	Aluminum	Titanium
Young's modulus E (N/m^2)	$21\ 10^{10}$	$4.14\ 10^{10}$	$6.9\ 10^{10}$	$10.7\ 10^{10}$
Density ρ (kg/m^3)	7840	1800	2770	4510
E / ρ ratio (N m / kg)	$2.64\ 10^7$	$2.3\ 10^7$	$2.49\ 10^7$	$2.38\ 10^7$

Table 1.5. *Comparison of the mechanical characteristics of the most commonly used materials for the design of assemblies*

Table 1.6 compares the main ways to build an assembly; the assembly usually being machined or welded.

Mode of manufacture	Advantages	Disadvantages
Machining	Easy to manufacture Good fixture (no joints) Used for small specimens	Costly for large specimens
Casting	Monolithic and homogeneous construction Less handling to be carried out	Only of interest for a small number of assemblies (cost of mold manufacture)
Bolting		Not recommended (behavior of bolts under vibration, loss of rigidity)
Laminating strips of material	Simple to manufacture Possibility of including layers of a damping material (rubber, plastic)	Cost linked to time spent on construction
Welding	Best solution	

Table 1.6. *Advantages and disadvantages of main methods for fabrication of assemblies*

Chapter 2

Basic Mechanics

2.1. Basic principles of mechanics

2.1.1. *Principle of causality*

The state of the universe at a given moment determines its state at any later moment.

2.1.2. *Concept of force*

A *force* can be defined as any external cause able to modify the rest state or the movement of a material point.

A force is characterized by:

– its point of application (material point on which it acts);

– its line of action, which is that of the straight line whereby it is applied;

– its direction, which is that of the movement that it tends to produce;

– its size (or intensity).

2.1.3. *Newton's first law (inertia principle)*

In the absence of any force, a material point, if it is at rest, remains at rest; if it is moving, it preserves a rectilinear and uniform motion.

2.1.4. *Moment of a force around a point*

Given a force $\vec{F}$ and an arbitrary point O, the *moment of the force* around point O is defined as the product $M = F\,d$, where d is the perpendicular distance from point O to F (d is called the *lever arm*).

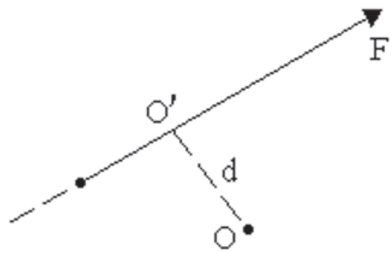


Figure 2.1. *Lever arm for the calculation of the moment of a force*

Let us set O' as the foot of the perpendicular to the support of $\vec{F}$ drawn from O. The moment M is positive if $\vec{F}$ tends to make O' turn clockwise around O, negative in the contrary case.

2.1.4.1. *Couple – moment of a couple*

Two forces form a *couple* if they are parallel, of opposite direction and equal in size.

The *moment of the couple* is equal to $M = Fd$, where F is the amplitude of each force and d is the distance which separates them (*couple lever arm*).

2.1.5. *Fundamental principle of dynamics (Newton's second law)*

The application of a force $\vec{F}$ to the point of mass m involves a variation of its momentum, defined by the product of its mass by its instantaneous speed $\vec{\dot{x}}$, according to the relation:

$$\vec{F} = \frac{d\left(m\,\vec{\dot{x}}\right)}{dt} \qquad [2.1]$$

m is a coefficient characteristic of the body. If the mass m is invariable, the relation becomes:

$$\vec{F} = m\,\frac{d\,\vec{\dot{x}}}{dt} \qquad [2.2]$$

i.e.

$$\vec{F} = m\,\vec{\ddot{x}} \qquad [2.3]$$

where $\vec{\ddot{x}}$ is the acceleration undergone by the mass subjected to $\vec{F}$.

2.1.6. *Equality of action and reaction (Newton's third law)*

If two particles isolated from the remainder of the universe are brought into each other's presence, they exert upon each other two forces, carried by the line which joins them, of equal sizes and opposite directions. One is the action, the other the reaction.

2.2. Static effects/dynamic effects

In order to evaluate the mechanical characteristics of materials, it is important to be aware of the nature of stresses [HAU 65]. There are two main load types that need to be considered when doing this:

– those which can be considered as applied statically;

– those which are the subject of a dynamic analysis of signal versus time.

Materials exhibit different behaviors under static and dynamic loads. Dynamic loads can be evaluated according to the following two criteria:

– the load varies quickly if it communicates significant velocities to the particles of the body in deformation, so that the total kinetic energy of the moving masses constitutes a large part of the total work of the external forces; this first criterion is that used during the analysis of the oscillations of elastic bodies;

– the speed of variation of the load can be related to the velocity of evolution of the plastic deformation process occurring at a time of fast deformation whilst studying the mechanical properties of the material.

According to this last criterion, plastic deformation does not have time to be completed when the loading is fast. The material becomes more fragile as the deformation velocity grows; elongation at rupture decreases and the ultimate load increases (Figure 2.2).

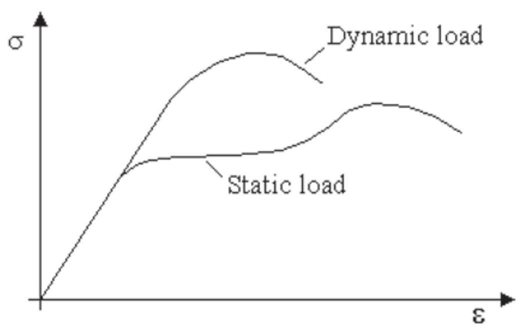


Figure 2.2. *Tension diagram for static and dynamic loads*

Thus, a material can sometimes sustain an important dynamic load without damage, whereas the same load, statically, would lead to plastic deformation or failure. Many materials subjected to short duration loads have ultimate strengths higher than those observed when they are static [BLA 56], [CLA 49], [CLA 54], [TAY 46]. The important parameter is in fact the strain rate, defined by:

$$\dot{\varepsilon} = \frac{1}{\ell_0} \frac{\Delta\ell}{\Delta t} \qquad [2.4]$$

where $\Delta\ell$ is the deformation observed in time Δt on a test-bar of initial length ℓ_0 subjected to stress.

If a test-bar of initial length 10 cm undergoes in 1 s a lengthening of 0.5 cm, the strain rate is equal to 0.05 s^{-1}. The observed phenomena vary according to the value of this parameter. Table 2.1 shows the principal fields of study and the usable test facilities [AST 01], [DAV 04], [DIE 88], [LIN 71], [MEN 05], [SIE 97]. This book will focus on the values in the region 10^{-1} to 10^{1} s^{-1} (these ranges being very approximate).

Certain dynamic characteristics require the data of the dynamic loads to be specified (the order of application being particularly important). Dynamic fatigue strength at any time t depends, for example, on the properties inherent in the material concerning the characteristics of the load over time, and on the previous use of the part (which can reflect a certain combination of residual stresses or corrosion).

Strain rate (s^{-1})

0 10^{-5} 10^{-1} 10^{1} 10^{5}

Phenomenon	Evolution of creep velocity in time	Constant strain rates	Response of structure, resonance	Elastic-plastic wave propagation	Shock wave propagation
Type of test	Creep	Quasi-statics	Slow dynamics	Fast dynamics (impact)	Very fast dynamics (hypervelocity)
Test facility	Constant load or stress machine	Hydraulic or screw driven machine	Hydraulic vibration machine Shakers	Impact metal–metal Pyrotechnic shocks	Explosives Gas guns
	Negligible inertia forces		Important inertia forces		
	Plane stress				Plane strain

Table 2.1. *Fields of strain rate*

2.3. Behavior under dynamic load (impact)

Hopkinson [HOP 04] noted that copper and steel wire can withstand stresses that are higher than their static elastic limit and are well beyond the static ultimate limit without separating proportionality between the stresses and the strains. This is provided that the length of time during which the stress exceeds the yield stress is of the order of 10^{-3} seconds or less.

From tests carried out on steel (annealed steel with a low percentage of carbon) it was noted that the initiation of plastic deformation requires a definite time when stresses greater than the yield stress are applied [CLA 49]. It was observed that this time can vary between 5 ms (under a stress of approximately 352 MPa) and 6 s (with approximately 255 MPa; static yield stress being equal to 214 MPa). Other tests carried out on five other materials showed that this delay exists only for materials for which the curve of static stress deformation presents a definite yield stress, and the plastic deformation then occurs for the load period.

Under dynamic loading, an elastic strain is propagated in a material with a velocity corresponding to the sound velocity c_0 in this material [CLA 54]. This velocity is a function of the modulus of elasticity, E, and of the density, ρ, of the material. For a long, narrow part, we have:

$$c_0 = \sqrt{\frac{E}{\rho}} \quad [2.5]$$

The longitudinal deflection produced in the part is given by:

$$\varepsilon = \frac{v_1}{c_0} \quad [2.6]$$

where v_1= velocity of the particles of the material. In the case of plastic deformation, we have [KAR 50]:

$$c(\varepsilon) = \sqrt{\frac{\partial\sigma/\partial\varepsilon}{\rho}} \quad [2.7]$$

where $\frac{\partial\sigma}{\partial\varepsilon}$ is the slope of the stress deformation curve for a given value of the deformation ε. The velocity of propagation c is therefore a function of ε. The relation between the impact velocity and the maximum deformation produced is given by:

$$v_1 = \int_0^{\varepsilon_1} c \, d\varepsilon \quad [2.8]$$

A velocity of impact v_1 produces a maximum deformation ε_1 that is propagated with low velocity since the deformation is small. This property makes it possible to determine the distribution of the deformations in a metal bar at a given time.

Most of the materials present a total ultimate elongation which is larger at impact than for static loading (Figure 2.3).

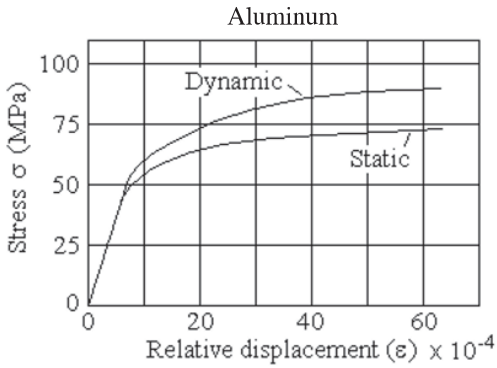


Figure 2.3. *Example of a stress–strain diagram [CAM 53]*

Some examples of static and dynamic ultimate strengths are given in Table 2.2.

Material	Ultimate strength (10^7 Pa)	
	Statics	Dynamics
SAE 5150 hardened and annealing	95.836	102.111
302 standard stainless steel	64.328	76.393
Annealing copper	20.615	25.304
2 S annealing aluminum	7.998	10.618
24S.T aluminum alloy	44.919	47.298
magnesium alloy (Dow J)	30.164	35.411

Table 2.2. *Properties of static and dynamic traction [KAR 50]*

2.4. Elements of a mechanical system

In this section, we will consider lumped parameter systems, in which each particular component can be identified according to its properties and can be distinguished from other elements (in distinction from distributed systems).

Three fundamental passive elements can be defined, each playing a role in linear mechanical systems which correspond to the coefficients of the expressions of the three types of forces which are opposed to the movement (these parameters can be identified for systems with rotary or translatory movements). These passive elements are frequently used in the modeling of structures to represent a physical system in simple terms [LEV 76].

2.4.1. *Mass*

A *mass* is a rigid body whose acceleration $\ddot{x}$ is, according to Newton's law, proportional to the resultant F of all the forces acting on this body [CRE 65]:

$$F = m\,\ddot{x} \tag{2.9}$$

This is a characteristic of the body.

In the case of rotational movement, the displacement has the dimension of an angle, and acceleration can be expressed in rad/s^2. The constant of proportionality is then called the *moment of inertia* of the body, not mass, although it obeys the same definition. The moment of inertia has the dimension M L. The inertia moment Γ is such that:

$$\Gamma = I_\theta\,\frac{d^2\theta}{dt^2} \tag{2.10}$$

where I_θ is the moment of inertia and θ the angular displacement. If $\Omega = \frac{d\theta}{dt}$ is the angular velocity we have:

$$\Gamma = I_\theta \frac{d\Omega}{dt} \tag{2.11}$$

In the SI system, which will be used throughout the book, mass is expressed in kilograms (kg), acceleration in m/s^2 and force in Newtons (N) (dimension MLT^{-2}).

The mass is schematically represented by a rectangle [CHE 66] (Figure 2.4).

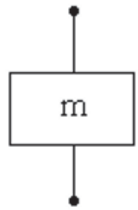


Figure 2.4. *Symbol used to represent mass*

2.4.2. *Stiffness*

2.4.2.1. *Definition*

In the case of linear movement, the *stiffness* of a spring is the ratio k of the variation of force ΔF to the spring deflection Δz which it produces: $k = -\frac{\Delta F}{\Delta z}$. The minus sign indicates that the force is opposed to the displacement (*restoring* force) (Figure 2.5).

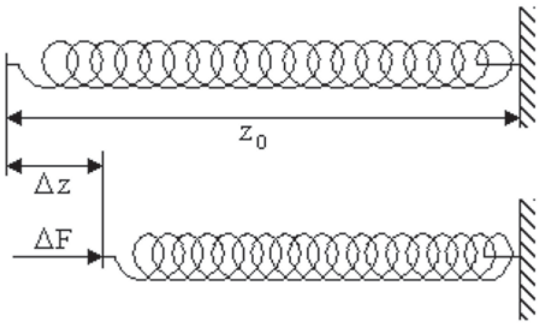


Figure 2.5. *Symbol used for spring*

This definition implicitly assumes that the spring obeys Hooke’s law when the deformation is weak.

In the SI system, stiffness k, the *spring constant,* is expressed in n Newtons per meter. It is assumed that the stiffness is that of a perfectly elastic massless spring [CRE 65], [CHE 66]. It is represented diagrammatically by the symbol

 or sometimes . The points at zero imposed displacement are shown as  (ground).

In the case of rotation around an axis, the restoring moment is defined by:

$$\Gamma = -C\ \alpha \tag{2.12}$$

with the same convention used for the negative sign. The constant C, which characterizes elasticity here, is expressed in Newtons per radian.

The stiffness of a perfectly rigid medium would thus be theoretically infinite. The input and output would be identical (the input being, for example, a force transmitted by the medium). The elongation would, of course, be zero. This is a theoretical case, since no material is perfectly rigid. When the stiffness decreases, the *response* of the spring (value function of time obtained at the end of the spring when an input excitation is applied at the other end) changes and can become different from the input.

2.4.2.2. *Equivalent spring constant*

Certain systems comprising several elastic elements can be reduced to the simple case of only one spring whose equivalent stiffness can easily be calculated from the general expression $F = -k\ z$. If the system can be moved in just one direction, it can be seen that the number of degrees of freedom is equal to the number of elements of mass. The number of elements of elasticity does not intervene and it is possible to *reduce* the system to a simple unit *mass–spring equivalent*; see Figure 2.6 for some examples.

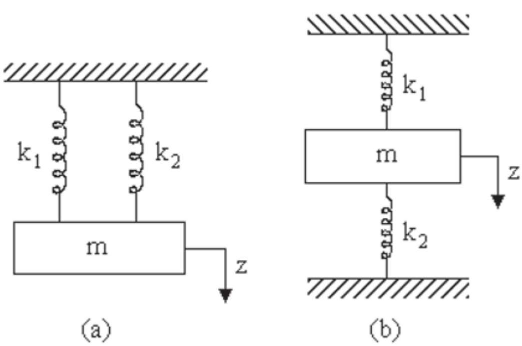


Figure 2.6. *Parallel stiffnesses*

The two diagrams in Figure 2.6 are equivalent to those in Figure 2.7. When the mass is moved by a quantity z, the restoring force is written [CLO 03], [HAB 68], [VER 67]:

$$\left|F_r\right| = k_1\, z + k_2\, z = k_{eq}\, z \qquad [2.13]$$

$$k_{eq} = k_1 + k_2 \qquad [2.14]$$

The stiffness elements have parallel configuration here.

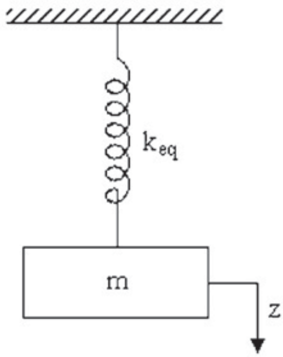


Figure 2.7. *Equivalent stiffness*

In *series* (Figure 2.8), the equivalent constant is calculated in a similar way.

F is a force directed downwards and produces an elongation of each spring respectively equal to:

$$z_1 = \frac{F}{k_1} \qquad [2.15]$$

and

$$z_2 = \frac{F}{k_2} \qquad [2.16]$$

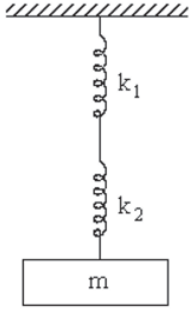


Figure 2.8. *Stiffnesses in series*

This yields

$$k_{eq} = \frac{F}{z} = \frac{F}{z_1 + z_2} = \frac{F}{\frac{F}{k_1} + \frac{F}{k_2}} \qquad [2.17]$$

i.e.

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} \qquad [2.18]$$

The equivalent stiffness of two springs in parallel is equal to the sum of their stiffnesses. The inverse of the stiffness of two springs in series is equal to the sum of the inverses of their stiffness [HAB 68], [KLE 71a].

It is easy to generalize this to n springs.

2.4.2.3. *Stiffness of various parts*

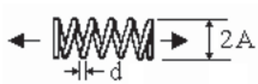
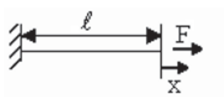
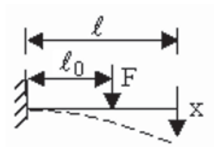
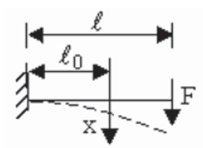
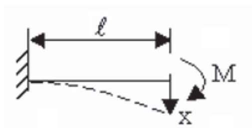
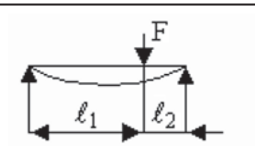
Springs in compression or axial tension D = average diameter of a coil d = diameter of the wire n = number of active coils G = shear elasticity modulus		$k = \dfrac{G\ d^4}{8\ n\ D^3}$ Deformation: $\delta = \dfrac{8\ F_y\ D^3\ n}{G\ d^4}$
Beam cantilever axial load S = area of the cross-section E = Young's modulus		$k = \dfrac{E S}{\ell} = \dfrac{F}{X}$
Cantilever beam I = moment of inertia of the section		$k = \dfrac{6\ E\ I}{\ell_0^3\left(3\ \ell - \ell_0\right)} = \dfrac{F}{X}$
Cantilever beam		$k = \dfrac{6\ E\ I}{\ell_0^3\left(3\ \ell - \ell_0\right)} = \dfrac{F}{X}$
Cantilever beam		$k = \dfrac{2\ E\ I}{\ell^2} = \dfrac{M}{X}$
Beam on two simple supports, charged at an arbitrary point		$k = \dfrac{3\ E\ I\ \ell}{\ell_1^2\ \ell_2^2}$

Table 2.3(a). *Examples of stiffnesses [DEN 56], [THO 65a]*

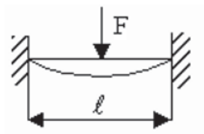
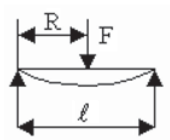
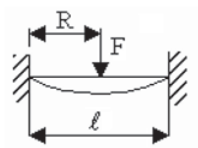
Fixed beam, loaded in its center		$k = \dfrac{192\ E\ I}{\ell^3}$
Circular plate thickness t, centrally loaded and circumferential edge simply supported		$k = \dfrac{16\ \pi\ D}{R^2}\ \dfrac{1+\nu}{3+\nu}$ where $D = \dfrac{E\ t^3}{12\left(1-\nu^2\right)}$ ν = Poisson's coefficient (≈ 0.3)
Circular plate centrally loaded, circumferential edge clamped		$k = \dfrac{16\ \pi\ D}{R^2}$

Table 2.3(b). *Examples of stiffnesses [DEN 56], [THO 65a]*

Stiffnesses in rotation

Twist of coil spring D = average coil diameter d = wire diameter n = number of turns		$k = \dfrac{E\ d^4}{64\ n\ D}$
Bending of coil spring		$k = \dfrac{E\ d^4}{32\ n\ D}\ \dfrac{1}{1+E/2\ G}$
Spiral spring ℓ = total length I = moment of inertia of cross section		$k = \dfrac{E\ I}{\ell}$

Table 2.4(a). *Examples of stiffnesses (in rotation)*

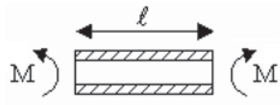
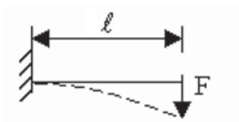
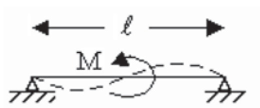
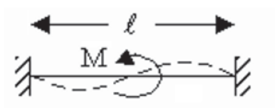
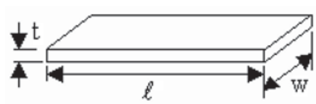
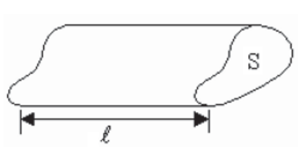
Twist of hollow circular tube D = outer diameter d = inner diameter ℓ = length		$k = \frac{G\,I}{\ell} = \frac{\pi\,G}{32}\,\frac{D^4 - d^4}{\ell}$ Steel: $k = 1.18\ 10^6\ \frac{D^4 - d^4}{\ell}$
Cantilever beam End moment		$k = \frac{M}{\theta} = \frac{E\,I}{\ell}$
Cantilever beam **End load**		$k = \frac{M}{\theta} = \frac{2\,E\,I}{\ell^2}$
Beam on two simple supports **Couple at its center**		$k = \frac{M}{\theta} = \frac{12\,E\,I}{\ell}$
Clamped-clamped **Couple at center**		$k = \frac{M}{\theta} = \frac{16\,E\,I}{\ell}$
Circular bar D = diameter ℓ= length		$k = \frac{\pi\,G\,D^4}{32\,\ell}$
Rectangular plate		$k = \frac{G\,w\,t^3}{3\,\ell}$
Bar of arbitrary form S = section I_P= polar inertia moment of the cross-section		$k = \frac{G\,S^4}{4\,\pi^2\,\ell\,I_P}$

Table 2.4(b). *Examples of stiffness (in rotation)*

2.4.2.4. *Non-linear stiffness*

A linear stiffness results in a linear force–deflection curve (Figure 2.9) [LEV 76]. Figures 2.10 to 2.12 show examples of non-linear stiffnesses.

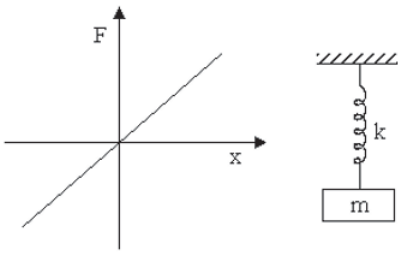


Figure 2.9. *Linear stiffness*

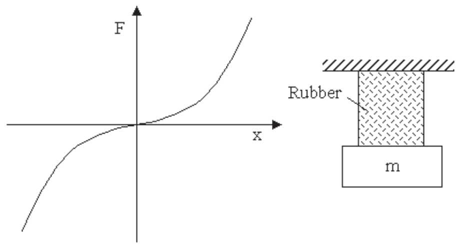


Figure 2.10. *Non-linear stiffness*

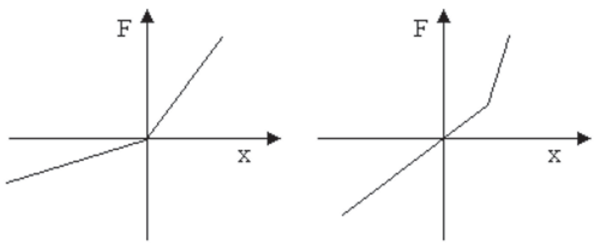


Figure 2.11. *Examples of bilinear stiffnesses*

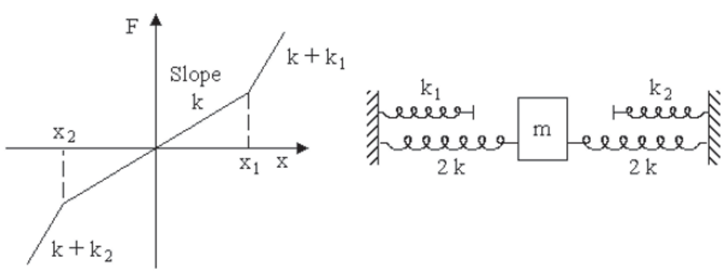


Figure 2.12. *An example of non-linear stiffness*

2.4.3. *Damping*

2.4.3.1. *Definition*

No system subjected to a dynamic stress maintains constant amplitude without an external input of energy. Materials do not behave in a perfectly elastic manner, even at low levels of stress. When cycles of alternate stress (stress varying between a positive maximum and a negative minimum) are carried out on a metal test-bar, we can distinguish the following [BAS 75]:

1. *Microelastic ultimate stress* σ_{me}, such that $\sigma \leq \sigma_{me}$, the stress–strain curve is perfectly linear (zero surface). The stress σ_{me} is, in general, very small.

2. *Anelastic stress* σ_{an} such that $\sigma_{me} < \sigma < \sigma_{an}$, the stress–strain cycle remains closed (without its surface being zero). In this case, the deformation remains "reversible", but is associated with a dissipation of energy.

3. *Accommodation ultimate stress* σ_{ac}, which is the strongest stress, although the first cycle is not closed. The repetition of several alternate stress cycles can still lead to the closing of the cycle ("accommodation" phenomenon).

4. For $\sigma > \sigma_{ac}$, the cycle is closed, leading to permanent deformation.

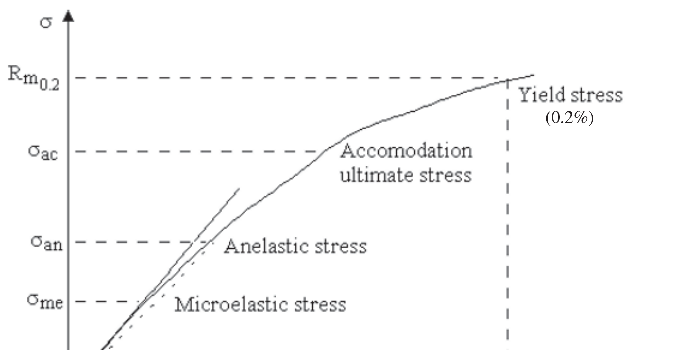


Figure 2.13. *Beginning of a stress–strain curve*

Figure 2.13 shows the beginning of a stress–strain curve. The *yield stress* $R_{m_{0.2}}$, which is defined in general as the stress that produces a deflection of 0.2%, is a conventional limit already located in the plastic range.

There is always a certain inelasticity that exists, although it is often very low and negligible. Due to this inelasticity, which can have several origins, the material or the structure dissipates part of the energy which it receives when a mechanical stress is applied to it. This is said to be *damping*.

Dissipated energy leads to a decrease in the amplitude of the free vibration of the system in the course of time, until it returns to its equilibrium position. This loss is generally connected to the relative movement between components of the system [HAB 68]. The energy communicated to the cushioning device is converted into heat. A damping device is thus a non-conservative device.

The inelastic behavior is underlined by plotting the stress–strain curve of a test-bar of material subjected to sinusoidal stress (for example in tension–compression) [LAZ 50], [LAZ 68].

Figure 2.14 shows such a curve (very deformed in order to show the phenomenon more clearly).

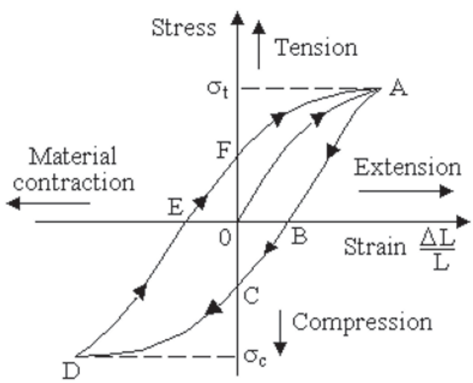


Figure 2.14. *Hysteresis loop*

At the time of the first tension loading, the stress–strain law is represented by arc OA. The passage of tension to compression is done along arc ABCD, while the return towards maximum tension follows arc DEFA.

Curve ABCDEFA is called a *hysteresis loop* and it occurs for completely alternating loads.

2.4.3.2. *Hysteresis*

A natural phenomenon observed in materials, hysteresis is related to partial relaxation of stress applied by means of plastic deformations and characterized by the absorption and the dissipation of energy [FEL 59]. This property of materials, studied since being highlighted by Lord Kelvin [THO 65b], has been given various names [FEL 59]:

– *damping capacity* [FÖP 36], the term most frequently used, which can be defined as the aptitude of a material to dissipate vibratory energy; this parameter, denoted by D, defined in 1923 by O. Föppl represents the work dissipated in heat per unit of volume of material during a complete cycle of alternated load, and is calculated by evaluating the area delimited by the hysteresis loop [FEL 59]:

$$D = \int_{1\,\text{cycle}} \sigma \, d\varepsilon \qquad [2.19]$$

Thus D is the energy absorbed by a macroscopically uniform material, per unit of volume and stress cycle (tension–compression, for example);

– *internal friction* [ZEN 40], relating to the capacity of a solid to convert its mechanical energy into internal energy;

– *mechanical hysteresis* [STA 53];

– *elastic hysteresis* [HOP 12].

Whether for a part comprised of a single material, which may or may not be part of a structure, or for a more complex structure, the hysteresis loop can be plotted by considering the variations of the deformation z due to the application of a sinusoidal force F. The energy dissipated by the cycle is then equal to:

$$\Delta E_d = \int_{1\,\text{cycle}} F \, dz \qquad [2.20]$$

ΔE_d is the *total damping energy* (equal to VD, where V is the volume of the part). ΔE_d is usually expressed in the following units:

– for a material: Joules per m^3 and cycle;

– for a structure: Joules per cycle.

The total plastic deformation can be permanent or anelastic, or a combination of both. Hysteresis thus appears as the non-coincidence of stress–strain loading and unloading curves on the diagram.

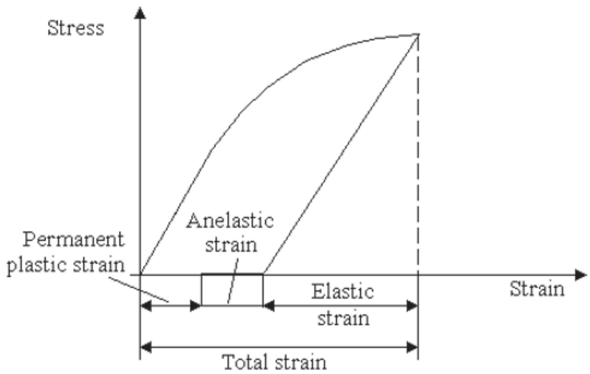


Figure 2.15. *Strain or hysteresis*

If the stress is sufficient to produce plastic deformation, the part will never return to its initial state ($\varepsilon = 0$, $\sigma = 0$). Even if the deformation is only anelastic, there is still the formation of a hysteresis loop. However, if the stress is maintained at zero for a sufficient period of time, the part can return to zero initial condition.

The anelastic strain is therefore not just a function of stress, but also of time (as well as temperature, magnetic field, etc.).

2.4.3.3. *Origins of damping*

Damping in materials has been studied for around 200 years. The principal motivation for this has been the analysis of the mechanisms which lead to inelastic behavior and the dissipation of energy, the control of certain characteristics of the materials during manufacturing (purity, fissures, etc.) and especially the design of structures, where it is interesting to attenuate the dynamic response stresses.

The damping of a complex structure is dependent on [HAY 72], [LAZ 68]:

– the *internal damping* of the materials which constitute each part;

– the damping introduced by connections or contacts between the various parts (*structural damping*).

Internal damping indicates the complex physical effects which transform the deformation energy in a vibrating mechanical system composed of a volume of macroscopically continuous matter into heat [GOO 76].

When a perfectly elastic system is deformed by the application of an external force, the energy exerted by the force during the deformation is stored in the material. When the external force is removed, the stored energy is released and the material oscillates around its equilibrium position (the system is not damped).

In a perfectly plastic material, all the energy exerted by the external force is dissipated and no energy is stored in the material. The suppression of the external force thus leaves the material in its deformed state (completely damped system).

Typical materials are neither perfectly elastic, nor perfectly plastic, but partly both. The ratio of the plasticity and the elasticity of a particular material, used to describe the behavior of this material, is the *damping* or *loss coefficient* of the material.

The origins of internal damping are multiple [CRA 62]: dislocations, phenomena related to the temperature, diffusion, magnetomechanical phenomena, etc. Damping depends on the level of stress to which the material is subjected, the distribution of the stresses in the specimen, sometimes the frequency, the static load, the temperature, etc. The external magnetic field can also be an important factor for ferromagnetic materials [BIR 77], [FÖP 36], [LAZ 68] and [MAC 58]. The effects of these different parameters vary according to whether the inelasticity belongs to the one of the following categories:

1. inelasticity function of the rate of setting in stress ($\frac{d\sigma}{dt}$ or $\frac{d\varepsilon}{dt}$);
2. inelasticity independent of the rate of setting in stress;
3. reversible strain under stress (Figures 2.16(a) and 2.16(b));
4. irreversible strain under stress.

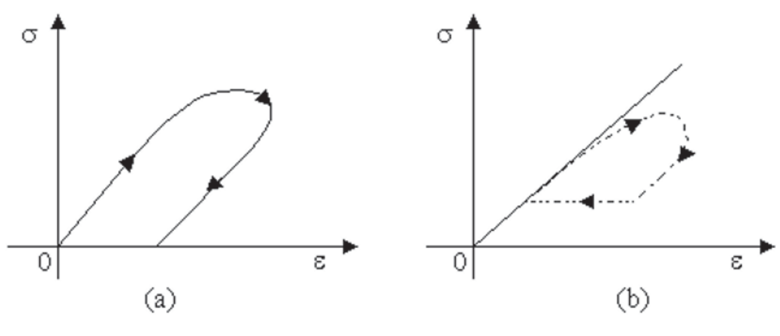


Figure 2.16. *Reversible strain under stress*

These four cases can combine in pairs according to [LAZ 68]:

– case (1 and 3): the material undergoes a strain known as *anelastic strain*. The anelasticity is characterized by:

- linear behavior: to double the stress results in doubling the strain,

- the existence of a single stress–strain relation, on the condition of allowing sufficient time to reach equilibrium;

– case (1 and 4): the material in this case is known as *viscoelastic.* The viscoelastic strain can be reversible or not. Case (1 and 3) is a particular case of (1 and 4) (recoverable strain);

– case (2 and 4): the material works in a field of plastic strain (under strong stresses in general).

The energy dissipated in cases 1 and 2 can be a function of the amplitude of the stress, but only (2 and 4) is independent of the stress frequency (i.e. of the rate of setting in stress).

Cases (1 and 3) and (1 and 4), for which damping is a function of the loading rate, thus lead to equations which involve the first derivatives $\frac{d\sigma}{dt}$ or $\frac{d\varepsilon}{dt}$. These cases of damping can be encountered in metals (anelasticity), in polymers (elastomers) (molecular interaction phenomena), in structures with various names: dynamic hysteresis; rheological damping; and internal friction [LAZ 68].

The relation between applied stress and damping is often complex; we can, however, in a great number of cases, approach this satisfactorily by a relation of the form [LAZ 50], [LAZ 53], [LAZ 68]:

$$D = J\ \sigma^n \tag{2.21}$$

where:

J and n are constants for the material;

J = damping constant (or dissipated energy at an unity amplitude stress);

n = damping exponent whose value varies (from 2 to 8) according to the behavior of the material, related to the stress amplitude, according to the temperature.

The exponent n is a constant for many materials when the stress amplitude is below a certain critical stress which is close to the stress of the ultimate resistance of the material.

Above this limit, damping becomes a function of the stress according to time [CRA 62].

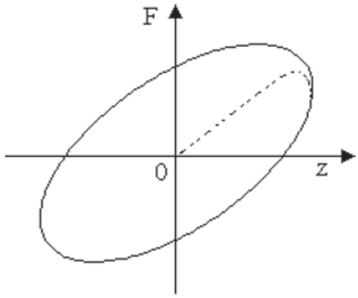


Figure 2.17. *Elliptical cycle (n = 2)*

For low stress amplitudes and ambient temperature, n is equal to 2 (*quadratic damping*) and its hysteresis loop has an elliptical form.

We define the case $n = 2$ as that of a *linear* damping, because it is observed in the case of a viscosity phenomenon, for which the differential equations describing the movement are linear.

At intermediate and high amplitudes, non-linear behaviors characterized by non-elliptic hysteresis loops and exponents n generally greater than 2 (observed up to 30 on a material with high stress) are observed.

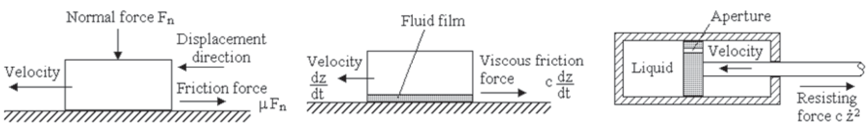
The damping capacity D is defined for a material under uniform stress. Relation [2.21] is generally valid up to a limit called the *limit of sensitivity to the cyclic stresses,* which is in the fatigue limit zone of the material [MOR 63a].

Structural damping, the least well-known, is the dominant phenomenon [NEL 80]. The phenomenology of the dissipation of energy at the ideal simple junction is reasonably well understood [BEA 82], [UNG 73], in particular at low frequencies. The problem is more difficult at high frequencies (much higher than the fundamental resonance frequency of the component). We can schematically distinguish three principal types of interface:

– *interfaces with dry friction:* metal–metal or more generally material–material (Coulomb friction); the frictional force is directly proportional to the normal force and the friction coefficient μ and independent of the sliding velocity. Dissipated energy is equal to the work spent against the frictional force;

– *lubricated interfaces* (fluid film, plastic, etc.) [POT 48]. In this mechanism, the friction is known as *viscous*. The amplitude of the damping force is directly proportional to the velocity of the relative movement, and its direction is opposite to that of the displacement;

– interfaces that are *bolted*, *welded*, *stuck*, *riveted*, etc.



Dry friction (Coulomb friction) — The frictional force is proportional (coefficient μ) to the normal force F_n of application of the moving part on its support.

Viscous friction — The frictional force is proportional to the relative velocity of the two parts in contact.

Case of an aperture — The frictional force is proportional to the square of the velocity of the piston relative to the cylinder.

Figure 2.18. *Examples of damping forces*

In the first two categories, the forces can be applied in the direction normal to the plane of interface or according to a direction located in the plane of the interface (shearing). It is in this last case that the energy dissipation can be strongest.

There are many other mechanisms of energy dissipation, such as:

– damping due to the environment (air), the moving part activating the air or the ambient fluid (damping force F_d is in general proportional to $\dot{z}^2$);

– magnetic damping (passage of a conductor in a magnetic field; the damping force is then proportional to the velocity of the conductor);

– the passage of a fluid through an aperture, etc.

2.4.3.4. *Specific damping energy*

The specific damping energy [FEL 59] is the ratio;

$$\phi = \frac{\Delta E_d}{U_s} \qquad [2.22]$$

where:

ΔE_d = damping capacity (area under the hysteresis loop);

$U_s = \dfrac{\sigma^2}{2\,E_d}$ = maximum strain energy in the specimen during the cycle;

E_d = dynamic modulus of elasticity.

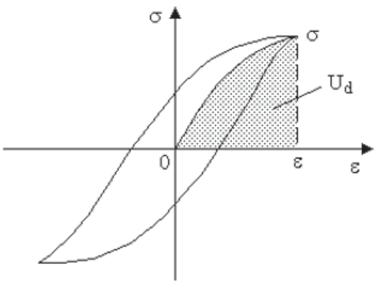


Figure 2.19. *Strain energy*

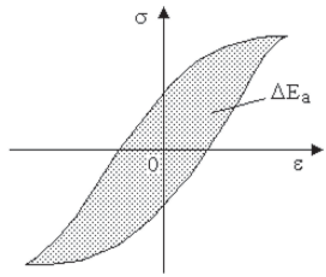


Figure 2.20. *Damping capacity*

The damping of a material can also be defined as the ratio of the dissipated energy to the total strain energy (by cycle and unit of volume):

$$\eta = \frac{D}{2\,\pi\,U_{ts}} \qquad [2.23]$$

For a linear material, $D = J\,\sigma^2$ and $U_{ts} = \dfrac{U_s}{V} = \dfrac{1}{2}\dfrac{\sigma^2}{E_d}\left(= \dfrac{1}{2}\dfrac{F}{S}\dfrac{\Delta\ell}{\ell}\right)$,

yielding:

$$\eta = \frac{J\,E_d}{\pi} \qquad [2.24]$$

Constants similar to those used above for viscoelasticity can be defined for anelastic materials. For anelastic materials, η lies between 0.001 and 0.1, while for viscoelastic materials, η varies between 0.1 and 1.5.

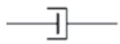
2.4.3.5. *Viscous damping constant*

The theory of viscous internal friction is very old and has been employed for a long time. Proposed by Coulomb, it was developed by W. Voight and E.J. Rought, and then used by other authors. It supposes that, in solid bodies, there are certain viscous attributes which can be compared with the viscosity of a fluid and which are proportional to the first derivative of the deformation [VOL 65]. This yields the damping force:

$$F_d = -c \frac{dz}{dt} \tag{2.25}$$

The factor c, which we will suppose to be constant at first approximation, can vary more or less in practice according to the material, and with the frequency of the excitation. This parameter, the "viscous damping coefficient" (N s / m), is a function of the geometry of the damping device and the viscosity of the liquid used. It is encountered at the time of the slip between lubricated surfaces in damping devices with fluid, or certain types of laminar flows through an aperture. Damping can be considered to be viscous as long as the flow velocity is not too large.

It is estimated in general that elastomers and rubber bladders (with low velocities) have comparable behavior to viscous damping. This type of damping is very often used in studies of the behavior of structures under vibration [JON 69], [JON 70], because it leads to linear equations which are relatively easy to treat analytically.

Viscous damping will be represented on the diagrams by the symbol  [JON 69].

In the case of a linear system in rotation, the damping torque Γ_d is:

$$\Gamma_d = D_\alpha \; \Omega = D_\alpha \frac{d\alpha}{dt} \tag{2.26}$$

where:

D_α = viscous damping constant in rotation;

$\Omega = \frac{d\alpha}{dt}$ = angular velocity.

2.4.3.6. *Rheology*

Rheology relates to the study of the flow and deformation of matter [ENC 73]. Theoretical rheology attempts to define mathematical models accounting for the behavior of solids under stresses. The simplest models are those with only one parameter:

– elastic solid following Hooke's law, with force varying linearly with the displacement, without damping (Figure 2.21);

Figure 2.21. *Elastic solid*

– damping *shock absorber* type of device, with force linearly proportional to the velocity (Figure 2.22).

Among the models, which account for the behavior of real solids better, are those models with two parameters [BER 73] such as:

– the Maxwell model, adapted to represent the behavior of the viscoelastic liquids rather well (Figure 2.23);

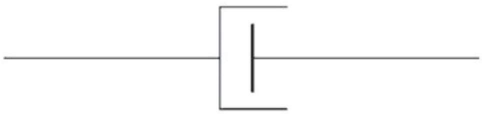


Figure 2.22. *Shock absorber-type damping device*

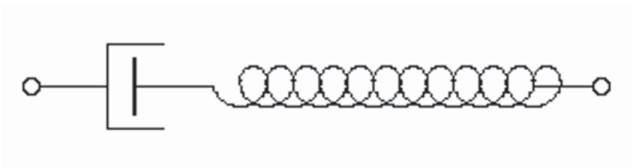


Figure 2.23. *Maxwell model*

– the Kelvin–Voigt model, better adapted to the case of viscoelastic solids. It allows a complex representation of the stiffness and damping for a sine wave excitation of the form:

$$k^* = k + i\,\Omega\,c \qquad [2.27]$$

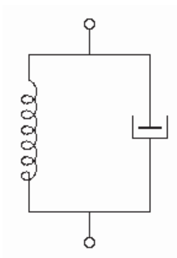


Figure 2.24. *Kelvin–Voigt model*

2.4.3.7. *Damper combinations*

Dampers in parallel

The force necessary to produce a displacement z between the ends of the dampers is equal to:

$$F = F_1 + F_2 = c_1\ z + c_2\ z \qquad [2.28]$$

$$F = \left(c_1 + c_2\right) z = c_{eq}\ z \qquad [2.29]$$

$$c_{eq} = c_1 + c_2 \qquad [2.30]$$

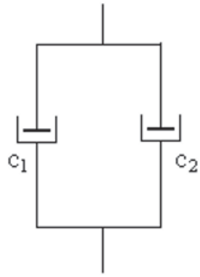


Figure 2.25. *Dampers in parallel*

Dampers in series [CLO 03], [VER 67]

$$F = c_1 \; z_1 + c_2 \; z_2 \qquad [2.31]$$

$$z = z_1 + z_2 = \frac{F}{c_1} + \frac{F}{c_2} = \frac{F}{c} \qquad [2.32]$$

$$c_{eq} = \frac{1}{1/c_1 + 1/c_2} \qquad [2.33]$$

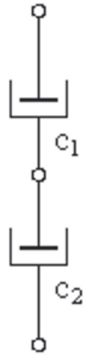


Figure 2.26. *Dampers in series*

2.4.3.8. *Non-linear damping*

Types of non-linear damping are described in Chapter 7 and their effect on the response of a one-degree-of-freedom mechanical system is examined. As an example, the case of dry friction (Coulomb damping) and that of an elastoplastic strain [LEV 76] are described.

Dry friction (or Coulomb friction)

The damping force here is proportional to the normal force N between the two moving parts (Figure 2.27):

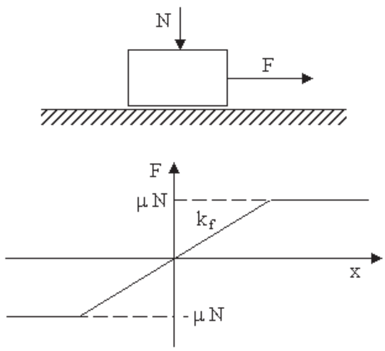


Figure 2.27. *Dry friction*

$$F = \mu\ N \quad [2.34]$$

if $k_f\ x > \mu\ N$

$$F = k_f\ x \quad [2.35]$$

if $-\mu\ N < k_f\ x < \mu\ N$

$$F = -\mu\ N \quad [2.36]$$

if $k_f\ s < -\mu\ N$.

This case will be detailed in Chapter 6.

Elements with plastic deformation

Figures 2.28 and 2.29 show two examples of force–displacement curves where plastic behavior intervenes.

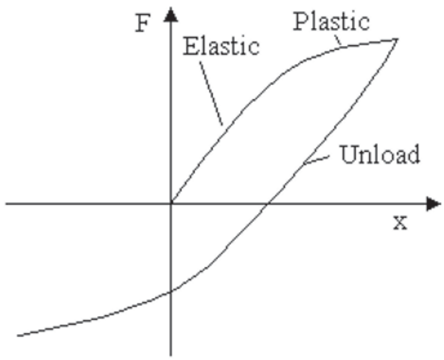


Figure 2.28. *Example of plastic deformation*

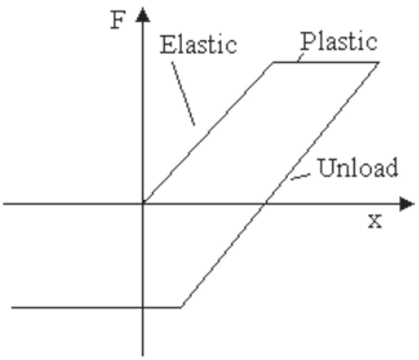


Figure 2.29. *Example of plastic deformation*

2.4.4. *Static modulus of elasticity*

The static modulus of elasticity of a material, which is dependent on the stiffness under the static load of the parts in which they are cut, is defined as the ratio of the variation of stress $\Delta\sigma$ to the resulting strain $\varepsilon = \frac{\Delta\ell}{\ell}$.

Linear materials have a single modulus even with very strong damping. For phenomena which are independent of the rate of setting in stress, such as those observed for metals working under the usual conditions of stress and temperature, the hysteresis loop no longer has an elliptical form which would make it possible to separate the elastic strain component which stores the energy and the component of energy dissipation. Two types of static modules are thus defined [LAZ 50], [LAZ 68]:

– *the tangent modulus of elasticity,* for a given value of the stress; this modulus is proportional to the slope of the stress–strain curve measured for this given stress;

– *the secant modulus of elasticity,* which is proportional to the slope of a straight line segment joining two given points of the stress–strain curve.

As an example, the tangent modulus at the origin is given (Figure 2.30) by the slope of tangent OG to arc OA at the origin and the secant modulus by the slope of segment OA (for a viscous linear material thus having an elliptic hysteresis loop, the secant modulus is none other than the static modulus of elasticity). The tangent OG corresponds to a material which would be perfectly elastic.

In practice, materials in the stress domain have similar tangents and secant moduli of elasticity where they follow Hooke's law reasonably well. In general, the secant modulus decreases when the maximum stress amplitude grows.

2.4.5. *Dynamic modulus of elasticity*

The dynamic modulus of elasticity of a material is the modulus of elasticity calculated from a stress–strain diagram plotted under cyclic dynamic stress. A tangent dynamic modulus and a secant dynamic modulus are defined in the same way. The values measured in dynamics often differ from static values.

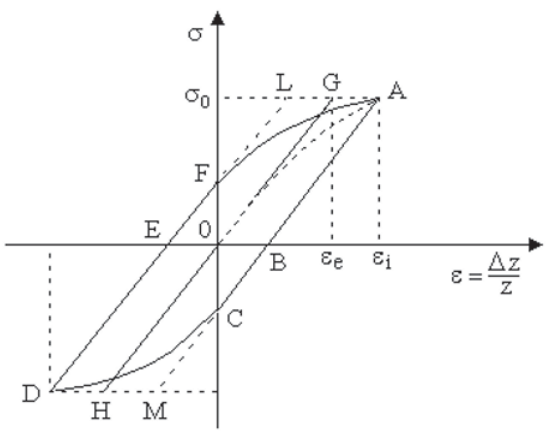


Figure 2.30. *Tangent modulus*

The stress–strain curve can be modified in dynamics by:

– a change in the initial tangent modulus (at the origin) (modification of the slope of OG or any other arc of curve and possibly even by rotation of the hysteresis loop);

– a variation in the surface delimited by the curve, i.e. of the damping capacity of the material.

A perfectly elastic material which has a stress–strain curve such as HOG undergoes a strain ε_e under the maximum stress σ_0, whereas the inelastic material takes a deformation ε_i under the same stress. The difference $\Delta\varepsilon = \varepsilon_i - \varepsilon_e$ is a measurement of the dynamic elasticity reduction (to which an increase in the damping capacity corresponds).

When the damping capacity of a material grows, the material becomes more deformed (for the same stress) and its dynamic modulus of elasticity decreases.

These variations can be represented by writing the dynamic modulus (*secant modulus*) in the form:

$$E_d = \frac{\sigma_0}{\varepsilon_i} \qquad [2.37]$$

$$E_d = \frac{\sigma_0}{\varepsilon_e + \Delta\varepsilon} = \frac{1}{\dfrac{\varepsilon_e}{\sigma_0} + \dfrac{\Delta\varepsilon}{\sigma_0}} \qquad [2.38]$$

$$E_d = \left(\frac{1}{E_e} + \frac{\Delta\varepsilon}{\sigma_0} \right)^{-1} \qquad [2.39]$$

The initial tangent dynamic modulus E_e is assumed to be equal to the static module. Since the specific damping capacity D is equal to the area under the curve of the hysteresis loop, we can set:

$$D = K\,\Delta\varepsilon\,\sigma_0 \qquad [2.40]$$

where K is a constant function of the shape of the cycle (for example, $K = 4$ for a trapezoidal cycle such as LAMDL). This yields:

$$E_d = \left(\frac{1}{E_e} + \frac{D}{K\,\sigma_0^2} \right)^{-1} \qquad [2.41]$$

The value of K depends on the shape of the loop, as well as on the stress amplitude. An average value is $K = 3$ [LAZ 50].

The modulus E_d is thus calculable from [2.41] provided that the initial tangent modulus (or the slope of arcs DF or AC) does not vary (with the velocity of loading, according to the number of cycles, etc.). B.J. Lazan [LAZ 50] has shown that in a particular case this variation is weak and that expression [2.41] is sufficiently precise.

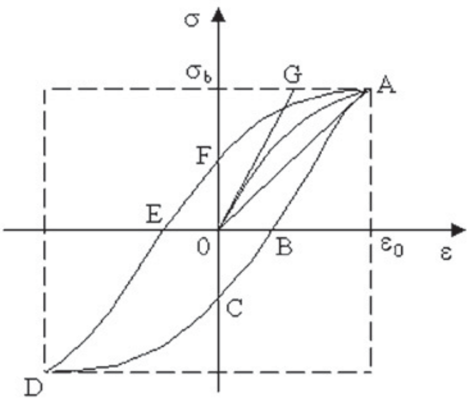


Figure 2.31. *Tangent and secant moduli:*
– slope of OG = initial tangent modulus of elasticity
(i.e. approximately the slope of arcs AB and of which are approximately linear);
– slope of OA = secant modulus of elasticity;
– OB = remanent deformation;
– OF = coercive force

2.5. Mathematical models

2.5.1. *Mechanical systems*

A *system* is a unit made up of mechanical elements having properties of mass, stiffness and of damping. The mass, stiffness and damping of a structure are important parameters since they determine its dynamic behavior.

The system can be:

– a *lumped parameter system* when the components can be isolated by distinguishing the masses, stiffnesses and dampings, by assuming them to be lumped in separate elements. In this case, the position at a given time depends on a finite number of parameters;

– a *distributed system,* when this number is infinite. The movement is then a function of time and space [GIR 08].

2.5.2. *Lumped parameter systems*

In practice, and generally for a real structure, these elements are distributed continuously, uniformly or not, with the properties of mass, stiffness and damping not being separate. The structure is made up of an infinite number of infinitesimal particles. The behavior of such a system with distributed constants must be studied using complete differential equations with partial derivatives.

It is often interesting to simplify the structure to be studied in order to be able to describe its movement using complete ordinary differential equations, by dividing it into a discrete number of specific masses connected by elastic massless elements and of energy dissipative elements, so as to obtain a lumped parameter system [HAB 68], [HAL 78].

The transformation of a physical system with distributed constants into a model with localized constants is generally a delicate operation, with the choice of the points having an important effect on the results of the calculations carried out thereafter with the model.

The procedure consists of:

1. Choosing a certain number of points (nodes) by which the mass of the structure is affected. The number of nodes and number of directions in which each node can be driven determines the number of degrees of freedom of the model.

The determination of the number of nodes and their position can be a function of:

– the nature of the study to be carried out: to define a problem roughly, it is often enough to be limited to a model with a few degrees of freedom;

– the complexity of the structure studied;

– available calculation means: if the complexity of the structure and the precision of the results justify it, then a model with several hundred nodes can be considered.

The choice of the number of nodes is therefore, in general, a compromise between a sufficient representativeness of the model and a simple analysis, leading to the shortest possible computing time.

2. Distributing the total mass of the structure between the various selected points. This task must be carried out carefully, particularly when the number of nodes is limited.

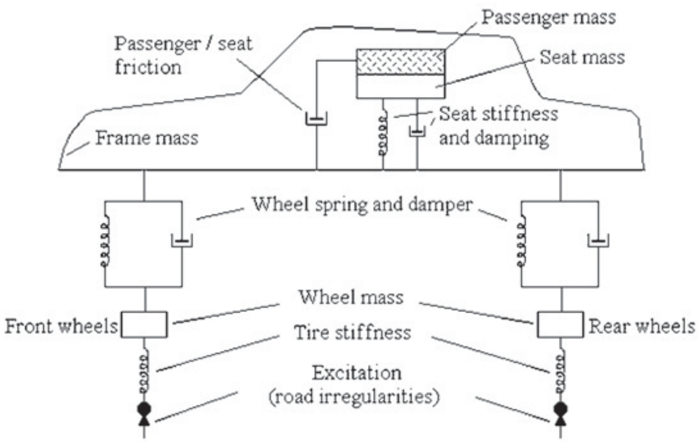


Figure 2.32. *Mathematical model of a car*

This type of modeling makes it easier to study more complicated structures such as a car–passenger unit (Figure 2.32) [CRE 65]. Such a model is sometimes called a *mathematical model*.

In these models, according to the preceding definitions, the element mass is assumed to be perfect, i.e. perfectly rigid and non-dissipative of energy, the elasticity element is massless and perfectly elastic, and finally the dissipative energy element is assumed to be perfectly massless and rigid.

Computer programs have been developed to study the dynamic behavior of structures modeled in this way numerically [GAB 69], [MAB 84], [MUR 64].

2.5.3. *Degrees of freedom*

The number of degrees of freedom of a material system is equal to the number of parameters necessary to determine the state of this system at any time. The simplest system, a material point, has three degrees of freedom in general: three coordinates are necessary at every moment to define its position in space. The number of equations necessary to know the movement of the system must be equal to the number of degrees of freedom.

A solid has six degrees of freedom in general. This number depends on:

– the complexity of the solid;

– the connections to which it is subjected.

If each element of mass of a model can be driven only in only one direction, the number of degrees of freedom is equal to the number of elements of mass. A very complex system can thus have a limited number of degrees of freedom.

NOTE.– *A deformable system has an infinite number of degrees of freedom.*

2.5.4. *Mode*

The exploitation of these models with lumped or distributed constants shows that the system can vibrate in a certain number of ways, called *modes*. Each one corresponds to a specific natural frequency. This number of frequencies is therefore equal to the number of modal shapes, and is therefore equal to the number of coordinates necessary at any moment to determine the position of the system, i.e., according to section 2.5.3, to the number of degrees of freedom of the system.

In the case of a system with distributed masses, the number of degrees of freedom is infinite. Each frequency corresponds to a single oscillatory mode, which is determined by its characteristic function or normal function. A transient or permanent forced excitation will excite, in general, some or all of these frequencies, the response in each point being a combination of the corresponding modal shapes. In the case of a linear system, we shall be able to use the principle of superposition to calculate this response.

This concept of a mode is important and deserves further development. The chapters which follow are limited to systems with only one degree of freedom.

Example 2.1.

1. Case of a beam fixed at one end, length L, uniform section and bending stiffness EI (E = modulus of elasticity and I = inertia moment of the section).

The natural pulsation ω_0 is given by [CRE 65], [KAR 01]:

$$\omega_0 = n^2 \pi^2 \sqrt{\frac{g\, E\, I}{P\, L^4}} = n^2 \pi^2 \sqrt{\frac{E\, I}{m\, L^4}} \qquad [2.42]$$

where n is an integer: n = 1, 2, 3... and g is the acceleration of gravity (9.81 m/s^2)

yielding frequencies

$$f_0 = \frac{n^2 \pi}{2} \sqrt{\frac{g\,E\,I}{P\,L^4}} = K \sqrt{\frac{g\,E\,I}{P\,L^4}} \text{ (Hertz)} \qquad [2.43]$$

where P is the weight of the beam per unit of length. Each value of n corresponds a frequency f_0. Figure 2.33 shows the first five modes.

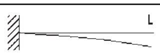
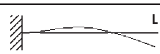
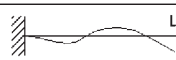
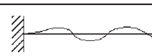
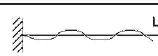
Mode 1	Mode 2	Mode 3	Mode 4	Mode 5
 K = 0.56	 K = 3.51	 K = 9.82	 K = 19.24	 K = 31.81

Figure 2.33. *First five modes of a fixed beam*

2. Beam fixed at two ends [STE 78]:

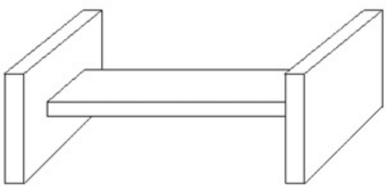


Figure 2.34. *Beam fixed at both ends*

Natural frequency

$$f_0 = \frac{22{,}44}{2\,\pi} \sqrt{\frac{E\,I\,g}{P\,L^3}} \qquad [2.44]$$

where

E = Young's modulus (units SI)　　I = moment of inertia
P = weight of the beam　　L = length of the beam
g = 9.81 m/s^2

Coupled modes

In a system with several degrees of freedom, the mode of one of the degrees can influence the movement corresponding to that of another degree.

It is important to distinguish between coupled and uncoupled movements. When two movements of a mass, horizontally and vertically for example, are not coupled, and can coexist simultaneously and independently, the system is not regarded as one with several degrees of freedom, but as being composed of several systems with only one degree of freedom, whose movements are collectively used to obtain the total resulting movement [CRE 65].

2.5.5. *Linear systems*

A vibrating *linear system* is any system whose positional variables follow, in the absence of an external exciting force, a system of linear differential equations, with constant coefficients, and no second members, in a number equal to that of the unknowns [MAZ 66].

In a linear structure, the characteristics of the response are additive and homogeneous [PIE 64]:

– the response to a sum of excitations is equal to the sum of the responses to each individual excitation;

– the response to k times the excitation (k = constant) is equal to k times the response to the excitation.

This concept of linearity generally imposes an assumption of weak displacements (for example, small relative displacement response of the mass of a one-degree-of-freedom system).

2.5.6. *Linear one-degree-of-freedom mechanical systems*

The simplest mechanical system consists of mass, stiffness and a damping device (Voigt model) (Figure 2.35). The response is calculated using a linear differential equation of the second order. Due to its simplicity, the results can be expressed in concise form, with a limited number of parameters.

The one-degree-of-freedom system is a model used for the analysis of mechanical shocks and vibrations (comparison of the severity of several excitations of the same nature or different nature, development of specifications, etc.). The implicit idea is that if a vibration (or a shock A) leads to a relative displacement response larger than a vibration B on a one-degree-of-freedom system, vibration A will be more severe than B on a more complex structure.

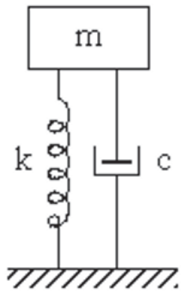


Figure 2.35. *Voigt model*

The displacement of an arbitrary system subjected to a stress being, in general, primarily produced by the response of the lowest frequency, this one-degree-of-freedom model very often makes it possible to obtain a good approximation to the result. For more precise stress calculations, the use of a more complicated mathematical model is sometimes necessary.

2.6. Setting an equation for n degrees-of-freedom lumped parameter mechanical system

Various methods can be used to write the differential equations of the movement of a several-degrees-of-freedom mechanical system with localized constants.

2.6.1. *Lagrange equations*

2.6.1.1. *General survey – definitions*

The differential equations describing the movement of a material point or a system can be established either starting from Newton's laws, or by using the Lagrange equations. There are two basically different approaches for the problems of dynamics.

Without rewriting the theory behind the Lagrange equations we will provide an overview with the aim of highlighting the definitions of the terms used. We will also show the approach that should be used when it comes to writing the equations.

The position of a system in space can be characterized by using s arbitrary parameters q_i where i, the *number of degrees of freedom of the system*, is an integer that ranges from 1 to s. q_i are the *generalized coordinates* and their derivatives $\dot{q}_i$ are the *generalized velocities*. The different s functions $q_i(t)$ will vary independently. The state of the system is completely and univocally determined by its coordinates and velocities.

The Hamilton principle (or the *principle of least action*) leads to the creation of the Lagrange equations: if the system considered is characterized by a function $L(q_i, \dot{q}_i, t)$, the system is then moved between two given positions for the times t_1 and t_2 so that the action

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt \qquad [2.45]$$

has the smallest value possible [LAN]. $L(q_i, \dot{q}_i, t)$ is then referred to as the system's Lagrange function. The principle of least action is written as:

$$\delta S = \delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt = 0 \qquad [2.46]$$

yielding

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}_i}\right) - \frac{\delta L}{\delta q_i} = 0 \qquad [2.47]$$

These equations are the Lagrange equations. For a material point in free movement, the Lagrange function can be written as follows:

$$L = \frac{m\, v^2}{2} \qquad [2.48]$$

where m is the mass of the weighted point and v is the velocity module.

For a system of n weighted points (which do not react with one another), which have a mass m_j and a velocity v_j, the Lagrange function can then be written as:

$$L = \sum_{j=1}^{n} \frac{m_j\, v_j^2}{2} \qquad [2.49]$$

(j = 1, 2, ..., n). In a closed system, in other words where the weighted points react with one another, and are isolated from any foreign body, the Lagrange function L needs to take into consideration the fact that these different weighted points react with one another:

$$L = \sum_{j} \frac{m_j\, v_j^2}{2} - E_P\left(\vec{r}_1, \vec{r}_2, \ldots\right) \qquad [2.50]$$

where E_P is a function of the points' coordinates, and depends on the interaction that occurs between the different points. r_j is the radius vector of the j^{th} point.

Definitions

The quantity $\sum_{j} \frac{m_j\, v_j^2}{2}$ is the *kinetic energy* of the system and the E_P function is the *potential energy*.

The Lagrange equation is therefore written as[1]:

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \vec{v}_j}\right) = \frac{\delta L}{\delta \vec{r}_j} \qquad [2.51]$$

1 The notation $\frac{\delta L}{\delta \vec{v}_j}$ or $\frac{\delta L}{\delta \vec{r}_j}$ does not mean a derivative of the scalar L with respect to the vector $\vec{v}_j$ or $\vec{r}_j$ (which does not have significance). By convention, this is the notation however which is used to represent a vector whose components are equal to the derivatives of L with respect to the corresponding components of the vector $\vec{v}_j$ or $\vec{r}_j$.

yielding, if L is replaced by its expression given in [2.50],

$$m_j \frac{d\vec{v}_j}{dt} = -\frac{\delta E_P}{\delta \vec{r}_j} = \vec{F}_j \qquad [2.52]$$

$\vec{F}_j$ is a force, whose components are $-\frac{\delta E_P}{\delta x_j}$, $-\frac{\delta E_P}{\delta y_j}$, $-\frac{\delta E_P}{\delta z_j}$, if the components of $\vec{r}_j$ are x_j, y_j, z_j

Whenever the system S_1 is moving in a given exterior field, thus interacting with another system S_2, the Lagrange function of the overall system $S = S_1 + S_2$ is calculated and it is thus the obtained closed system which is studied.

Certain properties of time and space (uniformity and isotropy) make it possible to establish laws which are known as *conservation laws*.

Uniformity of time allows us to write the following equation from the Lagrange equations of a closed system:

$$\sum_i \dot{q}_i \frac{\delta L}{\delta \dot{q}_i} - L = \text{constant} = E \qquad [2.53]$$

The energy of a system E, which is defined by [2.53], remains constant whenever a closed system moves.

In the case of a non-closed system, this law is also applicable if the exterior field does not depend on time.

The mechanical systems whose energy is conserved are known as conservative systems, and as a result the following equation can be written:

$$E = E_c\left(q, \dot{q}\right) + E_P\left(q\right) \qquad [2.54]$$

E_c, which is previously defined kinematic energy, is a function of the square root of the velocities. Its value is written $\sum_j \frac{m_j v_j^2}{2}$ in Cartesian coordinates. m_j is the mass of the point j, which has a velocity of v_j.

In addition to the Lagrange equation, the homogeneous properties of space show that in a closed mechanical system, the vector

$$\vec{P} = \sum_j \frac{\delta L}{\delta v_j} = \sum_j m_j \vec{v_j} \qquad [2.55]$$

remains unchanged during movement. The vector $\vec{P}$ is known as the *impulse* or the system's *movement quantity*. As far as the generalized coordinates q_i are concerned, *the generalized impulse* is as follows:

$$p_i = \frac{\delta L}{\delta \dot{q}_i} \qquad [2.56]$$

and the generalized force:

$$F_i = \frac{\delta L}{\delta q_i} \qquad [2.57]$$

NOTE.– p_i *are only components of* $\vec{P}$ *for Cartesian coordinates. They are not generally represented in the simple product of mass by velocity. They are linear functions of* $\dot{q}_i$ *[LAN 60].*

The isotropy of space makes it possible to demonstrate the conservation of a parameter which is known as a system's *kinetic moment*.

If movement is carried out in areas which offer any resistance and which tend to slow down the system, then part of the system's energy is converted into heat. This type of system is known as a *non-conservative system*. There is *dissipation of energy* or *damping*. In this case, if dissipation forces are proportional to the velocity, and if they are derived from a potential, then the Lagrange equation can be written as,

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}_i}\right) - \frac{\delta L}{\delta q_i} + \frac{\delta E_a}{\delta \dot{q}_i} = 0 \qquad [2.58]$$

where E_a is the *damping energy* (or *dissipation function*) [LAN 60].

2.6.1.2. *Application*

Let us consider a linear one-degree-of-freedom system. If the movement of such a system occurs along axis Ox, then the Lagrange function can be written as:

$$L = \frac{m\,\dot{x}^2}{2} - E_P(x) \qquad [2.59]$$

in the case of a closed system or for a system which is subject to constant exterior conditions. This yields:

$$E = \frac{m\,\dot{x}^2}{2} + E_P(z) \qquad [2.60]$$

The Lagrange equation is written as:

$$\frac{d}{dt}\left(\frac{\delta E_c}{\delta \dot{x}}\right) - \frac{\delta E_c}{\delta x} = -\frac{\delta E_P}{\delta x} \qquad [2.61]$$

This equation can be used to create equations of free oscillations for a (one-degree-of-freedom) undamped system.

If the system is made up of s degrees of freedom then the following equations are used:

$$\frac{d}{dt}\left(\frac{\delta E_c}{\delta \dot{q}_i}\right)-\frac{\delta E_c}{\delta q_i}=-\frac{\delta E_P}{\delta q_i}=F_i \quad [2.62]$$

in generalized coordinates, where each value of i corresponds to one degree of freedom. It is said that the system moves in a *potential force field*.

If the system is damped, there are forces which act against the free movement of the system that was initially excited. These forces are linear and non-linear functions of velocity.

Sometimes a *damping potential* E_a is defined. It is possible to introduce this potential into the Lagrange equations. In the general, and if the damping is viscous, we have:

$$\frac{d}{dt}\left(\frac{\delta E_c}{\delta \dot{q}_i}\right)-\frac{\delta E_c}{\delta q_i}=-\frac{\delta E_P}{\delta q_i}-\frac{\delta E_a}{\delta \dot{q}_i} \quad [2.63]$$

and if the system is linear:

$$\frac{d}{dt}\left(\frac{\delta E_c}{\delta \dot{x}}\right)-\frac{\delta E_c}{\delta x}=-\frac{\delta E_P}{\delta x}-\frac{\delta E_a}{\delta \dot{x}} \quad [2.64]$$

Whenever the system is not closed, the Lagrange equations can generally be written as follows whenever the value of E_a exists,

$$\frac{d}{dt}\left(\frac{\delta E_c}{\delta \dot{q}_i}\right)-\frac{\delta E_c}{\delta q_i}=-\frac{\delta E_P}{\delta q_i}-\frac{\delta E_a}{\delta \dot{q}_i}+F_i \quad [2.65]$$

(F_i = the generalized forces external with the system. These forces are not included in E_P).

Example 2.2. Using the Lagrange equations

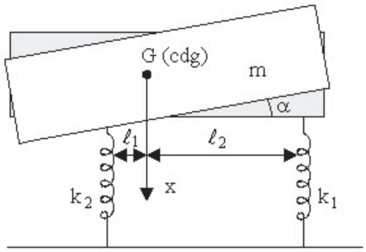


Figure 2.36. *Suspended mass with two springs*

Weight: m g

ρ = radius of gyration

$I = m\rho^2$ = moment of inertia of m with respect to the axis passing by the center of gravity

α = angle of rotation

x = vertical displacement of the center of gravity

Kinetic energy

$$E_c = \frac{m}{2}\dot{x}^2 + \frac{1}{2} I \dot{\alpha}^2 \qquad [2.66]$$

Potential energy

$$E_P = \frac{k_1}{2}(x - \ell_1 \alpha)^2 + \frac{k_2}{2}(x + \ell_2 \alpha)^2 \qquad [2.67]$$

Lagrange equation

$$L = \frac{1}{2}\left[m \dot{x}^2 + I \dot{\alpha}^2 - k_1 (x - \ell_1 \alpha)^2 - k_2 (x + \ell_2 \alpha)^2\right] \qquad [2.68]$$

Yielding

$$m \frac{d^2 x}{dt^2} + (k_1 + k_2) x + (k_2 \ell_2 - k_1 \ell_1) \alpha = 0 \qquad [2.69]$$

$$I \frac{d^2 \alpha}{dt^2} + \left(k_1 \ell_1^2 + k_2 \ell_2^2\right) \alpha + (k_2 \ell_2 - k_1 \ell_1) x = 0 \qquad [2.70]$$

α and x are independent if $k_1 \ell_1 = k_2 \ell_2$ [VOL 65], [WAL 84].

2.6.2. *D'Alembert's principle*

Using this principle, static equilibrium conditions can be applied to dynamic problems by considering the external exiting forces and the forces of reaction which are opposed to movement at the same time [CHE 66]:

– for any solid, the algebraic sum of the external applied forces and forces opposed to the movement are null in any direction.

This principle has an equivalent for systems in rotation:

– for any solid body, the algebraic sum of the external applied couples and resistive torques around an arbitrary axis are null.

2.6.3. *Free-body diagram*

One of the most useful tools which can be used to resolve problems linked to statics is the free-body diagram (FBD). The FBD relies on the fundamental principles of statics. If an entire system is in equilibrium then each of its individual components is also in equilibrium.

The FBD is a diagram which represents an element taken from a dynamic system. Such an element is taken away from its original environment, and from all the elements which surround it. Any interactions with these elements are replaced by force vectors. The FBD is therefore a simplified representation of an often complex system, where the system is divided into smaller, simpler elements to be studied. All the physical attributes of the structure are removed and are only represented for an element given the forces to which it is subjected.

The links with the neighboring elements are not directly represented in the FBD (which is where the name free-body diagram comes from). These links are only created because of the forces that are transmitted.

The drawing of an FBD is an important phase when it comes to finding a solution to mechanical problems. The FBD helps visualize all of the forces that act on a simple object and also helps resolve any equilibrium problems.

Components of the free-body diagram

Some components are necessary if the aim is to create a worthwhile FBD. The first and most important component is the object, i.e. the part of the structure which is represented on the diagram as a rectangle.The size and shape of the actual object are not important.

The second most important component to be included in the diagram is the force. The force is represented by a single arrow (→). The direction of the arrow and its size are important for working out calculations:

– the direction of the arrow shows the direction in which the force acts. More often than not, the direction is unknown. An arbitrary direction needs to be chosen. Resolving the equations which determine the position of equilibrium makes it possible to verify if the direction which is chosen is the correct one or not. If the answer is negative, then the direction is reversed;

– the size of the arrow represents a force's amplitude. Each arrow in the diagram must be labeled uniquely so that it is possible to see what exact type of force has been represented in the diagram. All of the forces which act on an object in a given situation must be represented in the FBD, unless they are specifically and voluntarily ignored.

There are, of course, forces whose characteristics are not known when an FBD is being created, in particular those which act on the contact point between the studied object and other close parts not being reproduced on the diagram.

Types of forces used in an FBD

Several different forces can be represented in an FBD. The most common forces include:

– contact forces, which include:

- normal forces,
- friction forces,
- aerodynamic resistance forces,
- forces which are applied by a person or by another object (traction, thrust, etc.),
- tension;

– forces having a remote action, which include

- gravity,
- electric forces,
- magnetic forces.

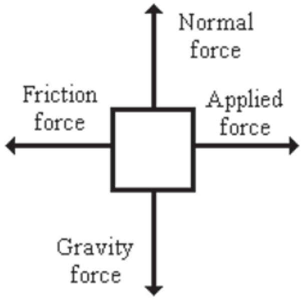


Figure 2.37. *Free-body diagram*

The first force to be considered, the most frequently observed, is the force of gravity. The acceleration which is due to gravity (on Earth) is roughly $g = 9.8 \text{ m/s}^2$.

The normal force is the force which prevents an object resting on a surface from falling; it is always perpendicular to the surface on which the object rests. If an object is resting on a non-horizontal surface, the normal force is perpendicular to this surface.

If the contact surface is smooth then there will be no friction and the reaction force acts in its normal position to the tangent of the surface at the point of contact. If this surface is flat, then the reaction force is always perpendicular to the surface.

Friction is a force which is linked to the normal force. This is because friction is also linked to the surface with which an object is in contact. Unlike the normal force, which acts perpendicularly to the surface on which the object is placed, friction always acts parallel to the surface on which the object is placed. Friction prevents or acts against movement; the vector which is used to represent friction having the same direction as the friction force.

There are two types of friction which can act on an object:

– static friction, which is produced when an object is at rest. This is the force which makes it difficult for an object to start moving;

– dynamic friction, which takes place when an object is moving. It is this force which slows down or even stops the movement of an object.

Thrust and traction: thrust is created by a liquid or by the wind, traction of an object by a cable. A flexible object which has little or no rigidity (such as a rope or a chain, etc.) only creates traction on another object according to the axis of the flexible body.

The last of these common forces is the force of tension. Tension occurs when two forces act on the extremities of an object (for example, the force which is transmitted when pulling on a spring).

Not all of these forces are generally present at the same time.

NOTES.–

Masses are not forces.

Do not confuse movement and force.

Do not include fictional forces such as centrifugal forces in an FBD.

Identify pairs of forces, such as those which are grouped together in Newton's third law, clearly.

Do not forget forces, and do not add forces that do not exist.

There are no forces on elements of a mechanical device that are still connected together.

Reaction forces cannot be produced anywhere other than at a point of contact.

Example 2.3. Linear one degree-of-freedom system

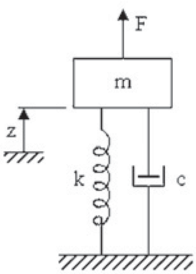


Figure 2.38. *Linear one degree-of freedom system*

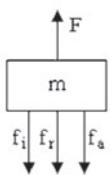


Figure 2.39. *Free-body diagram of the one degree-of-freedom system*

The system is initially assumed to be in equilibrium.

Inertial force

$$F_i = -m \frac{d^2z}{dt^2} \quad [2.71]$$

Damping force

$$F_d = -c\,\dot{z} \qquad [2.72]$$

Restoring force

$$F_r = -k\,z \qquad [2.73]$$

External force: F.

According to d'Alembert's theorem, the sum of all forces acting on a body, including the inertial force, is equal to zero. This yields

$$m\,\frac{d^2z}{dt^2} + c\,\frac{dz}{dt} + k\,z = F \qquad [2.74]$$

To avoid the possible errors of sign during the evaluation of these forces when the system is complex, the following rule can be used [STE 73]:

For each mass m_i of the model, it is assumed that all the forces associated with mass m_i are positive and that all the forces associated with the other masses m_j $(j \neq i)$ are negative.

In practice, for each mass m_i, the sum of the damping, spring and inertia forces is made equal to zero as follows:

– *inertial force*: positive, equal to $m_i\ \ddot{y}_i$;

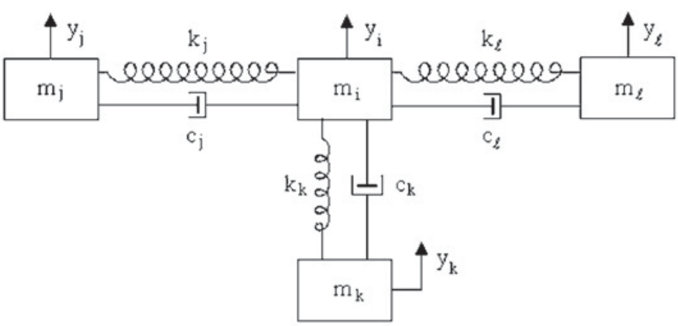


Figure 2.40. *Example of a lumped parameter system*

– *restoring force*: equal to $k_i\left(y_i - y_j\right)$, with k_i being the stiffness of each elastic element connected to the mass m_i, $y_i - y_j$ being written while starting with coordinate y_i of mass m_i, and y_j being the coordinate of the other end of each spring;

– *damping force*: same rule as for the stiffnesses, with the first derivative of $c_i\left(\dot{y}_i - \dot{y}_j\right)$.

Mass m_i (Figure 2.40) is as follows:

$$m_i\,\ddot{y}_i + c_j\left(\dot{y}_i - \dot{y}_j\right) + k_j\left(y_i - y_j\right) + c_k\left(\dot{y}_i - \dot{y}_k\right) + k_k\left(y_i - y_k\right)$$

$$+c_\ell\left(\dot{y}_i - \dot{y}_\ell\right) + k_\ell\left(y_i - y_\ell\right) = 0 \qquad [2.75]$$

Example 2.4. System with five degrees of freedom

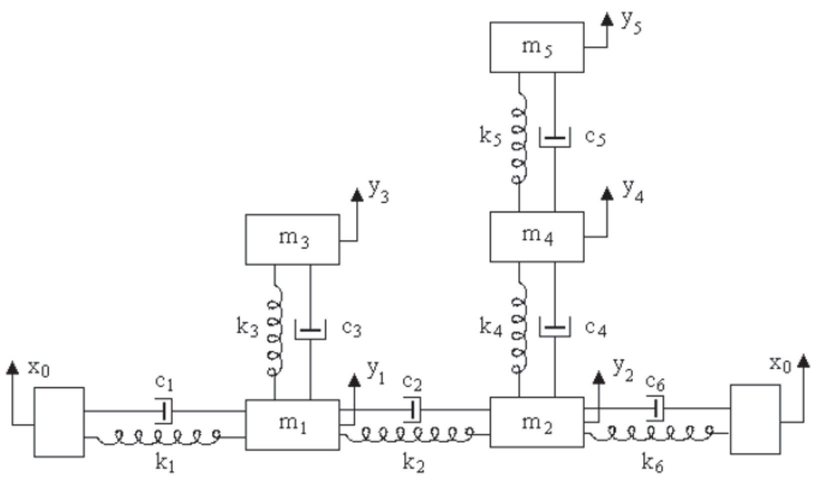


Figure 2.41. *System with five degrees of freedom*

Mass m_1

$$m_1\ddot{y}_1 + c_1(\dot{y}_1 - \dot{x}_0) + k_1(y_1 - x_0) + k_2(y_1 - y_2) + c_2(\dot{y}_1 - \dot{y}_2) + k_3(y_1 - y_3) + c_3(\dot{y}_1 - \dot{y}_3) = 0 \qquad [2.76]$$

Mass m_2

$$m_2\ddot{y}_2 + c_2(\dot{y}_2 - \dot{y}_1) + k_2(y_2 - y_1) + c_4(\dot{y}_2 - \dot{y}_4) + k_4(y_2 - y_4) + c_6(\dot{y}_2 - \dot{x}_0) + k_6(y_2 - x_0) = 0 \qquad [2.77]$$

Mass m_3

$$m_3\ddot{y}_3 + c_3(\dot{y}_3 - \dot{y}_1) + k_3(y_3 - y_1) = 0 \qquad [2.78]$$

Mass m_4

$$m_4\ddot{y}_4 + c_4(\dot{y}_4 - \dot{y}_2) + k_4(y_4 - y_2) + c_5(\dot{y}_4 - \dot{y}_5) + k_5(y_4 - y_5) = 0 \qquad [2.79]$$

Mass m_5

$$m_5\ddot{y}_5 + c_5(\dot{y}_5 - \dot{y}_4) + k_5(y_5 - y_4) = 0 \qquad [2.80]$$

Hence the system of equations:

$$\begin{cases} m_1\ddot{y}_1 + (c_1 + c_2 + c_3)\dot{y}_1 + (k_1 + k_2 + k_3)y_1 - c_2\dot{y}_2 - k_2 y_2 - c_3\dot{y}_3 - k_3 y_3 = c_1\dot{x}_0 + k_1 x_0 \\ -c_2\dot{y}_1 - k_2 y_1 + m_2\ddot{y}_2 + (c_2 + c_4 + c_6)\dot{y}_2 + (k_2 + k_4 + k_6)y_2 - c_4\dot{y}_4 - k_4 y_4 = c_6\dot{x}_0 + k_6 x_0 \\ -c_3\dot{y}_1 - k_3 y_1 + m_3\ddot{y}_3 + c_3\dot{y}_3 + k_3 y_3 = 0 \\ -c_4\dot{y}_2 - k_4 y_2 + m_4\ddot{y}_4 + (c_4 + c_5)\dot{y}_4 + (k_4 + k_5)y_4 - c_5\dot{y}_5 - k_5 y_5 = 0 \\ -c_5\dot{y}_4 - k_5 y_4 + m_5\ddot{y}_5 + c_5\dot{y}_5 + k_5 y_5 = 0 \end{cases} \qquad [2.81]$$

Use of the Lagrange equation

This differential equation of movement can also be obtained from Lagrange equation [2.63]:

$$\frac{d}{dt}\left(\frac{\delta E_c}{\delta \dot{y}_i}\right) - \frac{\delta E_c}{\delta y_i} + \frac{\delta E_P}{\delta y_i} + \frac{\delta E_a}{\delta \dot{y}_i} = 0$$

where

$$E_c = \frac{1}{2}\sum_i m_i\,\dot{y}_i \qquad [2.82]$$

$$E_P = \frac{1}{2}k_1(y_1 - x_0)^2 + \frac{1}{2}k_2(y_2 - y_1)^2 + \frac{1}{2}k_3(y_3 - y_1)^2 + \frac{1}{2}k_4(y_4 - y_2)^2 + \frac{1}{2}k_5(y_5 - y_4)^2 + \frac{1}{2}k_6(y_2 - x_0)^2 \qquad [2.83]$$

$$E_a = \frac{1}{2} c_1 (\dot{y}_1 - \dot{x}_0)^2 + \frac{1}{2} c_2 (\dot{y}_2 - \dot{y}_1)^2 + \frac{1}{2} c_3 (\dot{y}_3 - \dot{y}_1)^2 + \frac{1}{2} c_4 (\dot{y}_4 - \dot{y}_2)^2 + \frac{1}{2} c_5 (\dot{y}_5 - \dot{y}_4)^2 + \frac{1}{2} c_6 (\dot{y}_2 - \dot{x}_0)^2 \quad [2.84]$$

$$\frac{d}{dt}\left(\frac{\delta E_c}{\delta \dot{y}_i}\right) = m_i \ddot{y}_i \quad [2.85]$$

$$\begin{cases} \dfrac{\delta E_p}{\delta y_1} = k_1 (y_1 - x_0) - k_2 (y_2 - y_1) - k_3 (y_3 - y_1) \\ \dfrac{\delta E_p}{\delta y_2} = k_2 (y_2 - y_1) - k_4 (y_4 - y_2) + k_6 (y_2 - x_0) \\ \dfrac{\delta E_p}{\delta y_3} = k_3 (y_3 - y_1) \\ \dfrac{\delta E_p}{\delta y_4} = k_4 (y_4 - y_2) - k_5 (y_5 - y_4) \\ \dfrac{\delta E_p}{\delta y_5} = k_5 (y_5 - y_4) \end{cases} \quad [2.86]$$

$$\frac{\delta E_c}{\delta y_i} = 0 \quad [2.87]$$

$$\begin{cases} \dfrac{\delta E_a}{\delta \dot{y}_1} = c_1 (\dot{y}_1 - \dot{x}_0) - c_2 (\dot{y}_2 - \dot{y}_1) - c_3 (\dot{y}_3 - \dot{y}_1) \\ \dfrac{\delta E_a}{\delta \dot{y}_2} = c_2 (\dot{y}_2 - \dot{y}_1) - c_4 (\dot{y}_4 - \dot{y}_2) + c_6 (\dot{y}_2 - \dot{x}_0) \\ \dfrac{\delta E_a}{\delta \dot{y}_3} = c_3 (\dot{y}_3 - \dot{y}_1) \\ \dfrac{\delta E_a}{\delta \dot{y}_4} = c_4 (\dot{y}_4 - \dot{y}_2) - c_5 (\dot{y}_5 - \dot{y}_4) \\ \dfrac{\delta E_a}{\delta \dot{y}_5} = c_5 (\dot{y}_5 - \dot{y}_4) \end{cases} \quad [2.88]$$

Hence [2.81]

$$\begin{cases} m_1\,\ddot{y}_1 + k_1\,(y_1 - x_0) - k_2\,(y_2 - y_1) - k_3\,(y_3 - y_1) + c_1\,(\dot{y}_1 - \dot{x}_0) - c_2\,(\dot{y}_2 - \dot{y}_1) - c_3\,(\dot{y}_3 - \dot{y}_1) = 0 \\ m_2\,\ddot{y}_2 + k_2\,(y_2 - y_1) - k_4\,(y_4 - y_2) + k_6\,(y_2 - x_0) + c_2\,(\dot{y}_2 - \dot{y}_1) - c_4\,(\dot{y}_4 - \dot{y}_2) + c_6\,(\dot{y}_2 - \dot{x}_0) = 0 \\ m_3\,\ddot{y}_3 + k_3\,(y_3 - y_1) + c_3\,(\dot{y}_3 - \dot{y}_1) = 0 \\ m_4\,\ddot{y}_4 + k_4\,(y_4 - y_2) - k_5\,(y_5 - y_4) + c_4\,(\dot{y}_4 - \dot{y}_2) - c_5\,(\dot{y}_5 - \dot{y}_4) = 0 \\ m_5\,\ddot{y}_5 + k_5\,(y_5 - y_4) + c_5\,(\dot{y}_5 - \dot{y}_4) = 0 \end{cases}$$

Chapter 3

Response of a Linear One-Degree-of-Freedom Mechanical System to an Arbitrary Excitation

3.1. Definitions and notation

Any mechanical system can be represented by a combination of the three pure *elements:* mass, spring and damping device (Chapter 2). This chapter examines the movement of the simplest possible systems comprising one, two or three of these different elements when they are displaced from their rest position at an initial instant of time. The movement of mass alone is commonplace and without practical or theoretical interest for our applications. The cases of a spring alone, a damping device alone or a damper–spring system are really not of much more practical interest than any real system just having a mass. The simplest systems which are of interest are those composed of:

– mass and spring;

– mass, spring and damping.

We will consider the spring and damping to be linear and that the mass m can move in a frictionless manner along a vertical axis (for example) [AKA 69]. This system can be excited by:

– a force applied to the mass m, with the spring and the damping device being fixed to a rigid support (Figure 3.1(a));

– a movement (displacement, velocity or acceleration) of a massless rigid moving support (Figure 3.1(b)).

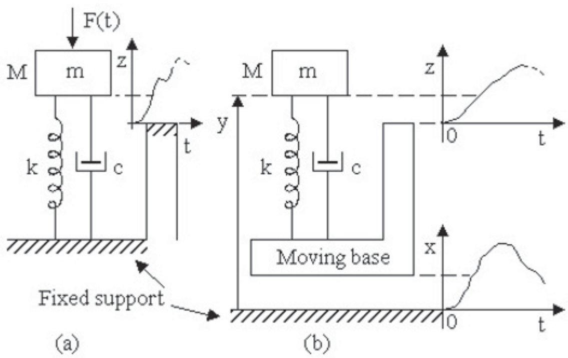


Figure 3.1. *Mass–spring–damping system*

One variable is enough at each instant of time to define the position of mass m on the axis 0z since it is *a one-degree-of-freedom system.* The origin of the abscissa of mass m is the point where the mass is at rest (unstretched spring). Gravity is ignored, even if the axis 0z is vertical (as in the case of Figure 3.1). It can be seen that the movement of m around its new static equilibrium position is the same as that around the rest position, excluding gravity.

Note that:

$x(t)$	is the absolute displacement of the support with respect to a fixed reference (Figure 3.1(b));
$\dot{x}(t)$ and $\ddot{x}(t)$	are the corresponding velocity and acceleration;
$y(t)$	is the absolute displacement of mass m with respect to a fixed reference;
$\dot{y}(t)$ and $\ddot{y}(t)$	are the corresponding velocity and acceleration;
$z(t)$	is the relative displacement of the mass relative to the support. To consider only the variations of z around this position of equilibrium (point 0) and to eliminate length from the spring at rest, the support was drawn so that it goes up to point 0 in Figure 3.1;
$\dot{z}(t)$ and $\ddot{z}(t)$	are the corresponding velocity and acceleration;
$F(t)$	is the force applied directly to mass m (Figure 3.1(a)).

NOTE.– *In the case of Figure 3.1(a), we have* $y \equiv z$.

The movement considered is a small excitation around the equilibrium position of the system. The excitations $x(t)$, $\dot{x}(t)$, $\ddot{x}(t)$ or $F(t)$ can be of a different nature, i.e. sinusoidal, swept sine, random or shock.

3.2. Excitation defined by force versus time

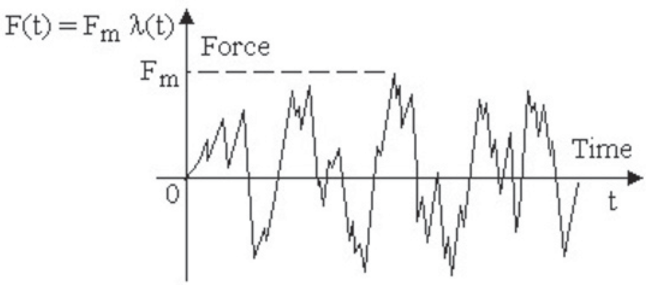


Figure 3.2. *Force versus time*

Let $F(t)$ be the force applied to mass m of a mass–spring–damping oscillator (one-degree-of-freedom system) [BAR 61], [FUN 58] (Figure 3.3).

We will express the excitation $F(t)$ in a dimensionless form $\lambda(t)$ such that:

$$F(t) = F_m \; \lambda(t)$$

$$\lambda(t) = 0 \quad \text{for } t \le 0 \qquad [3.1]$$

$$\max \lambda(t) = \lambda(t_m) = 1$$

The spring is assumed to be linear in the elastic range, with one end fixed and the other connected to the mass.

The forces which act on the mass m are:

– inertia $m \dfrac{d^2 z}{dt^2}$;

– an elastic force due to the spring, of value $-k\ z$ (restoring force), so long as it follows Hooke's law of proportionality between the forces and deformations. This force is directed in the opposite direction to the displacement;

– a resistant damping force, proportional and opposing the velocity $\frac{dz}{dt}$ of the mass, $-c\ \frac{dz}{dt}$;

– the imposed external force $F(t)$.

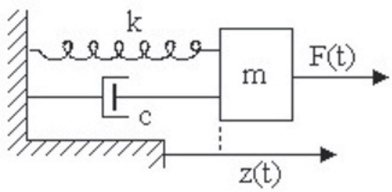


Figure 3.3. *Force on a one-degree-of-freedom system*

The result of the forces acting on mass m (the spring and the damping device of our model being supposed to be ideal and without mass), $-k\ z(t) - c\ \frac{dz}{dt} + F(t)$, thus obeys Newton's second law:

$$m\frac{d^2z}{dt^2} = -k\ z - c\ \frac{dz}{dt} + F(t) \qquad [3.2]$$

This is the *differential equation of the movement of the vibrating one-degree-of-freedom system [DEN 60],* yielding:

$$\frac{d^2z}{dt^2} + \frac{c}{m}\frac{dz}{dt} + \frac{k}{m}\ z = \frac{F(t)}{m} \qquad [3.3]$$

If we set

$$\xi = \frac{c}{2\ m\ \omega_0} \qquad [3.4]$$

and

$$\omega_0^2 = \frac{k}{m} \tag{3.5}$$

it becomes:

$$\frac{d^2 z}{dt^2} + 2\,\xi\,\omega_0\,\frac{dz}{dt} + \omega_0^2\,z = \frac{F_m}{m}\,\lambda(t) \tag{3.6}$$

ω_0 is *the natural pulsation* of the system or *angular frequency* and is expressed in radians per second (the pulsation of the undamped oscillator when the mass is moved away from its equilibrium position). It is only a characteristic of the system as long as the *small oscillation* assumption is checked (i.e. as long as it can be assumed that the potential energy is a function of the square of the coordinate) [POT 48].

The *natural period* of the system is defined as:

$$T_0 = \frac{2\,\pi}{\omega_0} \tag{3.7}$$

and the *natural frequency*

$$f_0 = \frac{\omega_0}{2\,\pi} \tag{3.8}$$

where T_0 is expressed in seconds (or its submultiples) and f_0 in Hertz (1 Hz = 1 cycle/s).

ξ is the *damping factor* or *damping ratio*: $\xi = \dfrac{c}{2\,m\,\omega_0} = \dfrac{c}{2\sqrt{k\,m}}$.

NOTE.–

When the mass moves horizontally without friction on a perfectly smooth surface, we do not have to consider other forces. The rest position is then both the equilibrium position of the mass and the unstretched position of the spring.

If we assume that the mass is suspended on the spring and is moving along a vertical axis, an application of the equation can be made, either by only considering the equilibrium position, or by considering the rest position of the spring.

If we count the amplitude z starting from the equilibrium position, i.e. starting from position 0, where the force of gravity $m\,g$ *is balanced by the spring force* $k\,z_{eq}$ *(*z_{eq} *being the deflection of the spring due to gravity g, measured from the point 0), its inclusion in the equation is absolutely identical to that of the preceding paragraphs.*

If, however, we count amplitude z from the end of the spring in its rest position 0, z_1 *is equal to* $z + z_{eq}$*; it is thus necessary to replace z by* $z + z_{eq}$ *in equation [3.6] and to add in the second member, a force m g. After simplification (*$k\,z_{eq} = m\,g$*), the final result is, of course, the same. In all the following sections, regardless of excitation, the force m g will not be taken into account.*

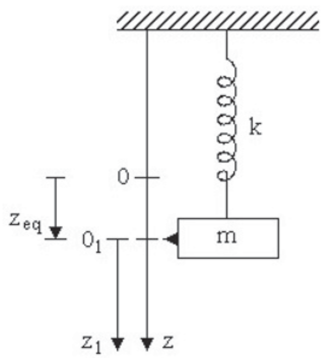


Figure 3.4. *Equilibrium position*

If we set $u = z$ and

$$\ell(t) = \frac{F(t)}{k} = \frac{F_m}{k}\,\lambda(t) \qquad [3.9]$$

then equation [3.6] can be written in the form:

$$\ddot{u} + 2\,\xi\,\omega_0\,\dot{u} + \omega_0^2\,u = \ell(t)\,\omega_0^2 \qquad [3.10]$$

3.3. Excitation defined by acceleration

The base (support of the one-degree-of-freedom system) receives *an excitation*, which we will assume to be defined by a known acceleration $\ddot{x}(t)$. The excitation is propagated towards the mass through elements k and c. The disturbance which m undergoes is translated by a *response movement*.

Excitation and response are not independent entities, but are mathematically related (Figure 3.5).

We will assume that:

– the simple one-degree-of-freedom system (Figure 3.6) is such that mass and base are driven in the same direction;

– the movement $x(t)$ of the support is not affected by the movement of the equipment which it supports.

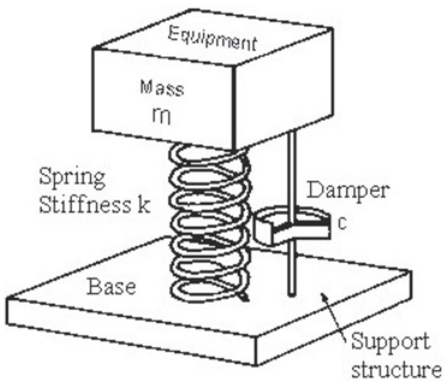


Figure 3.5. *One-degree-of-freedom system*

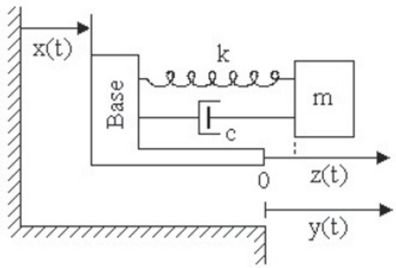


Figure 3.6. *Acceleration on the one-degree-of-freedom system*

The excitation is this known movement $x(t)$ of the support or the acceleration $\ddot{x}(t)$ communicated to this support. The equation for the movement is written:

$$m \frac{d^2y}{dt^2} = -k\ (y - x) - c \left(\frac{dy}{dt} - \frac{dx}{dt} \right) \qquad [3.11]$$

i.e. using the same notation as before:

$$\frac{d^2y(t)}{dt^2} + 2\ \xi\ \omega_0\ \frac{dy(t)}{dt} + \omega_0^2\ y(t) = \omega_0^2\ x(t) + 2\ \xi\ \omega_0\ \frac{dx(t)}{dt} \qquad [3.12]$$

The relative displacement of the mass relative to the base is equal to

$$z(t) = y(t) - x(t) \qquad [3.13]$$

yielding, after the elimination of y (absolute displacement of m) and its derivative,

$$\frac{d^2z}{dt^2} + 2\ \xi\ \omega_0\ \frac{dz}{dt} + \omega_0^2\ z(t) = -\frac{d^2x}{dt^2} \qquad [3.14]$$

If we set $u = z$ and $\ell(t) = -\frac{\ddot{x}}{\omega_0^2}$ (generalized excitation), the equation above can be written as:

$$\ddot{u}(t) + 2\ \xi\ \omega_0\ \dot{u} + \omega_0^2\ u = \omega_0^2\ \ell(t) \qquad [3.15]$$

This similar equation for an excitation by force or acceleration is known as the *generalized form.*

3.4. Reduced form

3.4.1. *Excitation defined by a force on a mass or by an acceleration of support*

Let us set, according to case, $z_s = -\frac{\ddot{x}_m}{\omega_0^2}$ or $z_s = \frac{F_m}{k}$. This parameter is the maximum static relative displacement, indeed:

$$z_s = \frac{F_m}{k} = \frac{\max |F(t)|}{k} = \frac{\max |F(t)|}{m\,\omega_0^2} \qquad [3.16]$$

$$z_s = -\frac{\ddot{x}_m}{\omega_0^2} = -\frac{m\,\ddot{x}_m}{m\,\omega_0^2} = \frac{\text{max. of static force corresponding to the max. of } \ddot{x}(t)}{m\,\omega_0^2} \qquad [3.17]$$

Let us note that $\ell_m = z_s = \dfrac{\ell(t)}{\lambda(t)}$.

$$\left(\ell_m = \text{maximum of } \ell(t) = \begin{bmatrix} -\dfrac{\ddot{x}_m}{\omega_0^2} \\ \dfrac{F_m}{k} \end{bmatrix}\right),$$

from [3.6] or [3.14]:

$$\frac{\ddot{u}(t)}{\ell_m} + 2\,\xi\,\omega_0\,\frac{\dot{u}(t)}{\ell_m} + \omega_0^2\,\frac{u(t)}{\ell_m} = \omega_0^2\,\frac{\ell(t)}{\ell_m} \qquad [3.18]$$

NOTE.– $\ell(t)$ *has the dimension of a displacement.*

Let us set $q = \dfrac{u}{\ell_m}$:

$$\ddot{q}(t) + 2\,\xi\,\omega_0\,\dot{q} + \omega_0^2\,q = \omega_0^2\,\lambda(t) \qquad [3.19]$$

and

$$\omega_0\,t = \theta \qquad [3.20]$$

$$\frac{dq}{dt} = \frac{dq}{d\theta}\,\frac{d\theta}{dt} = \omega_0\,\frac{dq}{d\theta}$$

$$\frac{d^2q}{dt^2} = \frac{d^2q}{d\theta^2}\left(\frac{d\theta}{dt}\right)^2 = \omega_0^2\,\frac{d^2q}{d\theta^2}$$

yielding:

$$\frac{d^2q}{d\theta^2} + 2\,\xi\,\frac{dq}{d\theta} + q(\theta) = \lambda(\theta) \qquad [3.21]$$

System	Excitation		Amplitude of the excitation	Reduced response
	Real	Generalized $\ell(t)$	ℓ_m	$q(t)$
Fixed base	$F(t)$	$\frac{F(t)}{k}$	$z_s = \frac{F_m}{k}$	$\frac{z(t)}{\ell_m}$
Moving base	$\ddot{x}(t)$	$-\frac{\ddot{x}(t)}{\omega_0^2}$	$z_s = -\frac{\ddot{x}_m}{\omega_0^2}$	$\frac{z(t)}{\ell_m}$

Table 3.1. *Reduced variables*

A problem of vibration or shock transmitted to the base can thus be replaced by the problem of force applied to the mass of the resonator.

3.4.2. *Excitation defined by velocity or displacement imposed on support*

We showed in equation [3.12] that the equation of the movement of the system can be put in the form:

$$\ddot{y} + 2\,\xi\,\omega_0\,\dot{y} + \omega_0^2\,y = 2\,\xi\,\omega_0\,\dot{x} + \omega_0^2\,x \qquad [3.22]$$

By double differentiation we obtain:

$$\frac{d^2\ddot{y}}{dt^2} + 2\,\xi\,\omega_0\,\frac{d\ddot{y}}{dt} + \omega_0^2\,\ddot{y} = 2\,\xi\,\omega_0\,\frac{d^2\dot{x}}{dt^2} + \omega_0^2\,\frac{d^2x}{dt^2} \qquad [3.23]$$

If the excitation is a displacement $x(t)$ and if the response is characterized by the absolute displacement $y(t)$ of the mass, the differential equation of movement [3.22] can be written as [3.24], while setting:

$$\ell(t) = x(t)$$

$$u(t) = y(t)$$

$$\ddot{u} + 2\,\xi\,\omega_0\,\dot{u} + \omega_0^2\,u = 2\,\xi\,\omega_0\,\dot{\ell} + \omega_0^2\,\ell \qquad [3.24]$$

If the excitation is the velocity $\dot{x}(t)$ and if $\dot{y}(t)$ is the response, equation [3.23] is written as [3.25], while noting:

$$\ell(t) = \dot{x}(t)$$

$$u(t) = \dot{y}(t)$$

$$\ddot{u} + 2\,\xi\,\omega_0\,\dot{u} + \omega_0^2\,u = 2\,\xi\,\omega_0\,\dot{\ell} + \omega_0^2\,\ell \qquad [3.25]$$

In the same way, if the input is acceleration $\ddot{x}(t)$ and response $\ddot{y}(t)$, we have, with:

$$\ell(t) = \ddot{x}(t)$$

$$u(t) = \ddot{y}(t)$$

$$\ddot{u} + 2\,\xi\,\omega_0\,\dot{u} + \omega_0^2\,u = 2\,\xi\,\omega_0\,\dot{\ell} + \omega_0^2\,\ell \qquad [3.26]$$

This equation is thus another generalized form applicable to a movement imposed on the base and an *absolute response.*

Reduced form

Let us set, as before, $\ell(t) = \ell_m\,\lambda(t)$ (ℓ_m = maximum of $\ell(t)$) and $\omega_0\,t = \theta$, this then becomes:

$$\ddot{q}(\theta) + 2\,\xi\,\dot{q}(\theta) + q(\theta) = 2\,\xi\,\dot{\lambda}(\theta) + \lambda(\theta) \qquad [3.27]$$

NOTE.– *If $\xi = 0$, then equations [3.21] and [3.27] take the single form*

$$\ddot{q}(\theta) + q(\theta) = \lambda(\theta)$$

The excitation for the relative motion is simply the inertial force $m\,\ddot{x}(t)$ required by the adoption of an accelerating frame of reference [CRA 58]. We will find an application of this property in the study of shock response spectra [LAL 75]. These reduced forms could be used for the solution of the equations. The following table indicates the input and response parameters corresponding to the variables $\ell(t)$ and $u(t)$ [SUT 68].

System	**Excitation $\ell(t)$**		**Response $u(t)$**
Fixed base	Force on the mass	$F(t)/k$	Mass relative displacement $z(t)$
		$F(t)$	Reaction force on base $F_T(t)$
Moving base	Base displacement $x(t)$		Mass absolute displacement $y(t)$
	Base velocity $\dot{x}(t)$		Mass absolute velocity $\dot{y}(t)$
	Base acceleration	$\ddot{x}(t)$	Mass absolute acceleration $\ddot{y}(t)$
		$-\dfrac{\ddot{x}(t)}{\omega_0^2}$	Relative displacement of spring $z(t)$
		$m\ \ddot{x}(t)$	Reaction force on base $F_T(t)$

Table 3.2. *Values of the reduced variables*

The solution of these two types of differential equation will make it possible to solve all the problems set by these inputs and responses. In practice, however, we will have to choose between the two formulations according to the parameter response desired, which is generally the relative displacement, related to the stress in the simple system.

The more usual case is when the excitation is an acceleration. Equation [3.21] is then essential. If the excitation is characterized by a base displacement, the differential equation will in response provide the absolute mass displacement. To return to the stresses we will have to calculate the relative displacement $y - x$.

3.5. Solution of the differential equation of movement

3.5.1. *Methods*

When the excitation can be expressed in a suitable analytical form, the differential equation of the movement can be explicitly solved for $q(\theta)$ or $u(t)$. When this is not the case, the response must be sought using analog or digital techniques.

The solution $q(\theta)$ can be obtained either by the traditional method of the variation of constants method, or by using the properties of Fourier or Laplace transforms. It is this last, faster method that we will generally use in the following sections and chapters.

Duhamel integral

A more general method consists of solving the differential equation in the case of an arbitrary excitation $\lambda(\theta)$ with, for example, the Laplace transform. The solution $q(\theta)$ can then be expressed in the form of an integral which, according to the nature of $\lambda(\theta)$ (numerical data, function leading to an analytically integrable expression), can be calculated numerically or analytically.

3.5.2. *Relative response*

3.5.2.1. *General expression for response*

The Laplace transform of the solution of a differential second order equation of the form:

$$\frac{d^2q}{d\theta^2}+a\ \frac{dq}{d\theta}+b\ q(\theta)=\lambda(\theta) \qquad [3.28]$$

can be written as (see Appendix) [LAL 75]:

$$Q(p)=\frac{\Lambda(p)+p\ q_0+a\ q_0+\dot{q}_0}{p^2+a\ p+b} \qquad [3.29]$$

where $\Lambda(p)$ is the Laplace transform of $\lambda(\theta)$

$$\Lambda(p)=L[\lambda(\theta)] \qquad [3.30]$$

$$q_0 = q(0)$$

$$\dot{q}_0 = \dot{q}(0)$$

and, in our case:

$$a = 2\xi$$

$$b = 1$$

After the solution of the rational fraction $\dfrac{p\, q(0) + 2\,\xi\, q(0) + \dot{q}(0)}{p^2 + 2\,\xi\, p + 1}$ into simple elements, [3.29] becomes, with p_1 and p_2 the roots of $p^2 + 2\,\xi\, p + 1 = 0$,

$$Q(p) = \frac{\Lambda(p)}{p_1 - p_2}\left[\frac{1}{p - p_1} - \frac{1}{p - p_2}\right] + \frac{1}{p_1 - p_2}\left[\frac{q_0\, p_1 + 2\,\xi\, q_0 + \dot{q}_0}{p - p_1} - \frac{q_0\, p_2 + 2\,\xi\, q_0 + \dot{q}_0}{p - p_2}\right]$$

[3.31]

The response $q(\theta)$ is obtained by calculating the original of $Q(p)$ [LAL 75]:

$$q(\theta) = \int_0^\theta \frac{\lambda(\alpha)}{p_1 - p_2}\left[e^{p_1(\theta-\alpha)} - e^{p_2(\theta-\alpha)}\right] d\alpha$$

$$+ \frac{1}{p_1 - p_2}\left[\left(q_0\, p_1 + 2\,\xi\, q_0 + \dot{q}_0\right) e^{p_1\,\theta} - \left(q_0\, p_2 + 2\,\xi\, q_0 + \dot{q}_0\right) e^{p_2\,\theta}\right] \quad [3.32]$$

where α is a variable of integration.

Particular case

For a system initially at rest:

$$q_0 = \dot{q}_0 = 0 \quad [3.33]$$

Then:

$$q(\theta) = \int_0^\theta \frac{\lambda(\alpha)}{p_1 - p_2}\left[e^{p_1(\theta-\alpha)} - e^{p_2(\theta-\alpha)}\right] d\alpha \quad [3.34]$$

Movement $q(\theta)$ is different according to the nature of roots p_1 and p_2 of $p^2 + 2\,\xi\,p + 1 = 0$.

3.5.2.2. *Subcritical damping*

In this case, roots p_1 and p_2 of the denominator $p^2 + 2\,\xi\,p + 1$ are complex:

$$p_{1,\,2} = -\xi \pm i\,\sqrt{1-\xi^2} \quad \text{(i.e. } 0 \le \xi < 1\text{)} \tag{3.35}$$

While replacing p_1 and p_2 with these expressions, response $q(\theta)$ given by equation [3.32] becomes:

$$q(\theta) = \frac{1}{\sqrt{1-\xi^2}} \int_0^\theta \lambda(\alpha)\, e^{-\xi\,(\theta-\alpha)} \sin\sqrt{1-\xi^2}\,(\theta-\alpha)\, d\alpha$$

$$+ e^{-\xi\,\theta}\left[q_0 \cos\sqrt{1-\xi^2}\,\theta + \frac{q_0\,\xi + \dot{q}_0}{\sqrt{1-\xi^2}} \sin\sqrt{1-\xi^2}\,\theta\right] \tag{3.36}$$

For a system initially at rest, $q(\theta)$ is reduced to:

$$q(\theta) = \frac{1}{\sqrt{1-\xi^2}} \int_0^\theta \lambda(\alpha)\, e^{-\xi\,(\theta-\alpha)} \sin\sqrt{1-\xi^2}\,(\theta-\alpha)\, d\alpha \tag{3.37}$$

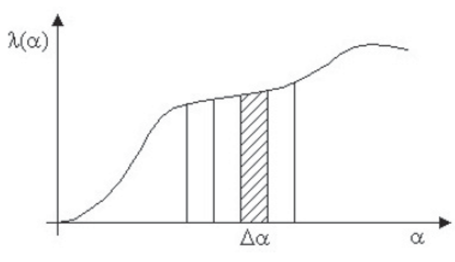


Figure 3.7. *Elemental impulses*

This integral is called *Duhamel's integral* or the *superposition integral* or *convolution integral.* We will indeed see that the excitation can be regarded as a series of impulses of duration $\Delta\alpha$ and that the total response can be calculated by superimposing the responses onto all these impulses.

For a small damping $\xi << 1$

$$q(\theta) \approx \int_0^\theta \lambda(\alpha)\ e^{-\xi(\theta-\alpha)}\ \sin(\theta-\alpha)\ d\alpha \qquad [3.38]$$

and for zero damping:

$$q(\theta) = \int_o^\theta \lambda(\alpha)\ \sin\ (\theta-\alpha)\ d\alpha \qquad [3.39]$$

3.5.2.3. *Critical damping*

Roots p_1 and p_2 are both equal to –1. This case occurs for $\xi = 1$. The differential equation of movement is then written:

$$\ddot{q}(\theta) + 2\ \dot{q}(\theta) + q(\theta) = \lambda(\theta) \qquad [3.40]$$

$$L(\ddot{q}) = p^2\ Q(p) - p\ q_0 - \dot{q}_0 \qquad [3.41]$$

$$L(\dot{q}) = p\ Q(p) - q_0 \qquad [3.42]$$

$$p^2\ Q(p) - p\ q_0 - \dot{q}_0 + 2\ p\ Q(p) - 2\ q_0 + Q(p) = \Lambda(p) \qquad [3.43]$$

yielding:

$$Q(p) = \frac{\Lambda(p)}{p^2 + 2p + 1} + \frac{pq_0 + \dot{q}_0 + 2q_0}{p^2 + 2p + 1} \qquad [3.44]$$

$$Q(p) = \frac{\Lambda(p)}{(p+1)^2} + \frac{p\ q_0 + \dot{q}_0 + 2\ q_0}{(p+1)^2} = \frac{\Lambda(p)}{(p+1)^2} + \frac{q_0}{p+1} + \frac{\dot{q}_0 + q_0}{(p+1)^2} \qquad [3.45]$$

i.e.

$$q(\theta) = \int_0^\theta \lambda(\alpha)\ (\theta-\alpha)\ e^{-(\theta-\alpha)}\ d\alpha + \left[q_0 + \left(q_0 + \dot{q}_0\right)\ \theta\right]\ e^{-\theta} \qquad [3.46]$$

3.5.2.4. *Supercritical damping*

The roots are real $(\xi > 1)$:

$$q(\theta) = \int_0^\theta \frac{\lambda(\alpha)}{p_1 - p_2} \left[e^{p_1 (\theta-\alpha)} - e^{p_2 (\theta-\alpha)} \right] d\alpha$$

$$+ \frac{1}{p_1 - p_2} \left[\left(p_1 q_0 + 2 \xi q_0 + \dot{q}_0 \right) e^{p_1 \theta} - \left(p_2 q_0 + 2 \xi q_0 + \dot{q}_0 \right) e^{p_2 \theta} \right] \quad [3.47]$$

where

$$p_{1,2} = \xi \pm \sqrt{\xi^2 - 1} \quad [3.48]$$

since

$$p^2 + 2 \xi p + 1 = p^2 + 2 \xi p + \xi^2 - \xi^2 + 1 \quad [3.49]$$

$$p^2 + 2 \xi p + 1 = (p + \xi)^2 - \left(\xi^2 - 1\right) = \left(p + \xi + \sqrt{\xi^2 - 1} \right) \left(p + \xi - \sqrt{\xi^2 - 1} \right) \quad [3.50]$$

$$p_1 - p_2 = 2 \sqrt{\xi^2 - 1} \quad [3.51]$$

yielding

$$q(\theta) = \frac{1}{\sqrt{\xi^2 - 1}} \int_0^\theta \lambda(\alpha) e^{-\xi(\theta-\alpha)} \sinh\left[\sqrt{\xi^2 - 1} (\theta - \alpha) \right] d\alpha$$

$$+ \frac{e^{-\xi\theta}}{\sqrt{\xi^2 - 1}} \left[(\xi q_0 + \dot{q}_0) \sinh\left(\sqrt{\xi^2 - 1}\, \theta \right) + q_0 \sqrt{\xi^2 - 1} \cosh\left(\sqrt{\xi^2 - 1}\, \theta \right) \right] \quad [3.52]$$

3.5.3. *Absolute response*

3.5.3.1. *General expression for response*

The solution of a differential second order equation of the form:

$$\frac{d^2 q}{d\theta^2} + a \frac{dq}{d\theta} + b\, q(\theta) = \lambda(\theta) + b \frac{d\lambda}{d\theta} \quad [3.53]$$

has as a Laplace transform $Q(p)$ [LAL 75]:

$$Q(p) = \frac{\Lambda(p)\,(1 + a\,p) + p\,q_0 + a\,(q_0 - \lambda_0) + \dot{q}_0}{p^2 + a\,p + b} \tag{3.54}$$

where

$$q_0 = q(0) \qquad \lambda_0 = \lambda(0) \qquad a = 2\,\xi$$

$$\dot{q}_0 = \dot{q}(0) \qquad \Lambda(p) = L[\lambda(\theta)] \qquad b = 1$$

As previously:

$$Q(p) = \frac{\Lambda(p)}{p^2 + 2\,\xi\,p + 1} + \frac{2\,\xi\,\Lambda(p)}{p^2 + 2\,\xi\,p + 1}$$

$$+ \frac{1}{p_1 - p_2}\left[\frac{q_0\,p_1 + 2\,\xi\,(q_0 - \lambda_0) + \dot{q}_0}{p - p_1} - \frac{q_0\,p_2 + 2\,\xi\,(q_0 - \lambda_0) + \dot{q}_0}{p - p_2}\right] \tag{3.55}$$

$q(\theta)$ is obtained by searching the original of $Q(p)$

$$q(\theta) = \int_0^\theta \frac{\lambda(\alpha)}{p_1 - p_2}\left[(1 + 2\,\xi\,p_1)\,e^{p_1\,(\theta-\alpha)} - (1 + 2\,\xi\,p_2)\,e^{p_2\,(\theta-\alpha)}\right] d\alpha$$

$$+ \frac{1}{p_1 - p_2}\left\{\left[q_0\,p_1 + 2\,\xi\,(q_0 - \lambda_0) + \dot{q}_0\right] e^{p_1\,\theta} - \left[q_0\,p_2 + 2\,\xi\,(q_0 - \lambda_0) + \dot{q}_0\right] e^{p_2\,\theta}\right\} \tag{3.56}$$

(α = variable of integration).

Particular case

$$\lambda_0 = q_0 = \dot{q}_0$$

$$q(\theta) = \int_0^\theta \frac{\lambda(\alpha)}{p_1 - p_2}\left[(1 + 2\,\xi\,p_1)\,e^{p_1\,(\theta-\alpha)} - (1 + 2\,\xi\,p_2)\,e^{p_2\,(\theta-\alpha)}\right] d\alpha \tag{3.57}$$

3.5.3.2. *Subcritical damping*

The roots of $p^2 + 2\,\xi\,p + 1$ are complex $(0 \le \xi < 1)$

$$p_{1,\,2} = -\xi \pm i\,\sqrt{1-\xi^2} \qquad [3.58]$$

While replacing p_1 and p_2 with their expressions in $q(\theta)$, it becomes:

$$q(\theta) = \frac{1}{\sqrt{1-\xi^2}} \int_0^\theta \lambda(\alpha)\, e^{-\xi(\theta-\alpha)} \left\{ \left(1 - 2\,\xi^2\right) \sin\sqrt{1-\xi^2}\,(\theta-\alpha) \right.$$

$$\left. +2\,\xi\,\sqrt{1-\xi^2}\,\cos\sqrt{1-\xi^2}\,(\theta-\alpha) \right\} d\alpha$$

$$+e^{-\xi\,\theta} \left[q_0 \cos\sqrt{1-\xi^2}\,\theta + \frac{\dot{q}_0 + \xi\left(q_0 - 2\,\lambda_0\right)}{\sqrt{1-\xi^2}} \sin\sqrt{1-\xi^2}\,\theta \right] \qquad [3.59]$$

If $\lambda_0 = q_0 = \dot{q}_0 = 0$

$$q(\theta) = \frac{1}{\sqrt{1-\xi^2}} \int_0^\theta \lambda(\alpha)\, e^{-\xi(\theta-\alpha)} \left\{ 2\,\xi \cos\sqrt{1-\xi^2}\,(\theta-\alpha) \right.$$

$$\left. +\left(1 - 2\,\xi^2\right) \sin\sqrt{1-\xi^2}\,(\theta-\alpha) \right\} d\alpha \qquad [3.60]$$

If moreover $\xi = 0$

$$q(\theta) = \int_0^\theta \lambda(\alpha)\,\sin(\theta-\alpha)\,d\alpha \qquad [3.61]$$

3.5.3.3. *Critical damping*

The equation $p^2 + 2\,\xi\,p + 1 = 0$ has a double root $(p = -1)$. In this case $\xi = 1$ and

$$q(\theta) = \frac{\Lambda(p)\,(1 + 2\,p) + p\,q_0 + \dot{q}_0 + 2\left(q_0 - \lambda_0\right)}{(p+1)^2} \qquad [3.62]$$

$$q(\theta) = \int_0^\theta \lambda(\alpha)\,[2-\theta+\alpha]\,e^{-(\theta-\alpha)}\,d\alpha + \left[q_0 + \theta\left(q_0 + \dot{q}_0 - 2\,\lambda_0\right)\right] e^{-\theta} \quad [3.63]$$

3.5.3.4. *Supercritical damping*

The equation $p^2 + 2\,\xi\,p + 1 = 0$ has two real roots. This condition is carried out when $\xi > 1$. Let us replace $p_1 = -\xi + \sqrt{\xi^2 - 1}$ and $p_2 = -\xi - \sqrt{\xi^2 - 1}$ by their expressions in equation [3.56] [KIM 26]:

$$q(\theta) = \int_0^\theta \frac{\lambda(\alpha)}{2\sqrt{\xi^2-1}} \left\{\left[1 + 2\,\xi\left(-\xi + \sqrt{\xi^2-1}\right)\right] e^{\left(-\xi+\sqrt{\xi^2-1}\right)(\theta-\alpha)}\right.$$

$$\left. - \left[1 + 2\,\xi\left(-\xi - \sqrt{\xi^2-1}\right)\right] e^{\left(-\xi-\sqrt{\xi^2-1}\right)(\theta-\alpha)}\right\} d\alpha$$

$$+ \frac{1}{2\sqrt{\xi^2-1}} \left\{\left[\left(-\xi + \sqrt{\xi^2-1}\right) q_0 + 2\,\xi\left(q_0 - \lambda\right) + \dot{q}_0\right] e^{\left(-\xi+\sqrt{\xi^2-1}\right)\theta}\right.$$

$$\left. - \left[\left(-\xi - \sqrt{\xi^2-1}\right) q_0 + 2\,\xi\left(q_0 - \lambda_0\right) + \dot{q}_0\right] e^{\left(-\xi-\sqrt{\xi^2-1}\right)\theta}\right\} \quad [3.64]$$

yielding

$$q(\theta) = \int_0^\theta \frac{\lambda(\theta)}{\sqrt{\xi^2-1}}\, e^{-\xi(\theta-\alpha)} \left\{\left(1 - 2\,\xi^2\right) \sinh\left[\sqrt{\xi^2-1}\,(\theta-\alpha)\right]\right.$$

$$\left. + 2\,\xi\sqrt{\xi^2-1}\,\cosh\left[\sqrt{\xi^2-1}(\theta-\alpha)\right]\right\} d\alpha + C(\theta) \quad [3.65]$$

where

$$C(\theta) = \frac{e^{-\xi\,\theta}}{\sqrt{\xi^2-1}} \left\{\left[\xi\left(q_0 - 2\,\lambda_0\right) + \dot{q}_0\right] \frac{e^{\theta\sqrt{\xi^2-1}} - e^{-\theta\sqrt{\xi^2-1}}}{2}\right.$$

$$+\sqrt{\xi^2-1}\; q_0\, \frac{e^{\theta\sqrt{\xi^2-1}} + e^{-\theta\sqrt{\xi^2-1}}}{2} \Bigg\} \qquad [3.66]$$

Another form

$$C(\theta) = \frac{e^{-\xi\,\theta}}{2\sqrt{\xi^2-1}} \left\{ e^{\sqrt{\xi^2-1}\,\theta} \left[\xi\left(q_0 - 2\,\lambda_0\right) + \dot{q}_0 + \sqrt{\xi^2-1}\; q_0 \right] \right.$$

$$\left. + e^{-\sqrt{\xi^2-1}\,\theta} \left[\sqrt{\xi^2-1}\, q_0 - \xi\left(q_0 - 2\,\lambda_0\right) - \dot{q}_0 \right] \right\} \qquad [3.67]$$

$$C(\theta) = a\, e^{\left(-\xi+\sqrt{\xi^2-1}\right)\theta} + b\, e^{\left(-\xi-\sqrt{\xi^2-1}\right)\theta} \qquad [3.68]$$

with

$$a = \frac{\xi\left(q_0 - 2\,\lambda_0\right) + \dot{q}_0 + \sqrt{\xi^2-1}\, q_0}{2\sqrt{\xi^2-1}} \qquad [3.69]$$

$$b = \frac{\sqrt{\xi^2-1}\, q_0 - \xi\left(q_0 - 2\,\lambda_0\right) - \dot{q}_0}{2\sqrt{\xi^2-1}} \qquad [3.70]$$

If $q_0 = \dot{q}_0 = \lambda_0 = 0$:

$$q(\theta) = \int_0^\theta \frac{\lambda(\alpha)}{\sqrt{\xi^2-1}}\, e^{-\xi\,(\theta-\alpha)} \left\{ \left(1 - 2\,\xi^2\right) \sinh\left[\sqrt{\xi^2-1}\,(\theta-\alpha)\right] \right.$$

$$\left. 2\xi\sqrt{\xi^2-1}\cosh\left[\sqrt{\xi^2-1}\,(\theta-\alpha)\right] \right\} d\alpha \qquad [3.71]$$

If moreover $\xi = 0$:

$$q(\theta) = \int_0^\theta \lambda(\alpha) \sinh(\theta-\alpha)\, d\alpha \qquad [3.72]$$

3.5.4. *Summary of main results*

Zero initial conditions:

Relative response

$0 \le \xi < 1$

$$q(\theta) = \frac{1}{\sqrt{1-\xi^2}} \int_0^\theta \lambda(\alpha) e^{-\xi(\theta-\alpha)} \sin \sqrt{1-\xi^2}(\theta-\alpha)\, d\alpha \qquad [3.73]$$

$\xi = 1$

$$q(\theta) = \int_0^\theta \lambda(\alpha)\,(\theta-\alpha)\, e^{-(\theta-\alpha)}\, d\alpha \qquad [3.74]$$

$\xi > 1$

$$q(\theta) = \frac{1}{\sqrt{1-\xi^2}} \int_0^\theta \lambda(\alpha) e^{-\xi(\theta-\alpha)} \sinh\left[\sqrt{\xi^2-1}(\theta-\alpha)\right] d\alpha \qquad [3.75]$$

Absolute response

$0 \le \xi < 1$

$$q(\theta) = \frac{1}{\sqrt{1-\xi^2}} \int_0^\theta \lambda(\alpha)\, e^{-\xi(\theta-\alpha)} \left\{2\,\xi \cos\sqrt{1-\xi^2}\,(\theta-\alpha) + \left(1-2\,\xi^2\right)\sin\sqrt{1-\xi^2}\,(\theta-\alpha)\right\} \qquad [3.76]$$

$\xi = 1$

$$q(\theta) = \int_0^\theta \lambda(\alpha)\,(2-\theta+\alpha)\, e^{-(\theta-\alpha)}\, d\alpha \qquad [3.77]$$

$\xi > 1$

$$q(\theta) = \frac{1}{\sqrt{1-\xi^2}} \int_0^\theta \lambda(\alpha) e^{-\xi(\theta-\alpha)} \left\{\left(1-2\,\xi^2\right)\sinh\left[\sqrt{\xi^2-1}(\theta-\alpha)\right] + 2\,\xi\sqrt{\xi^2-1}\cosh\left[\sqrt{\xi^2-1}(\theta-\alpha)\right]\right\} d\alpha \qquad [3.78]$$

If the initial conditions are not zero, we have to add to these expressions, according to the nature of the response, the following relations:

For $0 \le \xi < 1$

Relative response

$$C(\theta) = e^{-\xi \theta} \left[q_0 \cos \sqrt{1-\xi^2}\,\theta + \frac{q_0\, \xi + \dot{q}_0}{\sqrt{1-\xi^2}} \sin \sqrt{1-\xi^2}\,\theta \right] \quad [3.79]$$

Absolute response

$$C(\theta) = e^{-\xi \theta} \left[q_0 \cos \sqrt{1-\xi^2}\,\theta + \frac{\dot{q}_0 + \xi\,(q_0 - 2\,\lambda_0)}{\sqrt{1-\xi^2}} \sin \sqrt{1-\xi^2}\,\theta \right] \quad [3.80]$$

For $\xi = 1$

Relative response

$$C(\theta) = \left[q_0 + \theta\left(q_0 + \dot{q}_0\right) \right] e^{-\theta} \quad [3.81]$$

Absolute response

$$C(\theta) = \left[q_0 + \left(q_0 + \dot{q}_0 - 2\,\lambda_0\right)\theta \right] e^{-\theta} \quad [3.82]$$

For $\xi > 1$

Relative response

$$C(\theta) = e^{-\xi \theta} \left\{ \frac{\xi\, q_0 + \dot{q}_0}{\sqrt{\xi^2 - 1}} \sinh\left[\sqrt{\xi^2 - 1}\,\theta \right] + q_0 \cosh\left[\sqrt{\xi^2 - 1}\,\theta \right] \right\} \quad [3.83]$$

Absolute response

$$C(\theta) = e^{-\xi \theta} \left\{ \frac{\xi\, q_0 + \dot{q}_0 - 2\,\xi\,\lambda_0}{\sqrt{\xi^2 - 1}} \sinh\left(\sqrt{\xi^2 - 1}\,\theta \right) + q_0 \cosh\left(\sqrt{\xi^2 - 1}\,\theta \right) \right\} \quad [3.84]$$

In all these relations, the only difference between cases $0 \le \xi < 1$ and $\xi > 1$ resides in the nature of the sine and cosine functions (hyperbolic for $\xi > 1$).

3.6. Natural oscillations of a linear one-degree-of-freedom system

We have just shown that the response $q(\theta)$ can be written for non-zero initial conditions:

$$q_{IC}(\theta) = q(\theta) + C(\theta) \quad [3.85]$$

The response $q_{IC}(\theta)$ is equal to the sum of the response $q(\theta)$ obtained for zero initial conditions and the term $C(\theta)$ corresponds to a damped oscillatory response produced by non-zero initial conditions. So $\lambda(\theta) = 0$ is set in the differential equation of movement [3.21]:

$$\ddot{q}(\theta) + 2\,\xi\,\dot{q}(\theta) + q(\theta) = 0$$

It then becomes, after a Laplace transformation,

$$Q(p) = \frac{\dot{q}_0 + 2\,\xi\,q_0 + p\,q_0}{p^2 + 2\,\xi\,p + 1} \qquad (0 \le \xi < 1) \tag{3.86}$$

The various cases related to the nature of the roots of:

$$p^2 + 2\,\xi\,p + 1 = 0 \tag{3.87}$$

are considered here.

3.6.1. *Damped aperiodic mode*

In this case $\xi > 1$. The two roots of $p^2 + 2\,\xi\,p + 1 = 0$ are real. Suppose that the response is defined by an absolute movement. If this were not the case, it would be enough to make $\lambda_0 = 0$ in the relations of this section. The response $q(\theta)$ of the system around its equilibrium position is written:

$$q(\theta) = e^{-\xi\,\theta}\left[\frac{\dot{q}_0 + \xi\,(q_0 - 2\,\lambda_0)}{\sqrt{\xi^2 - 1}}\,\sinh\left(\sqrt{\xi^2 - 1}\;\theta\right) + q_0\,\cosh\left(\sqrt{\xi^2 - 1}\;\theta\right)\right] \tag{3.88}$$

$q(\theta)$ can also be written in the form:

$$q(\theta) = a\,e^{\left(-\xi + \sqrt{\xi^2 - 1}\right)\theta} + b\,e^{-\left(\xi + \sqrt{\xi^2 - 1}\right)\theta} \tag{3.89}$$

where

$$a = \frac{\xi\,(q_0 - 2\,\lambda_0) + \dot{q}_0 + \sqrt{\xi^2 - 1}\,q_0}{2\sqrt{\xi^2 - 1}} \tag{3.90}$$

and

$$b = \frac{q_0 \sqrt{\xi^2 - 1} - \xi \left(q_0 - 2\,\lambda_0\right) - \dot{q}_0}{2\sqrt{\xi^2 - 1}} \qquad [3.91]$$

It should be noted that the roots $-\xi + \sqrt{\xi^2 - 1}$ and $-\xi - \sqrt{\xi^2 - 1}$ of the equation $p^2 + 2\,\xi\,p + 1 = 0$ are both negative, their sum being negative and their product positive. Thus, the two exponential terms are decreasing functions of time, like $q(\theta)$.

The velocity $\frac{dq}{d\theta}$, which is equal to:

$$\frac{dq}{d\theta} = a\left(-\xi + \sqrt{\xi^2 - 1}\right) e^{\left(-\xi+\sqrt{\xi^2-1}\right)\theta} - b\left(\xi + \sqrt{\xi^2 - 1}\right) e^{-\left(\xi+\sqrt{\xi^2-1}\right)\theta} \qquad [3.92]$$

is also, for the same reason, a decreasing function of time. Therefore, the movement cannot be oscillatory. It is *a damped exponential motion.* $q(\theta)$ can also be written:

$$q(\theta) = a\, e^{\left(-\xi+\sqrt{\xi^2-1}\right)\theta}\left[1 + \frac{b}{a}\, e^{\left(-\xi-\sqrt{\xi^2-1}+\xi-\sqrt{\xi^2-1}\right)\theta}\right] \qquad [3.93]$$

$$q(\theta) = a\, e^{\left(-\xi+\sqrt{\xi^2-1}\right)\theta}\left[1 + \frac{b}{a}\, e^{-2\sqrt{\xi^2-1}\,\theta}\right] \qquad [3.94]$$

When θ tends towards infinity, $e^{-2\sqrt{\xi^2-1}}$ tends towards zero $(\xi > 1)$. After a certain time, the second term thus becomes negligible in comparison with one and $q(\theta)$ then behaves like:

$$a\, e^{\left(-\xi+\sqrt{\xi^2-1}\right)\theta} \qquad [3.95]$$

As $\left(-\xi + \sqrt{\xi^2 - 1}\right)$ is always negative, $q(\theta)$ decreases constantly with time.

If the system is moved away from its equilibrium position and released with a zero velocity $\dot{q}_0$ with an elongation q_0 at time $t = 0$, coefficient a becomes, for $\lambda_0 = 0$,

$$a = q_0 \frac{\xi + \sqrt{\xi^2 - 1}}{2\sqrt{\xi^2 - 1}} \quad [3.96]$$

q_0 being assumed positive, a being positive and $q(\theta)$ always remaining positive: the system returns towards its equilibrium position without crossing it.

The velocity can also be written:

$$\frac{dq}{d\theta} = a\left(-\xi + \sqrt{\xi^2 - 1}\right) e^{\left(-\xi+\sqrt{\xi^2-1}\right)\theta} \left[1 - \frac{b}{a}\,\frac{\xi + \sqrt{\xi^2 - 1}}{-\xi + \sqrt{\xi^2 - 1}}\, e^{-2\sqrt{\xi^2-1}\,\theta}\right] \quad [3.97]$$

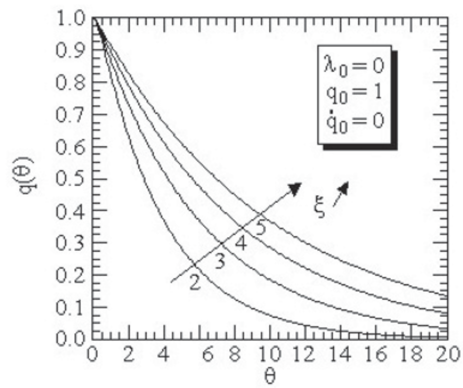


Figure 3.8. *Damped aperiodic response*

We have:

$$\frac{dq}{d\theta} \approx a\left(-\xi + \sqrt{\xi^2 - 1}\right) e^{\left(-\xi+\sqrt{\xi^2-1}\right)\theta} \quad [3.98]$$

when θ is sufficiently large. The velocity is then always negative.

Variations of the roots p_1 and p_2 according to ξ

Characteristic equation [3.87] $p^2 + 2\,\xi\,p + 1 = 0$ is that of a hyperbole (in the axes p, ξ) whose asymptotic directions are:

$$p^2 + 2\,\xi\,p = 0$$

i.e.

$$\begin{cases} p = 0 \\ p + 2\,\xi = 0 \end{cases} \tag{3.99}$$

The tangent parallel with the axis 0p is given by $2\,p + 2\,\xi = 0$, i.e. $p = -\xi$.

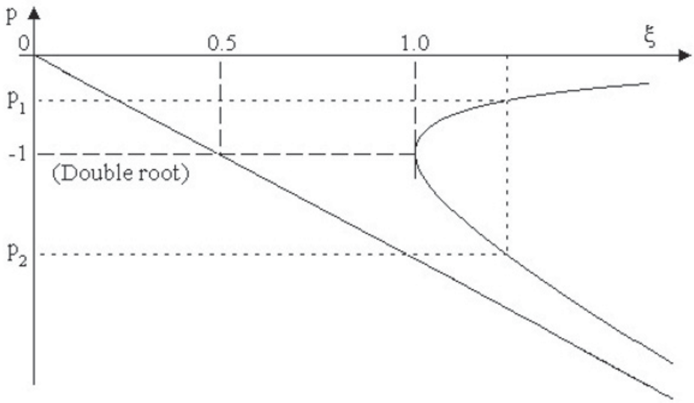


Figure 3.9. *Characteristic equation*

This yields $\xi = 1$ while using equation [3.87] (since ξ is positive or zero), i.e.

$$c = c_c = 2\,\sqrt{k\,m} \tag{3.100}$$

Any parallel line with the axis 0p such as $\xi > 1$ crosses the curve at two points corresponding to the values p_1 and p_2 of p.

The system returns all the more quickly to its equilibrium position, as $q(\theta)$ decreases quickly, therefore $|p_1|$ is larger (the time-constant, in the expression $q(\theta) = a\ e^{\left(-\xi+\sqrt{\xi^2-1}\right)\theta}$, is of value $\left|\frac{1}{p_1}\right| = \frac{1}{\left|-\xi + \sqrt{\xi^2 - 1}\right|}$), i.e. the relative damping ξ (or the coefficient of energy dissipation c) is still smaller.

$\left|p_1\right|$ has the greatest possible value when the equation $p^2 + 2\,\xi\,p + 1 = 0$ has a double root, i.e. when $\xi = 1$.

NOTE.– *If the system is released from its equilibrium position with a zero initial velocity, the resulting movement is characterized by a velocity which changes only once in sign. The system tends towards its equilibrium position without reaching it. It is said that the motion is 'damped aperiodic' ($\xi > 1$). Damping is supercritical.*

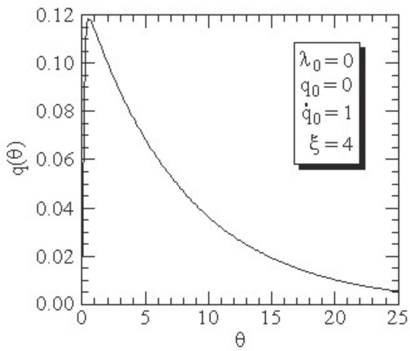


Figure 3.10. *Aperiodic damped response (for $q_0 = 0$)*

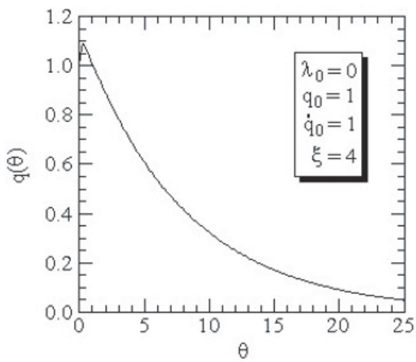


Figure 3.11. *Aperiodic damped response (for $q_0 = 1$)*

3.6.2. *Critical aperiodic mode*

On the assumption that $\xi = 1$, the two roots of $p^2 + 2\,\xi\,p + 1 = 0$ are equal to -1. By definition

$$\xi = \frac{c}{2\sqrt{k\,m}} = 1 \tag{3.101}$$

yielding $c = c_c = 2\sqrt{km}$. Parameter c_c, the *critical damping coefficient,* is the smallest value of c for which the damped movement is non-oscillatory. This is the reason why ξ is also defined as the *fraction of critical damping* or *critical damping ratio*.

Depending on whether the response is relative or absolute, the response $q(\theta)$ is equal to:

$$q(\theta)=\left[q_0+\left(q_0+\dot{q}_0\right)\theta\right]e^{-\theta} \qquad [3.102]$$

or

$$q(\theta)=\left[q_0+\left(q_0+\dot{q}_0-2\lambda_0\right)\theta\right]e^{-\theta} \qquad [3.103]$$

As an example, Figures 3.12–3.14 show $q(\theta)$, respectively, for $\lambda_0=0$, $q_0=\dot{q}_0=1$, $q_0=0$ and $\dot{q}_0=1$, then $q_0=1$ and $\dot{q}_0=0$.

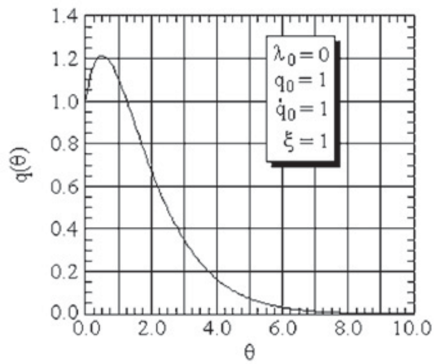


Figure 3.12. *Critical aperiodic response* $(q_0=1,\ \dot{q}_0=1)$

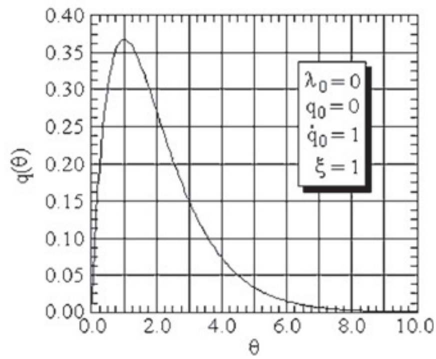


Figure 3.13. *Critical aperiodic response* $(q_0=0,\ \dot{q}_0=1)$

$q(\theta)$ can be written $q(\theta)=\left[\dfrac{q_0}{\theta}+\left(q_0+\dot{q}_0\right)\right]\theta\, e^{-\theta}$. For quite large θ, $\dfrac{q_0}{\theta}$ becomes negligible and $q(\theta)$ behaves like $\left(q_0+\dot{q}_0\right)\theta\, e^{-\theta}$; $q(\theta)$ thus tends towards zero when θ tends towards infinity. This mode, known as *critical,* is not oscillatory. It corresponds to the fastest possible return of the system towards the equilibrium position from all the damped exponential movements.

If $q_c(\theta)$ is written as the expression of $q(\theta)$ corresponding to the critical mode, this proposal can be verified while calculating:

$$\frac{q_c(\theta)}{q_{\xi>1}(\theta)}$$

If we consider the expression of $q(\theta)$ [3.89] given for $\xi > 1$ in the form of a sum, the exponential terms eventually become:

$$q(\theta) \approx a\ e^{\left(\sqrt{\xi^2-1}-\xi\right)\theta} \qquad [3.104]$$

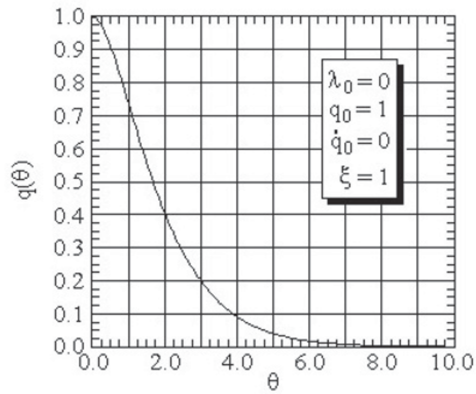


Figure 3.14. *Critical aperiodic response ($q_0 = 1$, $\dot{q}_0 = 0$)*

where:

$$a = \frac{\xi\left(q_0 - 2\ \lambda_0\right) + \dot{q}_0 + q_0\sqrt{\xi^2 - 1}}{2\ \sqrt{\xi^2 - 1}} \qquad [3.105]$$

whereas:

$$q_c(\theta) \approx \left(q_0 + \dot{q}_0\right)\theta\ e^{-\theta} \qquad [3.106]$$

yielding:

$$\frac{q_c(\theta)}{q(\theta)} \approx \frac{q_0 + \dot{q}_0}{a}\ \theta\ e^{-\left(1+\sqrt{\xi^2-1}-\xi\right)\theta} \qquad [3.107]$$

With the coefficient $1-\xi+\sqrt{\xi^2-1}$ always being positive for $\xi > 1$, this means that the exponential term tends towards zero when θ tends towards infinity and consequently:

$$\frac{q_c(\theta)}{q(\theta)} \to 0$$

This return towards zero is thus performed more quickly in critical mode than in damped exponential mode.

3.6.3. *Damped oscillatory mode*

It is assumed that $0 \le \xi < 1$.

3.6.3.1. *Free response*

The equation $p^2 + 2\,\xi\,p + 1 = 0$ has two complex roots. Let us suppose that the response is defined by an absolute movement ($\lambda_0 = 0$ for a relative movement). The response

$$q(\theta) = e^{-\xi\,\theta}\left[q_0\,\cos\sqrt{1-\xi^2}\;\theta + \frac{\dot{q}_0 + \xi\,(q_0 - 2\,\lambda_0)}{\sqrt{1-\xi^2}}\,\sin\sqrt{1-\xi^2}\;\theta\right] \qquad [3.108]$$

can also be written:

$$q(\theta) = q_m\; e^{-\xi\,\theta} \sin\left(\sqrt{1-\xi^2}\;\theta + \phi\right) \qquad [3.109]$$

with

$$q_m = \sqrt{q_0^2 + \frac{\left[\dot{q}_0 + \xi(q_0 - 2\,\lambda_0)\right]^2}{1-\xi^2}} \qquad [3.110]$$

$$\tan\phi = \frac{q_0\,\sqrt{1-\xi^2}}{\dot{q}_0 + \xi\,(q_0 - 2\,\lambda_0)} \qquad [3.111]$$

The response is of the damped oscillatory type with a pulsation equal to $P = \sqrt{1-\xi^2}$, which corresponds to a period $\Theta = \frac{2\pi}{P}$. It is said that the movement is *damped sinusoidal* or *pseudo-sinusoidal.* The *pseudo-pulsation* P is always lower than 1. For the usual values of ξ, the pulsation is equal to 1 at first approximation $(\xi < 0.1)$.

The envelopes of the damped sinusoid have as equations:

$$q = q_m \; e^{-\xi\,\theta} \qquad [3.112]$$

and

$$q = -q_m \; e^{-\xi\,\theta} \qquad [3.113]$$

The free response of a mechanical system around its equilibrium position is named "*simple harmonic*".

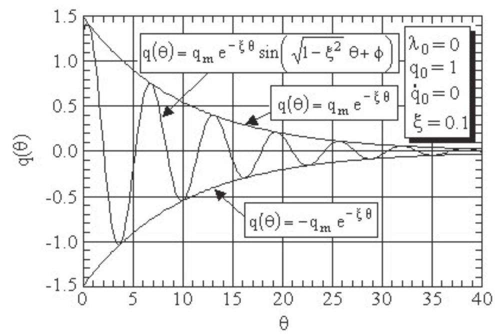


Figure 3.15. *Damped oscillatory response*

The exponent of the exponential term can be written as:

$$\xi\,\theta = \xi\,\omega_0\,t = \frac{t}{t_0}$$

where t_0 is the time constant of the system and

$$t_0 = \frac{1}{\xi\omega_0} = \frac{\omega_0}{2\,Q} = \frac{c}{2\,m} \qquad [3.114]$$

Application

If we return to the non-reduced variables, equation [3.108] can be written for the relative displacement response as:

$$u(t) = e^{-\xi\,\omega_0\,t}\left[u_0\,\cos\omega_0\sqrt{1-\xi^2}\;t + \frac{\dot{u}_0/\omega_0 + u_0\,\xi}{\sqrt{1-\xi^2}}\sin\omega_0\sqrt{1-\xi^2}\;t\right] \tag{3.115}$$

The relative velocity, calculated by deriving u(t), is equal to

$$\dot{u}(t) = e^{-\xi\,\omega_0\,t}\left[\dot{u}_0\,\cos\omega_0\sqrt{1-\xi^2}\;t - \frac{\dot{u}_0\,\xi + u_0\,\omega_0}{\sqrt{1-\xi^2}}\sin\omega_0\sqrt{1-\xi^2}\;t\right] \tag{3.116}$$

The *pseudo-pulsation* is equal to

$$\omega = \omega_0\,\sqrt{1-\xi^2} \tag{3.117}$$

with ω always being equal to or lower than ω_0.

Figure 3.16 shows the variations of the ratio $\dfrac{\omega}{\omega_0}$ with ξ.

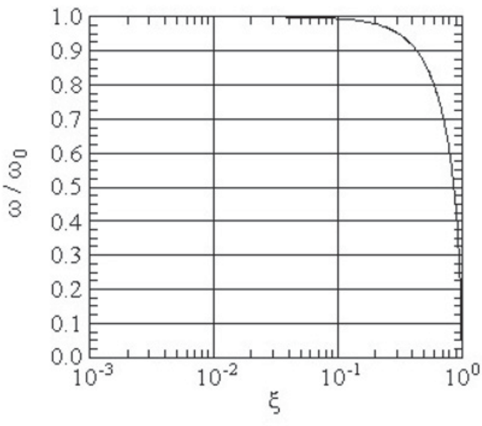


Figure 3.16. *Influence of damping on the pseudo-pulsation*

The pseudo-period

$$T = \frac{2\,\pi}{\omega} \qquad [3.118]$$

which separates two successive instants from the time axis crossing in the same direction is always higher than the period of the undamped motion.

Figure 3.17 represents, as an example, the variations of $q(\theta)$ with θ for $\dot{q}_0$ and q_0 equal to 1 and for $\xi = 0.1$. Figures 3.18 and 3.19 show $q(\theta)$ for $\xi = 0.1$ and for ($q_0 = 1$, $\dot{q}_0 = 0$) and ($q_0 = 0$, $\dot{q}_0 = 1$), respectively. Figure 3.20 gives the absolute response $q(\theta)$ for $\xi = 0.1$, $q_0 = 1$, $\dot{q}_0 = 1$ and $\lambda_0 = 1$.

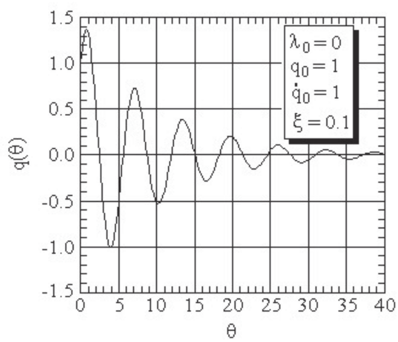


Figure 3.17. *Example of relative response for ($q_0 = 1$, $\dot{q}_0 = 1$)*

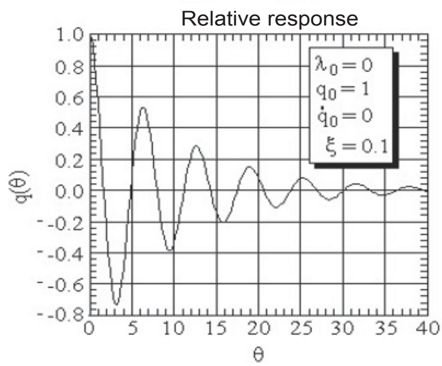


Figure 3.18. *Example of relative response for ($q_0 = 1$, $\dot{q}_0 = 0$)*

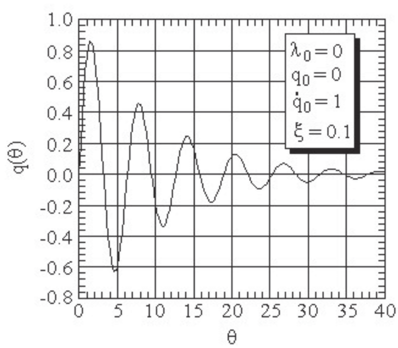


Figure 3.19. *Example of relative response for ($q_0 = 0$, $\dot{q}_0 = 1$)*

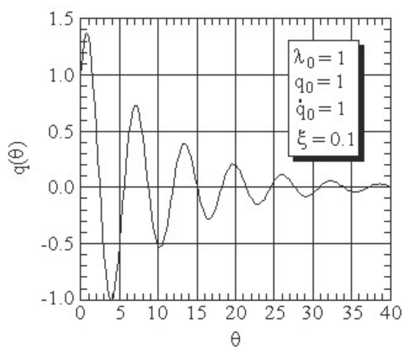


Figure 3.20. *Example of absolute response for ($q_0 = 1$, $\dot{q}_0 = 1$)*

3.6.3.2. *Points of contact of the response with its envelope*

From $q(\theta) = q_m \, e^{-\xi\theta} \sin\left(\sqrt{1-\xi^2}\ \theta + \phi\right)$, the points of contact of the curve with its envelope can be determined by seeking θ solutions of $\sin\left(\sqrt{1-\xi^2}\ \theta + \phi\right) = 1$. These points are separated by time intervals equal to $\frac{\Theta}{2}$.

The points of intersection of the curve with the time axis are such that $\sin\left(P\ \theta + \phi\right) = 0$.

The maximum response is located a little before the point of contact of the curve with its envelope.

The system needs a little more than $\frac{\Theta}{4}$ to pass from a maximum to the next position of zero displacement and little less than $\frac{\Theta}{4}$ to pass from this equilibrium position to the position of maximum displacement.

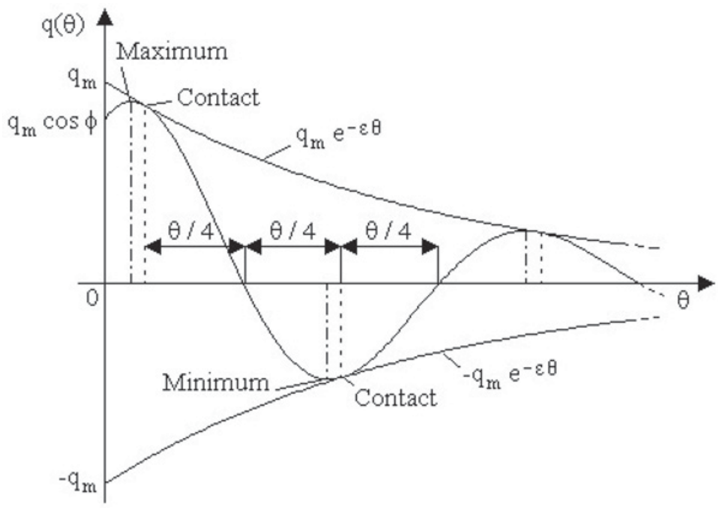


Figure 3.21. *Points of contact with the envelope*

3.6.3.3. *Reduction of amplitude: logarithmic decrement*

Considering two successive maximum displacements q_{1_M} and q_{2_M}:

$$q_{1_M} = q_m\ e^{-\xi\,\theta_1}\ \sin\left(P\ \theta_1 + \phi\right)$$

$$q_{2_M} = q_m\ e^{-\xi\,\theta_2}\ \sin\left(P\ \theta_2 + \phi\right)$$

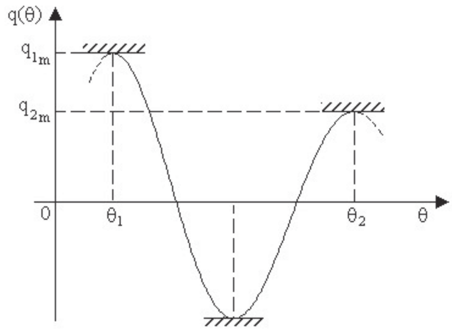


Figure 3.22. *Successive maxima of the response*

where the times θ_1 and θ_2 are such that $\dfrac{dq}{d\theta} = 0$:

$$\frac{dq}{d\theta} = q_m(-\xi)\ e^{-\xi\,\theta}\ \sin(P\,\theta + \phi) + q_m\ P\,e^{-\xi\,\theta}\ \cos(P\,\theta + \phi) \qquad [3.119]$$

$$\frac{dq}{d\theta} = 0 \text{ if}$$

$$\tan\left(P\,\theta + \phi\right) = \frac{P}{\xi} = \frac{\sqrt{1-\xi^2}}{\xi} \qquad [3.120]$$

i.e. if

$$\sin\left(P\,\theta+\phi\right)=\pm\left(\frac{\dfrac{1-\xi^2}{\xi^2}}{1+\dfrac{1-\xi^2}{\xi^2}}\right)^{1/2} \qquad [3.121]$$

$$\sin\left(P\,\theta+\phi\right)=\pm\sqrt{1-\xi^2} \qquad [3.122]$$

However, $\pm\sin\left(P\,\theta_1+\phi\right)=\pm\sin\left(P\,\theta_2+\phi\right)$ ($\pm$ according to whether they are two maxima or two minima, but the two signs are taken to be identical), yielding

$$\frac{q_{1M}}{q_{2M}}=e^{-\xi\,(\theta_1-\theta_2)} \qquad [3.123]$$

The difference $\theta_2-\theta_1$ is the *pseudo-period* Θ.

$$\frac{q_{1M}}{q_{2M}}=e^{\xi\,\Theta} \qquad [3.124]$$

Since $\dfrac{q_{1M}}{q_{2M}}=e^{\delta}$, we can connect the relative damping, ξ, and the logarithmic decrement, δ, by:

$$\xi\,\Theta=\delta \qquad [3.125]$$

δ is called the *logarithmic decrement.*

$$\delta=\ln\frac{q_{1M}}{q_{2M}} \qquad [3.126]$$

This is a quantity which is accessible experimentally. In practice, if damping is weak, the measurement of δ is imprecise when carried out from two successive positive peaks. It is better to consider n pseudo-periods and δ is then given by [HAB 68]:

$$\delta = \frac{1}{n} \ln \frac{q_{1_M}}{q_{(n+1)_M}} \qquad [3.127]$$

Here n is the number of positive peaks. In fact, the ratio of the amplitude of any two consecutive peaks is [HAL 78], [LAZ 68]:

$$\frac{q_1}{q_2} = \frac{q_2}{q_3} = \frac{q_3}{q_4} = \cdots\cdots = \frac{q_n}{q_{n+1}} = e^{\delta} \qquad [3.128]$$

yielding

$$\frac{q_1}{q_{n+1}} = \frac{q_1}{q_2} = \frac{q_2}{q_3} = \cdots\cdots = \frac{q_n}{q_{n+1}} = e^{n\delta} \qquad [3.129]$$

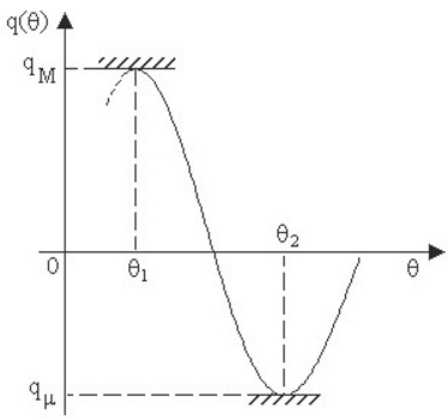


Figure 3.23. *Successive peaks of the response*

NOTES.–

1. *It is useful to be able to calculate the logarithmic decrement δ starting from a maximum and a consecutive minimum. In this case, we can write:*

$$q_M = q_m \, e^{-\xi \theta_1} \sin\left(P\theta_1 + \phi\right)$$

$$q_\mu = q_m \, e^{-\xi \theta_2} \sin\left(P\theta_2 + \phi\right)$$

$$\sin\left(P\,\theta_1+\phi\right)=-\sin\left(P\,\theta_2+\phi\right)$$

$$\frac{q_M}{q_\mu}=e^{-\xi\left(\theta_1-\theta_2\right)}=e^{\xi\frac{\Theta}{2}}=e^{\frac{\delta}{2}}$$

yielding

$$\delta=2\,\ln\left|\frac{q_M}{q_\mu}\right| \qquad [3.130]$$

If n is even (positive or negative peaks), which corresponds to the first positive peak and the last negative peak (or the reverse), we have:

$$\delta=\frac{2}{n-1}=\ln\left|\frac{q_{1M}}{q_\mu}\right| \qquad [3.131]$$

2. *The decrement δ can also be expressed according to the difference of two successive peaks:*

$$\frac{q_{1M}-q_{2M}}{q_{1M}}=1-\frac{q_{2M}}{q_{1M}}=1-e^{-\delta} \qquad [3.132]$$

(indices 1 and 2 or more generally n and $n+1$).

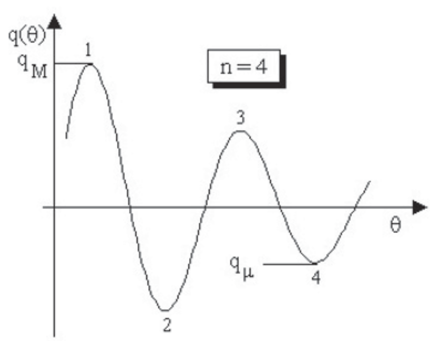


Figure 3.24. *Different peaks*

If damping is weak, q_{1_M} and q_{2_M} are not very different and if we set:

$$\Delta q = q_{1_M} - q_{2_M} \tag{3.133}$$

Δq can be considered as infinitely small:

$$\delta = \ln \frac{q_{1_M}}{q_{2_M}} = \ln\left(1 + \frac{\Delta q}{q_{2_M}}\right)$$

$$\delta \approx \frac{\Delta q}{q_{2_M}}$$

In the case of several peaks, we have:

$$\delta \approx \frac{\Delta q}{n\ q_{2_M}} \tag{3.134}$$

Knowing that [HAB 68]:

$$\Theta = \frac{2\pi}{P} \tag{3.135}$$

$$\delta = \frac{2\ \pi\ \xi}{\sqrt{1 - \xi^2}} \tag{3.136}$$

or

$$\xi = \frac{\delta}{\sqrt{\delta^2 + 4\ \pi^2}} \tag{3.137}$$

Example 3.1. [LAZ 50]

Material	δ	ξ
Concrete	0.06	0.010
Bolted steel	0.05	0.008
Welded steel	0.03	0.005

Table 3.3. *Examples of decrement and damping values*

NOTE.– *If ξ is very small, in practice less than 0.10, ξ^2 can, at first approximation, be neglected.*

Then:

$$\delta \approx 2\,\pi\,\xi \tag{3.138}$$

yielding

$$\xi \approx \frac{\Delta q}{2\pi\; q_{2M}} \tag{3.139}$$

and

$$\xi \approx \frac{\Delta q}{2\,\pi\,n\;q_{2M}} \tag{3.140}$$

Figure 3.25 represents the variations in the decrement δ with damping ξ and shows how, in the vicinity of the origin, we can, at first approximation, confuse the curve with its tangent [THO 65a].

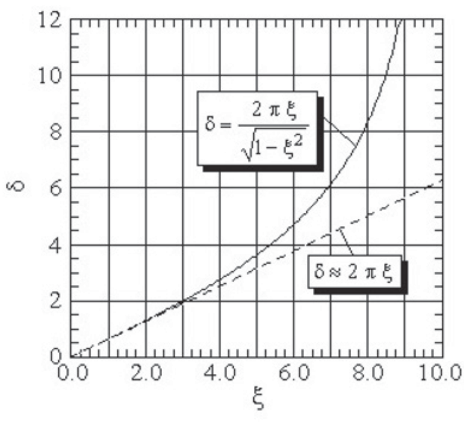


Figure 3.25. *Variations in decrement with damping*

We defined δ starting from [3.128]:

$$\frac{q_{1_M}}{q_{(n+1)_M}} = e^{n\,\delta}$$

yielding, by replacing δ with expression [3.138],

$$\frac{q_{1M}}{q_{(n+1)M}} = e^{n \frac{2\,\pi\,\xi}{\sqrt{1-\xi^2}}} \qquad [3.141]$$

i.e.

$$\frac{q_{(n+1)_M}}{q_{1_M}} = e^{-2\,\pi\,n\,\xi / \sqrt{1-\xi^2}} \qquad [3.142]$$

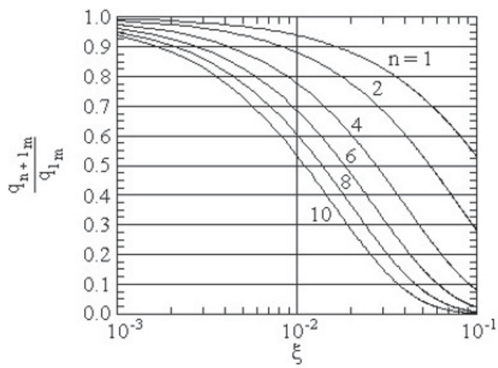


Figure 3.26. *Reduction of amplitude with damping*

The curves in Figure 3.26 give the ratio $\frac{q_{(n+1)_M}}{q_{1_M}}$ versus ξ, for various values of n. For very small ξ, we have, at first approximation:

$$\frac{q_{(n+1)_M}}{q_{1_M}} \approx e^{-2\,\pi\,n\,\xi} \qquad [3.143]$$

3.6.3.4. *Number of cycles for a given reduction in amplitude*

Amplitude reduction of 50%

On the assumption that

$$q_{(n+1)_M} = \frac{q_{1_M}}{2}$$

relation [3.126] becomes

$$\delta = \frac{1}{n} \ln 2 = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \qquad [3.144]$$

If ξ is small:

$$2\pi\ \xi \approx \frac{1}{n} \ln 2$$

$$n\,\xi \approx \frac{0.693}{2\pi} \approx 0.110$$

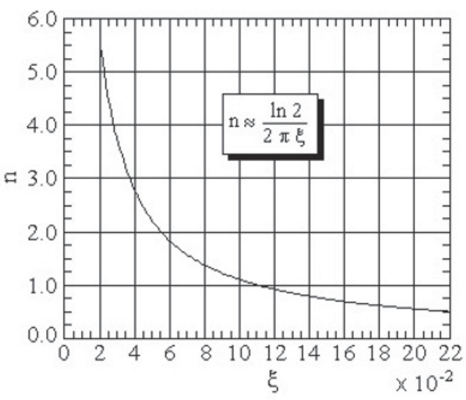


Figure 3.27. *Number of cycles for amplitude reduction of 50%*

The curve in Figure 3.27 shows the variations of n versus ξ, in the domain where the approximation $\delta \approx 2\pi\ \xi$ is correct [THO 65a].

Amplitude reduction of 90%

In the same way, we have:

$$q_{(n+1)_M} = \frac{q_{1M}}{10}$$

$$\delta = \frac{1}{n} \ln 10$$

and for small values of ξ:

$$n\,\xi \approx \frac{1}{2\pi} \ln 10$$

$$n\,\xi \approx 0.366$$

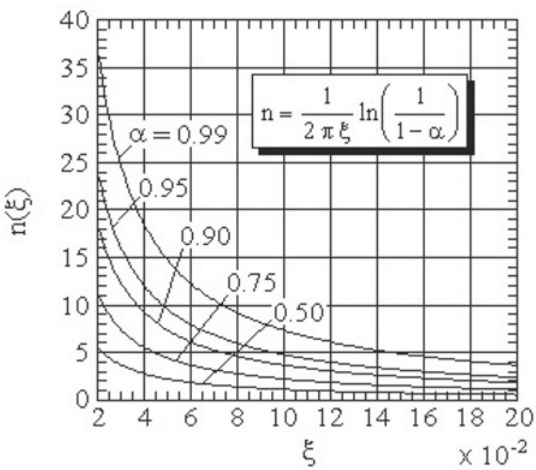


Figure 3.28. *Number of cycles for an amplitude reduction of 90%*

Reduction of α%

More generally, the number of cycles described for an amplitude reduction of α% will be, for small ξ:

$$n\,\xi \approx \frac{1}{2\pi}\ln\left[\frac{1}{1-\dfrac{\alpha}{100}}\right] \qquad [3.145]$$

3.6.3.5. *Influence of damping on period*

Unless it is very large, damping generally has little influence over the period; we have $\Theta_0 = \dfrac{2\pi}{Po}$ for $\xi = 0$ $(P_0 = 1)$ and, for small ξ,

$$\frac{\Delta\Theta}{\Theta_0} = \frac{\Theta - \Theta_0}{\Theta_0} = \frac{\Theta}{\Theta_0} - 1 = \frac{\dfrac{2\,\pi}{P}}{\dfrac{2\,\pi}{P_0}} - 1 = \frac{P_0}{P} - 1 = \frac{1}{\sqrt{1-\xi^2}} - 1 \approx \frac{\delta^2}{8\,\pi^2} \qquad [3.146]$$

For most current calculations, it is possible to confuse Θ with Θ_0. For the first order, the pulsation and the period are not modified by damping. For the second order, the pulsation is modified by a corrective term that is always negative and the period is increased:

$$\Theta = \frac{\Theta_0}{\sqrt{1-\xi^2}} \approx \Theta_0\left(1 + \frac{\xi^2}{2}\right) \qquad [3.147]$$

and

$$P = P_0\,\sqrt{1-\xi^2} \approx P_0\left(1 - \frac{\xi^2}{2}\right) \qquad [3.148]$$

$\left(P_0 = 1\right)$ or

$$\omega = \omega_0\,\sqrt{1-\xi^2} \approx \omega_0\left(1 - \frac{\xi^2}{2}\right) \qquad [3.149]$$

NOTE.– *The logarithmic decrement also represents the energy variation during a cycle of decrease. For sufficiently small δ we have [LAZ 50]:*

$$\delta \approx \frac{1}{2}\left(\frac{Energy(n-1) - Energy(n)}{Energy(n-1)}\right) \qquad [3.150]$$

$$\left[Energy(n-1) = Energy\ to\ the\ (n-1)^{th}\ cycle\right]$$

In practical cases where ξ lies between 0 and 1, the energy initially provided to the system dissipates itself little by little to the external medium in various forms (friction between solid bodies, with air or another fluid, internal slips in the metal during elastic strain, radiation, energy dissipation in electromagnetic form).

Consequently, the amplitude of the oscillations decreases constantly with time. If we wanted to keep a constant amplitude, it would be necessary to put back into the system the energy which it loses every time. The system is then no longer free: the oscillations are maintained or forced. We will study this case in Chapter 6.

3.6.3.6. *Particular case of zero damping*

In this case, $q(\theta)$ becomes:

$$q(\theta) = q_0 \cos \theta + \dot{q}_0 \sin \theta \qquad [3.151]$$

which can also be written as:

$$q(\theta) = q_m \sin(\theta + \varphi) \qquad [3.152]$$

where

$$q_m = \sqrt{q_0^2 + \dot{q}_0^2} \qquad [3.153]$$

$$\tan \varphi = \frac{\dot{q}_0}{q_0} \qquad [3.154]$$

If it is assumed that the mechanical system has moved away from its equilibrium position and then released in the absence of any external forces at time $t = 0$, the response is then of the non-damped oscillatory type for $\xi = 0$.

In this (theoretical) case, the movement of natural pulsation ω_0 should last indefinitely, since the characteristic equation does not contain a first order term. This is the consequence of the absence of a damping element. The potential energy of the spring decreases by increasing the kinetic energy of the mass and vice versa; the system is known as *conservative.*

To summarize, when the relative damping ξ varies continuously, the mode passes without discontinuity to one of the following:

- $\xi = 0$ undamped oscillatory mode;
- $0 \le \xi \le 1$ damped oscillatory mode. The system moves away from its equilibrium position, oscillating around the equilibrium point before stabilizing;
- $\xi = 1$ critical aperiodic mode, corresponding to the fastest possible return of the system without crossing the equilibrium position;
- $\xi > 1$ damped aperiodic mode. The system returns to its equilibrium position without any oscillation, all the more quickly because ξ is closer to 1.

In the following chapters, we will focus more specifically on the case $0 \le \xi \le 1$, which corresponds to the values observed in the majority of real structures.

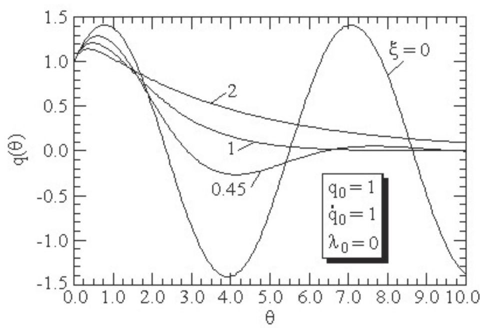


Figure 3.29. *Various modes for* $q_0 = 1$, $\dot{q}_0 = 1$

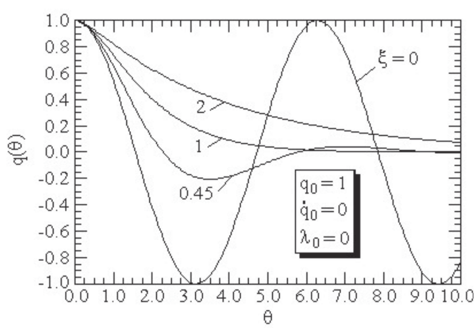


Figure 3.30. *Various modes for* $q_0 = 1$, $\dot{q}_0 = 0$

3.6.3.7. *Quality factor*

The *quality factor* or *Q factor* of the oscillator is the number Q defined by:

$$\frac{1}{Q} = \frac{c}{2\,\omega_0} = \frac{c}{\sqrt{k\,m}} = \frac{\omega_0\,c}{k} = 2\,\xi \qquad [3.155]$$

The properties of this factor will be considered in more detail in the following chapters.

Chapter 4

Impulse and Step Responses

4.1. Response of a mass–spring system to a unit step function (step or indicial response)

4.1.1. *Response defined by relative displacement*

4.1.1.1. *Expression for response*

Let us consider a damped mass–spring system. Before the initial time $t = 0$ the mass is assumed to be at rest. At time $t = 0$, a constant excitation of unit amplitude continuously acts for all $t > 0$ [BRO 53], [KAR 40]. We have seen that, for zero initial conditions, the Laplace transform of the response of a one-degree-of-freedom system is given by [3.29]:

$$Q(p) = \frac{\Lambda(p)}{p^2 + 2\,\xi\,p + 1} \qquad [4.1]$$

Here $\Lambda(p) = \dfrac{1}{p}$ (unit step transform), yielding the response:

$$q(\theta) = L^{-1}\left[\frac{1}{p\left(p^2 + 2\,\xi\,p + 1\right)}\right] = L^{-1}\left[\frac{1}{p}\right] - L^{-1}\left[\frac{p}{p^2 + 2\,\xi\,p + 1}\right] - L^{-1}\left[\frac{2\,\xi}{p^2 + 2\,\xi\,p + 1}\right]$$

[4.2]

$$q(\theta) = 1 - \frac{e^{-\xi\theta}}{\sqrt{1-\xi^2}}\left[\sqrt{1-\xi^2}\cos\sqrt{1-\xi^2}\;\theta - \xi\sin\sqrt{1-\xi^2}\;\theta\right]$$

$$-2\,\xi\frac{e^{-\xi\theta}}{\sqrt{1-\xi^2}}\sin\sqrt{1-\xi^2}\;\theta \qquad [4.3]$$

$(\xi \neq 1)$

$$q(\theta) = 1 - e^{-\xi\theta}\left[\cos\sqrt{1-\xi^2}\;\theta + \frac{\xi}{\sqrt{1-\xi^2}}\sin\sqrt{1-\xi^2}\;\theta\right] \qquad [4.4]$$

i.e.

$$u(t) = A(t) = \ell_m\left[1 - e^{-\xi\omega_0 t}\cos\omega_0\sqrt{1-\xi^2}\;t - \frac{\xi}{\sqrt{1-\xi^2}}e^{-\xi\omega_0 t}\sin\omega_0\sqrt{1-\xi^2}\;t\right]$$

[4.5]

with $\ell_m = 1$ [HAB 68], [KAR 40].

NOTE.– *This calculation is identical to that carried out to obtain the primary response spectrum to a rectangular shock [LAL 75].*

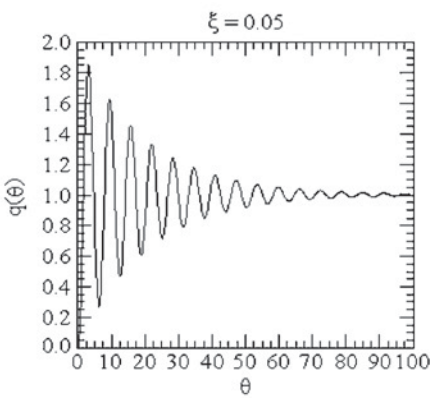


Figure 4.1. *Example of relative displacement response to a unit step excitation*

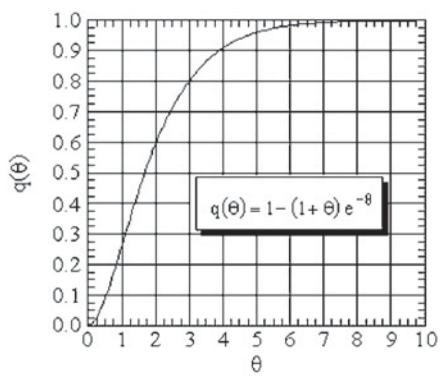


Figure 4.2. *Response to a unit step excitation for $\xi = 1$*

Specific cases

1. If $\xi = 1$

$$Q(p) = \frac{1}{p\,(p+1)^2} \qquad [4.6]$$

$$Q(p) = \frac{1}{p} - \frac{1}{p+1} - \frac{1}{(p+1)^2} \qquad [4.7]$$

$$q(\theta) = 1 - (1+\theta)\, e^{-\theta} \qquad [4.8]$$

and, for $\ell_m = 1$,

$$u(t) = 1 - e^{-\omega_0 t} - \omega_0\, t\, e^{-\omega_0 t} \qquad [4.9]$$

2. Zero damping

In the reduced form, the equation of movement can be written using the notation of the previous sections:

$$\frac{d^2 q(\theta)}{d\theta^2} + q(\theta) = \lambda(\theta) \qquad [4.10]$$

or

$$\ddot{u}(t) + \omega_0^2\, u(t) = \omega_0^2\, \ell(t) \qquad [4.11]$$

with the initial conditions being constant, namely, for $\theta = 0$

$$q(0) = \left(\frac{dq}{d\theta}\right)_{\theta=0} = 0,$$

or, according to the case, $t = 0$ and

$$u(0) = \left(\frac{du}{dt}\right)_{t=0} = 0.$$

After integration, this becomes as before:

$$q(\theta) = 1 - \cos\theta \tag{4.12}$$

and

$$u(t) = \ell_m \left(1 - \cos\omega_0\, t\right) \tag{4.13}$$

the expression in which, by definition of the excitation, $\ell_m = 1$:

$$u(t) = 1 - \cos\omega_0\, t \tag{4.14}$$

Example 4.1.

If the excitation is a force, the equation of the movement is, for $t \geq 0$, $m\dfrac{d^2z}{dt^2} + k\, z = 1$, with, for initial conditions at $t = 0$, $z(0) = \left(\dfrac{dz}{dt}\right)_{t=0} = 0$. This yields, after integration,

$$z(t) = \frac{1}{k}\left(1 - \cos\sqrt{\frac{k}{m}}\, t\right) \tag{4.15}$$

NOTE.– *The dimensions of [4.15] do not seem correct. It should be remembered that the excitation used is a force of amplitude equal to one and is thus homogeneous with a displacement.*

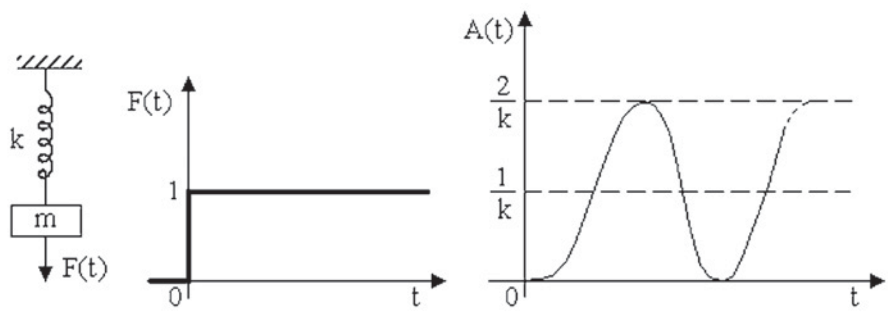


Figure 4.3. *Step response for $\xi = 0$*

The function z(t), a response with the step unit function, is often termed the *indicial admittance* or *step response* and is written A(t).

It can be seen in this example that if we set, according to our notation, $z_s = \frac{F_m}{k} = \frac{1}{k}$, the ratio of the maximum elongation z_m to the static deflection z_s, which the mass would take if the force were statically applied, reached a value of two. The spring, in dynamics, is deformed two times more than in statics, and there is a fear that it may undergo stresses that are twice as large.

Often, however, the materials resist transient stresses better than static stresses (Chapter 2). This remark relates to the initial moments, during which F(t) is transitory and is raised from 0 to 1. For this example, where F(t) remains equal to one for all positive values of t and where the system is undamped, the effect of shock would be followed by a fatigue effect.

4.1.1.2. *Extremum for response*

The expression for the response

$$q(\theta) = 1 - e^{-\xi\theta}\left[\cos\sqrt{1-\xi^2}\;\theta + \frac{\xi}{\sqrt{1-\xi^2}}\sin\sqrt{1-\xi^2}\;\theta\right] \qquad [4.16]$$

has a zero derivative $\frac{dq}{d\theta}$ for $\theta = \theta_m$ such that

$$-\xi\, e^{-\xi\theta_m}\left[\cos\sqrt{1-\xi^2}\;\theta_m + \frac{\xi}{\sqrt{1-\xi^2}}\sin\sqrt{1-\xi^2}\;\theta_m\right]$$

$$+\, e^{-\xi\theta_m}\left[-\sqrt{1-\xi^2}\sin\sqrt{1-\xi^2}\;\theta_m + \xi\cos\sqrt{1-\xi^2}\;\theta_m\right] = 0$$

$$\sin\sqrt{1-\xi^2}\;\theta_m = 0$$

$$\theta_m = \frac{k\,\pi}{\sqrt{1-\xi^2}} \qquad [4.17]$$

The first maximum (which would correspond to the point of the positive primary shock response spectrum at the natural frequency f_0 of the resonator) occurs for $\theta_m = \dfrac{\pi}{\sqrt{1-\xi^2}}$ at time $t_m = \dfrac{1}{2\, f_0 \sqrt{1-\xi^2}}$ [HAL 78].

From this the value $q(\theta)$ is deduced:

$$q(\theta_m) = q_m = 1 - e^{-\frac{\xi\,\pi}{\sqrt{1-\xi^2}}} \left[\cos \pi + \frac{\xi}{\sqrt{1-\xi^2}} \sin \pi \right] \qquad [4.18]$$

$$q_m = 1 + e^{-\frac{\pi\,\xi}{\sqrt{1-\xi^2}}} \qquad [4.19]$$

(always a positive quantity). The first maximum amplitude q_m tends towards 1 when ξ tends towards 1.

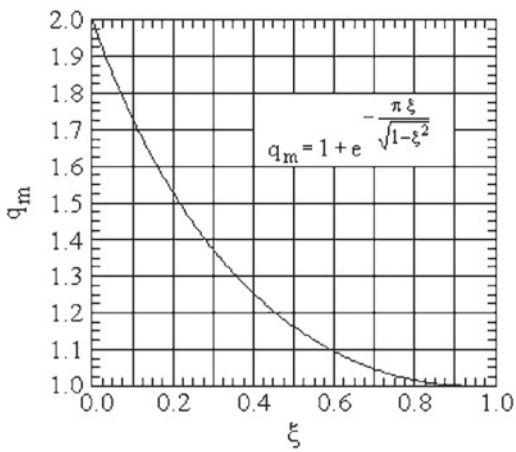


Figure 4.4. *First maximum amplitude versus* ξ

NOTES.–

1. q_m *is independent of the natural frequency of the resonator.*

2. *For* $\xi = 0$, $q_m = 2$.

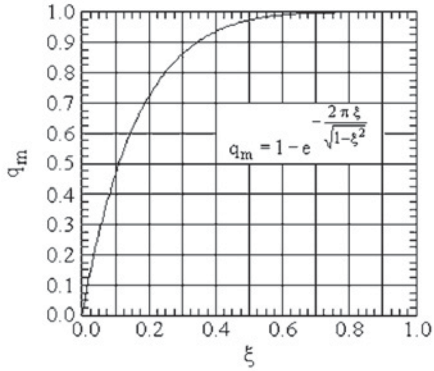


Figure 4.5. *Amplitude of the first minimum versus ξ*

For $k = 2$,

$$\theta_m = \frac{2\pi}{\sqrt{1-\xi^2}} \tag{4.20}$$

and

$$t_m = \frac{1}{f_0\sqrt{1-\xi^2}},$$

$$q(\theta_m) = q_m = 1 - e^{-\frac{2\pi\xi}{\sqrt{1-\xi^2}}} \tag{4.21}$$

q_m is negative for all $\xi \in [0, 1]$.

$q_m = 0$ for $\xi = 0$

$q_m = 1$ for $\xi = 1$

4.1.1.3. *First up-crossing of the unit value by the response*

θ_1 is searched such that:

$$q(\theta) \equiv 1 = 1 - e^{-\xi\,\theta_1}\left[\cos\sqrt{1-\xi^2}\;\theta_1 + \frac{\xi}{\sqrt{1-\xi^2}}\sin\sqrt{1-\xi^2}\;\theta_1\right] \tag{4.22}$$

As $e^{-\xi\,\theta_1} \neq 0$ is assumed

$$\cos\sqrt{1-\xi^2}\;\theta_1 = -\frac{\xi}{\sqrt{1-\xi^2}}\,\sin\sqrt{1-\xi^2}\;\theta_1$$

i.e.

$$\tan\sqrt{1-\xi^2}\;\theta_1 = -\frac{\sqrt{1-\xi^2}}{\xi} \qquad [4.23]$$

This yields, since $\tan\sqrt{1-\xi^2}\;\theta_1 \le 0$ and $\sqrt{1-\xi^2}\;\theta_1 \ge 0$ must be present simultaneously:

$$\sqrt{1-\xi^2}\;\theta_1 = \pi - \text{arc tan}\frac{\sqrt{1-\xi^2}}{\xi} \qquad [4.24]$$

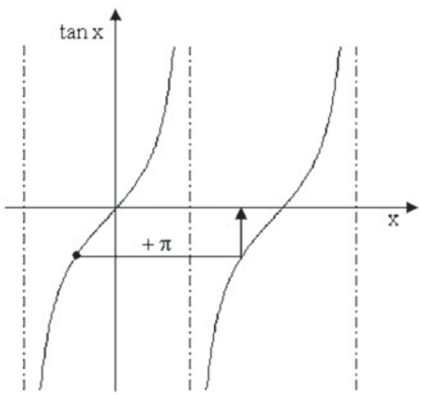


Figure 4.6. *Resolution of [4.23]*

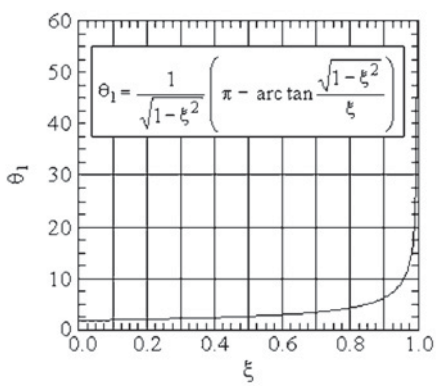


Figure 4.7. *Time of first up-crossing of the unit value by the response*

$$\theta_1 = \frac{1}{\sqrt{1-\xi^2}}\left[\pi - \text{arc tan}\frac{\sqrt{1-\xi^2}}{\xi}\right] \qquad [4.25]$$

If $\xi = 0$,

$$q(\theta) = 1 - \cos\theta \quad [4.26]$$

If $q(\theta) = 1$

$$\theta = \left(k + \frac{1}{2}\right)\pi \quad [4.27]$$

If $\xi = 1$

$$1 = 1 - e^{-\theta} - \theta\, e^{-\theta} \quad [4.28]$$

The only positive root exists for infinite θ.

4.1.2. *Response defined by absolute displacement, velocity or acceleration*

4.1.2.1. *Expression for response*

In this case, for any ξ and zero initial conditions,

$$Q(p) = \frac{\Lambda(p)\,(1 + 2\,\xi\,p)}{p^2 + 2\,\xi\,p + 1} \quad [4.29]$$

with $\Lambda(p) = \dfrac{1}{p}$.

$$q(\theta) = L^{-1}\left[\frac{1 + 2\,\xi\,p}{p\left(p^2 + 2\,\xi\,p + 1\right)}\right] = L^{-1}\left[\frac{1}{p}\right] - L^{-1}\left[\frac{p}{p^2 + 2\,\xi\,p + 1}\right] \quad [4.30]$$

$$q(\theta) = 1 - e^{-\xi\theta}\left[\cos\sqrt{1-\xi^2}\;\theta - \frac{\xi}{\sqrt{1-\xi^2}}\sin\sqrt{1-\xi^2}\;\theta\right] = A(\theta) \quad [4.31]$$

$(\xi \neq 1)$.

If $\xi = 0$

$$q(\theta) = 1 - \cos\theta \qquad [4.32]$$

If $\xi = 1$

$$Q(p) = \frac{1}{p}\frac{1 + 2\,p}{(p+1)^2} \qquad [4.33]$$

$$Q(p) = \frac{1}{p} - \frac{1}{p+1} + \frac{1}{(p+1)^2} \qquad [4.34]$$

$$q(\theta) = 1 - e^{-\theta} + \theta\, e^{-\theta} = A(\theta) = 1 + (\theta - 1)\, e^{-\theta} \qquad [4.35]$$

$$u(t) = 1 + \left(\omega_0\, t - 1\right) e^{-\omega_0\, t} \qquad [4.36]$$

($\ell_m = 1$).

4.1.2.2. *Extremum for response*

The extremum of the response $q(\theta) = A(\theta)$ occurs for $\theta = \theta_m$ such that $\frac{dA}{d\theta} = 0$, which leads to

$$-\xi\, e^{-\xi\,\theta_m}\left[\cos\sqrt{1-\xi^2}\,\theta_m - \frac{\xi}{\sqrt{1-\xi^2}}\sin\sqrt{1-\xi^2}\,\theta_m\right]$$

$$+\, e^{-\xi\,\theta_m}\left[-\sqrt{1-\xi^2}\sin\sqrt{1-\xi^2}\,\theta_m - \xi\cos\sqrt{1-\xi^2}\,\theta_m\right] = 0$$

i.e. to

$$\tan\sqrt{1-\xi^2}\,\theta_m = \frac{2\,\xi\sqrt{1-\xi^2}}{2\,\xi^2 - 1} \qquad [4.37]$$

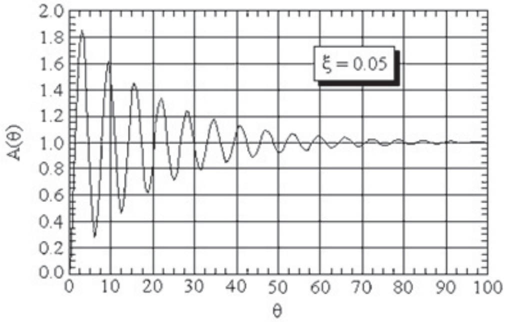


Figure 4.8. *Example of absolute response*

For $2\,\xi^2 - 1 \geq 0$ (and since θ_m is positive):

$$\theta_m = \frac{1}{\sqrt{1-\xi^2}} \arctan \frac{2\,\xi\sqrt{1-\xi^2}}{2\,\xi^2 - 1} \qquad [4.38]$$

and if $2\,\xi^2 - 1 < 0$

$$\theta_m = \frac{1}{\sqrt{1-\xi^2}} \left[\pi + \arctan \frac{2\,\xi\sqrt{1-\xi^2}}{2\,\xi^2 - 1} \right] \qquad [4.39]$$

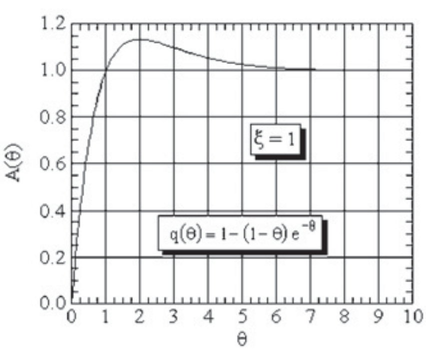


Figure 4.9. *Amplitude of the absolute response for $\xi = 1$*

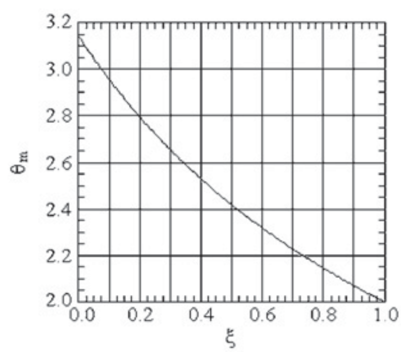


Figure 4.10. *Time of extremum versus ξ*

For $\theta = \theta_m$,

$$A(\theta_m) = 1 - e^{-\xi\,\theta_m}\left[\cos\sqrt{1-\xi^2}\;\theta_m - \frac{\xi}{\sqrt{1-\xi^2}}\sin\sqrt{1-\xi^2}\;\theta_m\right]$$

i.e.

$$A(\theta_m) = 1 + e^{-\xi\,\theta_m} \qquad [4.40]$$

If $\xi = 1$,

$$q(\theta) = 1 - e^{-\theta} + \theta\, e^{-\theta} \qquad [4.41]$$

Then

$$\frac{dq}{d\theta} = (2-\theta)\, e^{-\theta} = 0$$

if $\theta = 2$ or if $\theta \to \infty$.

This yields

$$q(\theta_m) = 1 + e^{-2}$$

or

$$q(\theta_m) = 1$$

If $\xi = \dfrac{1}{\sqrt{2}}$, $\theta_m = \dfrac{\pi}{2}\sqrt{2}$ and

$$q(\theta_m) = 1 + e^{-\pi/2}$$

$$q(\theta) = 1.20788\ldots$$

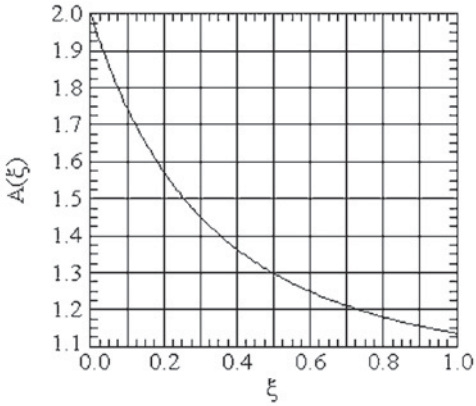


Figure 4.11. *Amplitude of absolute response versus ξ*

4.1.2.3. *First up-crossing of the unit value by the response*

The first up-crossing of the unit value happens at time θ_1 such that

$$A(\theta_1) = 1 - e^{-\xi\,\theta_1}\left[\cos\sqrt{1-\xi^2}\;\theta_1 - \frac{\xi}{\sqrt{1-\xi^2}}\sin\sqrt{1-\xi^2}\;\theta_1\right] \qquad [4.42]$$

$$\tan\sqrt{1-\xi^2}\;\theta_1 = \frac{\sqrt{1-\xi^2}}{\xi} \qquad [4.43]$$

$$\theta_1 = \frac{1}{\sqrt{1-\xi^2}}\,\arctan\frac{\sqrt{1-\xi^2}}{\xi} \qquad [4.44]$$

If $\xi = 0$,

$$\theta_1 = \frac{\pi}{2}$$

If $\xi = 1$,

$$q(\theta) = 1 = 1 - e^{-\theta} + \theta\, e^{-\theta} \qquad [4.45]$$

yielding $\theta_1 = 1$ or $\theta_1 =$ infinity.

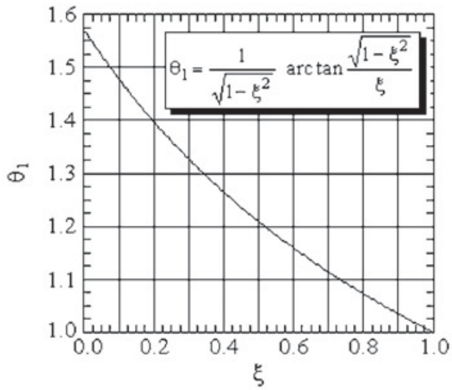


Figure 4.12. *Time of first up-crossing of the unit value versus ξ*

4.2. Response of a mass–spring system to a unit impulse excitation

4.2.1. *Response defined by relative displacement*

4.2.1.1. *Expression for response*

Let us consider a Dirac delta function $\delta_g(\theta)$ obeying

$$\left|\begin{array}{ll} \delta_g(\theta) = 0 & \text{for } \theta \neq 0 \\ \delta_g(0) & \text{infinite} \\ \int_{-\infty}^{+\infty} \delta_g(\theta)\, dt = 1 & \end{array}\right. \qquad [4.46]$$

such that, with $\theta = 0$, $q = 0$ and $m \dfrac{dq}{dt} = 1$. The quantity $m \dfrac{dq}{dt}$ is the impulse transmitted to the mass m by a force acting for a small interval of time $\Delta\theta$ [KAR 40]. The contribution of the restoring force of the spring to the impulse is negligible during the very short time interval $\Delta\theta$.

Depending on whether the impulse is defined by a force or an acceleration,

$$\frac{d^2 z}{dt^2} + 2\,\xi\,\omega_0 \frac{dz}{dt} + \omega_0^2\, z = \omega_0^2 \frac{\delta_F}{k} \qquad [4.47]$$

where $\delta_F = F\ \delta(t)$, $F = 1$ (force) and $\delta(t)$ is a Dirac delta function obeying the same definition, or

$$\frac{d^2 z}{dt^2} + 2\ \xi\ \omega_0\ \frac{dz}{dt} + \omega_0^2\ z = -\omega_0^2\ \frac{\delta_{AC}}{\omega_0^2} \qquad [4.48]$$

with $\delta_{AC} = \ddot{x}\ \delta(t)$ and $\ddot{x} = 1$ (acceleration). If $\delta_g(t)$, the generalized delta function, is equal, according to the case, to $\frac{\delta_F}{k}$ or to $-\frac{\delta_{AC}}{\omega_0^2}$, then the generalized equation is obtained as follows:

$$\frac{d^2 u}{dt^2} + 2\ \xi\ \omega_0\ \frac{du}{dt} + \omega_0^2\ u = \omega_0^2\ \delta_g(t) \qquad [4.49]$$

Then

$$\Im = \int_0^{\Delta t} \delta_g(t)\ dt = \begin{bmatrix} \int_0^{\Delta t} \frac{\delta_F}{k}\ dt = \frac{1}{k}\ I \\ \int_0^{\Delta t} \left(-\frac{\delta_{AC}}{\omega_0^2} \right) dt = -\frac{1}{\omega_0^2}\ I \quad \left(= -\frac{\delta V}{\omega_0^2} \right) \end{bmatrix} \qquad [4.50]$$

(δV = velocity change due to the acceleration impulse). To make the differential equation dimensionless, each member is divided by the quantity $\Im\ \omega_0$ homogeneous with length, and set $q = \frac{u}{\Im\ \omega_0}$ and $\theta = \omega_0\ t$. This becomes:

$$\frac{d^2 q}{d\theta^2} + 2\ \xi\ \frac{dq}{d\theta} + q(\theta) = \delta_g(\theta) \qquad [4.51]$$

The transform of a Dirac delta function being equal to the unit [LAL 75], the Laplace transform of this equation is written with the notation already used,

$$Q(p)\ \left(p^2 + 2\ \xi\ p + 1\right) = 1 \qquad [4.52]$$

From where

$$q(\theta) = \frac{e^{-\xi\,\theta}}{\sqrt{1-\xi^2}} \sin\sqrt{1-\xi^2}\,\theta = h(\theta) \qquad [4.53]$$

($\xi \neq 1$) and

$$u(t) = \omega_0\, \Im\, \frac{e^{-\xi\,\omega_0\,t}}{\sqrt{1-\xi^2}} \sin\omega_0 \sqrt{1-\xi^2}\,t \qquad [4.54]$$

and if $\Im = 1$,

$$h(t) = \omega_0\, \frac{e^{-\xi\,\omega_0\,t}}{\sqrt{1-\xi^2}} \sin\omega_0 \sqrt{1-\xi^2}\,t$$

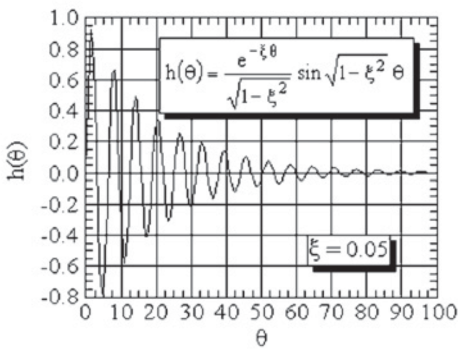


Figure 4.13. *Impulse response*

For an acceleration Dirac function,

$$u(t) = -\frac{\delta V}{\omega_0} \frac{e^{-\xi\,\omega_0\,t}}{\sqrt{1-\xi^2}} \sin\omega_0 \sqrt{1-\xi^2}\,t \qquad [4.55]$$

and

$$\omega_0^2\, z(t) = -\omega_0\, \delta V\, \frac{e^{-\xi\,\omega_0\,t}}{\sqrt{1-\xi^2}} \sin\omega_0 \sqrt{1-\xi^2}\,t \qquad [4.56]$$

Specific cases

1. For $\xi = 0$,

$$q(\theta) = h(\theta) = \sin\theta$$

and

$$u(t) = \omega_0\ \Im \sin\omega_0\ t \tag{4.57}$$

Example 4.2.

If the impulse is defined by a force, $\Im = \frac{1}{k}$ then

$$u(t) = z(t) = \frac{\omega_0}{k}\sin\omega_0\ t = \frac{1}{\sqrt{k\ m}}\sin\omega_0\ t \tag{4.58}$$

This relation is quite homogeneous, since the "number" 1 corresponds to the impulse $\frac{1}{\sqrt{k\ m}} = \frac{I}{\sqrt{k\ m}}$, which has the dimension of a displacement.

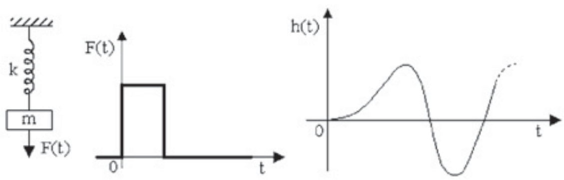


Figure 4.14. *Impulse response*

The unit impulse response is denoted by $h(t)$ [BRO 53], [KAR 40]. It is called the *impulse response, impulsive response* or *weight function* [GUI 63].

2. If $\xi = 1$,

$$Q(p) = \frac{1}{(p+1)^2}$$

$$q(\theta) = h(\theta) = \theta\ e^{-\theta} \tag{4.59}$$

$$u(t) = h(t) = \omega_0^2 \; t \; e^{-\omega_0 \, t} \qquad [4.60]$$

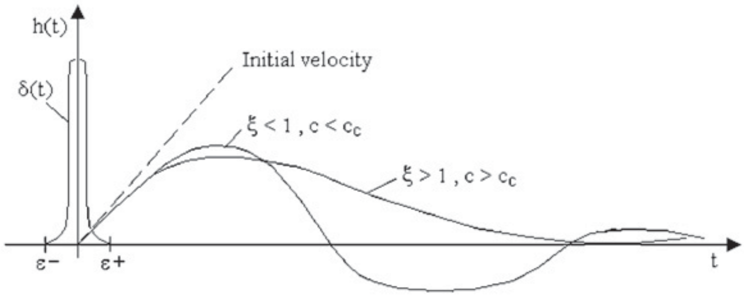


Figure 4.15. *Examples of impulse responses versus ξ*

4.2.1.2. *Extremum for response*

$q(\theta)$ presents a peak q_m when $\frac{dq}{d\theta} = 0$, i.e. for $\theta = \theta_m$ such that

$$-\xi \, e^{-\xi \, \theta_m} \sin\sqrt{1-\xi^2} \; \theta_m + e^{-\xi \, \theta_m} \sqrt{1-\xi^2} \cos\sqrt{1-\xi^2} \; \theta_m = 0$$

$$\tan\sqrt{1-\xi^2} \; \theta_m = \frac{\sqrt{1-\xi^2}}{\xi}$$

$$\theta_m = \frac{1}{\sqrt{1-\xi^2}} \arctan\frac{\sqrt{1-\xi^2}}{\xi} \qquad [4.61]$$

This yields

$$q_m = \frac{e^{-\xi \, \theta_m}}{\sqrt{1-\xi^2}} \sin\sqrt{1-\xi^2} \; \theta_m$$

i.e.

$$q_m = e^{-\frac{\xi}{\sqrt{1-\xi^2}} \arctan\frac{\sqrt{1-\xi^2}}{\xi}} \qquad [4.62]$$

For $\xi = 1$,

$$q(\theta) = \theta\, e^{-\theta} \qquad [4.63]$$

$$\frac{dq}{d\theta} = 0 \text{ if } \theta = 1 \text{, yielding}$$

$$q_m = \frac{1}{e} \qquad [4.64]$$

For $\xi = 0$,

$$h(\theta) = \sin\theta \qquad [4.65]$$

$$\frac{dh}{d\theta} = \cos\theta = 0 \text{ if } \theta = \left(k + \frac{1}{2}\right)\pi \text{. If } k = 0,\ h = 1.$$

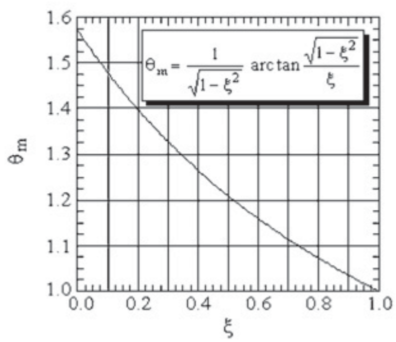


Figure 4.16. *Time of the first maximum versus ξ*

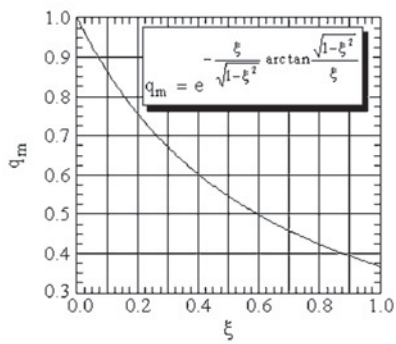


Figure 4.17. *Amplitude of the first maximum versus ξ*

NOTE.– *The comparison of the Laplace transforms of the response with the unit step function*

$$Q(p) = \frac{1}{p} \frac{1}{p^2 + 2\,\xi\, p + 1} \qquad [4.66]$$

and the unit impulse

$$Q(p) = \frac{1}{p^2 + 2\,\xi\,p + 1} \tag{4.67}$$

shows that these two transforms differ by a factor $\frac{1}{p}$ *and that, consequently [BRO 53], [KAR 40],*

$$h(t) = \frac{dA(t)}{dt} \tag{4.68}$$

4.2.2. *Response defined by absolute parameter*

4.2.2.1. *Expression for response*

$$Q(p) = \frac{1 + 2\,\xi\,p}{p^2 + 2\,\xi\,p + 1} \tag{4.69}$$

$$q(\theta) = h(\theta) = \frac{e^{-\xi\,\theta}}{\sqrt{1-\xi^2}} \sin\sqrt{1-\xi^2}\,\theta + 2\,\xi\,e^{-\xi\,\theta}\left[\cos\sqrt{1-\xi^2}\,\theta - \frac{\xi}{\sqrt{1-\xi^2}}\sin\sqrt{1-\xi^2}\,\theta\right]$$

[4.70]

$(\xi \neq 1)$

$$h(\theta) = e^{-\xi\theta}\left[2\,\xi\cos\sqrt{1-\xi^2}\,\theta + \frac{1-2\,\xi^2}{\sqrt{1-\xi^2}}\sin\sqrt{1-\xi^2}\,\theta\right] \tag{4.71}$$

i.e.

$$h(\theta) = \frac{e^{-\xi\,\theta}}{\sqrt{1-\xi^2}}\sin\left(\sqrt{1-\xi^2}\,\theta + \varphi\right)$$

with

$$\tan\varphi = \frac{2\,\xi\,\sqrt{1-\xi^2}}{1-2\,\xi^2}$$

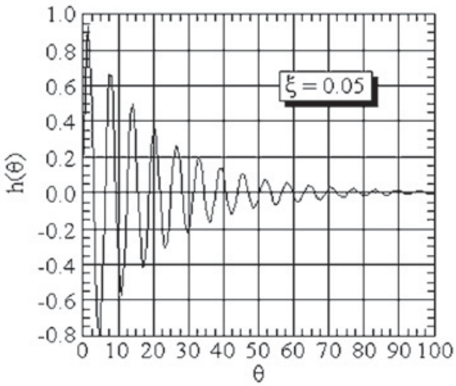


Figure 4.18. *Absolute response*

NOTE.– *For* $\theta = 0$,

$$h(0) = 2\,\xi = \frac{1}{Q} \qquad [4.72]$$

If $\xi = 0$, the preceding case is found

$$h(\theta) = \sin\,\theta \qquad [4.73]$$

In non-reduced coordinates the impulse response is written [BRO 62]:

$$u(t) = \omega_0\,\Im\,e^{-\xi\,\omega_0\,t}\left[2\,\xi\cos\omega_0\sqrt{1-\xi^2}\,t + \frac{1-2\,\xi^2}{\sqrt{1-\xi^2}}\sin\omega_0\sqrt{1-\xi^2}\,t\right] \qquad [4.74]$$

If $\Im = 1$, $h(t) = \omega_0\;e^{-\xi\,\omega_0\,t}\left[2\,\xi\cos\omega_0\sqrt{1-\xi^2}\,t + \frac{1-2\,\xi^2}{\sqrt{1-\xi^2}}\sin\omega_0\sqrt{1-\xi^2}\,t\right]$

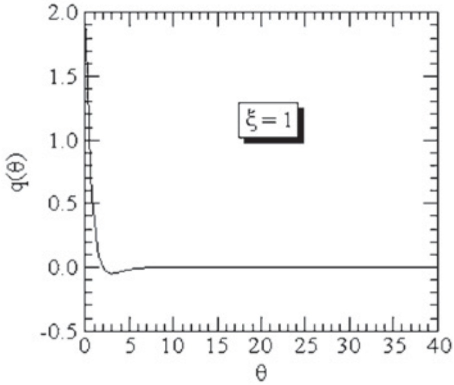


Figure 4.19. *Absolute response for $\xi = 1$*

If $\xi = 1$,

$$Q(p) = \frac{1 + 2\,p}{(p+1)^2} = \frac{2}{p+1} - \frac{1}{(p+1)^2} \qquad [4.75]$$

$$q(\theta) = 2\,e^{-\theta} - \theta\,e^{-\theta} = (2 - \theta)\,e^{-\theta} \qquad [4.76]$$

$$u(t) = \Im\,\omega_0\,(2 - \omega_0\,t)\,e^{-\omega_0\,t} \qquad [4.77]$$

4.2.2.2. *Peaks of response*

The response $h(\theta)$ presents a peak when $\dfrac{dh}{d\theta} = 0$, i.e. for $\theta = \theta_m$ such that

$$-\xi\,e^{-\xi\,\theta_m}\left[2\,\xi\cos\sqrt{1-\xi^2}\,\theta_m + \frac{1 - 2\,\xi^2}{\sqrt{1-\xi^2}}\sin\sqrt{1-\xi^2}\,\theta_m\right]$$

$$+\,e^{-\xi\,\theta_m}\left[-\,2\,\xi\sqrt{1-\xi^2}\,\sin\sqrt{1-\xi^2}\,\theta_m + \left(1 - 2\,\xi^2\right)\cos\sqrt{1-\xi^2}\,\theta_m\right] = 0$$

i.e., after simplification, if

$$\tan\left(\theta_m \sqrt{1-\xi^2}\right) = \frac{\sqrt{1-\xi^2}\left(1-4\,\xi^2\right)}{\xi\left(3-4\,\xi^2\right)} \quad [4.78]$$

If $3-4\,\xi^2 < 0$ (i.e. $\xi > \frac{\sqrt{3}}{2}$),

$$\theta_m = \frac{1}{\sqrt{1-\xi^2}} \arctan\left[\frac{\sqrt{1-\xi^2}}{\xi}\frac{1-4\,\xi^2}{3-4\,\xi^2}\right] \quad [4.79]$$

and if $3-4\,\xi^2 \geq 0$, i.e. $\xi \leq \frac{\sqrt{3}}{2}$,

$$\theta_m = \frac{1}{\sqrt{1-\xi^2}}\left[\pi + \arctan\left(\frac{\sqrt{1-\xi^2}}{\xi}\frac{1-4\,\xi^2}{3-4\,\xi^2}\right)\right] \quad [4.80]$$

this yields

$$h(\theta_m) = h_m = e^{-\xi\,\theta_m}\left[2\,\xi \cos\sqrt{1-\xi^2}\,\theta_m + \frac{1-2\,\xi^2}{\sqrt{1-\xi^2}}\sin\sqrt{1-\xi^2}\,\theta_m\right] \quad [4.81]$$

i.e.

$$h_m = -e^{-\xi\,\theta_m} \quad [4.82]$$

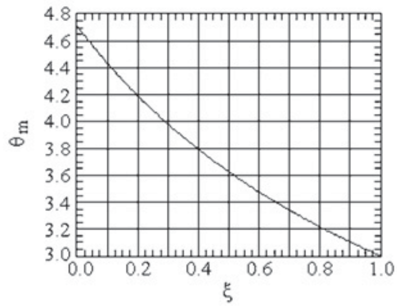


Figure 4.20. *Time of the first maximum of the absolute response versus ξ*

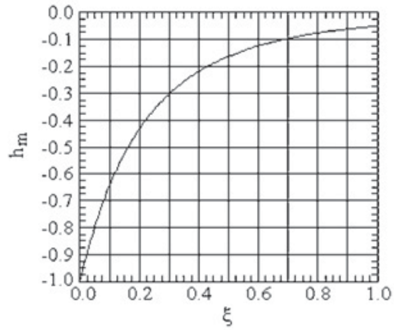


Figure 4.21. *Amplitude of the first maximum of the absolute response versus ξ*

If $\xi = 1$,

$$\frac{dq}{d\theta} = -2\, e^{-\theta} - e^{-\theta} + \theta\, e^{-\theta} = 0 \qquad [4.83]$$

Since $e^{-\theta} \neq 0$, we obtain $\theta_m = 3$ and $h_m = e^{-3} \approx -0.049788...$

If $\xi = 0$,

$$h(\theta) = \sin\theta \qquad [4.84]$$

$\dfrac{dh}{d\theta} = \cos\theta = 0$ if $\theta_m = \pi\left(k + \dfrac{1}{2}\right)$. If $k = 0$, $\theta_m = \dfrac{\pi}{2}$ and $h_m = 1$.

NOTES.–

1. *Equation [3.37], which can be written*

$$u(t) = \frac{\omega_0}{\sqrt{1-\xi^2}} \int_0^t \ell(\alpha)\, e^{-\xi\,\omega_0\,(t-\alpha)} \sin \omega_0 \sqrt{1-\xi^2}\,(t-\alpha)\, d\alpha\,,$$

is none other than a convolution integral applied to the functions $\ell(t)$ *and*

$$h(t) = \frac{\omega_0}{\sqrt{1-\xi^2}}\, e^{-\xi\,\omega_0\,t} \sin \omega_0 \sqrt{1-\xi^2}\; t \qquad [4.85]$$

($h(t)$ *= impulse response or weight function).*

2. *The Fourier transform of a convolution product of two functions* ℓ *and H is equal to the product of their Fourier transforms [LAL 75]. If* $u = \ell * h$

$$U(\Omega) = FT(U) = FT(\ell * h) = L(\Omega).H(\Omega) \quad [4.86]$$

The function $H(\Omega)$*, Fourier transform of the impulse response, is the transfer function of the system [LAL 75].*

3. *In addition, the Laplace transformation applied to a linear one-degree-of-freedom differential equation leads to a similar relation:*

$$U(p) = A(p)\, L(p) \quad [4.87]$$

$A(p)$ *is termed the* operational admittance *and* $Z(p) = \dfrac{1}{A(p)}$ the generalized impedance *of the system.*

4. *In the same way, relation [3.60] can be considered as the convolution product of the two functions* $\ell(t)$ *and*

$$h(t) = \frac{\omega_0}{\sqrt{1-\xi^2}}\, e^{-\xi\,\omega_0\,t}\left[\left(1-2\,\xi^2\right)\sin \omega_0\sqrt{1-\xi^2}\ t + 2\,\xi\,\sqrt{1-\xi^2}\ \cos \omega_0\,\sqrt{1-\xi^2}\ t\right]$$

[4.88]

4.3. Use of step and impulse responses

The preceding results can be used to calculate the response of the linear one-degree-of-freedom system (k, m) to an arbitrary excitation $\ell(t)$. This response can be considered in two ways [BRO 53]:

– either as the sum of the responses of the system to a succession of impulses of very short duration (the envelope of these impulses corresponding to the excitation) (Figure 4.22);

– or as the sum of the responses of the system to a series of step functions (Figure 4.23).

The application of the superposition principle assumes that the system is linear, i.e. described by linear differential equations [KAR 40], [MUS 68].

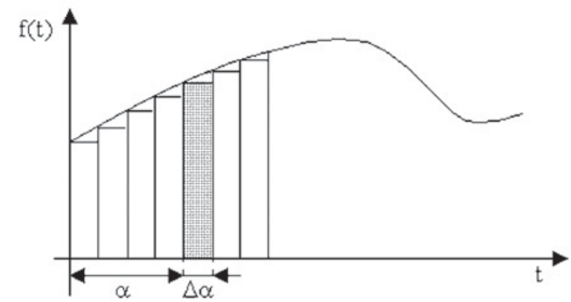


Figure 4.22. *Arbitrary pulse as a series of impulses*

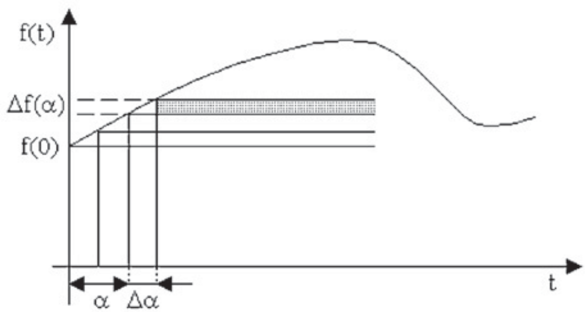


Figure 4.23. *Arbitrary pulse as a series of step functions*

Let us initially regard the excitation $\ell(t)$ as a succession of very short duration $\Delta\alpha$ impulses and let $\ell(\alpha)$ be the impulse amplitude at time α. By hypothesis, $\ell(\alpha) = 0$ for $\alpha < 0$.

Set $h(t - \alpha)$ as the response of the system at time t, resulting from the impulse at the time α pertaining to the time interval (0, t) (section 4.2.1). The response z(t) of the system to all the impulses occurring between $\alpha = 0$ and $\alpha = t$ is:

$$u(t) = \sum_{\alpha=0}^{\alpha=t} \ell(\alpha)\, h(t-\alpha)\, \Delta\alpha \qquad [4.89]$$

If the excitation is a continuous function, the intervals $\Delta\alpha$ can tend towards zero; it then becomes (Duhamel's formula):

$$u(t) = \int_0^t \ell(\alpha)\ h(t-\alpha)\ d\alpha \qquad [4.90]$$

The calculation of this integral requires knowledge of the excitation function $\ell(t)$ and of the response $h(t-\alpha)$ to the unit impulse at time α.

Integral [4.90] is none other than a convolution integral [LAL 75]; this can then be written as:

$$\ell(t) * h(t) = \int_0^t \ell(\alpha)\ h(t-\alpha)\ d\alpha \qquad [4.91]$$

According to properties of the convolution [LAL 75]:

$$\ell(t) * h(t) = \int_0^t \ell(t-\alpha)\ h(\alpha)\ d\alpha \qquad [4.92]$$

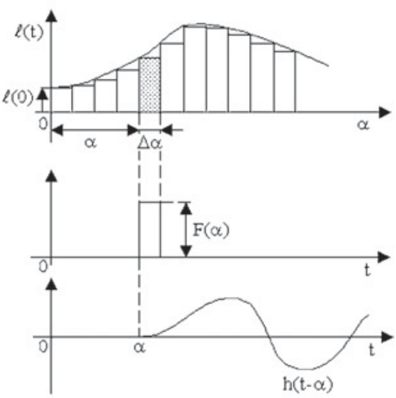


Figure 4.24. *Summation of impulse responses*

NOTE.– *It is presumed in this calculation that at time t the response to an impulse applied at time α is observed, so that this response is only one function of the time interval $t-\alpha$, but not of t or of α separately. This is the case if the coefficients of the differential equation of the system are constant. This assumption is in general not justified if these coefficients are functions of time [KAR 40].*

Consider the excitation as a sum of step functions separated by equal time intervals Δα (Figure 4.25).

The amplitude of each step function is $\Delta\ell(\alpha)$, i.e. $\dfrac{\Delta\ell(\alpha)}{\Delta\alpha}\Delta\alpha$. Set $A(t-\alpha)$ as the step response at time t, resulting from the application of a unit step function at time α (with $0 < \alpha < t$).

Set $\ell(0)$ as the value of the excitation at time $\alpha = 0$ and $A(t)$ as the response of the system at time t corresponding to the application of the unit step at the instant $\alpha = 0$.

The response of the system to a single unit step function is equal to

$$\frac{\Delta\ell(\alpha)}{\Delta\alpha}\Delta\alpha\ A(t-\alpha)$$

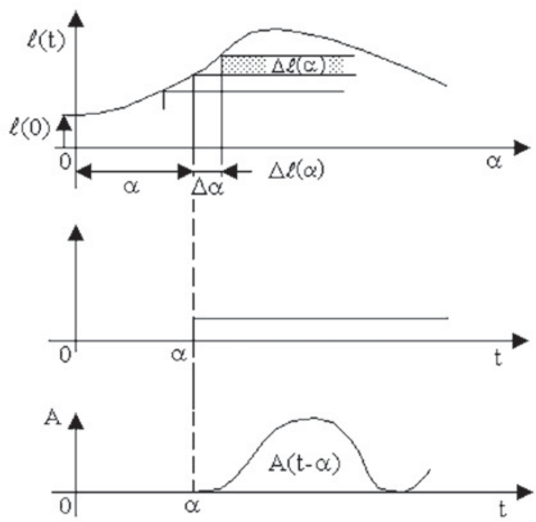


Figure 4.25. *Summation of step responses*

The response of the linear system to all the step functions applied between the times $\alpha = 0$ and $\alpha = t$ and separated by $\Delta\alpha$ is thus:

$$u(t) = \ell(0)\,.\,A(t) + \sum_{\alpha=0}^{\alpha=t}\frac{\Delta\ell(\alpha)}{\Delta\alpha}\Delta\alpha\ A(t-\alpha) \qquad [4.93]$$

If the excitation function is continuous, the response tends, when $\Delta\alpha$ tends towards zero, towards the limit

$$u(t) = \ell(0) \,.\, A(t) + \int_0^t \dot{\ell}(\alpha)\; A(t-\alpha)\; d\alpha \qquad [4.94]$$

where

$$\dot{\ell}(\alpha) = \frac{d\ell(\alpha)}{d\alpha}$$

This is the *superposition integral* or *Rocard integral.* In the majority of practical cases, and according to our assumptions, $\ell(0) = 0$ and

$$u(t) = \int_0^t \dot{\ell}(0)\; A(t-\alpha)\; d\alpha \qquad [4.95]$$

NOTES.–

1. *Expression [4.95] is sometimes called Duhamel's integral and sometimes [4.90] is called Rocard's integral [RID 69].*

2. *Integral [4.90] can be obtained by the integration by parts of [4.94] while setting* $U = A(t-\alpha)$ *and* $dV = \dot{\ell}(\alpha)\; d\alpha$ *using integration by parts, by noting that* $A(0)$ *is often zero in most current practical problems (knowing moreover that* $h(t) = \dfrac{dA(t)}{dt}$*).*

$$u(t) = \ell(0) \,.\, A(t) + \int_0^t \dot{\ell}(\alpha)\; A(t-\alpha)\; d\alpha \qquad [4.96]$$

If u is a continuous and derivable function in (0, t), integration by parts gives

$$\int_0^t \dot{\ell}(\alpha)\; A(t-\alpha)\; d\alpha = \left[\ell(\alpha)\; A(t-\alpha)\right]_0^t - \int_0^t \ell(\alpha)\; \frac{dA(t-\alpha)}{d\alpha}\; d\alpha \qquad [4.97]$$

yielding, if $\ell_0 = \ell(0)$,

$$u(t) = \ell_0\; A(t) + \ell(t)\; A(0) - \ell_0\; A(t) + \int_0^t \ell(\alpha)\; \dot{A}(t-\alpha)\; d\alpha \qquad [4.98]$$

and, since $A(0) = 0$, by Duhamel's formula:

$$u(t) = \int_0^t \ell(\alpha)\, \dot{A}(t-\alpha)\, d\alpha \qquad [4.99]$$

Functions $h(t)$ and $A(t)$ were calculated directly in the preceding sections. Their expression could be obtained by starting from the general equation of movement in its reduced form. The next step will be to find, for example, $h(t)$. The unit impulse can be defined, in generalized form, by the integral:

$$\lim_{\theta \to 0} \int_0^\theta \lambda(\alpha)\, d\alpha = 1 \qquad [4.100]$$

α being a variable of integration ($\alpha \le \theta$). This relation defines an excitation where the duration is infinitely small and whose integral in the time domain is equal to 1. Since it corresponds to an excitation of duration tending towards zero, it can be regarded as an initial condition to the solution of the equation of motion

$$\ddot{q}(\theta) + q(\theta) = \lambda(\theta) \qquad [4.101]$$

(while assuming $\xi = 0$), i.e.

$$q(\theta) = C_1 \cos\theta + C_2 \sin\theta \qquad [4.102]$$

The initial value of the response $q(\theta)$ is equal to C_1 and, for a system initially at rest ($C_1 = 0$), the initial velocity is C_2. The amplitude of the response being zero for $\theta = 0$, the initial velocity change is obtained by setting $q = 0$ in the equation of movement [4.101], while integrating $\ddot{q} = \dfrac{d\dot{q}}{d\theta}$ over time and taking the limit when θ tends towards zero [SUT 68]:

$$\dot{q}(\theta \to 0) = \lim_{\theta \to 0}\left(\int_0^\theta \frac{d\dot{q}}{d\alpha}\, d\alpha\right) = \lim_{\theta \to 0}\left(\int_0^\theta \lambda(\alpha)\, d\alpha\right) \qquad [4.103]$$

yielding

$$C_2 = 1$$

This then gives the expression of the response to the generalized unit impulse of an undamped simple system:

$$q(\theta) = \sin\,\theta \qquad [4.104]$$

– For zero damping, the indicial admittance and the impulse response to the generalized excitation are written, respectively:

$$A(t) = 1 - \cos\,\omega_0\,t \qquad [4.105]$$

and

$$h(t) = \omega_0\,\sin\,\omega_0\,t \qquad [4.106]$$

This yields

$$u(t) = \int_0^t \ell(\alpha)\,h(t-\alpha)\,d\alpha \qquad [4.107]$$

$$u(t) = \omega_0 \int_0^t \ell(\alpha)\,\sin\,\omega_0\,(t-\alpha)\,d\alpha \qquad [4.108]$$

for arbitrary ξ damping,

$$A(t) = 1 - e^{-\xi\,\omega_0\,t} \cos\,\omega_0\sqrt{1-\xi^2}\;t - \frac{\xi}{\sqrt{1-\xi^2}}\,e^{-\xi\,\omega_0\,t} \sin\,\omega_0\sqrt{1-\xi^2}\;t \qquad [4.109]$$

and

$$h(t) = \frac{\omega_0}{\sqrt{1-\xi^2}}\,e^{-\xi\,\omega_0\,t}\,\sin\,\omega_0\,\sqrt{1-\xi^2}\;t \qquad [4.110]$$

yielding

$$u(t) = \frac{\omega_0}{\sqrt{1-\xi^2}} \int_0^t \ell(\alpha)\,e^{-\xi\,\omega_0\,(t-\alpha)}\,\sin\,\omega_0\,\sqrt{1-\xi^2}\;(t-\alpha)\,d\alpha \qquad [4.111]$$

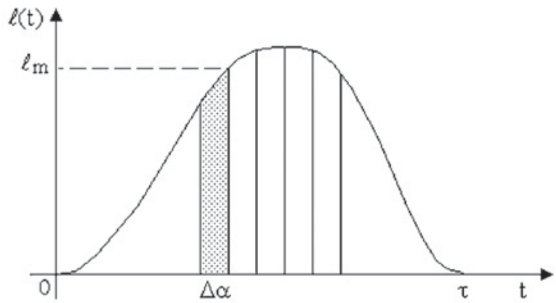


Figure 4.26. *Decline in impulses*

The response of the simple system of natural pulsation ω_0 can therefore be calculated after a decline of the excitation $\ell(t)$ in a series of impulses of duration $\Delta\alpha$. For a signal of given form, the displacement $u(t)$ is a function of t, ω_0 and ξ.

4.4. Transfer function of a linear one-degree-of-freedom system

4.4.1. *Definition*

It was shown in [4.90] that the behavior of a linear system can be characterized by its weight function (response of the system to a unit impulse function)

$$q(\theta) = \int_0^\theta \lambda(\alpha)\, h(\theta - \alpha)\, d\alpha$$

where, if the response is relative,

$$\begin{cases} h(\theta) = \dfrac{e^{-\xi\theta}}{\sqrt{1-\xi^2}} \sin\sqrt{1-\xi^2}\,\theta & (\xi \neq 1) \\ h(\theta) = \theta^{-\theta} & (\xi = 1) \end{cases} \qquad [4.112]$$

and, if it is absolute:

$$\begin{cases} h(\theta) = e^{-\xi\,\theta}\left[2\,\xi\cos\sqrt{1-\xi^2}\,\theta + \dfrac{1-2\,\xi^2}{\sqrt{1-\xi^2}}\sin\sqrt{1-\xi^2}\,\theta\right] & \text{for } \xi \neq 1 \\ h(\theta) = (2-\theta)\,e^{-\theta} & \text{for } \xi = 1 \end{cases} \qquad [4.113]$$

The function h( ) can be expressed versus time. We have, for example, for the relative response:

$$h(t) = \frac{\omega_0}{\sqrt{1-\xi^2}}\,e^{-\xi\,\omega_0\,t}\,\sin\sqrt{1-\xi^2}\,\omega_0\,t \qquad [4.114]$$

The Fourier transform of h(t) is the transfer function $H(\Omega)$ of the system [BEN 63]:

$$H(\Omega) = \int_0^{\infty} h(t)\,e^{-i\,\Omega\,t}\,dt \qquad [4.115]$$

Let us set $h = \dfrac{\Omega}{\omega_0}$. The variable h is defined as the *interval.* In reduced coordinates:

$$H(h) = \int_0^{\infty} h(\theta)\,e^{-i\,h\,\theta}\,d\theta \qquad [4.116]$$

NOTE.– *Rigorously,* $H(h)$ *is the response function in the frequency domain, whereas the transfer function is the Laplace transform of* $h(\theta)$ *[KIM 24]. Commonly,* $H(h)$ *is also known as the transfer function.*

The function $H(h)$[1] is complex and can be put in the form [BEN 63]

$$H(h) = \left|H(h)\right|\,e^{-i\,\phi(h)} \qquad [4.117]$$

---

1 The dimensionless term "h" is used throughout this and following chapters. This is equivalent to the frequency ratios f/f_0 or ω/ω_0.

Sometimes the modulus $\left|H(h)\right|$ is called the *gain factor* [KIM 24] or *gain,* or *amplification factor* when $h(\theta)$ is the relative response function; or *transmissibility* when $h(\theta)$ is the absolute response function and $\phi(h)$ is the associated phase (*phase factor*).

Taking into account the characteristics of real physical systems, $H(h)$ satisfies the following properties:

1. $H(-h) = H^*(h)$ [4.118]

where H^* is the complex conjugate of H

2. $\left|H(-h)\right| = \left|H(h)\right|$ [4.119]

3. $\phi(-h) = -\phi(h)$ [4.120]

4. If two mechanical systems having transfer functions $H_1(h)$ and $H_2(h)$ are put in series and if there is no coupling between the two systems thus associated, the transfer function of the unit is equal to [BEN 63]:

$$H(h) = H_1(h)\ H_2(h) \tag{4.121}$$

i.e.

$$\begin{cases} \left|H(h)\right| = \left|H_1(h)\right| \ . \ \left|H_2(h)\right| \\ \phi(h) = \phi_1(h) + \phi_2(h) \end{cases} \tag{4.122}$$

This can be found in references [LAL 75], [LAL 82], [LAL 95a] and examples of the use of this transfer function for the calculation of the response of a structure at a given point when it is subjected to a sinusoidal, random or shock excitation are given in the following chapters.

In a more general way, the transfer function can be defined as the ratio of the response of a structure (with several degrees of freedom) to the excitation, according to the frequency. The stated properties of $H(h)$ remain valid with this definition. Function $H(h)$ depends only on the structural characteristics.

4.4.2. *Calculation of* $H(h)$ *for relative response*

By definition,

$$H(h) = \int_0^\infty \frac{e^{-\xi\,\theta}}{\sqrt{1-\xi^2}} \sin\left(\sqrt{1-\xi^2}\ \theta\right) e^{-i\,h\,\theta}\, d\theta \qquad [4.123]$$

Knowing that

$$\int e^{a\,x} \sin b\,x\, dx = \frac{e^{a\,x}}{a^2+b^2}\left(a \sin b\,x - b \cos b\,x\right) \qquad [4.124]$$

it becomes

$$H(h) = \frac{1}{\sqrt{1-\xi^2}} \left\{ \frac{e^{-(\xi + i\,h)}}{1-\xi^2+(\xi+i\,h)^2} \right.$$

$$\left. \left[-(\xi+i\,h) \sin\sqrt{1-\xi^2}\ \theta - \sqrt{1-\xi^2}\ \cos\sqrt{1-\xi^2}\ \theta\right]\right\}_0^\infty \qquad [4.125]$$

$$H(h) = \frac{1}{\left(1-h^2\right)+2\,i\,h\,\xi} \qquad [4.126]$$

$$\left|H(h)\right| = \frac{1}{\sqrt{\left(1-h^2\right)^2+4\,\xi^2\,h^2}} \qquad [4.127]$$

$$\tan\phi = \frac{2\,\xi\,h}{1-h^2} \qquad [4.128]$$

If $0 \le h < 1$:

$$\phi = \arctan\frac{2\,\xi\,h}{1-h^2} \qquad [4.129]$$

If h = 1

$$\phi = \frac{\pi}{2} \tag{4.130}$$

If h > 1

$$\phi = \pi + \arc\tan\frac{2\xi h}{1-h^2} \tag{4.131}$$

4.4.3. *Calculation of $H(h)$ for absolute response*

In this case,

$$H(h) = \int_0^\infty e^{-\xi\theta}\left[2\xi\cos\sqrt{1-\xi^2}\,\theta + \frac{1-2\xi^2}{\sqrt{1-\xi^2}}\sin\sqrt{1-\xi^2}\,\theta\right]e^{-ih\theta}\,d\theta \tag{4.132}$$

$$H(h) = \int_0^\infty 2\,\xi\,e^{-(\xi+i\,h)\,\theta}\cos\sqrt{1-\xi^2}\;\theta\,d\theta + \frac{1-2\,\xi^2}{\sqrt{1-\xi^2}}\int_0^\infty e^{-(\xi+i\,h)\,\theta}\sin\sqrt{1-\xi^2}\;\theta\,d\theta \tag{4.133}$$

$$H(h) = \left\{2\xi\frac{e^{-(\xi+i\,h)\,\theta}}{(\xi+i\,h)^2+1-\xi^2}\left[-(\xi+i\,h)\cos\sqrt{1-\xi^2}\;\theta + \sqrt{1-\xi^2}\sin\sqrt{1-\xi^2}\;\theta\,d\theta\right]\right.$$

$$\left.+\frac{1-2\,\xi^2}{\sqrt{1-\xi^2}}\frac{e^{-(\xi+i\,h)\,\theta}}{(\xi+i\,h)^2+1-\xi^2}\left[-(\xi+i\,h)\sin\sqrt{1-\xi^2}\;\theta - \sqrt{1-\xi^2}\cos\sqrt{1-\xi^2}\;\theta\right]\right\}_0^\infty \tag{4.134}$$

$$H(h) = \frac{2\,\xi\,(\xi+i\,h)}{1-h^2+2\,i\,\xi\,h} + \frac{\left(1-2\,\xi^2\right)\sqrt{1-\xi^2}}{\sqrt{1-\xi^2}\,\left(1-h^2+2\,i\,\xi\,h\right)} \tag{4.135}$$

$$H(h) = \frac{1 + 2\,i\,\xi\,h}{1 - h^2 + 2\,i\,\xi\,h}\left(= \frac{1 - h^2 + 4\,h^2\xi^2 - 2\,i\,\xi\,h^3}{\left(1 - h^2\right)^2 + 4\,\xi^2\,h^2} \right) \qquad [4.136]$$

$$\left|H(h)\right| = \frac{\sqrt{1 + 4\,h^2\,\xi^2}}{\sqrt{\left(1 - h^2\right)^2 + 4\,\xi^2\,h^2}} \qquad [4.137]$$

$$\tan\,\phi = \frac{2\,\xi\,h^3}{1 - h^2 + 4\,h^2\,\xi^2} \qquad [4.138]$$

$$\phi = \arctan\frac{2\,\xi\,h^3}{1 - h^2 + 4\,h^2\,\xi^2} \qquad [4.139]$$

if $1 - h^2 + 4\,h^2\,\xi^2 > 0$, i.e. if $h^2 < \dfrac{1}{1 - 4\xi^2}$. For $h^2 = \dfrac{1}{1 - 4\,\xi^2}$,

$$\phi = \frac{\pi}{2} \qquad [4.140]$$

and for $h^2 > \dfrac{1}{1 - 4\,\xi^2}$

$$\phi = \pi + \arctan\frac{2\,\xi\,h^3}{1 - h^2 + 4\,\xi^2\,h^2} \qquad [4.141]$$

If $\xi = \dfrac{1}{2}$, $\tan\,\phi = h^3$

$$\phi = \arctan\,h^3 \qquad [4.142]$$

The complex transfer function can also be studied through its real and imaginary parts (Nyquist diagram):

$$H(f) = \frac{1 + i\;2\,\xi\,h}{1 - h^2 + i\;2\,\xi\,h} = \mathrm{Re}\left[H(f)\right] + i\,\mathrm{Im}\left[H(f)\right] \qquad [4.143]$$

$$\mathrm{Re}\left[H(f)\right] = \frac{1 - h^2 + \left(2\,\xi\,h\right)^2}{\left(1 - h^2\right)^2 + \left(2\,\xi\,h\right)^2} \tag{4.144}$$

$$\mathrm{Im}\left[H(f)\right] = \frac{-2\,\xi\,h^3}{\left(1 - h^2\right)^2 + \left(2\,\xi\,h\right)^2} \tag{4.145}$$

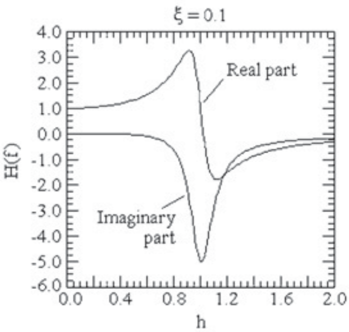


Figure 4.27. *Real and imaginary parts of H(h)*

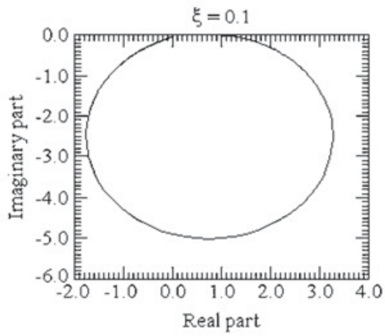


Figure 4.28. *Nyquist diagram*

4.4.4. *Other definitions of the transfer function*

4.4.4.1. *Notation*

According to the choice of parameters for excitation and response, the transfer function can be defined in different ways. In order to avoid any confusion, the two letters are placed as subscripts after the letter H; the first letter specifies the nature of the input and the second that of the response. The letter H will be used without subscript only in the case of reduced coordinates. We will use the same rule for the impedance $\frac{1}{H} = Z$.

4.4.4.2. *Relative response*

$$H_\rho = \left|H\right| = \frac{1}{\sqrt{\left(1 - h^2\right)^2 + \left(2\,\xi\,h\right)^2}} \tag{4.146}$$

Function $|H|$ is equal to function $\left|H_{\ell,u}\right| = \frac{u}{\ell}$. To distinguish it from the transfer function giving the absolute responses we will denote by H_R the transfer functions bringing into play the relative displacement and its derivatives.

Calculation of $\left|H_{\ddot{x},z}\right|$

$$|\ell(t)| = \frac{\ddot{x}(t)}{\omega_0^2} \tag{4.147}$$

$$\frac{u}{\ell} = \frac{z}{\ddot{x}}\omega_0^2 \tag{4.148}$$

$$\left|H_{x,z}\right| = \left|H_{\ell,u}\right|\frac{1}{\omega_0^2} \tag{4.149}$$

Calculation of $\left|H_{F,z}\right|$

$$\frac{u}{\ell} = \frac{z}{F/k} \tag{4.150}$$

$$\left|H_{F,z}\right| = \frac{\left|H_{\ell,u}\right|}{k} \tag{4.151}$$

Response defined by the relative velocity

$$\left|H_{\ell,\dot{u}}\right| = \frac{\dot{u}}{\ell} = \frac{\Omega\, u}{\ell} \tag{4.152}$$

It is assumed here that the excitation, and consequently the response, are sinusoidal and of frequency Ω, or that the excitation is resoluble into a Fourier series, with each component being a sinusoid. This yields

$$\left|H_{\ell,\dot{u}}\right| = \Omega\ \left|H_{\ell,u}\right| = \Omega\ |H| \tag{4.153}$$

Thus, if $|\ell(t)| = \frac{\ddot{x}(t)}{\omega_0^2}$,

$$\frac{\dot{u}}{\ell} = \frac{\dot{z}}{\ddot{x} / \omega_0^2} = \Omega \left|H\right| \tag{4.154}$$

$$H_{\ddot{x}, z} = \frac{\Omega}{\omega_0^2} \left|H\right| \tag{4.155}$$

4.4.4.3. *Absolute response*

In the same way, starting from [4.137],

$$H_\alpha = \left|H\right| = \sqrt{\frac{1 + 4 \xi^2 h^2}{\left(1 - h^2\right)^2 + \left(2 \xi h\right)^2}}$$

We will note H_A the expressions of the usual transfer functions of this nature.

4.4.4.4. *Summary tables*

Table 4.1 states the values of H_A and H_R for each parameter input and each parameter response.

Response ⇒ Excitation ⇓	**z**	$\dot{\mathbf{z}}$	$\ddot{\mathbf{z}}$	**y**	$\dot{\mathbf{y}}$	$\ddot{\mathbf{y}}$	**Reaction force F_T on base**
Force F on the mass m	$\frac{k\ z}{F}$	$\frac{k\ \dot{z}}{\Omega\ F}$	$\frac{k\ \ddot{z}}{\Omega^2\ F}$	/	/	/	$\frac{F_T}{F}$
$\ddot{\mathbf{x}}$	$\frac{\omega_0^2\ z}{\ddot{x}}$	$\frac{\omega_0^2\ \dot{z}}{\Omega\ \ddot{x}}$	$\frac{\omega_0^2\ \ddot{z}}{\Omega^2\ \ddot{x}}$	$\frac{\Omega^2\ y}{\ddot{x}}$	$\frac{\Omega\ \dot{y}}{\ddot{x}}$	$\frac{\ddot{y}}{\ddot{x}}$	/
$\dot{\mathbf{x}}$	$\frac{\omega_0^2\ z}{\Omega\ \dot{x}}$	$\frac{\omega_0^2\ \dot{z}}{\Omega^2\ \dot{x}}$	$\frac{\omega_0^2\ \ddot{z}}{\Omega^3\ \dot{x}}$	$\frac{\Omega\ y}{\dot{x}}$	$\frac{\dot{y}}{\dot{x}}$	$\frac{\ddot{y}}{\Omega\ \dot{x}}$	/
x	$\frac{\omega_0^2\ z}{\Omega^2\ x}$	$\frac{\omega_0^2\ \dot{z}}{\Omega^3\ x}$	$\frac{\omega_0^2\ \ddot{z}}{\Omega^4\ x}$	$\frac{y}{x}$	$\frac{\dot{y}}{\Omega\ x}$	$\frac{\ddot{y}}{\Omega^2\ x}$	/
Reduced transfer function	H_R			H_A			

Table 4.1. *Transfer function corresponding to excitation and response*

These results may also be presented as in Table 4.2.

Example 4.3.

Assume that the excitation and the response are, respectively, velocities $\dot{x}$ and $\dot{z}$. Table 4.1 indicates that the transfer function can be obtained from the relation

$$H_R = \frac{\omega_0^2\ \dot{z}}{\Omega^2\ \dot{x}} \qquad [4.156]$$

This yields

$$\frac{\dot{z}}{\dot{x}} = \frac{\Omega^2}{\omega_0^2} \frac{1}{\sqrt{\left(1-h^2\right)^2 + 4\ \xi^2\ h^2}} \qquad [4.157]$$

Table 4.2 gives this relation more directly. To continue to use reduced parameters, and in particular reduced transfer functions (which is not the case for the transfer functions in Table 4.2), these functions can be defined as follows.

For a given excitation, we obtain the acceleration and *velocity* and *acceleration* transfer function while multiplying respectively by h and h^2 the *displacement* transfer function (relative or absolute response).

This is used to draw the transfer function in a four-coordinate representation from which can be read (starting from only one curve plotted against the reduced frequency h) the transfer function for the displacement, velocity and acceleration (section 6.7).

NOTE.–

The transfer functions are sometimes expressed in decibels

$$H(dB) = 20 \log_{10} H(h) \qquad [4.158]$$

where H(h) is the amplitude of the transfer function as defined in the preceding tables. A variation of H(h) by a factor of 10 corresponds to an amplification of 20 dB.

$$= \frac{1}{\sqrt{\left(1-h^2\right)^2+\left(2\,\xi\,h\right)^2}}$$

$$= \sqrt{\frac{1+4\,\xi^2\,h^2}{\left(1-h^2\right)^2+\left(2\,\xi\,h\right)^2}}$$

Response ⇒ / Excitation ⇓		Displacement (m)		Velocity (m/s)		Acceleration (m/s^2)		R f t
		Absolute $y(t)$	Relative $z(t)$	Absolute $\dot{y}(t)$	Relative $\dot{z}(t)$	Absolute $\ddot{y}(t)$	Relative $\ddot{z}(t)$	
Base movement	**Displacement $x(t)$ (m)**	H_A	$\frac{\Omega^2}{\omega_0^2} H_R$	$\Omega\, H_A$	$\frac{\Omega^3}{\omega_0^2} H_R$	$\Omega^2 H_A$	$\frac{\Omega^4}{\omega_0^2} H_A$	
	Velocity $\dot{x}(t)$ (m/s)	$\frac{H_A}{\Omega}$	$\frac{\Omega}{\omega_0^2} H_R$	H_A	$\frac{\Omega^2}{\omega_0^2} H_R$	$\Omega\, H_A$	$\frac{\Omega^3}{\omega_0^2} H_R$	
	Acceleration $\ddot{x}(t)$ (m/s^2)	$\frac{H_A}{\Omega^2}$	$\frac{H_R}{\omega_0^2}$	$\frac{H_A}{\Omega}$	$\frac{\Omega}{\omega_0^2} H_R$	H_A	$\frac{\Omega^2}{\omega_0^2} H_R$	
Force on the mass m		$\frac{H_R}{k}$		$\frac{\Omega}{k} H_R$		$\frac{\Omega^2}{k} H_R$		

Table 4.2. *Transfer function corresponding to excitation and response*

Response ⇒ Excitation ⇓		**Displacement**		**Velocity**		**Acceleration**	
		Absolute $\mathbf{y(t)}$	**Relative** $\mathbf{z(t)}$	**Absolute** $\mathbf{\dot{y}(t)}$	**Relative** $\mathbf{\dot{z}(t)}$	**Absolute** $\mathbf{\ddot{y}(t)}$	**Relative** $\mathbf{\ddot{z}(t)}$
Base movement	**Displacement** $\mathbf{x(t)}$	$\frac{y}{x} = H_A$	$\frac{z}{x} = h^2\ H_R$	$\frac{\dot{y}}{\omega_0\ x} = h\ H_A$	$\frac{\dot{z}}{x} = h^3\ H_R$	$\frac{\ddot{y}}{\omega_0^2\ x} = h^2\ H_A$	$\frac{\ddot{z}}{\omega_0^2\ x} = h^4\ H_R$
	Velocity $\mathbf{\dot{x}(t)}$	$\frac{\omega_0\ y}{\dot{x}} = \frac{H_A}{h}$	$\frac{\omega_0\ z}{\dot{x}} = h\ H_R$	$\frac{\dot{y}}{\dot{x}} = H_A$	$\frac{\dot{z}}{\dot{x}} = h^2\ H_R$	$\frac{\ddot{y}}{\omega_0\ \dot{x}} = h\ H_A$	$\frac{\ddot{z}}{\omega_0\ \dot{x}} = h^3\ H_R$
	Acceleration $\mathbf{\ddot{x}(t)}$	$\frac{\omega_0^2\ y}{\ddot{x}} = \frac{H_A}{h^2}$	$\frac{\omega_0^2\ z}{\ddot{x}} = H_R$	$\frac{\omega_0\ \dot{y}}{\ddot{x}} = \frac{H_A}{h}$	$\frac{\omega_0\ \dot{z}}{\ddot{x}} = h\ H_R$	$\frac{\ddot{y}}{\ddot{x}} = H_A$	$\frac{\ddot{z}}{\ddot{x}} = h^2\ H_R$
Force $\mathbf{F(t)}$ **on the mass m**		$\frac{k\ z}{F} = H_R$		$\frac{\sqrt{k\ m}\ \dot{z}}{F} = h\ H_R$		$\frac{m\ \ddot{z}}{F} = h^2\ H_R$	

Table 4.3. *Transfer function corresponding to excitation and response*

4.5. Measurement of transfer function

The transfer function of a mechanical system can be defined:

– in steady state sinusoidal mode, by calculation of the amplitude ratio of the response to the amplitude of the excitation for several values of the frequency f of the excitation [TAY 77];

– in a slowly swept sine, the sweep rate being selected as slow enough that the transient aspect can be neglected when crossing the resonances. The frequency can be varied in one of two ways: either by increments or in a continuous way. The time spent at each frequency must be sufficient so that the response of the system can reach its permanent state (i.e. to reach its highest value);

– in a quickly swept sine (method developed by C.W. Skingle [SKI 66]);

– under random vibrations (the ratio of the power spectral density functions of the response and excitation, or the ratio of the cross-spectral density $G_{\ddot{x}\ddot{y}}$ and power spectral density of the excitation $G_{\ddot{x}}$) (see Volume 3);

– under shock (ratio of the Fourier transforms of the response and excitation) (see Volume 2). In this last case, a hammer equipped with a sensor measuring the input force and a sensor measuring acceleration response or, as with the preceding methods, an electrodynamic shaker can be used.

Most of the authors agree that the fast swept sine is one of the best methods of measurement of the transfer function of a system. Shock excitation can give good results provided that the amplitude of the Fourier transform of the shock used has a level far enough from zero in all the useful frequency bands. The random vibrations require longer tests [SMA 85], [TAY 75].

Chapter 5

Sinusoidal Vibration

5.1. Definitions

5.1.1. *Sinusoidal vibration*

A sinusoidal vibration is the simplest and most basic form of periodical movement. This movement can be represented as an analytical equation in the form:

$$\ell(t) = \ell_m \sin(\Omega t + \varphi) \qquad [5.1]$$

where:

- t is the instantaneous value of time (seconds);
- ℓ_m is the amplitude of the movement (maximum value of $\ell(t)$);
- $\ell(t)$ is the parameter used to define the movement;
- Ω is the pulsation (rad/sec), and is linked to a frequency f by $\Omega = 2\pi f$. Frequency f is expressed in Hertz (Hz) or in cycles per second (cps). The opposite of the frequency f is the period T;

φ is the phase (related to the value of ℓ for $t = 0$). φ is expressed in radians. In practice, it is assumed that $\varphi = 0$ if possible;

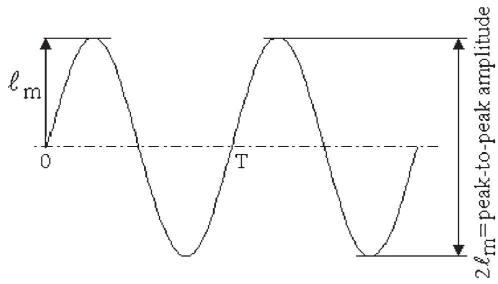


Figure 5.1. *Sinusoidal vibration*

$\ell(t)$ is generally acceleration, but it can be a velocity, a (linear or angular) displacement or a force.

Displacement refers to the variation in distance, or in position of an object from a particular point or reference axes. The unit of displacement is the meter (m) and its submultiples such as the micron (μm) and the millimeter (mm). The amplitude of the displacement can indicate the displacement's range of values, which are between zero (for the zero displacement of a resting system) and the maximum displacement value (zero – peak displacement). The amplitude of the displacement can also indicate the interval that exists between the minimum and maximum values (peak to peak displacement), meaning that the interval includes all possible displacement values.

Velocity refers to the variation in displacement over time (it is the first derivative of displacement). Velocity is expressed in meters per second (m/s) and its submultiples (cm/s and mm/s). As is the case for displacement, it is possible to consider the following values of velocity: zero – peak and peak to peak.

Acceleration refers to the variation in velocity over time. It is equal to the first derivative of velocity or to the second derivative of displacement. Acceleration is expressed in m/s^2 or more usually as the value g, where g is the acceleration due to gravity ($1\ g = 9.81\ m/s^2$).

These three parameters are derived from each other by integration or by differentiation:

$$\begin{cases} \dot{\ell}(t) = \dfrac{d\ell}{dt} = \ell_m \, \Omega \cos \Omega \, t = \dot{\ell}_m \cos \Omega \, t = \dot{\ell}_m \sin\left(\Omega \, t + \dfrac{\pi}{2}\right) \\ \ddot{\ell}(t) = \dfrac{d^2\ell}{dt^2} = -\ell_m \, \Omega^2 \sin \Omega \, t = -\ddot{\ell}_m \sin \Omega \, t = \ddot{\ell}_m \sin(\Omega \, t + \pi) \end{cases} \qquad [5.2]$$

From these expressions, it can be observed that acceleration, velocity and displacement are all sinusoidal, of period T, and that velocity and displacement have a difference of phase angle of $\frac{\pi}{2}$, like velocity and acceleration.

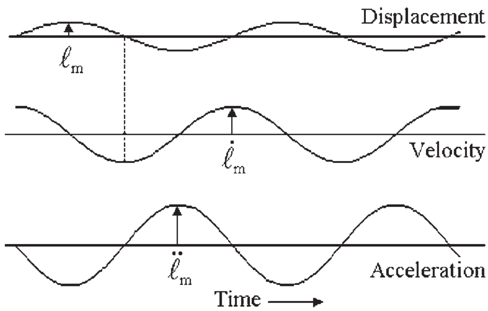


Figure 5.2. *Difference of phase between sinusoidal displacement, velocity and acceleration*

Let $\dot{\ell}_m$ and $\ddot{\ell}_m$ be the maximum values of velocity and acceleration respectively. It can be noticed that whenever the value of displacement has reached its maximum value, the velocity is zero. The acceleration reaches its maximum level when the velocity is zero. The acceleration varies as the square of the frequency. If the acceleration is constant, then the displacement varies as the inverse of the square of the frequency. The displacement thus rapidly decreases as frequency increases. Inversely, when frequency decreases, the displacement rapidly increases.

5.1.2. *Mean value*

The mean value of the quantity $\ell(t)$, which is defined over one period T by

$$\bar{\ell} = \frac{1}{T} \int_0^T \ell(t) \, dt \qquad [5.3]$$

is zero (over one period, there is symmetry of all the points with respect to the time axis). The surface under the positive part (between the curve and the time axis) is equal to the surface under the negative part. The average value of the signal on a half-period is more significant:

$$\bar{\ell} = \frac{2}{T} \int_0^{T/2} \ell(t)\, dt \qquad [5.4]$$

$$\bar{\ell} = \frac{2}{T}\, \ell_m \int_0^{T/2} \sin \Omega t\, dt$$

(yielding, since $\Omega\, T = 2\, \pi$)

$$\bar{\ell} = \frac{2\, \ell_m}{\pi} \approx 0.637\, \ell_m \qquad [5.5]$$

5.1.3. *Mean square value – rms value*

The *mean square value* is defined as

$$\overline{\ell^2} = \frac{1}{T} \int_0^{T} \ell^2(t)\, dt \qquad [5.6]$$

$$\overline{\ell^2} = \frac{1}{T} \int_0^{T} \ell_m^2 \sin^2 \Omega t\, dt$$

$$\overline{\ell^2} = \frac{\ell_m^2}{2} \qquad [5.7]$$

and the root mean square value (rms value) is

$$\ell_{rms} = \sqrt{\overline{\ell^2}} = \frac{\ell_m}{\sqrt{2}} \approx 0.707\, \ell_m \qquad [5.8]$$

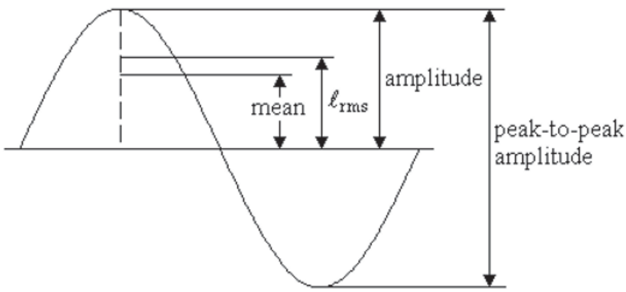


Figure 5.3. *Characteristics of a single sinusoid*

Thus

$$\ell_{rms} = \frac{\pi}{2\sqrt{2}}\,\bar{\ell} \qquad [5.9]$$

This can then be written in the more general form [BRO 84]:

$$\ell_{rms} = F_f\,\bar{\ell} = \frac{1}{F_c}\,\ell_m \qquad [5.10]$$

The F_f and F_c factors are, respectively, termed the *form factor* and *peak factor.* These parameters give, in real cases where the signal is not pure, some indications of its form and its resemblance to a sinusoid. For a pure sinusoid:

$$F_f = \frac{\pi}{2\sqrt{2}} \approx 1.11 \qquad [5.11]$$

and

$$F_c = \sqrt{2} \approx 1.414 \qquad [5.12]$$

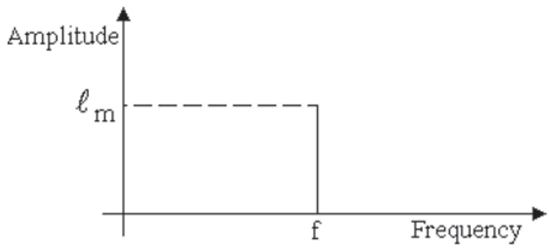


Figure 5.4. *Spectrum of a sinusoid (line spectrum)*

Such a signal is also termed *simple harmonic.* Its spectrum comprises only one line at a particular frequency.

The spectrum of a signal made up of several sinusoids is known as *discrete* (*spectrum of lines)* [BEN 71].

NOTE.– *The mean square value is, generally, a total measurement of the static and dynamic components of the vibratory signal. The continuous component can be separated by calculating the mean value [BEN 63], [PEN 65]:*

$$\overline{\ell} = \frac{1}{T} \int_0^T \ell(t)\, dt$$

This is zero for a perfect sinusoid, the time axis being centered, while the dynamic part is characterized by calculating the centered mean square value (variance).

$$s_\ell^2 = \frac{1}{T} \int_0^T \left[\ell(t) - \overline{\ell}\right]^2 dt \qquad [5.13]$$

We then have

$$\overline{\ell^2} = s_\ell^2 + \left(\overline{\ell}\right)^2 \qquad [5.14]$$

The variance is equal to the mean square value if $\overline{\ell} = 0$.

5.1.4. *Periodic vibrations*

Movements encountered in the real environment are seldom purely sinusoidal. Some are simply periodic, the signal being repeated at regular time intervals T_1 (period).

Its instantaneous amplitude can be written in the form:

$$\ell(t) = \ell(t + n\, T_1) \qquad [5.15]$$

where n is an integer positive constant.

With rare exceptions, a periodic signal can be represented by a Fourier series, i.e. by a sum of purely sinusoidal signals:

$$\ell(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos 2\,\pi\, n\, f_1\, t + b_n \sin 2\,\pi\, n\, f_1\, t \right) \qquad [5.16]$$

where

$$f_1 = \frac{1}{T_1} = \text{fundamental frequency}$$

$$a_n = \frac{2}{T_1} \int_0^{T_1} \ell(t) \cos 2\,\pi\, n\, f\, t \;\, dt$$

$$b_n = \frac{2}{T_1} \int_0^{T_1} \ell(t) \sin 2\,\pi\, n\, f\, t \;\, dt$$

(n = 0, 1, 2, 3...).

All the frequencies $f_n = n\, f_1$ are multiple integers of the fundamental frequency f_1.

For the majority of practical applications, it is sufficient to know the amplitude and the frequency of the various components, the phase being ignored. The representation of such a periodic signal can then be made, as in Figure 5.5, by a discrete spectrum giving the amplitude ℓ_{m_n} of each component according to its frequency.

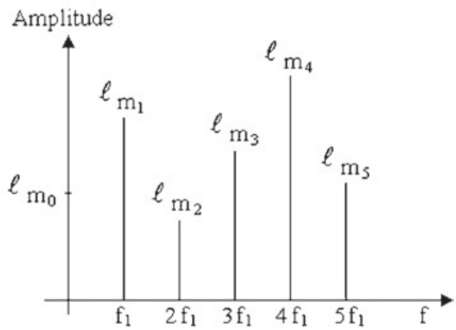


Figure 5.5. *Spectrum of a periodic signal*

With each component being sinusoidal, the rms value $\ell_{rms_n} = \dfrac{\ell_{m_n}}{\sqrt{2}}$ or the mean value of $\left|\ell_n(t)\right|$, $\ell_{n\,mean} = \dfrac{2}{\pi}\ell_{m_n}$ can easily be drawn against f. These various parameters give information on the excitation severity, but are insufficient to describe it alone since they do not give any idea of the frequency. $\ell(t)$ can also be written [PEN 65]:

$$\ell(t) = \ell_{m0} + \sum_{n=1}^{\infty} \ell_{mn} \sin(2\,\pi\,n\,f_1\,t - \varphi_n) \qquad [5.17]$$

where:

ℓ_{m_n} = amplitude of the n^{th} component;
ϕ_n = phase of the n^{th} component;
L_0 = continuous component.

$$\ell_n(t) = \ell_{m_n} \sin(2\,\pi\,n\,f_1 - \varphi_n)$$

$$\ell_{m_0} = \frac{a_0}{2}$$

$$\ell_{m_n} = \sqrt{a_n^2 + b_n^2} \quad (n = 1, 2, 3...)$$

$$\varphi = \arc\tan \frac{b_n}{a_n}$$

The periodic signal $\ell(t)$ can thus be regarded as the sum of a constant component and an infinite number (or not) of sinusoidal components, called *harmonics,* whose frequencies are multiple integers of f.

The Fourier series can be entirely characterized by coefficients a_n and b_n at frequencies $n\ f_1$ and can be represented by line spectra giving a_n and b_n versus the frequency. If we do not consider phases φ_n as is often the case in practice, we can also draw a line spectrum giving coefficients ℓ_{m_n} versus the frequency.

The vertical axis can indicate the amplitude of each component or its rms value. We have [FOU 64]:

$$\overline{\ell} = \ell_{m0} \qquad [5.18]$$

$$\ell(t) = \sum_{n=1}^{\infty} \ell_{m_n} \sin\left(2\ \pi\ f_n\ t + \varphi_n\right)$$

$$\ell_{rms}^2 = \frac{1}{T_1} \sum \int_0^{T_1} \ell_{mn}^2\ \sin^2\left(2\ \pi\ n\ f_1\ t + \phi_n\right)\ dt$$

$$+ \frac{2}{T_1} \sum \int_0^{T_1} \ell_{mp}\ \ell_{mq} \sin\left(2\ \pi\ p\ f_1\ t + \varphi_p\right) \sin\left(2\ \pi\ q\ f_1\ t + \varphi_q\right) dt$$

The second term, the integral over one period of the product of two sinusoidal functions, is zero:

$$\ell_{rms}^2 = \frac{1}{T_1} \sum \ell_{mn}^2 \int_0^{T_1} \frac{1}{2} \left\{1 - \cos\left[2\left(2\ \pi\ n\ f_1\ t + \varphi_n\right)\right]\right\} dt = \frac{1}{T_1} \sum \ell_{mn}^2 \int_0^{T_1} \frac{dt}{2}$$

If the mean value is zero

$$\ell_{rms}^2 = \frac{1}{T_1} \sum_{n=1}^{\infty} \int_0^{T_1} \ell_{mn}^2\ \frac{dt}{2} = \sum_{n=1}^{\infty} \frac{\ell_{mn}^2}{2} \qquad [5.19]$$

Each component has as a mean square value equal to

$$\overline{\ell_n^2} = \frac{1}{2}\,\ell_{m_n}^2 \qquad [5.20]$$

If the mean value is not zero

$$\overline{\ell^2} = \ell_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} \ell_{m_n}^2 \qquad [5.21]$$

the variance is given by

$$s_\ell^2 = \overline{\ell^2} - \left(\overline{\ell}\right)^2 = \frac{1}{2}\sum_{n=1}^{\infty} \ell_{m_n}^2 \qquad [5.22]$$

Relations [5.2] giving $\dot{\ell}(t)$ and $\ddot{\ell}(t)$ from $\ell(t)$ do not directly apply any more (it is necessary to derive each term from the sum). The forms of each one of these curves are different.

The mean value and the rms value of $\ell(t)$ can always be calculated from the general expressions [BRO 84], [KLE 71b].

5.1.5. *Quasi-periodic signals*

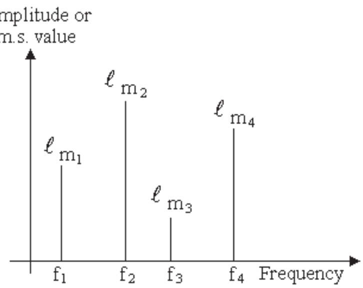


Figure 5.6. *Spectrum of a quasi-periodic signal*

A signal made up of the sum of several periodic signals will not in itself be periodic if all the possible ratios between the frequencies of the components are irrational numbers; the resulting signal can then be written

$$\ell(t) = \sum_{n=1}^{\infty} \ell_{m_n} \sin\left(2\pi f_n\ t + \varphi_n\right) \qquad [5.23]$$

If we also neglect the phases φ_n here, we can still represent $\ell(t)$ graphically by a line spectrum.

5.2. Periodic and sinusoidal vibrations in the real environment

Perfectly sinusoidal vibrations are seldom encountered in the real environment. In certain cases, however, the signal can be treated in the same way as a sinusoid in order to facilitate the analyses. Such vibrations are observed, for example, in rotating machines, and in badly balanced rotating parts (unbalanced shafts, defects in coaxiality in reducers (shafts speed changers) with the driving shafts, electric motor, gears) [RUB 64].

The more frequent case of periodic vibrations decomposable in Fourier series is reduced to a sinusoidal vibrations problem, by studying the effect of each harmonic component and by applying the superposition theorem (if the necessary assumptions, in particular that of linearity, are respected). They can be observed on machines generating periodic impacts (presses), in internal combustion engines with several cylinders and so on [BEN 71], [BRO 84], [KLE 71b], [TUS 72].

Quasi-periodic vibrations can be studied in the same manner, component by component, insofar as each component can be characterized. They are measured, for example, in plane structures propelled by several badly synchronized engines [BEN 71].

5.3. Sinusoidal vibration tests

The sinusoidal vibration tests carried out using electrodynamic shakers or hydraulic vibration machines can have several aims:

– the simulation of an environment of the same nature;

– the search for resonance frequencies (identification of the dynamic behavior of a structure). This research can be carried out by measuring the response of the structure at various points when it is subjected to random excitation, shocks or *swept* *frequency* sinusoidal vibrations. In this last case, the frequency of the sinusoid varies over time according to a law which is in general exponential, although sometimes linear. When the swept sine test is controlled by an analog control system, the frequency varies in a continuous way with time. When numerical control systems are used, the frequency remains constant at a certain time with each selected value, and varies between two successive values by increments that may or may not be constant depending on the type of sweeping selected;

Example 5.1.

The amplitude is assumed to be $x_m = 10$ cm at a frequency of 0.5 Hz.

Maximum velocity:

$$\dot{x}_m = 2\pi f\, x_m = 0.314 \text{ m/s}$$

Maximum acceleration:

$$\ddot{x}_m = (2\pi f)^2\, x_m = 0.987 \text{ m/s}^2$$

At 3 Hz, $x_m = 10$ cm

Velocity:

$$\dot{x}_m = 1.885 \text{ m/s}$$

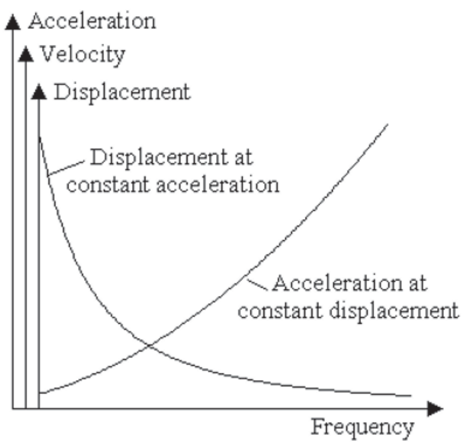


Figure 5.7. *Acceleration, velocity and displacement versus frequency*

Acceleration:

$$\ddot{x}_m = 35.53\ \text{m/s}^2$$

At 10 Hz, if $\ddot{x}_m = 5\ \text{m/s}^2$, the velocity is equal to $\dot{x}_m = \dfrac{\ddot{x}_m}{2\,\pi\,f} = 0.0796\ \text{m/s}$

and the displacement is $x_m = \dfrac{\ddot{x}_m}{(2\,\pi\,f)^2} = 1.27\ 10^{-3}\ \text{m}$.

– fatigue tests either on test-bars or directly on structures, the frequency of the sinusoid often being chosen equal to the resonance frequency of the structure. In this last case, the test is often intended to simulate the fatigue effects of a more complex real environment, generally random and making the assumption that induced fatigue is at a maximum around resonance [GAM 92]. The problems to be solved are then the following [CUR 71]:

- the determination of an equivalence between random and sinusoidal vibration. There are rules to choose the severity and the duration of an *equivalent sine test* [GAM 92],

- it is necessary to know the resonance frequencies of the material (determined by a preliminary test),

- for these frequencies, it is necessary to choose the number of test frequencies, in general lower than the number of resonances (in order for a sufficient fraction of the total testing time to remain at each frequency), and then to define the severity, and the duration of each sinusoid at each resonance frequency selected. The choice of the frequencies is very important. As far as possible, those for which rupture by fatigue is most probable are chosen, i.e. those for which the Q factor is higher than a given value (2 generally). This choice can be questioned since, being based on previously measured transfer functions, it is a function of the position of the sensors and can thus lead to errors,

- the frequent control of the resonance frequency, which varies appreciably at the end of the material’s lifetime.

For the sine tests, the specifications indicate the frequency of the sinusoid, its duration of application and its amplitude.

The amplitude of the excitation is generally defined by a zero-to-peak acceleration (sometimes peak-to-peak); for very low frequencies (less than a few Hertz), it is often preferable to describe the excitation by a displacement because the acceleration is, in general, very weak. With intermediate frequencies, velocity is sometimes used.

Chapter 6

Response of a Linear One-Degree-of-Freedom Mechanical System to a Sinusoidal Excitation

In Chapter 3, simple harmonic movements, both damped and undamped, were considered, where the mechanical system, displaced from its equilibrium position and released at the initial moment, was simply subjected to a restoring force and, possibly, to a damping force.

In this chapter, the movement of a system subjected to steady state excitation, whose amplitude varies sinusoidally with time and with its restoring force in the same direction will be studied. The two possibilities of an excitation defined by a force applied to the mass of the system, or by a movement of the support of the system, this movement itself being defined by a displacement, a velocity or an acceleration varying with time, will also be examined.

The two types of excitation focused on will be:

– the case close to reality where there are damping forces;

– the ideal case where damping is zero.

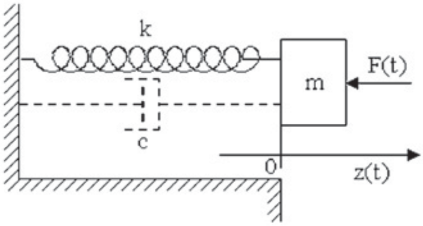


Figure 6.1. *Excitation by a force*

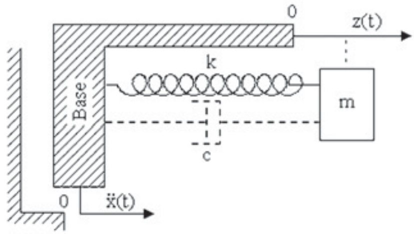


Figure 6.2. *Excitation by an acceleration*

6.1. General equations of motion

6.1.1. *Relative response*

The differential equation of movement was established in Chapter 3. The Laplace transform is written, for the relative response:

$$Q(p) = \frac{\Lambda(p)}{p^2 + 2\,\xi\,p + 1} + \frac{p\,q_0 + \left(\dot{q}_0 + 2\,\xi\,q_0\right)}{p^2 + 2\,\xi\,p + 1} \qquad [6.1]$$

q_0 and $\dot{q}_0$ being initial conditions. To simplify the calculations, and by taking account of the remarks of this chapter, it is supposed that $q_0 = \dot{q}_0 = 0$. If this were not the case, it would be enough to add to the final expression of $q(\theta)$ the term $C(\theta)$ previously calculated.

The transform of a sinusoid

$$\lambda(\theta) = \sin\,h\,\theta \qquad [6.2]$$

is given by

$$\Lambda(p) = \frac{h}{p^2 + h^2} \qquad [6.3]$$

where $h = \dfrac{\Omega}{\omega_0}$ (Ω being the pulsation of the sinusoid and ω_0 the natural pulsation of the undamped one-degree-of-freedom mechanical system), yielding

$$Q(p) = \frac{h}{\left(p^2 + h^2\right)\left(p^2 + 2\,\xi\,p + 1\right)} \tag{6.4}$$

Case 1: $0 \le \xi \le 1$ *(underdamped system)*

$$Q(p) = \frac{-h}{\left(1 - h^2\right)^2 + 4\,\xi^2\,h^2}\left[\frac{2\,\xi\,p + h^2 - 1}{p^2 + h^2} - \frac{2\,\xi\,p + 4\,\xi^2 + h^2 - 1}{p^2 + 2\,\xi\,p + 1}\right] \tag{6.5}$$

$$q(\theta) = \frac{\left(1 - h^2\right)\sin(h\,\theta) - 2\,\xi\,h\cos(h\,\theta)}{\left(1 - h^2\right)^2 + 4\,\xi^2 h^2} + h\,e^{-\xi\theta}\,\frac{2\,\xi\cos\sqrt{1 - \xi^2}\,\theta + \dfrac{2\,\xi^2 + h^2 - 1}{\sqrt{1 - \xi^2}}\sin\sqrt{1 - \xi^2}\,\theta}{\left(1 - h^2\right)^2 + 4\,\xi^2 h^2} \tag{6.6}$$

For non-zero initial conditions, this must be added to $q(\theta)$

$$C(\theta) = e^{-\xi\theta}\left[q_0\,\cos\sqrt{1 - \xi^2}\,\theta + \frac{\dot{q}_0 + q_0\,\xi}{\sqrt{1 - \xi^2}}\,\sin\sqrt{1 - \xi^2}\,\theta\right] \tag{6.7}$$

Case 2: $\xi = 1$ *(critical damping)*

For zero initial conditions,

$$Q(p) = \frac{h}{\left(p^2 + h^2\right)(p + 1)^2} \tag{6.8}$$

$$q(\theta) = L^{-1}\left\{\frac{-h}{\left(1 + h^2\right)^2}\left[\frac{2\,p + h^2 - 1}{p^2 + h^2} - \frac{2\,p + h^2 + 3}{(p + 1)^2}\right]\right\} \tag{6.9}$$

$$q(\theta) = -\frac{h}{\left(1 + h^2\right)^2}\left\{2\cos(h\,\theta) + \frac{h^2 - 1}{h}\sin(h\,\theta) - e^{-\theta}\left(2 + \theta + h^2\,\theta\right)\right\} \tag{6.10}$$

For non-zero initial conditions, we have to add to $q(\theta)$:

$$C(\theta) = \left[q_0 + \left(q_0 + \dot{q}_0\right) \theta\right] e^{-\theta} \tag{6.11}$$

Case 3: $\xi > 1$ *(overdamped system)*

$$Q(p) = \frac{-h}{\left(1-h^2\right)^2 + 4 \xi^2 h^2} \left[\frac{2 \xi p + h^2 - 1}{p^2 + h^2} - \frac{2 \xi p + 4 \xi^2 + h^2 - 1}{p^2 + 2 \xi p + 1}\right] \tag{6.12}$$

The denominator can be written $p^2 + 2 \xi p + 1$, for $\xi > 1$,

$$p^2 + 2 \xi p + 1 = \left(p + \xi + \sqrt{\xi^2 - 1}\right)\left(p + \xi - \sqrt{\xi^2 - 1}\right) \tag{6.13}$$

yielding

$$q(\theta) = \frac{-h}{\left(1-h^2\right)^2 + 4 \xi^2 h^2} \left\{ 2 \xi \cos(h \theta) + \frac{h^2 - 1}{h} \sin(h \theta) \right.$$

$$+ 2 \xi \frac{\left(\xi - \sqrt{\xi^2 - 1}\right) e^{-\left(\xi - \sqrt{\xi^2 - 1}\right)\theta} - \left(\xi + \sqrt{\xi^2 - 1}\right) e^{-\left(\xi + \sqrt{\xi^2 - 1}\right)\theta}}{2 \sqrt{\xi^2 - 1}}$$

$$\left. + \left(4 \xi^2 + h^2 - 1\right) \frac{e^{-\left(\xi + \sqrt{\xi^2 - 1}\right)\theta} - e^{-\left(\xi - \sqrt{\xi^2 - 1}\right)\theta}}{2 \sqrt{\xi^2 - 1}} \right\}$$

$$q(\theta) = \frac{\left(1-h^2\right) \sin(h \theta) - 2 \xi h \cos(h \theta)}{\left(1-h^2\right)^2 + 4 \xi^2 h^2} + h\, e^{-\xi \theta} \frac{2 \xi \cosh\left(\sqrt{\xi^2 - 1}\, \theta\right) + \dfrac{h^2 + 2 \xi^2 - 1}{\sqrt{\xi^2 - 1}} \sinh\left(\sqrt{\xi^2 - 1}\, \theta\right)}{\left(1-h^2\right)^2 + 4 \xi^2 h^2}$$

[6.14]

with, for non-zero initial conditions,

$$C(\theta)=\frac{e^{-\xi\theta}}{\sqrt{\xi^2-1}}\left[(\xi\ q_0+\dot{q}_0)\ \sinh\left(\sqrt{\xi^2-1}\ \theta\right)+q_0\sqrt{\xi^2-1}\ \cosh\left(\sqrt{\xi^2-1}\ \theta\right)\right] \qquad [6.15]$$

6.1.2. *Absolute response*

Case 1: $0 \le \xi < 1$

Zero initial conditions

$$Q(p)=\frac{h\,(1+2\,\xi\,p)}{\left(p^2+h^2\right)\left(p^2+2\,\xi\,p+1\right)} \qquad [6.16]$$

$$Q(p)=\frac{h}{\left(1-h^2\right)^2+4\,\xi^2\,h^2}\left\{\frac{2\,\xi\,h^2\,p+h^2-1}{p^2+2\,\xi\,p+1}+\frac{-2\,\xi\,h^2\,p+4\,\xi^2\,h^2+1-h^2}{p^2+h^2}\right\} \qquad [6.17]$$

$$q(\theta)=\frac{\left(1-h^2+4\xi^2h^2\right)\sin(h\,\theta)-2\,\xi\,h^3\cos(h\,\theta)}{\left(1-h^2\right)^2+4\,\xi^2h^2}$$
$$-\,h\,e^{-\xi\theta}\,\frac{\dfrac{1-h^2+2\,\xi^2h^2}{\sqrt{1-\xi^2}}\sin\sqrt{1-\xi^2}\,\theta-2\,\xi\,h^2\cos\sqrt{1-\xi^2}\,\theta}{\left(1-h^2\right)^2+4\,\xi^2h^2} \qquad [6.18]$$

If the initial conditions are not zero, it must be added to $q(\theta)$

$$C(\theta)=e^{-\xi\theta}\left[q_0\cos\sqrt{1-\xi^2}\,\theta+\frac{\dot{q}_0+\xi\,(q_0-2\lambda_0)}{\sqrt{1-\xi^2}}\sin\sqrt{1-\xi^2}\,\theta\right] \qquad [6.19]$$

Case 2: $\xi = 1$

$$Q(p)=\frac{h\,(1+2\,p)}{\left(p^2+h^2\right)(p+1)^2} \qquad [6.20]$$

$$Q(p) = \frac{h}{\left(1+h^2\right)^2}\left[\frac{2\ h^2}{p+1} - \frac{1+h^2}{(p+1)^2} - \frac{2\ h^2\ p}{p^2+h^2} + \frac{3\ h^2+1}{p^2+h^2}\right] \qquad [6.21]$$

$$q(\theta) = \frac{1}{\left(1+h^2\right)^2}\left\{h\left[2\ h^2 - \left(1+h^2\right)\theta\right]e^{-\theta} + \left(3\ h^2+1\right)\sin(h\,\theta) - 2\ h^3\cos(h\,\theta)\right\} \qquad [6.22]$$

Non-zero initial conditions

$$C(\theta) = \left[q_0 + \left(q_0 + \dot{q}_0 - 2\ \lambda_0\right)\theta\right]e^{-\theta} \qquad [6.23]$$

Case 3: $\xi > 1$

Zero initial conditions

$$Q(p) = \frac{h\left(1+2\ \xi\ p\right)}{\left(p^2+h^2\right)\left(p^2+2\ \xi\ p+1\right)} \qquad [6.24]$$

$$Q(p) = \frac{h}{\left(1-h^2\right)^2+4\ \xi^2 h^2}\left\{\frac{2\ \xi\ h^2 p + h^2 - 1}{p^2+2\ \xi\ p+1} + \frac{-2\ \xi\ h^2 p + 4\ \xi^2 h^2 + 1 - h^2}{p^2+h^2}\right\} \qquad [6.25]$$

i.e.

$$q(\theta) = \frac{h\ e^{-\xi\theta}}{\left(1-h^2\right)^2+4\,\xi^2 h^2}\left\{\frac{h^2-1-2\,\xi^2 h^2}{\sqrt{\xi^2-1}}\sinh\left(\sqrt{\xi^2-1}\ \theta\right) + 2\,\xi\,h^2\cosh\left(\sqrt{\xi^2-1}\ \theta\right)\right\}$$
$$+\frac{\left(4\,\xi^2 h^2+1-h^2\right)\sin(h\,\theta) - 2\,\xi\,h^3\cos(h\,\theta)}{\left(1-h^2\right)^2+4\,\xi^2 h^2} \qquad [6.26]$$

Non-zero initial conditions

$$C(\theta) = \frac{e^{-\xi\theta}}{\sqrt{\xi^2-1}}\left\{\left[\xi\left(q_0 - 2\ \lambda_0\right) + \dot{q}_0\right]\sinh\left(\sqrt{\xi^2-1}\ \theta\right) + \sqrt{\xi^2-1}\ q_0\cosh\left(\sqrt{\xi^2-1}\ \theta\right)\right\} \qquad [6.27]$$

This must be added to $q(\theta)$.

6.1.3. *Summary*

The principal relations obtained for zero initial conditions are brought together below.

Relative response

$0 \le \xi < 1$

$$q(\theta) = \frac{\left(1 - h^2\right)\sin(h\,\theta) - 2\,\xi\,h\cos(h\,\theta)}{\left(1 - h^2\right)^2 + 4\,\xi^2\,h^2} + h\,e^{-\xi\theta}\,\frac{2\xi\cos\left(\sqrt{1-\xi^2}\,\theta\right) + \dfrac{h^2 + 2\,\xi^2 - 1}{\sqrt{1-\xi^2}}\sin\left(\sqrt{1-\xi^2}\,\theta\right)}{\left(1 - h^2\right)^2 + 4\,\xi^2 h^2}$$

$\xi = 1$

$$q(\theta) = \frac{h}{\left(1 + h^2\right)^2}\left\{\frac{1 - h^2}{h}\sin(h\,\theta) - 2\cos(h\,\theta) + \left(2 + \theta + h^2\theta\right)e^{-\theta}\right\}$$

$\xi > 1$

$$q(\theta) = \frac{\left(1 - h^2\right)\sin(h\,\theta) - 2\,\xi\,h\cos(h\,\theta)}{\left(1 - h^2\right)^2 + 4\,\xi^2 h^2} + h\,e^{-\xi\,\theta}\,\frac{2\,\xi\cosh\left(\sqrt{\xi^2 - 1}\,\theta\right) + \dfrac{h^2 + 2\,\xi^2 - 1}{\sqrt{\xi^2 - 1}}\sinh\left(\sqrt{\xi^2 - 1}\,\theta\right)}{\left(1 - h^2\right)^2 + 4\,\xi^2\,h^2}$$

Absolute response

$0 \le \xi < 1$

$$q(\theta) = \frac{\left(1 - h^2 + 4\,\xi^2 h^2\right)\sin(h\,\theta) - 2\,\xi\,h^3\cos(h\,\theta)}{\left(1 - h^2\right)^2 + 4\,\xi^2\,h^2}$$

$$- h\,e^{-\xi\,\theta}\,\frac{\dfrac{1 - h^2 + 2\,\xi^2\,h^2}{\sqrt{1-\xi^2}}\sin\left(\sqrt{1-\xi^2}\;\theta\right) - 2\,\xi\,h^2\cos\left(\sqrt{1-\xi^2}\;\theta\right)}{\left(1 - h^2\right)^2 + 4\,\xi^2\,h^2}$$

$\xi = 1$

$$q(\theta) = \frac{1}{\left(1 + h^2\right)^2}\left\{h\left[2\,h^2 - \left(1 + h^2\right)\theta\right]e^{-\theta} + \left(3\,h^2 + 1\right)\sin(h\,\theta) - 2\,h^3\cos(h\,\theta)\right\}$$

$\xi > 1$

$$q(\theta) = \frac{\left(1 - h^2\right)\sin(h\,\theta) - 2\,\xi\,h\cos(h\,\theta)}{\left(1 - h^2\right)^2 + 4\,\xi^2 h^2}$$

$$+ h\,e^{-\xi\theta}\,\frac{2\,\xi\cosh\left(\sqrt{\xi^2 - 1}\;\theta\right) + \dfrac{h^2 + 2\,\xi^2 - 1}{\sqrt{\xi^2 - 1}}\sinh\left(\sqrt{\xi^2 - 1}\;\theta\right)}{\left(1 - h^2\right)^2 + 4\,\xi^2 h^2}$$

6.1.4. *Discussion*

Whatever the value of ξ, the response $q(\theta)$ is made up of three terms:

– the first, $C(\theta)$, related to initially non-zero conditions, which disappears when θ increases, because of the presence of the term $e^{-\xi\,\theta}$;

– the second, which corresponds to the transient movement at the reduced frequency $\sqrt{1-\xi^2}$ resulting from the sinusoid application at time $\theta = 0$. This oscillation attenuates and disappears after a while from ξ because of the factor $e^{-\xi\,\theta}$. In the case of the relative response, for example, for $0 \le \xi < 1$, this term is equal to

$$h\ e^{-\xi\,\theta}\ \frac{2\ \xi\ \cos\sqrt{1-\xi^2}\ \theta+\dfrac{h^2+2\ \xi^2-1}{\sqrt{1-\xi^2}}\sin\sqrt{1-\xi^2}\ \theta}{\left(1-h^2\right)^2+4\ \xi^2\ h^2}$$

– the third term corresponds to an oscillation of reduced pulsation h, which is that of the sinusoid applied to the system. The vibration of the system *is forced,* the frequency of the response being imposed on the system by the excitation. The sinusoid applied theoretically having one unlimited duration, it is said that the response, described by this third term, is *steady state*.

Response →		**Displacement**		**Velocity**		**Acceleration**		**Reaction force on**
Excitation ↓		**Absolute** $\mathbf{y(t)}$	**Relative** $\mathbf{z(t)}$	**Absolute** $\mathbf{\dot{y}(t)}$	**Relative** $\mathbf{\dot{z}(t)}$	**Absolute** $\mathbf{\ddot{y}(t)}$	**Relative** $\mathbf{\ddot{z}(t)}$	**base** $\mathbf{F_T(t)}$
Base movement	**Displacement** $\mathbf{x(t)}$	$\frac{y}{x_m}$	$\frac{z}{h^2\ x_m}$	$\frac{\dot{y}}{h\ \omega_0\ x_m}$	$\frac{\dot{z}}{h^2\ \omega_0\ x_m}$	$\frac{\ddot{y}}{h^2\ \omega_0^2\ x_m}$	$\frac{\ddot{z}}{h^4\ \omega_0^2\ x_m}$	
	Velocity $\mathbf{\dot{x}(t)}$	$\frac{h\ \omega_0\ y}{\dot{x}_m}$	$\frac{\omega_0\ z}{h\ \dot{x}_m}$	$\frac{\dot{y}}{\dot{x}_m}$	$\frac{\dot{z}}{h^2\ \dot{x}_m}$	$\frac{\ddot{y}}{h\ \omega_0\ \dot{x}_m}$	$\frac{\ddot{z}}{h^3\ \omega_0\ \dot{x}_m}$	
	Acceleration $\mathbf{\ddot{x}(t)}$	$\frac{h^2\ \omega_0^2\ y}{\ddot{x}_m}$	$\frac{\omega_0^2\ z}{\ddot{x}_m}$	$\frac{h\ \omega_0\ \dot{y}}{\ddot{x}_m}$	$-\frac{\omega_0\ \dot{z}}{h\ \ddot{x}_m}$	$\frac{\ddot{y}}{\ddot{x}_m}$	$\frac{\ddot{z}}{h^2\ \ddot{x}_m}$	
Force on the mass m (here, $\mathbf{z \equiv y}$ **)**		$\frac{k\ z}{F_m}$		$\frac{\sqrt{k\ m}\ \dot{z}}{h\ F_m}$		$\frac{m\ \ddot{z}}{h^2\ F_m}$		$\frac{F_T}{F_m}$

Table 6.1. *Expressions for reduced response*

The steady state response for $0 \le \xi < 1$ will be considered in detail in the following sections. The reduced parameter $q(\theta)$ is used to calculate the response of the mechanical system. This is particularly interesting because of the possibility of deducing expressions for relative or absolute response $q(\theta)$ easily, irrespective of the way the excitation (force, acceleration, velocity or displacement of the support) is defined, as in Table 6.1.

6.1.5. *Response to periodic excitation*

The response to a periodic excitation can be calculated by development of a Fourier series for the excitation [HAB 68]:

$$\ell(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\,\Omega\,t + b_n \sin n\,\Omega\,t\right) \qquad [6.28]$$

$$a_0 = \frac{2}{T} \int_0^T \ell(t)\,dt \qquad [6.29]$$

$$a_n = \frac{2}{T} \int_0^T \ell(t)\,\cos n\,\Omega\,t\;\,dt \qquad [6.30]$$

$$b_n = \frac{2}{T} \int_0^T \ell(t)\,\sin n\,\Omega\,t\;\,dt \qquad [6.31]$$

The response of a one-degree-of-freedom system obeys the differential equation

$$\ddot{u}(t) + 2\,\xi\,\omega_0\,\dot{u}(t) + \omega_0^2\,u(t) = \omega_0^2\,\ell(t) \qquad [6.32]$$

$$\ddot{u}(t) + 2\,\xi\,\omega_0\,\dot{u}(t) + \omega_0^2\,u(t) = \omega_0^2 \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\,\Omega\,t + b_n \sin n\,\Omega\,t\right)\right] \qquad [6.33]$$

This equation being linear, the solutions of the equation calculated successively for each term in sine and cosine can be superimposed. This yields

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n \cos\left(n\,\Omega\,t - \phi_n\right) + b_n \sin\left(n\,\Omega\,t - \phi_n\right)}{\sqrt{\left(1 - n^2 \dfrac{\Omega^2}{\omega_0^2}\right)^2 + \left(n \dfrac{\Omega}{\omega_0}\right)^2}} \qquad [6.34]$$

with

$$\phi_n = \arc\tan \frac{n \dfrac{\Omega}{\omega_0}}{1 - n^2 \dfrac{\Omega^2}{\omega_0^2}} \qquad [6.35]$$

6.1.6. *Application to calculation for vehicle suspension response*

Consider a vehicle rolling at velocity v on a sinusoidal road as shown in Figure 6.3.

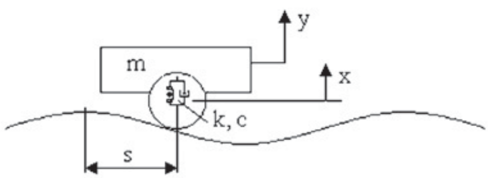


Figure 6.3. *Example of a vehicle on road*

$$x = X \cos \frac{2\,\pi\, s}{L} \qquad [6.36]$$

s = distance between a maximum of the sinusoid and the vehicle

L = sinusoid period

It is assumed that [VOL 65]:

– the wheels are small, so that the hub of each wheel is at a constant distance from the road;

– the tires have negligible deformation.

We have, with the notation already used in the preceding sections:

$$m\ \ddot{y} + c\left(\dot{y} - \dot{x}\right) + k\left(y - x\right) = 0 \qquad [6.37]$$

$$m\ \ddot{y} + c\ \dot{y} + k\ y = k\ x + c\ \dot{x} \qquad [6.38]$$

$$\ddot{y} + 2\,\xi\,\omega_0\,\dot{y} + \omega_0^2\,y = \omega_0^2\,x + 2\,\xi\,\omega_0\,\dot{x} \qquad [6.39]$$

Distance s is related to time by $s = v\,t$, yielding

$$x = X\cos\Omega\,t \qquad [6.40]$$

with

$$\Omega = \frac{2\,\pi\,v}{L} \qquad [6.41]$$

$$\ddot{y} + 2\,\xi\,\omega_0\,\dot{y} + \omega_0^2\,y = \omega_0^2\,X\cos\Omega\,t - 2\,\xi\,\omega_0\,\Omega\sin\Omega\,t \qquad [6.42]$$

$$\ddot{y} + 2\,\xi\,\omega_0\,\dot{y} + \omega_0^2\,y = \omega_0^2\,X\sqrt{1+(2\,\xi\,h)^2}\,\cos(\Omega\,t+\theta) \qquad [6.43]$$

where

$$\tan\theta = 2\,\xi\,h \qquad [6.44]$$

$$h = \frac{\Omega}{\omega_0}$$

$$y = x\,\cos(\Omega\,t + \theta - \varphi) \qquad [6.45]$$

$$y = x\,\sqrt{\frac{1+(2\,\xi\,h)^2}{(1-h^2)^2+(2\,\xi\,h)^2}} \qquad [6.46]$$

$$\tan\varphi = \frac{2\,\xi\,h}{1-h^2} \qquad [6.47]$$

Displacement y must be the smallest possible to make the suspension effective. It is necessary, therefore, that h or the velocity is large. If ξ tends towards zero, y tends towards infinity when h tends towards 1, with critical velocity

$$v_{cr} = \frac{\omega_0\,L}{2\,\pi} \qquad [6.48]$$

When ξ is non-zero, the value of y for $h = 1$ is

$$y = x \sqrt{1 + \frac{1}{(2\,\xi)^2}} \qquad [6.49]$$

6.2. Transient response

6.2.1. *Relative response*

For $0 \le \xi < 1$

The response

$$q(\theta) = h\, e^{-\xi\,\theta} \frac{2\,\xi \cos\sqrt{1-\xi^2}\;\theta + \dfrac{h^2 + 2\,\xi^2 - 1}{\sqrt{1-\xi^2}} \sin\sqrt{1-\xi^2}\;\theta}{\left(1-h^2\right)^2 + 4\,\xi^2\,h^2} \qquad [6.50]$$

can also be written

$$q(\theta) = e^{-\xi\,\theta}\, A(h)\, \sin\left(\sqrt{1-\xi^2}\;\theta - \alpha\right) \qquad [6.51]$$

where

$$A(h) = \frac{h}{\sqrt{1-\xi^2}\,\sqrt{\left(1-h^2\right)^2 + 4\,\xi^2\,h^2}} \qquad [6.52]$$

and

$$\tan\,\alpha = \frac{2\,\xi\sqrt{1-\xi^2}}{1 - h^2 - 2\,\xi^2} \qquad [6.53]$$

A *pseudo-sinusoidal movement* occurs. The total response $q(\theta)$ is zero for $\theta = 0$ since the term representing the transient response is then equal to

$$q_T(0) = \frac{2\,\xi\,h}{\left(1-h^2\right)^2 + 4\,\xi^2\,h^2} \tag{6.54}$$

This response q_T never takes place alone. It is superimposed on the steady state response $q_P(\theta)$ studied in the following section.

Amplitude $A(h)$ is maximum when $\frac{dA(h)}{dh} = 0$, i.e. when

$$\frac{dA(h)}{dh} = \frac{1}{\sqrt{1-\xi^2}}\,\frac{1-h^4}{\left[\left(1-h^2\right)^2 + 4\,\xi^2\,h^2\right]^{3/2}} = 0 \tag{6.55}$$

$\frac{dA(h)}{dh} = 0$ when $h = 1$ $(h \ge 0)$.

In this case,

$$A_m(h) = \frac{1}{2\,\xi\,\sqrt{1-\xi^2}} \tag{6.56}$$

The movement has a logarithmic decrement equal to [KIM 29]:

$$\delta = \frac{2\,\pi\,\xi}{\sqrt{1-\xi^2}} \tag{6.57}$$

and for the reduced pseudo-period $\frac{2\,\pi}{\sqrt{1-\xi^2}}$.

The transient response q_T has an amplitude equal to $\frac{1}{N}^{\text{th}}$ of the first peak after cycle number n such that $\frac{2\,\pi\,\xi}{\sqrt{1-\xi^2}} = \frac{1}{n}\ln N$.

I.e.

$$n = \frac{\sqrt{1-\xi^2}}{2\,\pi\,\xi}\,\ln\,N \qquad [6.58]$$

For ξ small, it becomes

$$n \approx \frac{\ln N}{2\,\pi\,\xi} = \frac{Q}{\pi}\ln N$$

i.e.

$$n \approx \frac{Q\ln N}{\pi} \qquad [6.59]$$

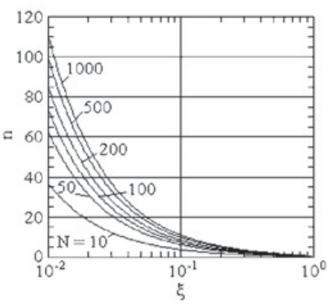


Figure 6.4. *Cycle number for attenuation N of the transient relative response*

If $N \approx 23$, we have $n \approx Q$. When a system is subjected to a sine wave excitation, the amplitude of the response is established gradually during the transitional stage up to a level proportional to that of the excitation and which corresponds to the steady state response. In section 6.5.2.1 it is seen that, if $h = \sqrt{1-\xi^2}$, the response tends in steady state mode towards

$$H_m = \frac{1}{2\,\xi\,\sqrt{1-\xi^2}}$$

The number of cycles necessary to reach this steady state response is independent of h. For ξ small, this number is roughly proportional to the Q factor of the system.

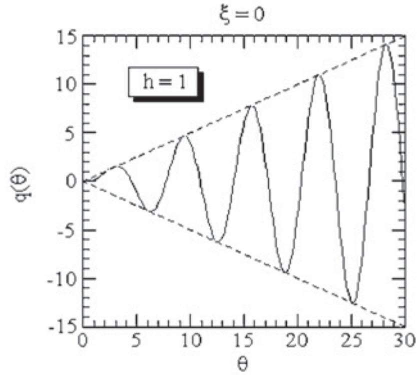


Figure 6.5. *Establishment of the relative response for $\xi = 0$*

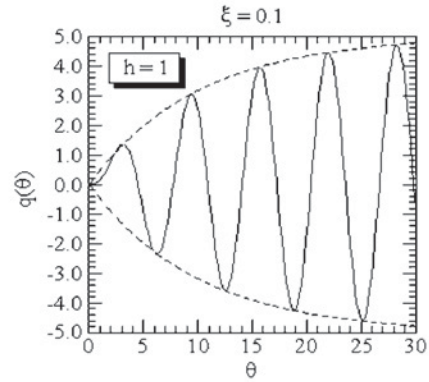


Figure 6.6. *Establishment of the relative response for $\xi = 0.1$*

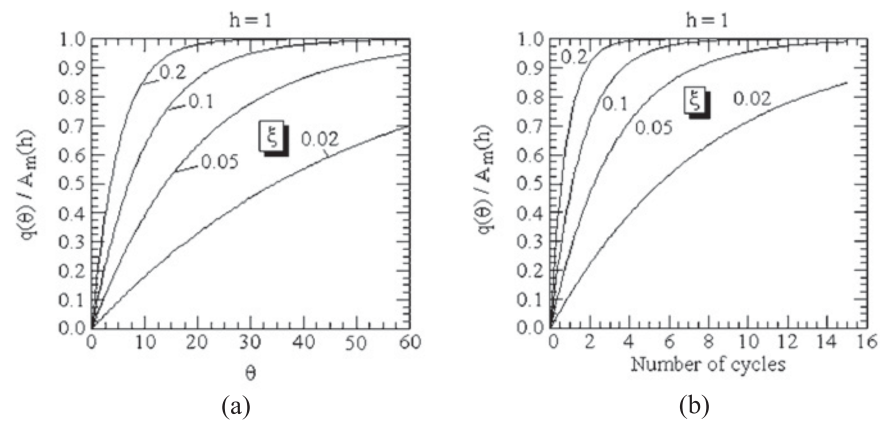


Figure 6.7. *Ratio of transient response/steady state response*

For the particular case where $\xi = 1$

$$q_T(\theta) = \frac{h}{\left(1 + h^2\right)^2} \left(2 + \theta + h^2\, \theta\right) e^{-\theta} \qquad [6.60]$$

6.2.2. *Absolute response*

For $0 \le \xi < 1$

$$q(\theta) = \frac{2\,\xi\,h^2\,\cos\sqrt{1-\xi^2}\;\theta + \dfrac{h^2 - 1 - 2\,\xi^2\,h^2}{\sqrt{1-\xi^2}}\sin\sqrt{1-\xi^2}\;\theta}{\left(1-h^2\right)^2 + 4\,\xi^2\,h^2}\;h\;e^{-\xi\,\theta} \qquad [6.61]$$

or

$$q(\theta) = e^{-\xi\,\theta}\;B(h)\;\sin\left(\sqrt{1-\xi^2}\;\theta - \beta\right) \qquad [6.62]$$

with

$$B(h) = \frac{h}{\sqrt{1-\xi^2}\;\sqrt{\left(1-h^2\right)^2 + 4\,\xi^2\,h^2}} = A(h) \qquad [6.63]$$

and

$$\tan\,\beta = \frac{2\,\xi\,\sqrt{1-\xi^2}\;h^2}{h^2 - 1 - 2\,\xi^2\,h^2} \qquad [6.64]$$

If $\xi = 1$

$$q_T(\theta) = \frac{h}{\left(1+h^2\right)^2}\left[2\,h^2 - \left(1+h^2\right)\,\theta\right]e^{-\theta} \qquad [6.65]$$

6.3. Steady state response

6.3.1. *Relative response*

For $0 \le \xi < 1$, the steady state response is written

$$q(\theta) = \frac{\left(1-h^2\right)\sin\left(h\;\theta\right) - 2\;\xi\;h\;\cos\left(h\;\theta\right)}{\left(1-h^2\right)^2 + 4\;\xi^2\;h^2} \qquad [6.66]$$

This expression can also be put in the form

$$q(\theta) = H(h)\,\sin(h\,\theta - \varphi) \qquad [6.67]$$

In the amplitude of this response H_{RD} it should also be noted that the first index (R) recalls that the response is relative and the second (D) is about a displacement. Therefore,

$$H(h) = \frac{1}{\sqrt{\left(1-h^2\right)^2 + 4\,\xi^2\,h^2}} = H_{RD}(h) \qquad [6.68]$$

The phase is such that

$$\tan\varphi = \frac{2\,\xi\,h}{1-h^2} \qquad [6.69]$$

6.3.2. *Absolute response*

For $0 \le \xi < 1$, the steady state response is expressed

$$q(\theta) = \frac{\left(1 - h^2 + 4\,\xi^2 h^2\right)\sin(h\,\theta) - 2\,\xi\,h^3 \cos(h\,\theta)}{\left(1-h^2\right)^2 + 4\,\xi^2 h^2} \qquad [6.70]$$

As previously, this response can be written

$$q(\theta) = \frac{\sqrt{1+4\,\xi^2\,h^2}\,\sin(h\,\theta-\phi)}{\sqrt{\left(1-h^2\right)^2 + 4\xi^2\,h^2}} = H_{AD}\,\sin(h\,\theta-\phi) \qquad [6.71]$$

where

$$H_{AD} = \sqrt{\frac{1+4\,\xi^2\,h^2}{\left(1-h^2\right)^2 + 4\,\xi^2\,h^2}} \qquad [6.72]$$

and

$$\tan\varphi = \frac{2\,\xi\,h^3}{1-h^2+4\,\xi^2\,h^2} \qquad [6.73]$$

H_{AD} is termed the *transmissibility factor* or *transmissibility* or *transmittance*.

6.4. Responses $\left|\dfrac{\omega_0\ \dot{z}}{\ddot{x}_m}\right|$, $\left|\dfrac{\omega_0\ z}{\dot{x}_m}\right|$ and $\dfrac{\sqrt{k\ m}\ \dot{z}}{F_m}$

6.4.1. *Amplitude and phase*

Starting with the study of the responses $\left|\dfrac{\omega_0\ \dot{z}}{\ddot{x}_m}\right|$, $\left|\dfrac{\omega_0\ z}{\dot{x}_m}\right|$ and $\dfrac{\sqrt{k\ m}\ \dot{z}}{F_m}$, some important definitions are introduced. These responses are equal to:

$$\dot{q}(\theta) = \frac{h}{\sqrt{\left(1-h^2\right)^2 + 4\,\xi^2\ h^2}} \sin\left(h\ \theta - \psi\right) = H_{RV}\ \sin\left(h\ \theta - \psi\right) \qquad [6.74]$$

where

$$H_{RV} = \frac{h}{\sqrt{\left(1-h^2\right)^2 + 4\,\xi^2\ h^2}} \qquad [6.75]$$

The case where the input is an acceleration $\ddot{x}_m$ is more interesting, and the reduced response $q(\theta)$ gives the relative displacement $z(t)$ yielding

$$\left|\dot{q}(\theta)\right| = \left|\frac{\omega_0\ \dot{z}}{\ddot{x}_m}\right| = H_{RD}\ h\ \cos\left(h\ \theta - \psi\right) \qquad [6.76]$$

$$\left|\dot{q}(\theta)\right| = H_{RV}\ h\ \sin\left(h\ \theta - \psi\right) \qquad [6.77]$$

with

$$H_{RV} = h\ H_{RD} \qquad [6.78]$$

and

$$\psi = \varphi - \frac{\pi}{2} \qquad [6.79]$$

6.4.2. *Variations of velocity amplitude*

6.4.2.1. *Quality factor*

The amplitude H_{RV} of the velocity passes through a maximum when the derivative $\frac{dH_{RV}}{dh}$ is zero.

$$\frac{dH_{RV}}{dh} = \frac{1 - h^4}{\left[\left(1 - h^2\right)^2 + 4\,\xi^2\,h^2\right]^{3/2}} \qquad [6.80]$$

This function is equal to zero when $h = 1$ $(h \geq 0)$. The response is thus at a maximum (whatever the value of ξ) for $h = 1$. There is then *velocity resonance*, and

$$H_{RV\,max} = \frac{1}{2\,\xi} = Q \qquad [6.81]$$

At resonance, the amplitude of the forced vibration $\dot{q}(\theta)$ is Q times that of the excitation (here the physical significance of the Q factor is seen). It should be noted that this resonance takes place for a frequency equal to the natural frequency of the undamped system, and not for a frequency equal to that of the free oscillation of the damped system. It tends towards 1 when h tends towards zero. The curve thus starts from the origin with a slope equal to 1 (whatever the value of ξ). For $h = 0$, $H_{RV} = 0$.

The slope tends towards zero when $h \to \infty$, like H_{RV}. The expression of H_{RV} does not change when h is replaced by $\frac{1}{h}$; thus, taking a logarithmic scale for the abscissae, the curves $H_{RV}(h)$ are symmetric with respect to the line $h = 1$.

For $\xi = 0$,

$$H_{RV} = \frac{h}{\left|1 - h^2\right|} \qquad [6.82]$$

$H_{RV} \to \infty$ when $h \to 1$.

Since $\tan\psi = \tan\left(\varphi - \frac{\pi}{2}\right) = \text{cotan}\ \varphi = \frac{h^2 - 1}{2\ \xi\ h}$, H_{RV} can be written in the form

$$H_{RV} = \frac{1}{2\ \xi\ \sqrt{1 + \tan^2\psi}} \tag{6.83}$$

Setting $y = 2\ \xi\ H_{RV}$ and $x = \tan\psi$, the curves

$$\begin{cases} y = \dfrac{1}{\sqrt{1 + x^2}} \\ \psi = \text{arc}\tan x \end{cases}$$

valid for all the systems m, k, and c are known as *universal.*

In the case of an excitation by force, the quantity $2\ \xi\ H_{RV}$ is equal to

$$2\ \frac{c}{2\ \sqrt{k\ m}}\ \frac{\sqrt{k\ m}\ \dot{z}}{F_m} = \frac{c\ \dot{z}}{F_m}$$

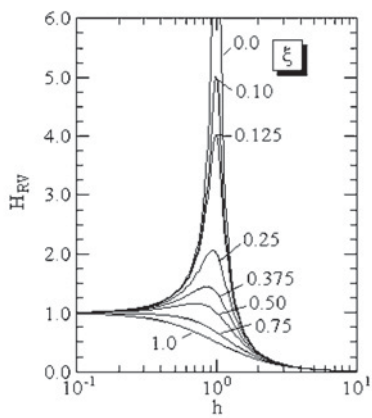


Figure 6.8. *Amplitude of velocity response*

6.4.2.2. *Hysteresis loop*

It has been assumed up to now that damping was viscous, with the damping force being proportional to the relative velocity between the ends of the damping device and of the form $F_d = -c\,\dot{z}$, where c is the damping constant, acting in a direction opposed to that of the movement. This damping, which leads to a linear differential equation of the movement, is itself known as *linear*. If the relative displacement response $z(t)$ has the form

$$z = z_m \sin(\Omega\,t - \varphi)$$

the damping force is equal to

$$F_d(t) = c\,z_m\,\Omega\cos(\Omega\,t - \varphi) = F_{d_m}\cos(\Omega\,t - \varphi) \qquad [6.84]$$

where

$$F_{d_m} = c\,\Omega\,z_m \qquad [6.85]$$

The curve $F_d(z)$ (hysteresis loop) has the equations, in parametric coordinates,

$$\begin{cases} z = z_m \sin(\Omega\,t - \varphi) \\ F_d = F_{d_m}\cos(\Omega\,t - \varphi) \end{cases}$$

i.e. after elimination of time t:

$$\frac{F_d^2}{F_{d_m}^2} + \frac{z^2}{z_m^2} = 1 \qquad [6.86]$$

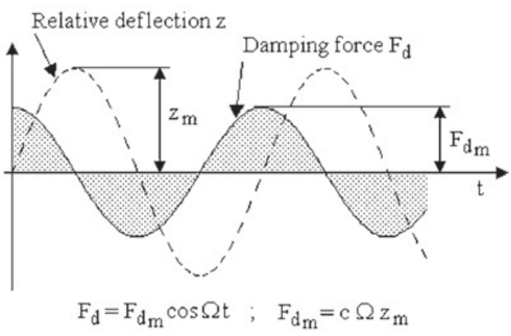


Figure 6.9. *Viscous damping force*

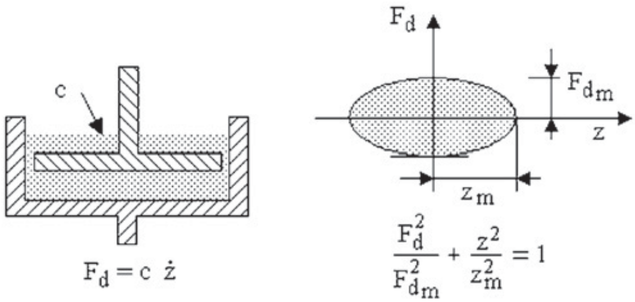


Figure 6.10. *Hysteresis loop for viscous damping [RUZ 71]*

The hysteresis loop is thus presented in the form of an ellipse of half the smaller axis $F_{d_m} = c\,\Omega\,z_m$ and half the larger axis z_m.

6.4.2.3. Energy dissipated during a cycle

The energy dissipated during a cycle can be written:

$$\Delta E_d = \int_{1\,\text{cycle}} |F_d|\ dz = \int_0^T |F_d|\ \frac{dz}{dt} dt$$

$$\Delta E_d = c \int_0^T \dot{z}^2\ dt$$

Knowing that $z(t) = z_m \sin(\Omega\,t - \varphi)$, we have

$$\Delta E_d = c\ z_m^2\ \Omega^2 \int_0^{2\pi/\Omega} \cos^2(\Omega\,t - \varphi)\,dt$$

i.e. since $\cos^2(\Omega\,t - \varphi) = \dfrac{1 + \cos[2\,(\Omega\,t - \varphi)]}{2}$

$$\Delta E_d = \pi\,c\,\Omega\,z_m^2 \qquad [6.87]$$

or [CRE 65]:

$$\Delta E_d = \pi\,z_m\,F_{d_m} \qquad [6.88]$$

For a viscously damped system, in which the damping constant c is independent of the frequency, the relative damping ξ is inversely proportional to the frequency:

$$\xi = \frac{c}{2\sqrt{k\,m}} = \frac{c}{2\,\pi\,m\,f_0} \qquad [6.89]$$

We can deduce the energy Δ consumed per time unit from this. If T is the period of the excitation $\left(T = \dfrac{2\,\pi}{\Omega}\right)$,

$$\Delta = \frac{\Delta E_d}{T} = \frac{\pi\,c\,\Omega}{T}\,z_m^2 = \frac{1}{2}\,c\,\Omega^2\,z_m^2 = \frac{1}{2}\,c\,\dot{z}_m^2$$

$$\Delta = \frac{1}{2} = c\,\Omega^2\,z_m^2 = \xi\,\omega_0\,m\,\Omega^2\,z_m^2 \qquad [6.90]$$

Since (Chapter 4) $z_m = \dfrac{z_s}{\sqrt{\left(1-h^2\right)^2 + 4\,\xi^2\,h^2}}$,

$$\Delta = \xi\,\omega_0\,m\,\Omega^2\,\frac{z_s^2}{\left(1-h^2\right)^2 + 4\,\xi^2\,h^2}$$

$$\Delta = \xi\,\omega_0^3\,m\,z_s^2\,\frac{h^2}{\left(1-h^2\right)^2 + 4\,\xi^2\,h^2} \qquad [6.91]$$

Energy consumed is at a maximum when the function $\dfrac{h^2}{\left(1-h^2\right)^2 + 4\,\xi^2\,h^2}$, equal to $H_{RV}^2(h)$, is at a maximum, i.e. for $h = 1$, yielding

$$\Delta_m = \frac{\omega_0^2\,m\,z_s^2}{4\,\xi} \qquad [6.92]$$

and

$$\frac{\Delta}{\Delta_m} = \frac{4\,\xi^2\,h^2}{\left(1-h^2\right)^2 + 4\,\xi^2\,h^2} = H_{RV}^2(h) \qquad [6.93]$$

The energy dissipated is thus inversely proportional to ξ. When ξ decreases, the resonance curve $\Delta(h)$ presents a larger and narrower peak [LAN 60]. However, it can be shown that the surface under the curve $\Delta(h)$ remains unchanged.

This surface S is described by:

$$S = \int_0^\infty \Delta(\Omega)\,d\Omega = \int_0^\infty \xi\,\omega_0^3\,m\,z_s^2\,\frac{h^2}{\left(1-h^2\right)^2 + 4\,\xi^2\,h^2}\,d\Omega \qquad [6.94]$$

$$S = \xi\,\omega_0^4\,m\,z_s^2 \int_0^\infty \frac{h^2}{\left(1-h^2\right)^2 + 4\,\xi^2\,h^2}\,dh$$

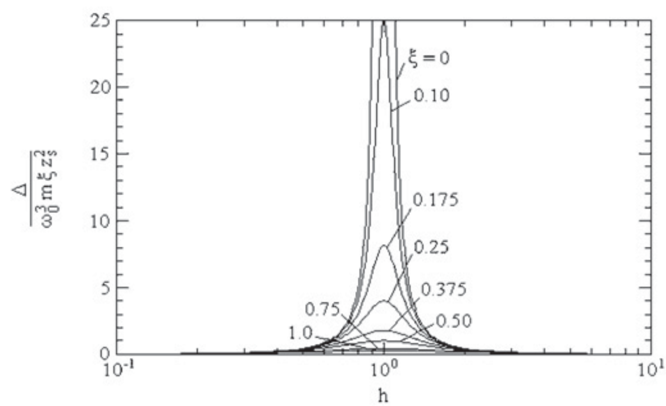


Figure 6.11. *Energy dissipated by damping*

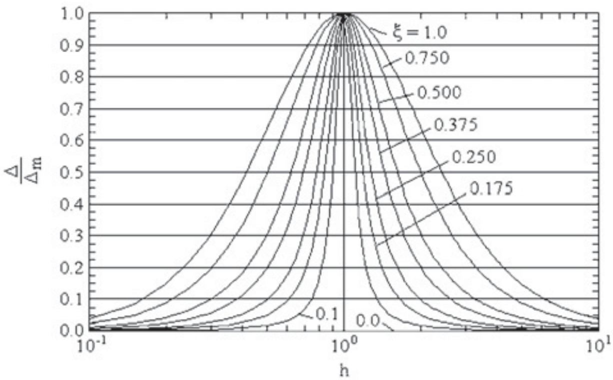


Figure 6.12. *Reduced energy dissipated by damping*

The integral is equal to $\frac{\pi}{4\,\xi}$ (Volume 3), yielding

$$S = \pi\ m\ \omega_0^4\ z_s^2 = \pi\ \omega_0^2\ k\ z_s^2 \qquad [6.95]$$

The surface S is thus quite independent of ξ. Therefore

$$\frac{S}{\Delta_m} = \pi\,\xi\,\omega_0 \qquad [6.96]$$

6.4.2.4. *Half-power points*

The *half-power points* are defined by the values of h such that the energy dissipated per unit time is equal to $\frac{\Delta_m}{2}$ yielding

$$\frac{1}{2}\ \frac{\omega_0^2\ m\ z_s^2}{4\,\xi} = \xi\,\omega_0^3\ m\ z_s^2\ \frac{h^2}{\left(1-h^2\right)^2 + 4\,\xi^2\ h^2}$$

$$\left(1-h^2\right)^2 + 4\,\xi^2\ h^2 = 8\,\xi^2\ h^2$$

$$\frac{h^2 - 1}{2\,\xi\,h} = \pm 1$$

i.e., since h and ξ are positive,

$$\begin{cases} h_1 = -\xi + \sqrt{1+\xi^2} \\ h_2 = -\xi + \sqrt{1+\xi^2} \end{cases} \qquad [6.97]$$

A logarithmic scale is sometimes used to represent the transmissibility, and a unit, the *Bel*, is introduced, or generally in practice, a subunit, the *decibel.* It is said that a power P_1 is higher by n decibels (dB) than a power P_0 if

$$10 \log \frac{P_1}{P_0} = n \qquad [6.98]$$

If $P_1 > P_0$, the system has a gain of n dB. If $P_1 < P_0$, the system produces an attenuation of n dB [GUI 63]. If instead of the powers, forces or velocities are considered here, the definition of the gain (or attenuation which is merely a negative gain) remains identical with the condition of replacing the constant 10 by a factor of 20 $(\log\ P = 2\ \log V_e + \text{Constant})$, since the power is proportional to the square of the rms velocity [LAL 95a].

The curve $2\,\xi\,H_{RV}$ or H_{RV} is close to a horizontal line for small values of ξ, passes through a maximum for $h = 1$, then decreases and tends towards zero when h becomes large compared to 1. By analogy with a resonant electrical system, the mechanical system can be characterized by the interval (bandwidth*)* between two frequencies h_1 and h_2 selected in such a way that $2\,\xi\,H_{RV}$ is either equal to $\frac{1}{\sqrt{2}}$, or for h such that

$$H_{RV}(h) = \frac{Q}{\sqrt{2}} \qquad [6.99]$$

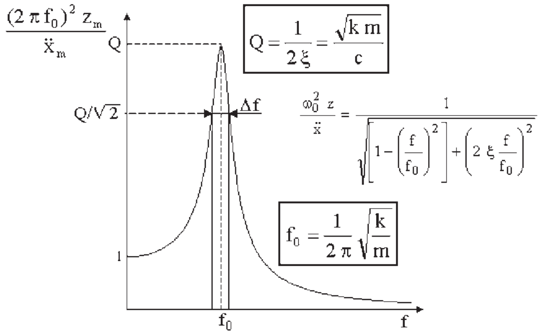


Figure 6.13. *The transfer function of a linear one-degree-of-freedom system*

Values h_1 and h_2 correspond to the abscissae of two points N_1 and N_2 named *half-power points* because the power which can be dissipated by the shock absorber during a simple harmonic movement at a given frequency is proportional to the square of the reduced amplitude H_{RV} [MEI 67].

6.4.2.5. *Bandwidth*

If $2\,\xi\,H_{RV} = \frac{1}{\sqrt{2}}$, $\sqrt{1+\tan^2\psi} = \sqrt{2}$, i.e. $\tan^2\psi = 1$, yielding

$$\frac{h^2-1}{2\,\xi\,h} = Q\left(h - \frac{1}{h}\right) = \pm 1 \qquad [6.100]$$

The quantity $Q\left(h-\frac{1}{h}\right)$ is the *dissonance.* It is zero with resonance and equivalent to $Q\,(h-1)$ in its vicinity [GUI 63]. The condition $\tan\psi = \pm 1$, is $\psi = \pm\frac{\pi}{4}$ (modulo π) which shows that ψ undergoes, when h varies from h_1 to h_2, a variation of $-\frac{\pi}{4}$ to $\frac{\pi}{4}$, i.e. of $\frac{\pi}{2}$.

h_1 and h_2 are calculated from [6.100]

$$\begin{cases} Q\left(h_2 - \dfrac{1}{h_2}\right) = 1 \\ Q\left(h_1 - \dfrac{1}{h_1}\right) = -1 \end{cases} \qquad [6.101]$$

This becomes

$$\begin{cases} \left(h_2 - h_1\right)\left(1 + \dfrac{1}{h_1\ h_1}\right) = \dfrac{2}{Q} \\ h_1\ h_2 = \dfrac{1}{Q} \end{cases} \qquad [6.102]$$

$$h_2 - h_1 = \frac{1}{Q} \qquad [6.103]$$

The *bandwidth* of the system $\Delta h = h_2 - h_1$ can also be written

$$\Omega_2 - \Omega_1 = \frac{\omega_0}{Q} \qquad [6.104]$$

This is all the narrower when Q is larger.

Selectivity

In the general case of a system having its largest response for a pulsation ω_m, the *selectivity* σ is defined by

$$\sigma = \frac{\omega_m}{\Delta\Omega}\ (= \frac{h_m}{\Delta h}) \qquad [6.105]$$

where $\Delta\Omega$ is the previously defined bandwidth. σ characterizes the function of the filter of the system, by its ability to allow through a single frequency, by eliminating near-frequencies (of $\Delta\Omega$). For a resonant system, $\omega_m = \omega_0$ and $\sigma = Q$.

In electricity, the interval $\Omega_2 - \Omega_1$ characterizes the selectivity of the system. The resonant system is a simple model of a filter where the selective transmissibility can make it possible to choose signals in the useful band (Ω_1, Ω_2) among other signals external to this band which are undesirable. The selectivity is improved as the peak becomes more acute. In mechanics, this property is used for protection against vibrations (filtering by choosing the frequency of resonance smaller than the frequency of the vibration).

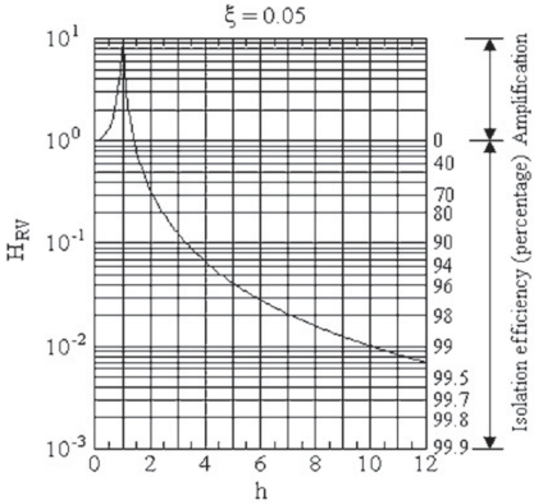


Figure 6.14. *Domains of the transfer function* H_{RV}

It can also be shown [LAL 95a], [LAL 95b] that the response of a one-degree-of-freedom system is primarily produced by the contents of the excitation in this frequency band.

From these relations, the expressions of h_1 and h_2 can be deduced:

$$h_1 = -\xi + \sqrt{1+\xi^2} \; (\approx 1-\xi \text{ if } \xi \text{ is small}) \qquad [6.106]$$

$$h_2 = +\xi + \sqrt{1+\xi^2} \; (\approx 1+\xi \text{ if } \xi \text{ is small}) \qquad [6.107]$$

The bandwidth $\Delta h = h_2 - h_1$ can also be written

$$\Delta h = 2\,\xi = \frac{1}{Q} \qquad [6.108]$$

yielding, since $h = \dfrac{\Omega}{\omega_0}$,

$$Q = \frac{\omega_0}{\Delta\Omega} = \frac{f_0}{\Delta f} \tag{6.109}$$

NOTE.– *The ratio* $\dfrac{\Omega - \omega_0}{\omega_0} = h - 1$ *is also sometimes considered. For the abscissae* Ω_1 *and* Ω_2 *of the half-power points, and for small* ξ*, this ratio is equal, respectively, to* $-\dfrac{1}{2\,Q}$ *and* $+\dfrac{1}{2\,Q}$.

The Q factor of mechanical systems does not exceed a few tens of units and those of electric circuits do not exceed a few hundred.

In [3.138] it was seen that

$$\delta \approx 2\,\pi\,\xi$$

yielding [GUR 59]:

$$\delta \approx \frac{\pi}{Q} \tag{6.110}$$

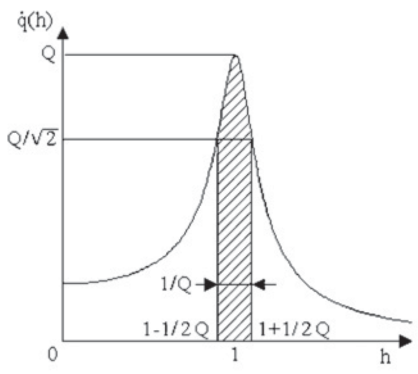


Figure 6.15. *Bandwidth*

The bandwidth can thus also be defined as the field of the frequencies transmitted with an attenuation of $10 \log 2 \approx 3.03$ dB below the maximum level (attenuation between the levels Q and $\frac{Q}{\sqrt{2}}$) [DEN 56], [THU 71].

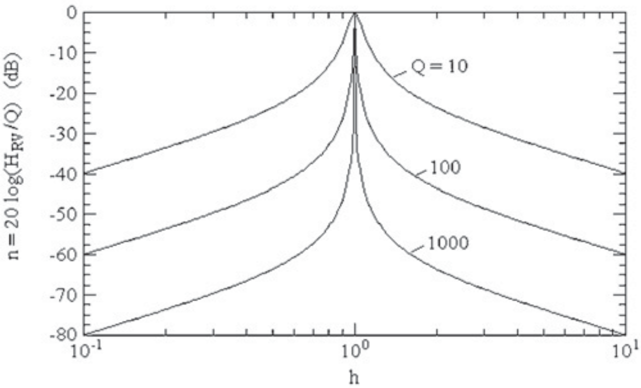


Figure 6.16. *Representation in dB of the transfer function* H_{RV}

Figure 6.16 represents some resonance curves, plotted versus variable h and for various values of the Q factor with the vertical axis being in dB [GUI 63].

6.4.3. *Variations in velocity phase*

In [6.77] it was seen that

$$\dot{q}(\theta) = H_{RV} \sin(h\,\theta - \psi)$$

where [6.79]

$$\psi = \varphi - \frac{\pi}{2}$$

yielding

$$\tan\psi = -\frac{1}{\tan\varphi} = \frac{h^2 - 1}{2\,\xi\,h} \qquad [6.111]$$

To obtain curves $\psi(h)$, it is therefore enough to shift by $\frac{\pi}{2}$ the already plotted curves $\varphi(h)$, while keeping ξ the same. The phase ψ varies from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$ since φ varies from 0 to π. It is zero for $h = 1$, i.e. when the frequency of the system is equal to that of the excitation (whatever value is taken by ξ).

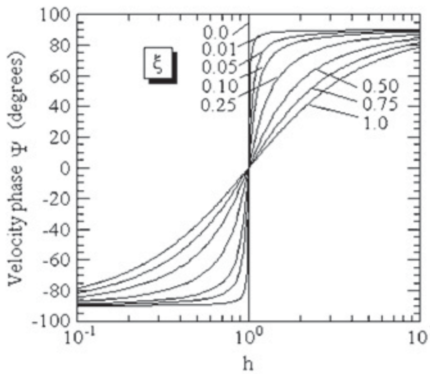


Figure 6.17. *Velocity phase versus h*

The velocity of the mass is thus always in phase with the excitation in this case.

When h is lower than 1, the velocity of the mass is in phase advance with respect to the excitation ($\psi < 0$, i.e. $-\psi > 0$). When h is larger than 1, the velocity of the mass has a phase lag with respect to excitation. In passing through resonance, the curve $\psi(h)$ presents a point of inflection. Around this point there is then a roughly linear variation of the phase varying with h (in an interval that is larger as ξ becomes smaller).

6.5. Responses $\frac{k\,z}{F_m}$ and $\frac{\omega_0^2\,z}{\ddot{x}_m}$

6.5.1. *Expression for response*

In these cases,

$$q(\theta) = H_{RD}(h)\ \sin(h\,\theta - \varphi) \tag{6.112}$$

The response $q(\theta)$ is at a maximum when $\sin(h\,\theta - \varphi) = 1$, i.e. for $h\,\theta - \varphi = (4\,k + 1)\,\dfrac{\pi}{2}$.

6.5.2. *Variation in response amplitude*

6.5.2.1. *Dynamic amplification factor*

Given that the excitation is a force applied to the mass, or an acceleration communicated to the support, the reduced response makes it possible to calculate the relative displacement z. The ratio H_{RD} between the amplitude of the relative displacement response and the equivalent static displacement ($\dfrac{F_m}{k}$ or $\dfrac{\ddot{x}_m}{\omega_0^2}$) is often called the *dynamic amplification factor*.

NOTE.– *Some authors [RUZ 71] call the amplification factor of the quantities* $\dfrac{k\,z}{F_m}$, $\dfrac{\sqrt{k\,m}\,\dot{z}}{F_m}$ *or* $\dfrac{m\,\ddot{z}}{F_m}$ *(amplification factor of the displacement, of the velocity and of the acceleration respectively) and relative transmissibility* $\dfrac{\ddot{z}}{\ddot{x}_m}$, $\dfrac{\dot{z}}{\dot{x}_m}$ *or* $\dfrac{z}{x_m}$ *(acceleration, velocity or displacement).*

The function $H_{RD}(h)$ depends on parameter ξ. This is always a positive function which passes through a maximum when the denominator passes through a minimum. The derivative of $\left(1 - h^2\right)^2 + 4\,\xi^2\,h^2$ is canceled when

$$h_m = \sqrt{1 - 2\,\xi^2} \qquad [6.113]$$

$(h \ge 0)$, provided that $1 - 2\xi^2 \ge 0$, i.e. $\xi \le \dfrac{1}{\sqrt{2}}$. When h tends towards zero, $H_{RD}(h)$ tends towards 1 whatever the value of ξ. There is resonance for $h = h_m$, the function $H_{RD}(h)$ is maximum and is then equal to

$$H_m = \frac{1}{2\,\xi\,\sqrt{1 - \xi^2}} \qquad [6.114]$$

When $h \to \infty$, $H_{RD}(h) \to 0$. In addition, $H_m \to \infty$ when $\xi \to 0$. In this case, $h_m = 1$. Resonance is all the more acute since the relative damping ξ is smaller; the damping has two effects: it lowers the maximum and makes the peak less acute.

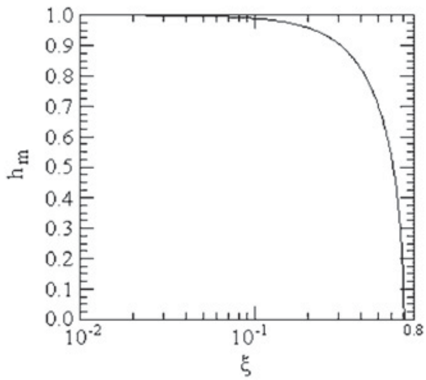


Figure 6.18. *Frequency of the maximum of H_{RD} versus ξ*

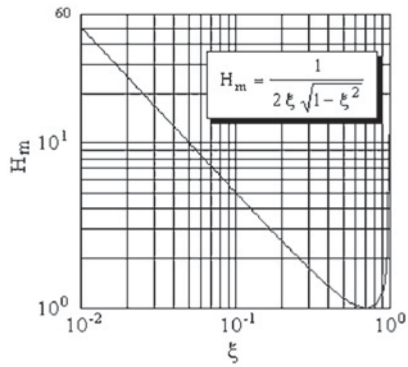


Figure 6.19. *Maximum of H_{RD} versus ξ*

It can be interesting to chart H_m versus h; it can be seen that the calculation of ξ versus h_m from [6.113] gives $\xi = \sqrt{\dfrac{1 - h_m^2}{2}}$.

This yields:

$$H_m = \frac{1}{\sqrt{1 - h_m^4}} \qquad [6.115]$$

where h_m can only be positive. Here interest will focus on the branch of the curve belonging to the interval $0 \le h \le 1$.

There can be a maximum only for $h \le 1$ (i.e. for a frequency of the excitation lower than that of the resonator ω_0); the condition $\xi \le \dfrac{1}{\sqrt{2}}$ being assumed to be realized.

If $h = 1$ is not a condition of resonance, then there is resonance only if at the same time $\xi = 0$. Otherwise, resonance takes place when $h < 1$.

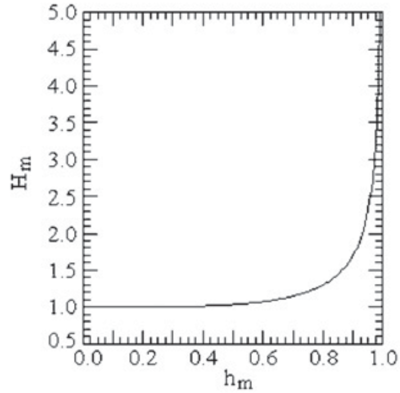


Figure 6.20. *Maximum of H_{RD} versus the peak frequency*

It can be seen that the condition $\xi = 1$ corresponds to the *critical modes.*

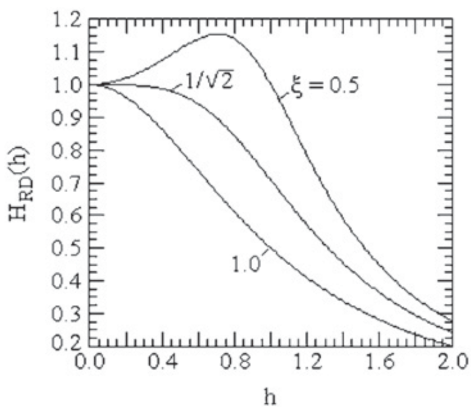


Figure 6.21. *Dynamic amplification factor around the critical mode*

Like all the curves of $H(h)$, the one corresponding to $\xi = \dfrac{1}{\sqrt{2}}$, which separates the domains of the curves with or without a maximum, has a horizontal level in the vicinity of the vertical axis ($h = 0$).

$\xi = \dfrac{1}{\sqrt{2}}$ gives optimum damping. It is for this value that H_m varies less versus h (an interesting property in electro-acoustics).

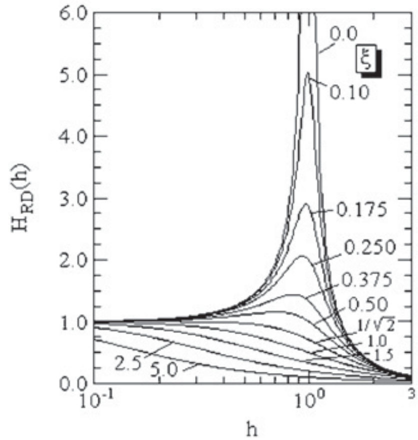


Figure 6.22. *Dynamic amplification factor for various values of ξ*

It can also be seen that when $\xi = \dfrac{1}{\sqrt{2}}$, the first three derivatives from H_m are zero for $h = 0$.

Finally it should be noted that this $\xi = \dfrac{1}{\sqrt{2}}$ value is lower than that for critical damping ($\xi = 1$). It could be thought that the existence of the transient state ($\xi < 1$) does not disturb the response, but in practice it has little influence. Setting δ, the logarithmic decrement, it was shown that

$$\delta = \frac{2\,\pi\,\xi}{\sqrt{1-\xi^2}}$$

For $\xi = \dfrac{1}{\sqrt{2}}$, thus $\delta = 2\,\pi$. This is an enormous damping: the ratio of two successive maximum displacements is then equal to $e^{\delta} = e^{2\,\pi} \approx 560$. The transient state disappears very quickly and is negligible as of the second oscillation.

6.5.2.2. *Width of $H(h)$ for $H_{RD} = \frac{H_{RD\,max}}{\sqrt{2}}$*

By analogy with the definition of the half-power points given for H_{RV} in section 6.4.2.4, we can calculate the width Δh of the peak of H_{RD} for the ordinate $H_{RD} = \frac{H_{RD_{max}}}{\sqrt{2}}$. It has been seen that $H_{RD_{max}} = \frac{Q}{\sqrt{1-\xi^2}}$ yielding

$$H_{RD} \equiv \frac{1}{\sqrt{\left(1-h^2\right)^2 + \frac{h^2}{Q^2}}} = \frac{Q}{\sqrt{2}\,\sqrt{1-\xi^2}} \qquad [6.116]$$

and

$$h^2 = 1 - \frac{1}{2\,Q^2} \pm \frac{1}{Q}\sqrt{1-\frac{1}{4\,Q^2}} \quad \left(Q \geq \frac{1}{2}\text{, i.e. } \xi \leq 1\right)$$

$$h^2 = 1 - 2\,\xi^2 \pm 2\,\xi\,\sqrt{1-\xi^2}$$

h^2 must be positive, which requires for the first root that $1 + 2\,\xi\sqrt{1-\xi^2} \geq 2\,\xi^2$. The other root leads to $2\,\xi^2 + 2\,\xi\,\sqrt{1-\xi^2} \leq 1$. Let us make h_1 and h_2 the two roots.

This gives

$$h_2^2 - h_1^2 = 1 - 2\,\xi^2 + 2\,\xi\,\sqrt{1-\xi^2} - 1 + 2\,\xi^2 + 2\,\xi\,\sqrt{1-\xi^2}$$

$$h_2^2 - h_1^2 = 4\,\xi\,\sqrt{1-\xi^2} \qquad [6.117]$$

If ξ is small, $h^2 \approx 1 \pm 2\xi$, $h \approx \sqrt{1 \pm 2\,\xi} \approx 1 \pm \xi$

$$h_2^2 - h_1^2 \approx 4\,\xi$$

$$h_2 - h_1 \approx 2\,\xi \text{ and } h_2 + h_1 \approx 2$$

Particular case

If ξ is small with respect to 1, we have, at first approximation,

$$h \approx \sqrt{1 \pm 2\,\xi} \qquad (h \geq 0)$$

$$h \approx 1 \pm \xi \qquad [6.118]$$

In the particular case where ξ is small, the abscissa of the points for which $H_{RD} = \dfrac{H_{RD_{max}}}{\sqrt{2}}$ is approximately equal to the abscissa of the half-power points (defined from H_{RV}). The bandwidth can be calculated from

$$\Delta h = h_2 - h_1 \qquad [6.119]$$

6.5.3. *Variations in phase*

The phase is given by

$$\tan\varphi = \frac{2\,\xi\,h}{1 - h^2} \qquad [6.120]$$

It should also be noted that:

– $\left|\tan\varphi\right|$ is unchanged when h is replaced by $\dfrac{1}{h}$;

– $\tan\varphi \to \infty$ when $h \to 1$, therefore $\varphi \to \dfrac{\pi}{2}$: the response is in quadrature advance with respect to the excitation;

– $\tan\varphi = 0$, i.e. $\varphi = 0$ when $h = 0$ (in the interval considered);

the derivatives below do not cancel

$$\frac{d\varphi}{dh} = \frac{2\,\xi\left(1 + h^2\right)}{\left(1 - h^2\right)^2 + 4\,\xi^2\,h^2} \qquad [6.121]$$

– $\tan \varphi \to 0$, i.e. $\varphi \to \pi$, when $h \to \infty$ (φ cannot tend towards zero since there is no maximum. The function which is canceled already when $h = 0$ cannot cancel a second time), the response and the excitation are in opposite phase;

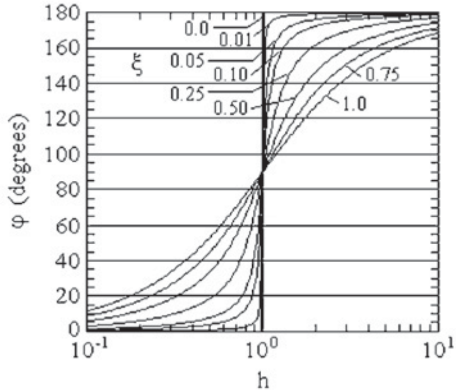


Figure 6.23. *Phase of response*

– for all values of ξ, φ is equal to $\frac{\pi}{2}$ when $h = 1$; all the curves thus pass through the point $h = 1$, $\varphi = \frac{\pi}{2}$;

– for $\xi < 1$, all the curves have a point of inflection in $h = 1$, $\varphi = \frac{\pi}{2}$. The slope at this point becomes greater as ξ becomes smaller.

Particular cases

– For $h = \sqrt{1 - 2\,\xi^2}$ (resonance) and $\xi \le \frac{1}{\sqrt{2}}$

$$\tan\varphi = \frac{2\,\xi\,h}{1 - h^2} = \frac{\sqrt{1 - 2\,\xi^2}}{\xi} \qquad [6.122]$$

$$\varphi = \text{arc}\ \tan \frac{\sqrt{1 - 2\,\xi^2}}{\xi} \qquad [6.123]$$

– When h is small, mass m practically has a movement in phase with the excitation ($\varphi \approx 0$). In this case, q_{max} being closer to 1 as h is smaller, the mass follows the movement of the support.

Values of angle φ ranging between 180° and 360° cannot exist because, in this case, the shock absorber would provide energy to the system instead of dissipating it [RUB 64].

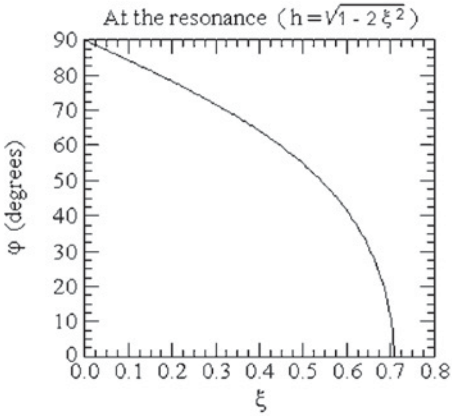


Figure 6.24. *Resonance phase*

Since $H_{RD} \approx 1$ for small values of h, for an excitation by force on the mass, $\frac{k\,z}{F_m} \approx 1$, i.e. $z \approx \frac{F_m}{k}$. The response is controlled in a dominating way by the stiffness of the system. In this domain, where h is small with respect to one, the calculations for the dimensioning of structure in statics can be carried out by taking the values of H_{RD} at the frequency of the vibration, in order to take account at the same time of the static load and of the small dynamic amplification. These calculations can possibly be supplemented by a fatigue analysis if this phenomenon is considered to be important [HAL 75].

– For h = 1, the maximum value of $q(\theta)$ is

$$q_{max} = \frac{1}{2\,\xi\,\sqrt{1-\xi^2}} \approx Q \qquad [6.124]$$

and the phase is

$$\varphi \to +\frac{\pi}{2} \qquad [6.125]$$

$$q(\theta) = \frac{\sin\left(h\,\theta + \frac{\pi}{2}\right)}{2\,\xi\,\sqrt{1-\xi^2}} \qquad [6.126]$$

$$q(\theta) = \frac{\cos(h\,\theta)}{2\,\xi\,\sqrt{1-\xi^2}} \qquad [6.127]$$

The amplitude of the response is a function of the damping ξ. It is larger if ξ is smaller. The movement is out of phase by $\frac{\pi}{2}$ with respect to the excitation.

If the excitation is a force, at the resonance, $H_{RD} = \frac{1}{2\,\xi\,\sqrt{1-\xi^2}}$, i.e.

$$z_m = \frac{-F_m}{2\,k\,\xi\,\sqrt{1-\xi^2}} \qquad [6.128]$$

$$z_m \approx \frac{-F_m}{2\,k\,\xi} = \frac{-F_m}{c\,\omega_0} \qquad [6.129]$$

Here, analysis must be of the dynamic type, the response being potentially several times the equivalent static excitation.

– For $h >> 1$,

$$q(\theta) \approx \frac{\sin(h\,\theta - \varphi)}{h^2} \qquad [6.130]$$

where $\varphi = -\pi$:

$$q(\theta) \approx -\frac{\sin(h\,\theta)}{h^2} \qquad [6.131]$$

If the excitation is a force, we have

$$H_{RD} \approx \frac{1}{h^2} \qquad [6.132]$$

i.e.

$$z_m \approx \frac{F_m}{k\ h^2} \qquad [6.133]$$

$$z_m \approx \frac{F_m}{m\ \Omega^2} \qquad [6.134]$$

where Ω = pulsation of the excitation.

The response is primarily a function of the mass m. It is smaller than the equivalent static excitation.

According to whether h satisfies one or the other of these three conditions, one of the three elements stiffness, damping or mass thus has a dominating effect on the resulting movement of the system [BLA 61], [RUB 64].

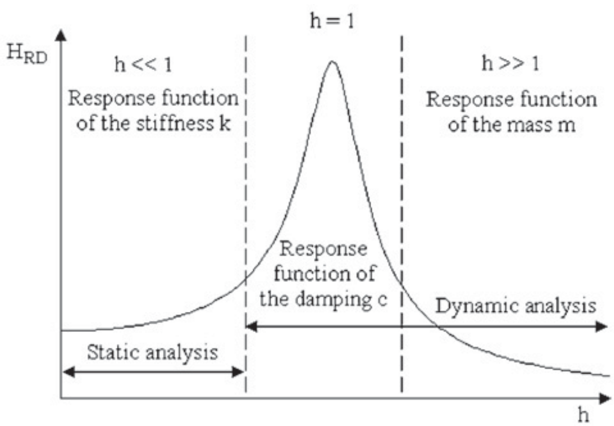


Figure 6.25. *Fields of dynamic amplification factor*

Particular case where $\xi = 0$

$$q(\theta) = \frac{\sin(h\,\theta - \varphi)}{1 - h^2} \qquad [6.135]$$

(The positive root for $h < 1$ is chosen in order to preserve at $q(\theta)$ the same sign for $\xi = 0$ rather than for ξ which is very small in expression [6.66].)

$$q_{max} = H_{RD} = \frac{1}{1 - h^2} \qquad [6.136]$$

The variations of q_{max} versus h are represented in Figure 6.26. It should be noted that, when h tends towards 1, q_{max} tends towards infinity. It is necessary here to return to the assumptions made, and to remember that the system is considered linear, which assumes that the amplitude of the variations of the response q remains small. This curve $q_{max}(h)$ thus does not make sense in the vicinity of the asymptote.

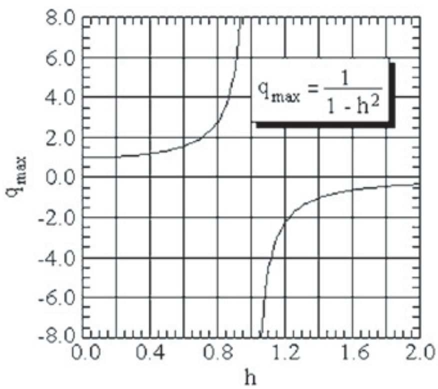


Figure 6.26. *Variations of q_{max} versus h*

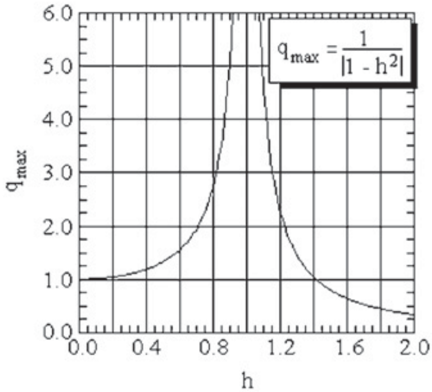


Figure 6.27. *Dynamic amplification factor for $\xi = 0$*

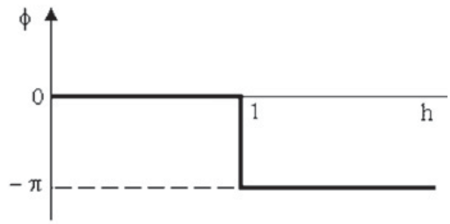


Figure 6.28. *Phase for $\xi = 0$*

The case where $\xi = 0$ is an ideal case: in practice, friction is never negligible in the vicinity of resonance (apart from resonance, it is sometimes neglected at first approximation to simplify the calculations).

As h varies, q_{max} changes sign while passing through infinity. To preserve the character of an always positive amplitude at the reduced amplitude (the temporal response being symmetrical with respect to the time axis), an abrupt phase shift of value π is introduced into the passage of $h = 1$.

The phase φ is zero in the interval $0 \le h \le 1$; it is then equal to $\pm\,\pi$ for $h > 1$ (the choice of the sign is unimportant). If the value $-\pi$ is taken in $(1, \infty)$, then for example, for $0 \le h \le 1$:

$$q_{max} = \frac{\sin(h\,\theta)}{1 - h^2} \qquad [6.137]$$

and, for $h > 1$:

$$q_{max} = \frac{\sin(h\,\theta - \pi)}{1 - h^2} \qquad [6.138]$$

Particular case where $\xi = 1$

Here

$$q(\theta) = \frac{h}{\left(1 + h^2\right)^2} \left[\frac{1 - h^2}{h} \sin(h\,\theta) - 2\cos(h\,\theta)\right] \quad [6.139]$$

or

$$q(\theta) = H_{RD}(h)\,\sin(h\,\theta - \varphi) \quad [6.140]$$

with

$$H_{RD}(h) = \frac{1}{1 + h^2} \quad [6.141]$$

and

$$\tan\varphi = \frac{2\,h}{1 - h^2} \quad [6.142]$$

NOTE.– *The resonance frequency, defined as the frequency for which the response is at a maximum, has the following values.*

Response	**Resonance frequency**	**Amplitude of the relative response**
Displacement	$h = \sqrt{1 - 2\,\xi^2}$	$\dfrac{1}{2\,\xi\,\sqrt{1 - \xi^2}}$
Velocity	$h = 1$	$\dfrac{1}{2\,\xi}$
Acceleration	$h = \dfrac{1}{\sqrt{1 - 2\,\xi^2}}$	$\dfrac{1}{2\,\xi\,\sqrt{1 - \xi^2}}$

Table 6.2. *Resonance frequency and maximum of the transfer function*

(The natural frequency of the system being equal to $h = \sqrt{1 - \xi^2}$. For the majority of real physical systems, ξ is small and the difference between these frequencies is negligible.)

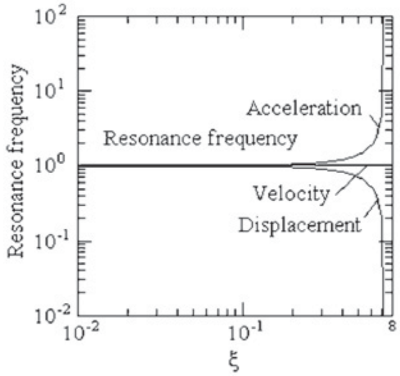


Figure 6.29. *Resonance frequency versus ξ*

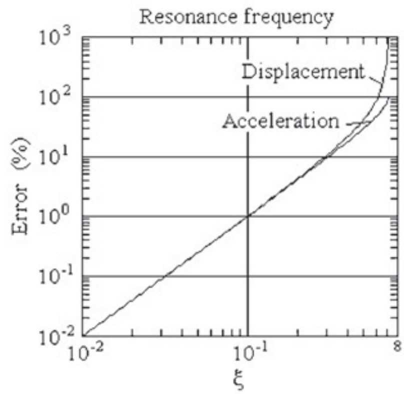


Figure 6.30. *Error made by always considering h = 1*

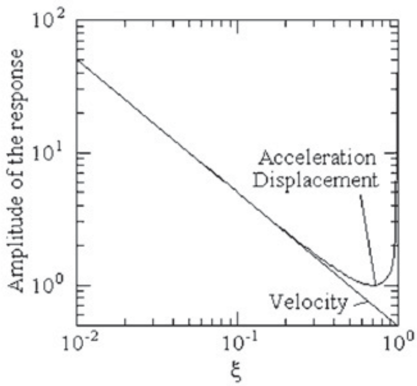


Figure 6.31. *Peak amplitude of the transfer function*

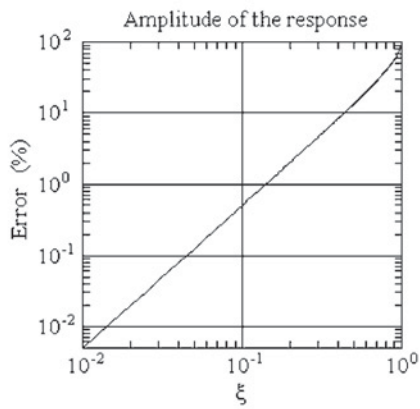


Figure 6.32. *Error made by always taking 1/2ξ*

6.6. Responses $\frac{y}{x_m}$, $\frac{\dot{y}}{\dot{x}_m}$, $\frac{\ddot{y}}{\ddot{x}_m}$ and $\frac{F_T}{F_m}$

6.6.1. *Movement transmissibility*

Here

$$q(\theta) = H_{AD} \sin(h\,\theta - \varphi) \tag{6.143}$$

The maximum amplitude of $q(\theta)$ obtained for $\sin(h\,\theta - \varphi) = 1$, occurring for $h\theta - \varphi = (4\ k + 1)\ \dfrac{\pi}{2}$, is equal to

$$H_{AD} = \sqrt{\frac{1 + 4\ \xi^2\ h^2}{\left(1 - h^2\right)^2 + 4\ \xi^2\ h^2}} \qquad [6.144]$$

If the excitation is an absolute displacement of the support, the response is the absolute displacement of mass m. The *movement transmissibility* is defined as the ratio of the amplitude of these two displacements:

$$T_m = \left|\frac{y_m}{x_m}\right| \qquad [6.145]$$

For certain applications, in particular in the case of calculations of vibration isolators or package cushioning, it is more useful to know the fraction of the force amplitude applied to m which is transmitted to the support through the system [BLA 61], [HAB 68]. Then a force transmission coefficient or *force transmissibility* T_f is defined by

$$T_f = \left|\frac{F_T}{F_m}\right| \qquad [6.146]$$

$T_f = T_m = H_{AD}$ is then obtained according to Table 6.1.

6.6.2. *Variations in amplitude*

The amplitude $H_{AD}(h)$ is at a maximum when $\dfrac{dH_{AD}(h)}{dh} = 0$, i.e. for h such that

$$\frac{dH_{AD}}{dh} = \frac{2\ h\left(1 - h^2 - 2\ \xi^2\ h^4\right)}{\sqrt{1 + 4\ \xi^2\ h^2}\left[\left(1 - h^2\right)^2 + 4\ \xi^2\ h^2\right]^{3/2}}$$

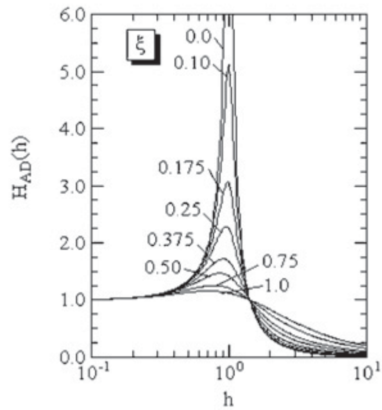


Figure 6.33. *Transmissibility*

This derivative is zero if $h = 0$ or if

$$1 - h^2 - 2\xi^2 h^4 = 0 \qquad [6.147]$$

i.e. for

$$h^2 = \frac{-1 + \sqrt{1 + 8\xi^2}}{4\xi^2}$$

or, since $h \geq 0$,

$$h = \frac{\sqrt{-1 + \sqrt{1 + 8\xi^2}}}{2\xi} \qquad [6.148]$$

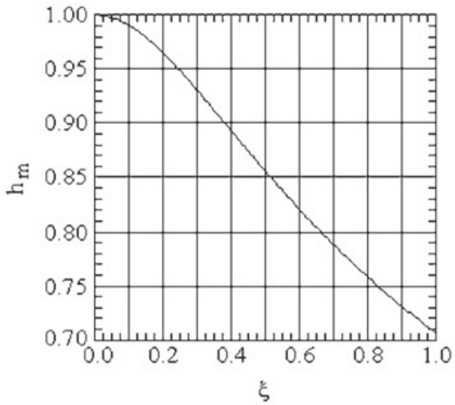


Figure 6.34. *Frequency of the maximum of transmissibility versus ξ*

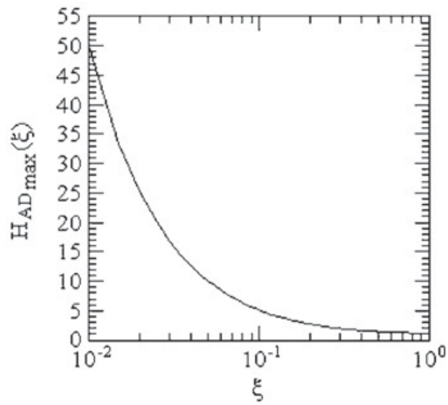


Figure 6.35. *Maximum of transmissibility versus ξ*

yielding

$$H_{AD_{max}} = \frac{4\,\xi^2}{\sqrt{16\,\xi^4 - 8\,\xi^2 - 2 + 2\sqrt{1 + 8\,\xi^2}}} \qquad [6.149]$$

When h tends towards zero, amplitude H_{AD} tends towards 1 (whatever the value of ξ). When $h \to \infty$, $H_{AD} \to 0$. From relation [6.147] is drawn

$$\xi^2 = \frac{1 - h^2}{2\,h^4} \qquad [6.150]$$

yielding $h \le 1$.

The locus of the maxima thus has as an equation

$$H_{AD} = \frac{1}{\sqrt{1 - h^4}} \qquad [6.151]$$

This gives the same law as that obtained for relative displacement.

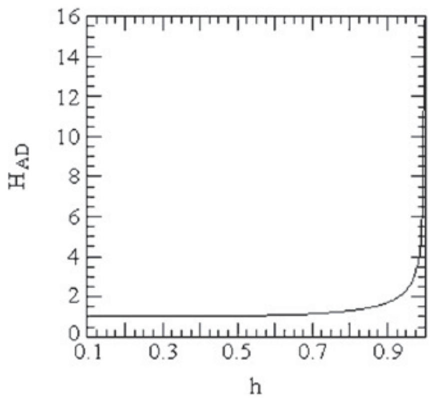


Figure 6.36. *Locus of maximum of transmissibility versus h*

Case of $\xi = 0$

With this assumption, $H_{AD} = H_{RD}$. For all values of ξ, all the curves $|H_{AD}(h)|$ pass through 1, for $h = 0$ and for $h = \sqrt{2}$. Indeed, $H_{AD}(h) = 1$ if $1 + 4\xi^2 h^2 = (1 - h^2)^2 + 4\xi^2 h^2$, i.e. $h^2 (h^2 - 2) = 0$ $(h \geq 0)$.

For $h < \sqrt{2}$, all the curves are above $H_{AD} = 1$. Indeed, the condition $1 + 4\xi^2 h^2 > (1 - h^2)^2 + 4\xi^2 h^2$ is carried out only if $1 > (1 - h^2)^2$, i.e. if $h < \sqrt{2}$.

In the same way, for $h > \sqrt{2}$, all the curves are below the straight line $H_{AD} = 1$.

6.6.3. *Variations in phase*

If

$$H_{AD}(h) = |H_{AD}(h)| \, e^{-j\,\phi(h)} \tag{6.152}$$

$$\tan \phi = \frac{2\,\xi\,h^3}{1 - h^2 + 4\,\xi^2\,h^2} \tag{6.153}$$

– $\tan\phi = 0$ when $\xi = 0$;

– $\tan\phi \to \infty$ if $\xi = 0$ and $h \to 1$ (thus $\phi \to \frac{\pi}{2}$);

– $\tan\phi = 0$ if $h = 0$, i.e. $\phi = 0$;

– $\tan\phi$ behaves like $-\frac{2\,\xi\,h}{1-4\,\xi^2}$ when $h \to \infty$.

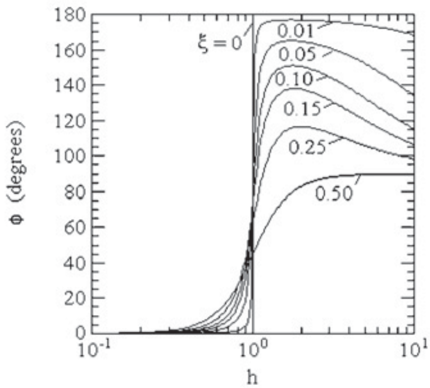


Figure 6.37. *Phase variations*

The denominator is zero if $1 - h^2 + 4\xi^2 h^2 = 0$, i.e. for $h^2 = \frac{1}{1-4\,\xi^2}$ ($\xi < 0.5$) or, since $h \geq 0$,

$$h = \frac{1}{\sqrt{1-4\,\xi^2}} \qquad [6.154]$$

In this case, $\tan\phi \to \infty$ and $\phi \to \frac{\pi}{2}$.

All the curves have, for $\xi < 1$, a point of inflection at $h = 1$. The slope at this point gets larger as ξ gets smaller.

For $\xi = 0.5$, $\tan\phi = h^3$ ($\phi \to \frac{\pi}{2}$ when $h \to \infty$).

For h = 1,

$$\tan \phi = \frac{1}{2\,\xi} \qquad [6.155]$$

ϕ then becomes smaller as ξ becomes larger.

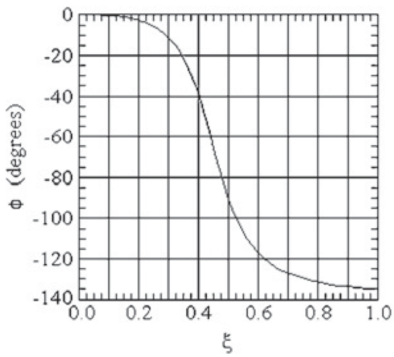


Figure 6.38. *Phase versus* ξ *for h = 1*

For $h = \dfrac{\sqrt{-1+\sqrt{1+8\,\xi^2}}}{2\,\xi}$,

$$\tan \phi = 2\,\xi\,\frac{\sqrt{1+8\,\xi^2}-1}{\left(2\,\xi-1\right)\sqrt{1+8\,\xi^2}+1} \qquad [6.156]$$

6.7. Graphical representation of transfer functions

The transfer functions can be plotted in a traditional way on linear or logarithmic axes, but also on a four-coordinate nomographic grid, which makes it possible to deduce the transfer functions of the displacements, the velocities and the accelerations directly. In this plane diagram, which has four inputs, the frequency is always carried on the abscissa.

Knowing that $H_{RV} = \Omega\, H_{RD}$ and that $H_{RA} = H_{RV}$, from the ordinate, the following can be read along the vertical axis:

– either the velocity (Figure 6.39). Accelerations are then located on an axis of negative slope (–45°) with respect to the axis of the velocities, while the amplitude of the displacements are on an axis at 45° with respect to the same vertical axis. Indeed (Figure 6.40):

$$\log H_{RA} = \log H_{RV} + \log f + \log 2\pi$$

However, a line at 45° with respect to the vertical axis,

$$O'K = O'J + JK = \left(\log H_{RV} + \log 2\pi\right)\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\log f$$

$$O'K = \frac{\sqrt{2}}{2}\left(\log H_{RV} + \log f + \log 2\pi\right) = \frac{\sqrt{2}}{2}\log H_{RA}$$

$O'K$ is thus proportional to $\log H_{RA}$;

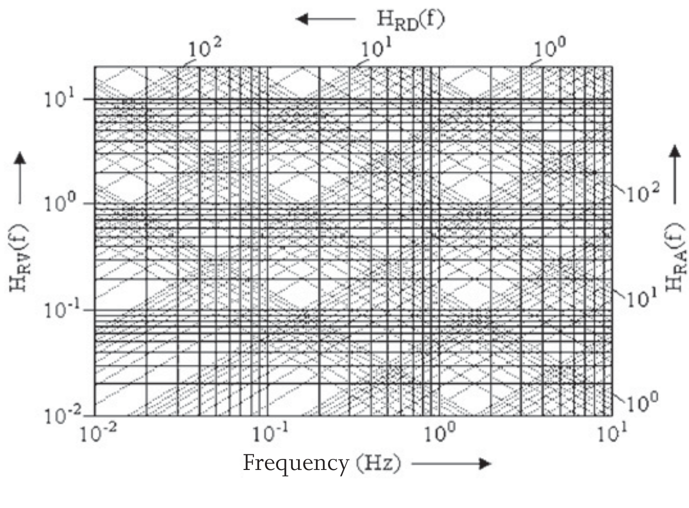


Figure 6.39. *Four-coordinate diagram*

– or the amplitude of the displacements. A similar calculation shows that the axis of the velocities forms an angle of + 45° with respect to the horizontal line and that of the accelerations an angle of 90° with respect to the axis of the velocities.

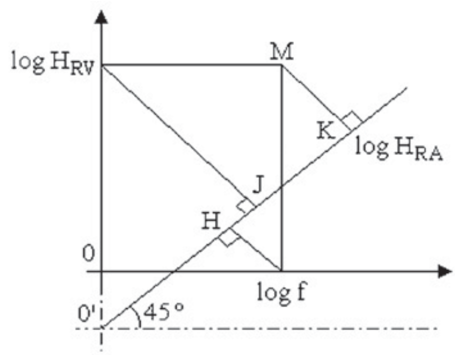


Figure 6.40. *Construction of the four input diagram*

6.8. Definitions

6.8.1. *Compliance – stiffness*

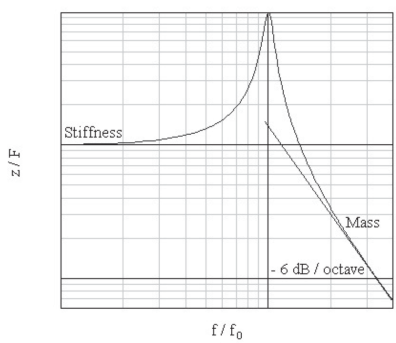


Figure 6.41. *Compliance*

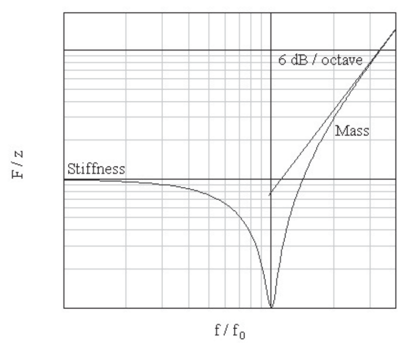


Figure 6.42. *Dynamic stiffness*

The complex transfer functions $\frac{z}{F}$ and $\frac{F}{z}$ are called *compliance* (or *receptance*) and *dynamic stiffness* respectively.

In the case of a one-degree-of-freedom system, these functions can be calculated from

$$H(\omega) = \frac{1}{\left(k - m\,\omega^2\right) + j\,c\,\omega} \qquad [6.157]$$

thus

$$H(f) = \frac{1}{k\left[\left(1 - \frac{f^2}{f_0^2}\right) + j\,2\,\xi\,\frac{f}{f_0}\right]} \qquad [6.158]$$

Figures 6.41 and 6.42 show module variations of $\frac{z}{F}$ and $\frac{F}{z}$ according to $\frac{f}{f_0}$. This curves are usually traced in the logarithmic axes. We can see the presence of 3 areas in each of which one of the parameters − stiffness, damping or mass − is of predominant importance. We can thus read on the asymptotes, at low frequency, the stiffness and at high frequency the mass.

6.8.2. *Mobility – impedance*

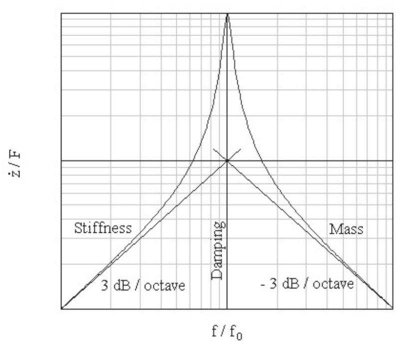


Figure 6.43. *Mobility*

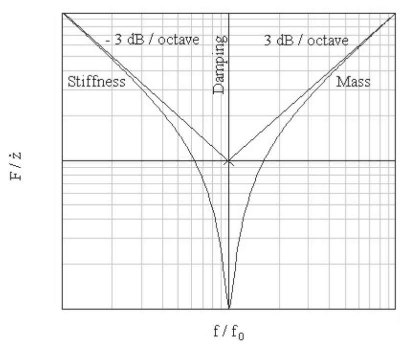


Figure 6.44. *Impedance*

In a similar way, *mobility* and *impedance* are the transfer functions $\frac{\dot{z}}{F}$ and $\frac{F}{\dot{z}}$ (Figures 6.43 and 6.44). They are calculated from the modulus of

$$H(\omega) = \frac{j\,\omega}{\left(k - m\,\omega^2\right) + j\,c\,\omega} \quad [6.159]$$

$$H(f) = \frac{j\,\dfrac{f}{f_0}}{\sqrt{k\,m}\left[\left(1 - \dfrac{f^2}{f_0^2}\right) + j\,2\,\xi\,\dfrac{f}{f_0}\right]} \quad [6.160]$$

6.8.3. *Inertance – mass*

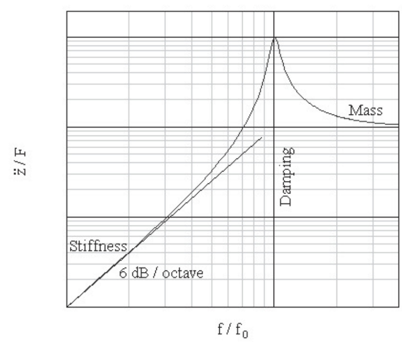


Figure 6.45. *Inertance*

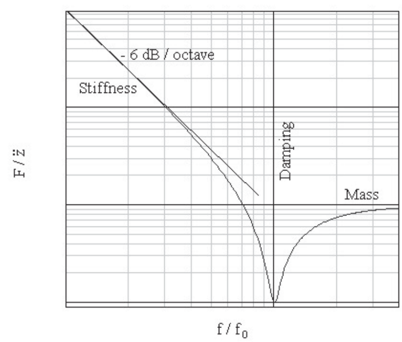


Figure 6.46. *Mass*

The *inertance* and *mass* transfer functions give variations with respect to acceleration over force $\frac{\ddot{z}}{F}$ and its inverse $\frac{F}{\ddot{z}}$ as a function of $\frac{f}{f_0}$ (or of f) (Figures 6.45 and 6.46), from

$$H(\omega) = \frac{-\omega^2}{\left(k - m\,\omega^2\right) + j\,c\,\omega} \quad [6.161]$$

$$H(f) = \frac{-\dfrac{f^2}{f_0^2}}{m\left[\left(1-\dfrac{f^2}{f_0^2}\right)+j\,2\,\xi\,\dfrac{f}{f_0}\right]} \qquad [6.162]$$

Chapter 7

Non-viscous Damping

7.1. Damping observed in real structures

In real structures, damping, which is not perfectly viscous, is actually a combination of several forms of damping. The equation of movement is as a consequence more complex, but the definition of damping ratio ξ remains $\frac{c}{c_c}$, where c_c is the critical damping of the mode of vibration considered. The exact calculation of ξ is impossible for several reasons [LEV 60]: probably insufficient knowledge of the exact mode of vibration, and of the effective mass of the system, the stiffnesses, the friction of the connections, the constant c and so on. It is therefore important to measure these parameters when possible.

In practice, non-linear damping can often be compared to one of the following categories, which will be considered in the following sections:

– damping force proportional to the power b of the relative velocity $\dot{z}$;

– constant damping force (Coulomb or dry damping), which corresponds to the case where $b = 0$;

– damping force proportional to the square of the velocity $(b = 2)$;

– damping force proportional to the square of the relative displacement;

– hysteretic damping, with force proportional to the relative velocity and inversely proportional to the excitation frequency.

Such damping produces a force which is opposed to the direction or the velocity of the movement.

7.2. Linearization of non-linear hysteresis loops – equivalent viscous damping

Generally, the differential equation of the movement can be written [DEN 56]:

$$m \frac{d^2 z}{dt^2} + f(z, \dot{z}) + k\, z = \begin{cases} F_m \sin \Omega\, t \\ -m\, \ddot{x}(t) \end{cases} \qquad [7.1]$$

with, for viscous damping, $f(z, \dot{z}) = c\, \dot{z}$. Because of the presence of this term, the movement is no longer harmonic in the general case and the equation of the movement is no longer linear. Such damping leads to non-linear equations which make calculations complex in a way seldom justified by the result obtained.

Except in some particular cases, such as Coulomb damping, there is no exact solution. The solution of the differential equation must be carried out numerically. The problem can sometimes be solved by using a development of the Fourier series of the damping force [LEV 60].

Damping is, fortunately, very often rather weak in practice, so the response can be approached using a sinusoid. This makes it possible to go back to a linear problem, which is easier to treat analytically, by replacing the term $f(z, \dot{z})$ by a force of viscous damping equivalent $c_{eq}\, \dot{z}$; by assuming that the movement response is sinusoidal, the *equivalent damping constant* c_{eq} of a system with viscous damping is calculated which would dissipate the same energy per cycle as non-linear damping.

The practice therefore consists of determining the nature and the amplitude of the dissipation of energy of the real damping device, then of replacing the mathematical models of the damping component by a viscous damping device having a dissipation of equivalent energy [CRE 65]. This is equivalent to saying that the hysteresis loop is modified.

In contrast to structures with viscous damping, non-linear structures have non-elliptic hysteresis loops $F_d(z)$ whose form approaches, for example, those shown in Figures 7.1 and 7.2 (dotted curve).

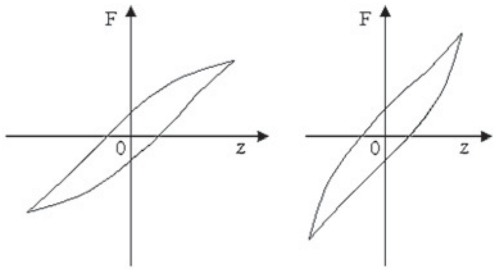


Figure 7.1. *Hysteresis loops of non-linear systems*

Linearization results in the transformation of the real hysteresis loop into an equivalent *ellipse* (Figure 7.2) [CAU 59], [CRE 65], [KAY 77], [LAZ 68].

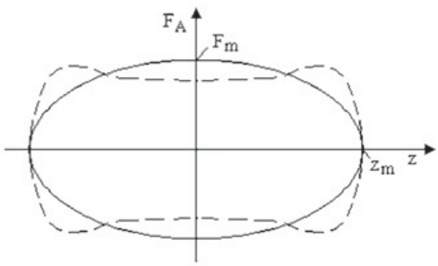


Figure 7.2. *Linearization of hysteresis loop*

Equivalence thus consists of seeking the characteristics of a viscous damping which include:

– the surface delimited by the cycle $F_d(z)$ (same energy dissipation);

– the amplitude of the displacement z_m.

The curve obtained is equivalent only for the selected criteria. For example, the remanent deformation and the coercive force are not exactly the same. Equivalence leads to results which are much better when the non-linearity of the system is lower.

This method, developed in 1930 by L. S. Jacobsen [JAC 30], is general in application and its author was able to show good correlation with the results calculated in a precise way when such calculations were possible (Coulomb damping [DEN 30a]) and with experimental results. This can, in addition, be extended to the case of systems with several degrees of freedom.

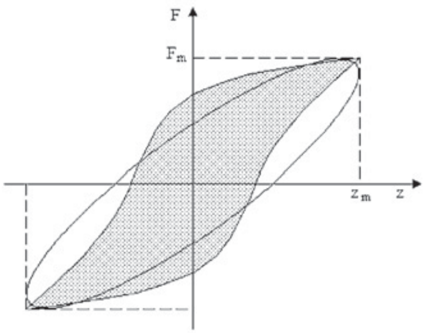


Figure 7.3. *Linearization of hysteresis loop*

If the response can be written in the form $z(t) = z_m \sin(\Omega t - \varphi)$, the energy dissipated per cycle can be calculated using

$$\Delta E_d = \int_{1\ \text{cycle}} F\ dz = \int_{1\ \text{cycle}} f(z,\ \dot{z})\ \frac{dz}{dt}\ dt \qquad [7.2]$$

$$\Delta E_d = z_m\ \Omega \int_0^{2\pi/\Omega} f(z,\ \dot{z})\ \cos(\Omega\ t - \phi)\ dt \qquad [7.3]$$

$$\Delta E_d = 4\ z_m\ \Omega \int_0^{\pi/2\Omega} f(z,\ \dot{z})\ \cos(\Omega\ t - \phi)\ dt \qquad [7.4]$$

Energy ΔE_d is equal to that dissipated by an equivalent viscous damping c_{eq} if [HAB 68]:

$$\Delta E_d = c_{eq}\ \Omega\ \pi\ z_m^2 = 4\ z_m\ \Omega \int_0^{\pi/2\Omega} f(z,\ \dot{z})\ \cos(\Omega\ t - \phi)\ dt \qquad [7.5]$$

i.e. if [BYE 67], [DEN 56], [LAZ 68], [THO 65a]:

$$c_{eq} = \frac{4}{\pi\ z_m} \int_0^{\pi/2\Omega} f(z,\ \dot{z})\ \cos(\Omega t - \phi)\ dt = \frac{\Delta E_d}{\Omega\ \pi\ z_m^2} \qquad [7.6]$$

The transfer function of a one-degree-of-freedom system $\frac{\omega_0^2 \, z_m}{\ddot{x}_m}$ (or in a more general way $\frac{z_m}{F_m / k} = \frac{z_m}{\ell_m}$) can be written while replacing c_{eq} by this value in the relation established, for viscous damping:

$$\frac{z_m}{\ell_m} = \frac{1}{\sqrt{\left(1 - h^2\right)^2 + \left(\frac{c_{eq} \, \Omega}{k}\right)^2}} \qquad [7.7]$$

(since $\frac{4 \, \xi^2}{\omega_0^2} = \frac{c^2}{k^2}$) and for the phase

$$\tan \, \phi = \frac{c_{eq} \, \Omega}{k \, \left(1 - h^2\right)} \qquad [7.8]$$

($h = \frac{\Omega}{\omega_0}$).

In addition, $c_{eq} \frac{\Omega}{k} = \frac{c_{eq}}{k} \frac{\Omega}{\omega_0} \omega_0 = \frac{c_{eq}}{\sqrt{k \, m}} h = 2 \, \xi_{eq} \, h$, yielding

$$\xi_{eq} = \frac{c_{eq} \, \Omega}{2 \, k \, h} = \frac{c_{eq} \, \omega_0}{2 \, k} \qquad [7.9]$$

$$\xi_{eq} = \frac{\Delta E_d}{2 \, \pi \, h \, k \, z_m^2} \qquad [7.10]$$

$$\frac{z_m}{\ell_m} = \frac{1}{\sqrt{\left(1 - h^2\right)^2 + \left(2 \, \xi_{eq} \, h\right)^2}} \qquad [7.11]$$

If ΔE_d is the energy dissipated by the cycle, the amplitude of the equivalent force applied is [CLO 03]:

$$F_m = \frac{\Delta E_d}{\pi \, z_m} \qquad [7.12]$$

7.3. Main types of damping

7.3.1. *Damping force proportional to the power b of the relative velocity*

Damping force	$F_d = \beta \lvert\dot{z}\rvert^b \dfrac{\dot{z}}{\lvert\dot{z}\rvert}$ or $F_d = \beta \lvert\dot{z}\rvert^b \operatorname{sgn}(\dot{z})$
Equation of the hysteresis loop	$\begin{cases} z = z_m \sin(\Omega t - \varphi) \\ F_d = \beta \left[\Omega z_m \cos(\Omega t - \varphi)\right]^b \operatorname{sgn}(\dot{z}) \end{cases}$ $\dfrac{F_d}{\beta \Omega^b z_m^b} = \operatorname{sgn}(\dot{z}) \left(1 - \dfrac{z^2}{z_m^2}\right)^{b/2}$
Energy dissipated by damping during a cycle	$\Delta E_d = \pi \beta \gamma_b \Omega^b z_m^{b+1}$ with $\gamma_b = \dfrac{2}{\sqrt{\pi}} \dfrac{\Gamma\left(1 + \dfrac{b}{2}\right)}{\Gamma\left(1 + \dfrac{b+1}{2}\right)}$
Equivalent viscous damping	$c_{eq} = \beta \gamma_b \Omega^{b-1} z_m^{b-1}$
Equivalent damping ratio	$\xi_{eq} = \dfrac{\beta z_m^{b-1} \gamma_b h^{b-1} \omega_0^b}{2 k}$
Amplitude of the response	obeys $z_m^{2b} + \dfrac{\left(1 - h^2\right)^2 \ell_m^{2(b-1)}}{\rho_b^2 h^{2b}} z_m^2 - \dfrac{\ell_m^{2b}}{\rho_b^2 h^{2b}} = 0$ where $\rho_b = \beta \gamma_b \omega_0^b k^{-1} \ell_m^{b-1}$
Phase of the response	$\tan \varphi = \dfrac{\rho_b h^b z_m^{b-1}}{\ell_m^{b-1} \left(1 - h^2\right)}$

Table 7.1. *Expressions for damping proportional to power b of relative velocity*

References in Table 7.1: [DEN 30b], [GAM 92], [HAB 68], [JAC 30], [JAC 58], [MOR 63a], [PLU 59], [VAN 57], [VAN 58].

Relation between b and parameter J – the B. J. Lazan expression

It has been shown [JAC 30], [LAL 96] that if the stress is proportional to the relative displacement z_m ($\sigma = K\, z_m$), coefficient J of the B.J. Lazan expression ($D = J\, \sigma^n$) is related to parameter b by

$$J = \frac{\pi\, \gamma_b\, \beta\, \omega_0^b}{K} \qquad [7.13]$$

J depends on parameters related to the dynamic behavior of the structure being considered (K and ω_0).

7.3.2. *Constant damping force*

If the damping force opposed to the movement is independent of displacement and velocity, the damping is known as *Coulomb* or *dry damping*. This damping is observed during friction between two surfaces (dry friction) applied one against the other with a normal force N (mechanical assemblies). It is [BAN 77], [BEA 96], [BYE 67], [NEL 80], [VOL 65]:

– a function of the materials in contact and of their surface quality;

– proportional to the force normal to the interface;

– mainly independent of the relative velocity of slipping between two surfaces;

– larger before the beginning of the relative movement than during the movement in steady state mode.

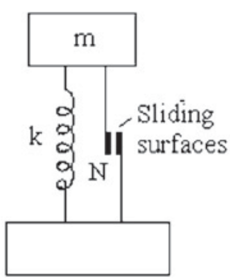


Figure 7.4. *One-degree-of-freedom system with dry friction*

The difference between the coefficients of static and dynamic friction is generally neglected, and force N is assumed to be constant and independent of the frequency and of the displacement.

A one-degree-of-freedom system damped by dry friction is represented in Figure 7.4.

Damping force	$F_d = \mu\ N\ \text{sgn}(\dot{z})$
Equation of the hysteresis loop	$F_d = \pm\mu\ N \quad (\lvert z\rvert \le z_m)$
Energy dissipated by damping during a cycle	$\Delta E_d = 4\ z_m\ \mu\ N$
Equivalent viscous damping	$c_{eq} = \dfrac{4\ \mu\ N}{\pi\ z_m\ \Omega}$
Equivalent damping ratio	$\xi_{eq} = \dfrac{2}{\pi}\ \dfrac{\mu\ N}{k\ h\ z_m}$
Amplitude of the response	$H = \dfrac{z_m}{\ell_m} = \dfrac{1}{\lvert 1-h^2\rvert}\sqrt{1-\rho_0^2} \quad \rho_0 = \dfrac{4}{\pi}\ \dfrac{\mu\ N}{k\ \ell_m} = \begin{cases} \dfrac{4\ \mu\ N}{\pi\ F_m} \\ -\dfrac{4\ \mu\ N}{\pi\ k\ \ddot{x}_m} \end{cases}$
Phase of the response	$\tan\varphi = \dfrac{\rho_0}{\sqrt{1-\rho_0^2}}$

Table 7.2. *Expressions for a constant damping force*

References in Table 7.2: [BEA 80], [CRE 61], [CRE 65], [DEN 29], [DEN 56], [EAR 72], [HAB 68], [JAC 30], [JAC 58], [LEV 60], [MOR 63b], [PAI 59], [PLU 59], [ROO 82], [RUZ 57], [RUZ 71], [UNG 73], [VAN 58].

The free response displacement of a one-degree-of-freedom system thus damped decreases following a linear law. The natural period remains constant. The oscillation frequency of the system damped using solid friction is the same as for the non-damped system. The stopping position can be different from the initial equilibrium position.

7.3.3. *Damping force proportional to the square of velocity*

A damping of this type is observed in the case of a body moving in a fluid (applications in fluid dynamics, the force of damping being of the form $C_x \ \rho \ A \ \frac{\dot{z}^2}{2}$) or during the turbulent flow of a fluid through an aperture (with high velocities of the fluid, from 2 to 200 m/s, resistance to the movement ceases to be linear with the velocity). When the movement becomes fast [BAN 77], the flow becomes turbulent and the resistance non-linear. Resistance varies with the square of the velocity [BAN 77], [BYE 67], [VOL 65].

Damping force	$F_d = \beta \, \dot{z} \, \lvert \dot{z} \rvert$ or $F_d = \beta \, \dot{z}^2 \, \operatorname{sgn}(\dot{z})$
Equation of the hysteresis loop	$\frac{z^2}{z_m^2} + \frac{F_d}{F_{dm}} = 1$
Energy dissipated by damping during a cycle	$\Delta E_d = \frac{8}{3} \beta \, \Omega^2 \, z_m^3$
Equivalent viscous damping	$c_{eq} = \frac{8 \beta \Omega z_m}{3 \pi}$
Equivalent damping ratio	$\xi_{eq} = \rho_2 \frac{h \, z_m}{2 \, \ell_m}$
Amplitude of the response	$z_m = \pm \frac{\ell_m \sqrt{-\left(1-h^2\right)^2 + \sqrt{\left(1-h^2\right)^4 + 4 \, \rho_2^2 \, h^4}}}{\sqrt{2} \, \rho_2 \, h^2}$ $\rho_2 = \beta \, \frac{8}{3 \pi} \, \frac{\omega_0^2}{k} \, \ell_m$
Phase of response	$\tan \varphi = \frac{\rho_2 \, h^2}{1 - h^2} \sqrt{\frac{2}{\sqrt{\left(1-h^2\right)^4 + 4 \, \rho_2^2 \, h^4} + \left(1-h^2\right)^2}}$

Table 7.3. *Expressions for quadratic damping*

References in Table 7.3: [CRE 65], [HAB 68], [JAC 30], [RUZ 71], [SNO 68], [UNG 73].

Constant β is termed the *quadratic damping coefficient.* It is characteristic of the geometry of the damping device and of the properties of the fluid [VOL 65].

7.3.4. *Damping force proportional to the square of displacement*

Damping force	$F_d = \gamma\, z^2 \dfrac{\dot{z}}{\lvert\dot{z}\rvert}$ or $F_d = \gamma\, z^2 \operatorname{sgn}(\dot{z})$
Equation of hysteresis loop	$\begin{cases} z(t) = z_m \sin(\Omega\, t - \varphi) \\ F_d(t) = \gamma\, z^2 \operatorname{sgn}(\dot{z}) = \gamma\, z_m^2 \sin^2(\Omega\, t - \varphi) \operatorname{sgn}(\dot{z}) \end{cases}$
Energy dissipated by damping during a cycle	$\Delta E_d = \pi\, \Omega\, c_{eq}\, z_m^2 = \dfrac{4}{3}\gamma\, z_m^3$
Equivalent viscous damping	$c_{eq} = \dfrac{4\, \gamma\, z_m}{3\, \pi\, \Omega}$
Equivalent damping ratio	$\xi_{eq} = \dfrac{2\, \gamma\, z_m}{3\, \pi\, k\, h}$ $\xi_{eq} = \dfrac{4\, \gamma}{3\, \pi} \dfrac{\ell_m}{k} \dfrac{z_m}{2\, \ell_m\, h} = \dfrac{\theta\, z_m}{2\, \ell_m\, h}$
Amplitude of response	$z_m^2 = \dfrac{-\left(1-h^2\right)^2 \left(\dfrac{3\,\pi}{4}\dfrac{k}{\gamma}\right)^2 + \sqrt{\left(\dfrac{3\,\pi}{4}\dfrac{k}{\gamma}\right)^4 \left(1-h^2\right)^4 + 4\, \ell_m^2 \left(\dfrac{3\,\pi}{4}\dfrac{k}{\gamma}\right)^2}}{2}$
Phase of response	$\tan\varphi = \dfrac{\theta}{1-h^2} \dfrac{\sqrt{2}}{\sqrt{\left(1-h^2\right)^2 + \sqrt{\left(1-h^2\right)^4 + 4\,\theta^2}}}$ $\beta = \dfrac{4\,\gamma}{3\,\pi\, k}$ $\theta = \beta\, \ell_m$

Table 7.4. *Expressions for damping force proportional to the square of the displacement*

Such damping is representative of the internal damping of materials, of the structural connections, and cases where the specific energy of damping can be expressed as a function of the level of stress, independent of the form and distribution of the stresses and volume of the material [BAN 77], [BYE 67], [KIM 26], [KIM 27].

7.3.5. *Structural or hysteretic damping*

	Damping coefficient function of Ω	Damping force proportional to the displacement	Complex stiffness
Damping force	$F_d = \frac{a}{\Omega}\dot{z}$	$F_d = d\left\lvert\frac{z}{\dot{z}}\right\rvert\dot{z} = d\lvert z\rvert \operatorname{sgn}(\dot{z})$	$F = k^* z = (k + i\ a)\ z$ or $F = k^* z = k\,(1 + i\,\eta)\,z$
Equation of the hysteresis loop	$\frac{z^2}{z_m^2} + \frac{F_d^2}{a^2 z_m^2} = 1$	$\frac{z^2}{z_m^2} + \frac{\pi^2 F_d^2}{4\,d^2\,z_m^2} = 1$	$\lvert F^*\rvert = k\ z \pm a\sqrt{z_m^2 - z^2}$
Energy dissipated by damping during a cycle	$\Delta E_d = \pi\,a\,z_m^2$	$\Delta E_d = 2\,d\,z_m^2$	$\Delta E_d = \pi\,k\,\eta\,z_m^2$ $(= \pi\ a\ z_m^2)$
Equivalent viscous damping	$c_{eq} = \frac{a}{\Omega}$	$c_{eq} = \frac{2\,d}{\pi\,\Omega}$	$c_{eq} = \frac{k\eta}{\Omega}\quad (= \frac{a}{\Omega})$
Equivalent damping ratio	$\xi_{eq} = \frac{a}{2\,m\,\omega_0^2}$	$\xi_{eq} = \frac{d}{\pi\,m\,\omega_0^2}$	$\xi_{eq} = \frac{a}{2\,k\,h} = \frac{\eta}{2\,h}$
Amplitude of response	$z_m = \frac{F_m}{k\sqrt{\left(1 - h^2\right)^2 + \frac{a^2}{k^2}}}$		
Phase of response	$\varphi = \arctan\frac{2\,a/k}{1 - h^2}$		

Table 7.5. *Expressions for structural damping*

This kind of damping is observed when the elastic material is imperfect, when the dissipation of energy is mainly obtained by deformation of material and slip, or friction in the connections of a system. Under a cyclic load, the curve σ, ε of the material forms a closed hysteresis loop rather than a single line [BAN 77]. The dissipation of energy per cycle is proportional to the surface enclosed by the hysteresis loop. This type of mechanism is observable when repeated stresses are applied to an elastic body, causing a rise in temperature of the material.

This is called internal friction, hysteretic damping, structural damping or displacement damping. Various formulations are used [BER 76], [BER 73], [BIR 77], [BIS 55], [CLO 03], [GAN 85], [GUR 59], [HAY 72], [HOB 76], [JEN 59], [KIM 27], [LAL 75], [LAL 80], [LAZ 50], [LAZ 53], [LAZ 68], [MEI 67], [MOR 63a], [MYK 52], [PLU 59], [REE 67], [REI 56], [RUZ 71], [SCA 63], [SOR 49], [WEG 35].

7.3.6. *Combination of several types of damping*

If several types of damping, as is often the case, are simultaneously present together with a linear stiffness [BEN 62], [DEN 30a], equivalent viscous damping can be obtained by calculating the energy ΔE_{d_i} dissipated by each damping device and by computing c_{eq} [JAC 30], [JAC 58]:

$$c_{eq} = \frac{\sum_i \Delta E_{d_i}}{\pi\, \Omega\, z_m^2} \qquad [7.14]$$

Example 7.1.

Viscous damping and Coulomb damping [JAC 30], [JAC 58], [LEV 60], [RUZ 71]

$$z = z_m \sin(\Omega\, t - \varphi)$$

$$z_m = \frac{\left\{F_m^2 \left[c^2\, \Omega^2 + \left(k - m\, \Omega^2\right)^2\right] - \frac{16}{\pi^2}\, F^2 \left(k - m\, \Omega^2\right)^2\right\}^{1/2} - \frac{4}{\pi}\, c\, F\, \Omega}{c^2\, \Omega^2 + \left(k - m\, \Omega^2\right)^2} \qquad [7.15]$$

$$\tan \varphi = \frac{\frac{4}{\pi} F \, z_m^{-1} \, \Omega^{-1} + c}{k - m \, \Omega^2} \, \Omega \qquad [7.16]$$

F_m = maximum $F(t)$ (excitation);
F = frictional force;
c = viscous damping ratio;
Ω = pulsation of the excitation

$$c_{eq} = \frac{4}{\pi} F \, z_m^{-1} \, \Omega^{-1} + c \qquad [7.17]$$

7.3.7. *Validity of simplification by equivalent viscous damping*

The cases considered above do not cover all the possibilities, but are representative of many situations.

The viscous approach supposes that although non-linear mechanisms of damping are present, their effect is relatively small. It is thus applicable if the term for viscous damping is selected to dissipate the same energy per cycle as the system with non-linear damping [BAN 77]. Equivalent viscous damping tends to underestimate the energy dissipated in the cycle and the amplitude of a steady state forced vibration: the real response can be larger than envisaged with this simplification.

The decrease of the transient vibration calculated for equivalent viscous damping takes a form different from that observed with Coulomb damping, with a damping force proportional to the square of the displacement or with structural damping. This difference should not be neglected if the duration of the decrease of the response is an important parameter in the problem being considered.

The damped natural frequency is itself different in the case of equivalent viscous damping and in the non-linear case. However, this difference is generally so small that it can be neglected.

When damping is sufficiently small (10%), the equivalent viscous damping method is a precise technique for the approximate solution of non-linear damping problems.

7.4. Measurement of damping of a system

All moving mechanical systems dissipate energy. This dissipation is often undesirable (in an engine, for example), but can be necessary in certain cases (vehicle suspension, isolation of a material to shocks and vibrations and so on).

Generally, mass and stiffness parameters can be calculated quite easily. It is much more difficult to evaluate damping by calculation because of ignorance of the phenomena concerned, and difficulties in modeling them. It is thus desirable to define this parameter experimentally.

The methods of measuring damping generally require the object being tested to be subjected to vibration and to measure the vibratory energy dissipated, or a parameter directly related to this energy. Damping is generally studied through the properties of the response of a one-degree-of-freedom mass–spring–damping system [BIR 77], [CLO 03], [PLU 59]. There are several possible methods for evaluating the damping of a system:

– amplitude of the response or amplification factor;

– quality factor;

– logarithmic decrement;

– equivalent viscous damping;

– complex modulus;

– bandwidth $\frac{\Delta f}{f}$.

7.4.1. *Measurement of amplification factor at resonance*

The damping of the one-degree-of-freedom system tends to reduce the amplitude of the response to a sine wave excitation. If the system were subjected to no external forces, the oscillations created in response to a short excitation would attenuate and disappear in some cycles. In order for the response to preserve a constant amplitude, the excitation must provide a quantity of energy equal to the energy dissipated by damping in the system.

The amplitude of the velocity response $\dot{z}$ is at a maximum when the frequency of the sine wave excitation is equal to the resonance frequency f_0 of the system. Since the response depends on the damping of the system and since the one-degree-of-freedom system is supposedly linear, this damping can be deduced from measurement of the amplitude of the response:

$$Q = \frac{\omega_0 \ \dot{z}_m}{\ddot{x}_m} \qquad [7.18]$$

or

$$Q = \frac{\sqrt{k \ m} \ \dot{z}}{F_m} \qquad [7.19]$$

For sufficiently small ξ, it has been seen that with a small error, the amplification factor, defined by $H_{RD} = \dfrac{\omega_0^2 \ z_m}{\ddot{x}_m}$, was equal to Q. The experimental determination of ξ can thus consist of plotting the curve H_{RD} or H_{RV} and of calculating ξ from the peak value of this function. If the amplitude of the excitation is constant, the sum of potential and kinetic energies is constant. The stored energy is thus equal to the maximum of one or the other; it will be, for example $U_s = \dfrac{1}{2} k \, z_m^2$. The energy dissipated during a cycle is equal to [6.87] $\Delta E_d = \pi \, c \, \Omega \, z_m^2$, yielding, since it is assumed that $\Omega = \omega_0$:

$$\frac{U_s}{\Delta E_d} = \frac{1}{2} \ \frac{k \ z_m^2}{\pi \ c \ \omega_0 \ z_m^2} = \frac{k}{2 \ \pi \ c \ \omega_0} = \frac{k \ Q \ \sqrt{m}}{2 \ \pi \ \sqrt{k \ m} \ \sqrt{k}} \qquad [7.20]$$

$$\frac{U_s}{\Delta E_d} = \frac{Q}{2 \ \pi} \qquad [7.21]$$

i.e.

$$Q = \frac{2 \ \pi \ U_s}{\Delta E_d} \qquad [7.22]$$

NOTE.– *The measurement of the response/excitation ratio depends on the configuration of the structure as much as the material. The system is therefore characterized by this rather than the basic properties of the material. This method is not applicable to non-linear systems, since the result is a function of the level of excitation.*

7.4.2. *Bandwidth or $\sqrt{2}$ method*

Another evaluation method (known as Kennedy–Pancu [KEN 47]) consists of measuring the bandwidth Δf between the half-power points relating to one peak of the transfer function [AER 62], with the height equal to the maximum of the curve H_{RD} (or H_{RV}) divided by $\sqrt{2}$ (Figure 7.5).

From the curve $H_{RV}(h)$, we will have, if h_1 and h_2 are the abscissae of the half-power points:

$$Q = \frac{1}{2\,\xi} = \frac{f_0}{\Delta f} \qquad [7.23]$$

where (f_0 = peak frequency, $h_1 = \frac{f_1}{f_0}$, $h_2 = \frac{f_2}{f_0}$)

and

$$\xi = \frac{c}{c_c} = \frac{\Delta f}{2\,f_0} \quad \left(= \frac{1}{2}\left(h_2 - h_1\right)\right) \qquad [7.24]$$

If T_0 is the natural period and T_1 and T_2 are the periods corresponding to an attenuation of $\frac{\sqrt{2}}{2}$, damping c is given by

$$c = 2\,\pi\,m\left(\frac{1}{T_2} - \frac{1}{T_1}\right) \qquad [7.25]$$

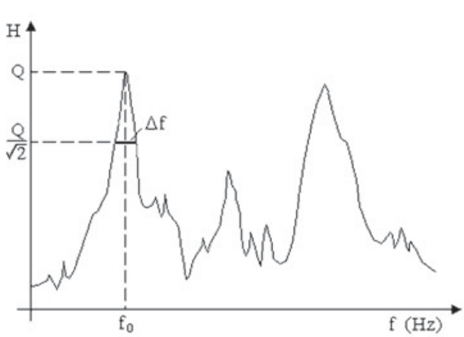


Figure 7.5. *Bandwidth associated with resonance*

since $c_c = 2\sqrt{km}$ and $k = m\,\omega_0^2$, and

$$\xi = \frac{T_0\left(T_2 - T_1\right)}{2\,T_1\,T_2} \qquad [7.26]$$

i.e. with the approximation $f_0 \approx \dfrac{f_1 + f_2}{2}$,

$$\xi \approx \frac{f_2 - f_1}{f_1 + f_2} \qquad [7.27]$$

From the curve H_{RD}, these relations are valid only if ξ is small. The curve H_{AD} could also be used for small ξ.

7.4.3. *Decreased rate method (logarithmic decrement)*

The precision of the bandwidth method is often limited by the non-linear behavior of the material or the reading of the curves. Sometimes it is better to use the traditional relation of logarithmic decrement, defined from the free response of the system after cessation of the exciting force (Figure 7.6).

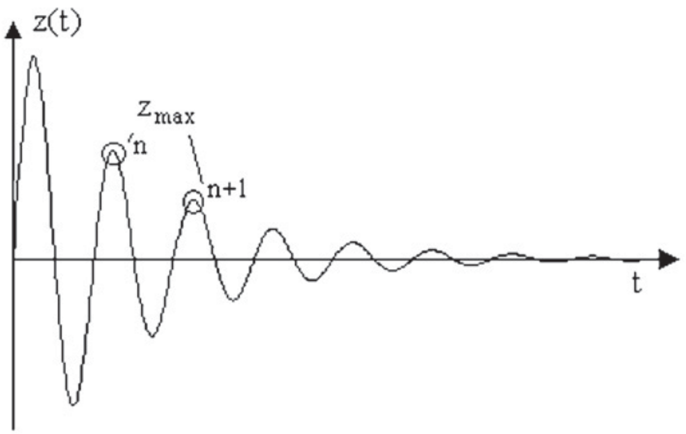


Figure 7.6. *Measurement of logarithmic decrement [BUR 59]*

The amplitude ratio of two successive peaks allows the calculation of the logarithmic decrement δ from

$$\frac{(z_m)_{n+1}}{(z_m)_n} = e^{-\delta} \qquad [7.28]$$

In addition, the existence of the following relation between this decrement and damping ratio is also shown

$$\delta = \frac{2\,\pi\,\xi}{\sqrt{1-\xi^2}} \qquad [7.29]$$

The measurement of the response of a one-degree-of-freedom system to an impulse load thus makes it possible to calculate δ or ξ from the peaks of the curve [FÖR 37] and [MAC 58]:

$$\xi = \frac{\delta}{\sqrt{\delta^2 + 4\,\pi^2}} \qquad [7.30]$$

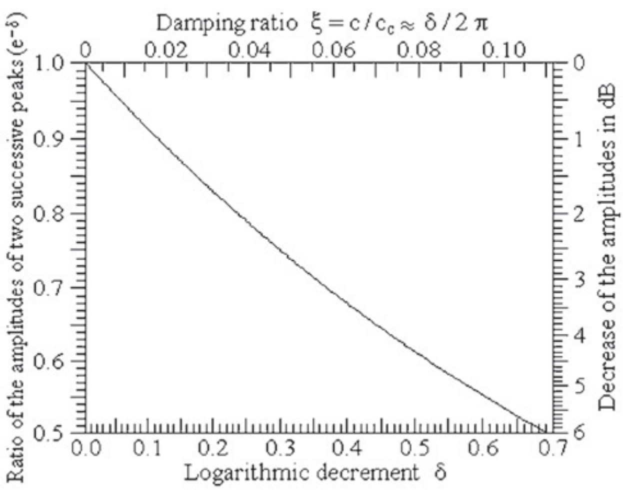


Figure 7.7. *Calculation of damping from δ*

The curve of Figure 7.7 can be used to determine ξ from δ. In order to improve the precision of the estimate of δ, it is preferable to consider two non-consecutive peaks. The definition then used is:

$$\delta = \frac{1}{n-1} \ln \frac{z_{m_1}}{z_{m_n}} \qquad [7.31]$$

where z_{m_1} and z_{m_n} are, respectively, the first and the n[th] peak of the response (Figure 7.8). In the particular case where ξ is much lower than 1, from [7.29] is obtained:

$$\xi \approx \frac{\delta}{2\,\pi}$$

yielding

$$\frac{\pi}{\delta} \approx Q$$

and

$$\frac{\pi}{\delta} \approx \frac{2\,\pi\,U_{ts}}{D} \qquad [7.32]$$

with

$$\delta = \ln \frac{z_{m_1}}{z_{m_2}} = \frac{1}{n} \ln \frac{z_{m_1}}{z_{m_{n+1}}} \qquad [7.33]$$

with ξ being small

$$\frac{z_{m_1}}{z_{m_{n+1}}} \approx 1 + n\,\delta = 1 + 2\,\pi\,n\,\xi_a \qquad [7.34]$$

yielding the approximate value ξ_a

$$\xi_a \approx \frac{z_1 - z_{m_{n+1}}}{2\,\pi\,n\,z_{m_{n+1}}} \qquad [7.35]$$

The error caused by using this approximate relation can be evaluated by plotting the curve $\frac{\xi_a - \xi}{\xi}$ according to ξ (Figure 7.8) or that giving the exact value of ξ according to the approximate value ξ_a (Figure 7.9). This gives

$$\xi_a = \frac{z_{m_1} - z_{m_2}}{2\,\pi\,z_{m_{m_2}}} = \frac{1}{2\,\pi}\left[\frac{z_{m_1}}{z_{m_2}} - 1\right] \qquad [7.36]$$

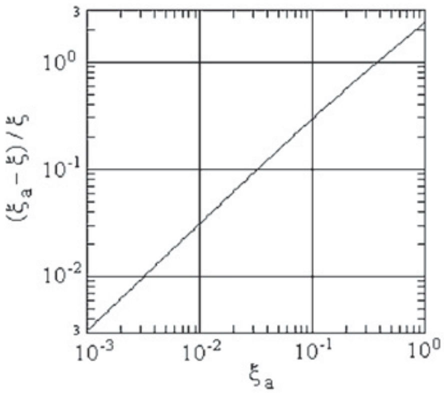


Figure 7.8. *Error related to the approximate relation ξ(δ)*

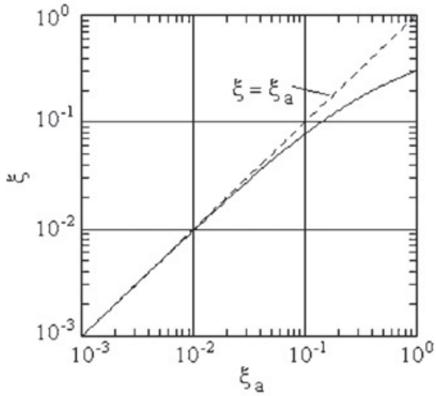


Figure 7.9. *Exact value of ξ versus approximate value ξ_a*

$$\xi = \frac{\delta}{\sqrt{\delta^2 + 4\,\pi^2}} = \frac{\ln \frac{z_{m_1}}{z_{m_2}}}{\sqrt{\ln^2 \frac{z_{m_1}}{z_{m_2}} + 4\,\pi^2}} \qquad [7.37]$$

yielding

$$\xi = \frac{\ln\left(1 + 2\,\pi\,\xi_a\right)}{\sqrt{\ln^2\left(1 + 2\,\pi\,\xi_a\right) + 4\,\pi^2}} \qquad [7.38]$$

and

$$\frac{\xi - \xi_a}{\xi} = 1 - \frac{\xi_a}{\xi} = 1 - \frac{1}{2\,\pi\,\xi}\left[e^{2\,\pi\,\xi / \sqrt{1-\xi^2}} - 1 \right] \qquad [7.39]$$

The specific damping capacity p, the ratio of the specific energy dissipated by damping to the elastic deformation energy per unit of volume, is thus equal to

$$p\,(\%) = 100\ \frac{D}{U_{ts}} \approx 200\ \delta \qquad [7.40]$$

In a more precise way, p can also be written

$$p = 100\,\frac{z_{m1}^2 - z_{mn+1}^2}{n\ z_{m1}^2} \qquad [7.41]$$

while assuming that U_{ts} is proportional to the square of the amplitude of the response. For a cylindrical test-bar,

$$W = S\ \ell\ U_{ts} = \frac{1}{2}\ k\ z^2 + \frac{1}{2}\ m\ \dot{z}^2 \qquad [7.42]$$

(potential energy + kinetic energy)

$$W = \frac{1}{2}\, m \left(z^2 + \Omega^2\ z^2 \right) \qquad [7.43]$$

i.e. since $z = z_m\ \sin(\Omega\ t - \varphi)$,

$$W = \frac{1}{2}\, m\, \Omega^2\, z_m^2 = \text{constant}\ z_m^2 \qquad [7.44]$$

U_{ts} is thus proportional to z_m^2 yielding, from [7.31] and [7.41] for two successive peaks:

$$p\,(\%) = 100 \left(1 - e^{-2\,\delta} \right) \qquad [7.45]$$

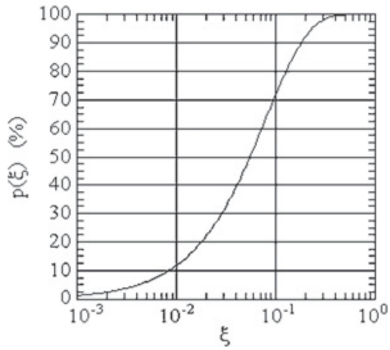


Figure 7.10. *Specific damping capacity versus ξ*

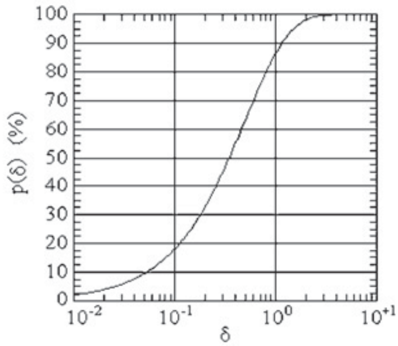


Figure 7.11. *Specific damping capacity versus δ*

The use of the decrement to calculate p from the experimental results assumes that δ is constant during n cycles. This is not always the case. It was seen that damping increases as a power of the stress, i.e. of the deformation, and it is thus desirable to use this method only for very low levels of stress.

For δ small, we can write [7.45] in the form of a series:

$$p(\%) = 100 \left[\frac{2\,\delta}{1!} - \frac{(2\,\delta)^2}{2!} + \frac{(2\,\delta)^3}{3!} - \cdots \right] \quad [7.46]$$

If $\delta < 0.01$, we find $p \approx 200\,\delta$.

The method of logarithmic decrement takes no account of non-linear effects. The logarithmic decrement δ can also be expressed according to the resonance peak amplitude H_{max} and its width Δf at an arbitrary height H [BIR 77], [PLU 59]. F. Förester [FÖR 37] showed that

$$\delta = \pi \, \frac{\Delta f}{f_0} \sqrt{\frac{H^2}{H_{max}^2 - H^2}} \quad [7.47]$$

$$\delta = \pi \, \frac{\Delta f}{f_0} \sqrt{\frac{H^2}{Q^2 - H^2}} \quad [7.48]$$

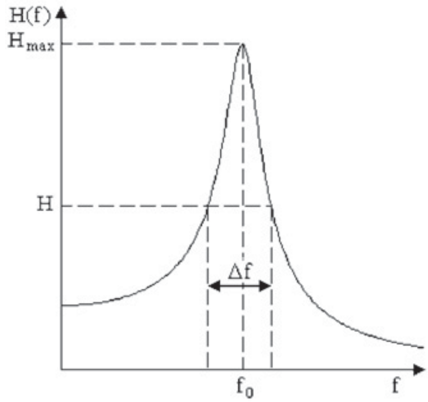


Figure 7.12. *Bandwidth at amplitude H*

If $H = \frac{Q}{2}$,

$$\delta = \frac{\pi}{\sqrt{3}} \frac{\Delta f}{f_0} \qquad [7.49]$$

Setting as n_e the number of cycles such that the amplitude decreases by a factor e (Neper's number), it becomes

$$\delta = -\frac{1}{(n_e - 1)} \ln \frac{1}{e} = \frac{1}{(n_e - 1)} = \frac{1}{f_0 \; t_e} \qquad [7.50]$$

where t_e= time to reach the amplitude $\frac{z_{m_1}}{e}$. If envelope $Z(t)$ of the response $z(t)$ (which is roughly a damped sinusoid) is considered, this gives

$$\delta = \frac{1}{f_0 \; Z} \frac{dZ}{dt} = -\frac{1}{f_0} \frac{d \ln Z}{dt} = -\frac{2.302}{f_0} \frac{d \ln Z}{dt} \qquad [7.51]$$

and if the amplitude in decibels is expressed as

$$y_{dB} = 20 \log Z$$

$$\delta = -\frac{0.115}{f_0}\frac{dy}{dt} \qquad [7.52]$$

For a value of H such that $H^2 = \frac{Q^2}{2}$,

$$\delta = \pi \frac{\Delta f}{f_0} \qquad [7.53]$$

If $\xi \le 0.1$, $\delta \approx \frac{\pi}{Q}$, yielding $Q = \frac{f_0}{\Delta f}$, a relation already obtained. The calculation of the Q factor from this result and from the curve $H(f)$ can lead to errors if the damping is not viscous.

In addition, it was assumed that the damping was viscous. If this assumption is not checked, different values of δ are obtained depending on the peaks chosen, particularly for peaks chosen at the beginning or end of the response [MAC 58].

Another difficulty can arise in the case of a several-degrees-of-freedom system for which it can be difficult to excite only one mode. If several modes are excited, the response of a combination of several sinusoids to various frequencies will be presented.

7.4.4. *Evaluation of energy dissipation under permanent sinusoidal vibration*

An alternative method can consist of subjecting the mechanical system to harmonic excitation and to evaluate, during a cycle, the energy dissipated in the damping device [CAP 82], this parameter being largely accepted as a measure of the damping.

This method can be applied to an oscillator whose spring is not perfectly elastic. This then leads to constants k and c of an equivalent simple oscillator.

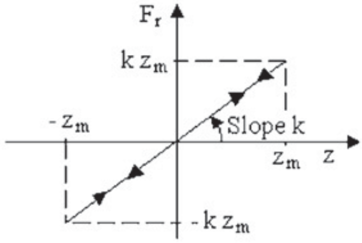


Figure 7.13. *Force in a spring*

It has been seen that, if a one-degree-of-freedom mechanical system is subjected to a sinusoidal force $F(t) = F_m \sin \Omega t$ such that the pulsation is equal to the natural pulsation of the system (ω_0), the displacement response is given by

$$z(t) = -z_m \cos \Omega t$$

where

$$z_m = \frac{F_m}{2 k \xi}$$

The force F_s in the spring is equal to $F_s = k z(t)$ and the force F_d in the damping device to $F_d = c \dot{z} = 2 m \xi \Omega \dot{z} = 2 k \xi z_m \sin \Omega t$, yielding F_d according to z:

$$\frac{F_d^2}{(2 k \xi z_m)^2} = \sin^2 \Omega t = 1 - \frac{z^2}{z_m^2} \qquad [7.54]$$

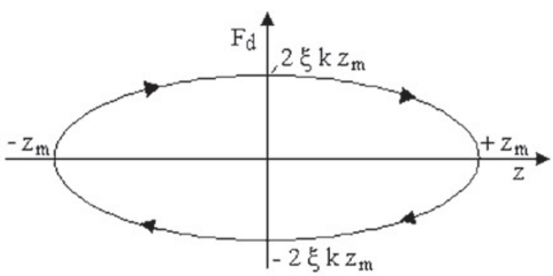


Figure 7.14. *Damping force versus displacement*

This function is represented by an ellipse. During a complete cycle the potential energy stored in the spring is entirely restored. On the other hand, energy ΔE_d is spent in the damping device, which is equal to the surface of the ellipse:

$$\Delta E_d = 2\,\pi\, z_m^2\, k\, \xi \qquad [7.55]$$

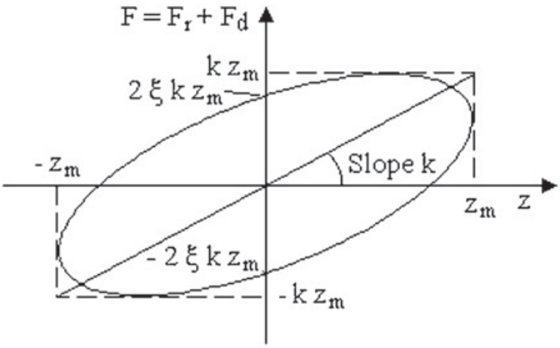


Figure 7.15. *Total force versus elongation*

The superposition of Figures 7.13 and 7.14 makes it possible to plot $F = F_s + F_d$ against z (Figure 7.15).

From these results, the damping constant c is measured as follows:

– by plotting the curve $F(z)$ after moving the system out of equilibrium (force F applied to the mass);

– by taking the maximum deformation z_m .

It is assumed here that stiffness k is linear and k is thus calculated from the slope of the straight line plotted at the centre of the ellipse (Figure 7.15).

The surface of the ellipse gives ΔE_d , yielding

$$\xi = \frac{\Delta E_d}{2\,\pi\, z_m^2\, k} \qquad [7.56]$$

NOTES.–

1. When z_m *increases, the spring has an increasingly non-linear behavior in general and the value of* ξ *obtained grows.*

2. The energy dissipated by the cycle (ΔE_d *) depends on the form, dimensions and the distribution of the dynamic stresses. It is preferable to consider the specific damping energy D, which is a basic characteristic of the material (damping energy per cycle and unit of volume by assuming a uniform distribution of the dynamic stresses in the volume V considered) [PLU 59].*

$$\Delta E_d = \int_V D \ dV \qquad [7.57]$$

where ΔE_d *is in Joules/cycle and D is in Joules/cycle/m*^{3}*.*

Some examples of different values of ξ are given in Table 7.6 [BLA 61] and [CAP 82].

Rubber-type materials with weak damping

The dynamic properties of Neoprene show a very weak dependence on the frequency. The damping ratio of Neoprene increases more slowly at high frequencies than the damping ratio of natural rubber [SNO 68].

Material	ξ
Welded metal frame	0.04
Bolted metal frame	0.07
Concrete	0.010
Pre-stressed concrete	0.05
Reinforced concrete	0.07
High-strength steel (springs)	$0.637\ 10^{-3}$ to $1.27\ 10^{-3}$
Mild steel	$3.18\ 10^{-3}$
Wood	$7.96\ 10^{-3}$ to $31.8\ 10^{-3}$
Natural rubber for damping devices	$1.59\ 10^{-3}$ to $12.7\ 10^{-3}$
Bolted steel	0.008
Welded steel	0.005

Table 7.6. *Examples of damping values*

Rubber-type materials with strong damping

The dynamic modulus of these materials increases very rapidly with the frequency. The damping ratio is large and can vary slightly with the frequency.

7.4.5. *Other methods*

Other methods have been developed to evaluate the damping of structures such as, for example, that using the derivative of the phase at the resonance with respect to the frequency (Kennedy–Pancu improved method) [BEN 71].

7.5. Non-linear stiffness

We considered in section 7.3 the influence of non-linear damping on the response of a one-degree-of-freedom system. The non-linearity was thus brought about by damping. Another possibility relates to the non-linearities due to stiffness. It can occur that the stiffness varies according to the relative displacement response. The restoring force, which has the form $F = -k\ z$, is no longer linear and can follow a law such as, for example, $F = k\ z + r\ z^3$ where k is the constant used before and where r determines the rate of non-linearity. The stiffness can increase with relative displacement (hardening spring) (Figure 7.16) or decrease (softening spring) (Figure 7.17) [MIN 45].

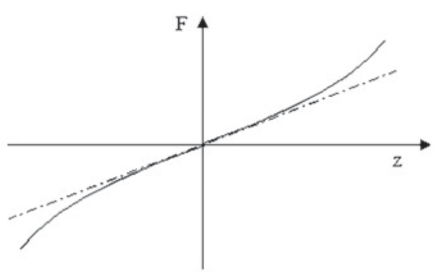


Figure 7.16. *Hardening spring*

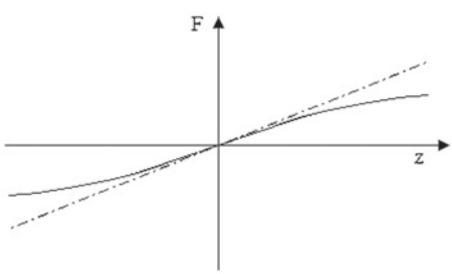


Figure 7.17. *Softening spring*

There is a "jump" from A to B, for example, [BEN 62] that can then be observed on the transfer function.

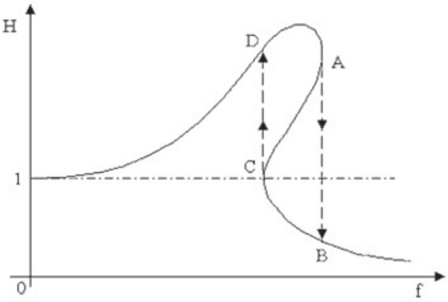


Figure 7.18. *Transmissibility for an increasing stiffness with frequency*

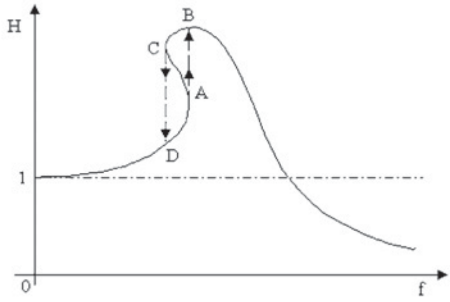


Figure 7.19. *Transmissibility for a decreasing stiffness with frequency*

When the frequency increases slowly from zero, the transmissibility increases from 1 up to point A while passing through D and then decreases to B (Figure 7.18) [TIM 74].

If, on the contrary, resonance is approached from high frequencies by a slow sinusoidal sweeping at decreasing frequency, the transfer function increases, passes through C and moves to D near the resonance, and then decreases to 1 as f tends towards zero (Figure 7.19).

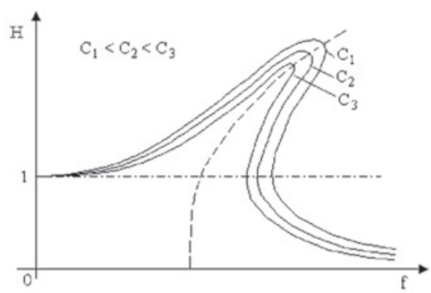


Figure 7.20. *Influence of damping*

It should be noted that the area CA is unstable and therefore cannot represent the transfer function of a physical system.

The shape of the curve depends, like the frequency of resonance, on the amplitude of the force of excitation. The mass can vibrate at its natural frequency with an excitation frequency that is much larger (a phenomenon known as *resonance of the n^{th} order*) [DUB 59].

Chapter 8

Swept Sine

8.1. Definitions

8.1.1. *Swept sine*

The swept sine is a logical extension of the sinusoidal vibration at *constant frequency* (this is a somewhat redundant expression because the definition of a sinusoid includes this assumption; this terminology is, however, commonly used for distinguishing between these two vibration types better. This test is also called the dwell test). This is a sinusoidal vibration at a given moment, whose frequency varies with time according to a certain law.

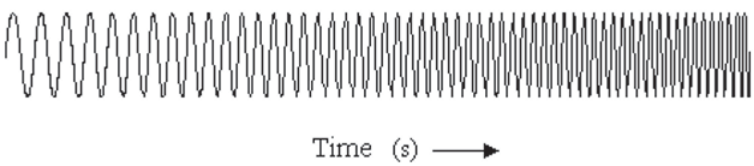


Figure 8.1. *Example of swept sine time history*

A *swept sine* can be defined as a function characterized by a relation of the form:

$$\ell(t) = \ell_m \sin[E(t) + \phi] \tag{8.1}$$

where:

– the phase ϕ is generally zero;

– $E(t)$ is a time function characteristic of the sweep mode;

– $\ell(t)$ is generally an acceleration, sometimes a displacement, a velocity or a force.

The pulsation of the sinusoid can be defined as the derivative of the function under the symbol *sine* [BRO 75], [HAW 64], [HOK 48], [LEW 32], [PIM 62], [TUR 54], [WHI 72], i.e. by:

$$\Omega = 2\,\pi\,f = \frac{dE}{dt} \tag{8.2}$$

We will see that the most interesting sweep modes are:

– the *linear sweep*, where f has the form $f = \alpha\,t + \beta$;

– the *logarithmic sweep* (which should rather be termed exponential) if $f = f_1\,e^{t/T_1}$;

– the *hyperbolic sweep* (or parabolic, or log-log) if: $\frac{1}{f_1} - \frac{1}{f} = a\,t$.

These sweeps can be carried out at an increasing frequency or a decreasing frequency.

The first two laws are the most frequently used in laboratory tests. Other laws can however be met, some of which have been the subject of other published work [SUZ 78a], [SUZ 79], [WHI 72].

Under this vibration, the material is thus subjected during a certain time interval (function of the sweeping rate) to a sinusoid whose frequency is lying in a specified range. This range must include *a priori* the resonance frequency (or frequencies) of the material. These frequencies of resonance will thus be necessarily excited.

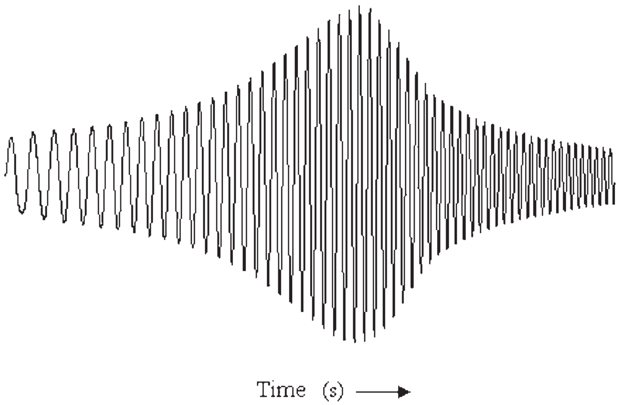


Figure 8.2. *Example of time history response to a swept sine*

To estimate the importance of a resonance, a number without dimension, the transmissibility, can be used. Transmissibility was previously defined as the ratio between the response acceleration of one point of the product and the system's input acceleration which is measured on the exciter table (or on the fixture).

A frequency with a local transmissibility peak that exceeds a predetermined value (typically two) is considered as a resonance frequency.

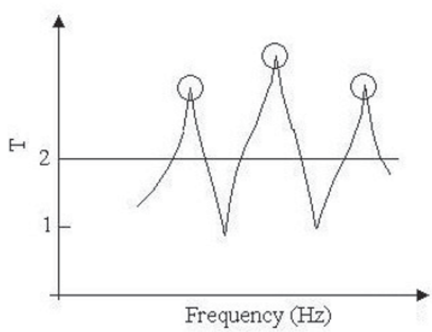


Figure 8.3. *Local transmissibility peaks which are considered as resonance peaks*

NOTE.– *After the resonance frequencies were determined, a common test was used consisting of applying sinusoidal vibration to these frequencies for a given duration. The vibration's amplitude is determined in relation to the future real environment of the product. The aim of this test is to ensure that the material is able to function under the harshest conditions, the highest stresses appearing at the resonance frequencies. The test duration varies largely, but five minutes is a common value.*

8.1.2. *Octave – number of octaves in frequency interval* (f_1, f_2)

An *octave* is the interval between two frequencies whose ratio is 2. The number of octaves ranging between two frequencies f_1 and f_2 is such that:

$$\frac{f_2}{f_1} = 2^n \qquad [8.3]$$

yielding

$$n = \frac{\ln \frac{f_2}{f_1}}{\ln 2} \qquad [8.4]$$

(logarithms in both cases being base e or base 10).

8.1.3. *Decade*

A *decade* is the interval ranging between two frequencies whose ratio is 10. The number of decades n_d ranging between two frequencies f_1 and f_2 is such that:

$$\frac{f_2}{f_1} = 10^{n_d} \qquad [8.5]$$

yielding

$$n_d = \log \frac{f_2}{f_1} = \frac{\ln f_2/f_1}{\ln 10} \qquad [8.6]$$

(ln 10 = 2.30258 ...).

The relation between the number of decades and the number of octaves ranging between two frequencies:

$$\ln \frac{f_2}{f_1} = n \ln 2 = n_d \ln 10 \qquad [8.7]$$

$$\frac{n}{n_d} = \frac{\ln 10}{\ln 2} \approx 3.3219\ldots \qquad [8.8]$$

8.2. "Swept sine" vibration in the real environment

Such vibrations are relatively rare. They are primarily measured on structures and equipment installed in the vicinity of rotating machines, at times of launching, stopping or changes of speed. They were more particularly studied to evaluate their effects during transition through the resonance frequency of a material [HAW 64], [HOK 48], [KEV 71], [LEW 32], [SUZ 78a], [SUZ 78b], [SUZ 79].

8.3. "Swept sine" vibration in tests

Tests on materials were and still are frequently carried out by applying a sine-type excitation to the specimen, the objectives being:

– identification of the material: the test is carried out by subjecting the material to a swept sine having in general a rather low and constant amplitude (not to damage the specimen), about 5 ms^{-2}, the variation of the frequency with time being rather small (close to one octave per minute) in order to study the response at various points of the specimen, to emphasize the resonance frequencies and to measure the amplification factors;

– the application of a test defined in a standard document (MIL STD 810 C, AIR 7304, GAM T 13, etc.), the test being intended to show that the material has a given standard robustness, independent or difficult to relate to the vibrations which the material will undergo in its service life;

– the application of a specification which, as well as being feasible, covers vibrations in its future real environment.

Swept sinusoidal vibration tests are badly adapted to the simulation of random vibrations, whose amplitude and phase vary in a random way and in which all the frequencies are excited simultaneously.

Many parameters are necessary to define a swept sine test.

So far in this chapter we have mentioned the physical quantities which were used to carry out the control, which can be, exactly as in the case of a *fixed frequency* sinusoidal vibration test, an acceleration, a displacement or a velocity. It is also necessary, however, to specify the frequency range to be swept.

The swept sine can have a constant level over all the frequency band studied (Figure 8.4(a)) or can be composed of several constant levels at various frequency intervals (Figure 8.4(b)).

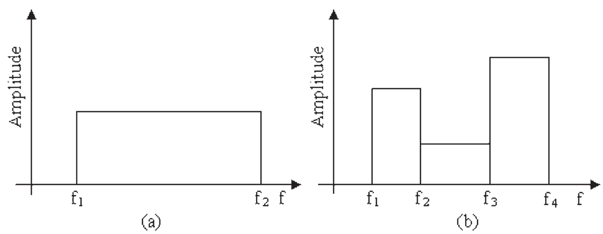


Figure 8.4. *Examples of swept sines*

In the same test, each frequency range can be characterized by a different quantity and/or a different amplitude: a displacement is sometimes specified at very low frequencies, this parameter being easier to measure in this frequency domain, more rarely a velocity, and, in general, an acceleration.

Example: a swept sine of between 5 and 500 Hz is defined:

– in the 5 to 15 Hz range by a displacement of 1 mm;

– in the 15 to 200 Hz range by an acceleration of 18 m/s^2;

– in the 200 Hz to 500 Hz range by an acceleration of 40 m/s^2.

Sweep is most frequently logarithmic. The specification sometimes specifies the direction of sweep: *increasing* or *decreasing frequency.*

Either the *sweep rate* (number of octaves per minute) or the *sweep duration* (from lowest frequency to highest or in each frequency band) is specified.

The level is defined by the peak value of the sinusoid or the peak-to-peak amplitude.

The sweep rate is generally, selected to be sufficiently low to enable the response of the equipment being tested to reach a very high percentage of the level obtained in steady operation under pure sinusoidal excitation.

If the sweep is fast, it can be estimated that each resonance is excited one after the other, in a transient way, when the frequency sweeps the interval ranging between the half-power points of each peak of the transfer function of the material. We will see (Volume 3) how this method can be used to measure the transfer functions.

In this approach, the swept sine is a vibration the effects of which can be compared with those of a shock (except the fact that under a shock, all the modes are excited simultaneously) [CUR 55].

Several questions arise in relation to how to carry out sweeps:

– How can we choose the direction of sweeping starting from the initial frequency (i.e. at increasing or decreasing frequency)? Should the sweep be carried out in one direction or both?

– How can we vary the frequency according to time (linearly or logarithmically)?

– How can we choose sweep rate?

– What should the duration of the test be? How many unidirectional sweeps are necessary?

Several parameters thus remain to be determined, their choice being function of the aims and constraints of the test to be realized.

8.4. Origin and properties of main types of sweepings

8.4.1. *The problem*

We know that for a linear one-degree-of-freedom mechanical system the damping ratio is given by:

$$\xi = \frac{c}{2\sqrt{k\,m}} \qquad [8.9]$$

the Q factor by:

$$Q = \frac{1}{2\xi} = \frac{\sqrt{k\,m}}{c} = 2\pi\, f_0 \frac{m}{c} \qquad [8.10]$$

the resonance frequency by:

$$f_0 = \frac{1}{2\,\pi}\sqrt{\frac{k}{m}}\left(= \frac{\omega_0}{2\,\pi}\right) \qquad [8.11]$$

and the width Δf of the peak of the transfer function between the half-power points by:

$$\Delta f = \frac{f_0}{Q} \qquad [8.12]$$

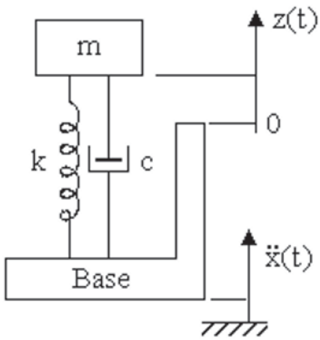


Figure 8.5. *One-degree-of-freedom system*

NOTE.–

We saw that the maximum of the transfer function $\left|H(f)\right| = \frac{\omega_0^2\, z_{max}}{\ddot{x}_{max}}$ *is actually equal to*

$$H_m = \frac{1}{2\,\xi\sqrt{1-\xi^2}} = \frac{Q}{\sqrt{1-\xi^2}} \qquad [8.13]$$

Mechanical systems have, in general, rather weak damping so that the approximation $H_m = Q$ *(which is the exact result for the maximum of the transfer function acceleration–relative velocity instead here of the function acceleration–relative displacement) can be used. It should be remembered that the half-power points are defined, because of a mechanical–electrical analogy, from the transfer function acceleration–relative velocity.*

Writing:

$$\dot{f} = \frac{df}{dt} \qquad [8.14]$$

for the sweep rate around the resonance frequency f_0, the time spent in the band Δf is given roughly by:

$$\Delta t = \frac{\Delta f}{\dot{f}} \qquad [8.15]$$

and the number of cycles performed by:

$$\Delta N = f_0\ \Delta t = \frac{\Delta f}{\dot{f}}\ f_0 \qquad [8.16]$$

When such a system is displaced from its equilibrium position and then released (or when the excitation to which it is subjected is suddenly stopped), the displacement response of the mass can be written in the form:

$$z(t) = z_m\ e^{-t/T}\ \cos\left(2\ \pi\ f_0\ \sqrt{1-\xi^2}\ t + \phi\right) \qquad [8.17]$$

where T is a time constant equal to:

$$T = \frac{2\ m}{c} \qquad [8.18]$$

i.e. according to [8.10]:

$$T = \frac{Q}{\pi\ f_0} = \frac{1}{\omega_0\ \xi} \qquad [8.19]$$

It will then be assumed that the Q factor is independent of the natural frequency f_0, in particular in the case of viscoelastic materials. Different reasoning can take into account a variation of Q with f_0 according to various laws [BRO 75]; this leads to the same sweeping laws.

In a swept sine test, with the frequency varying according to time, the response of a mechanical system is never perfectly permanent. It is closer to the response which the system would have under permanent stress at a given frequency when the sweep rate is slower. To approach as closely as possible this response in the vicinity of the resonance frequency, it is necessary that the time Δt spent in Δf is long compared to the constant T, a condition which can be written [MOR 76]:

$$\Delta t = \mu \, T \qquad [8.20]$$

$(\mu >> 1)$; yielding[1]

$$\left|\dot{f}\right| = \frac{\Delta f}{\Delta t} = \Delta f \, \frac{\pi \, f_0}{\mu \, Q} = \frac{f_0}{Q} \, \frac{\pi \, f_0}{\mu \, Q} \qquad [8.21]$$

$$\left|\dot{f}\right| = \frac{\pi \, f_0}{\mu \, Q^2} \qquad [8.22]$$

Natural frequency f_0 can be arbitrary in the band considered $(f_1,\ f_2)$ and, whatever its value, the response must be close to Q times the input to the resonance. To calculate the sweep law $f(t)$ let us generalize f_0 by writing f as:

$$\dot{f} = \pm \frac{\pi \, f^2}{\mu \, Q^2} \qquad [8.23]$$

It can be seen that the sweep rate varies as $1 / Q^2$.

NOTE.– *The derivative* $\dot{f}$ *is positive for increasing frequency sweep, negative for decreasing frequency sweep.*

1 It is assumed here that Δf is sufficiently small (i.e. ξ is small) to be able to approximate with a small error the slope of the tangent to the curve f(t) by the slope of the chord relating to the interval Δf. We will see that this approximation is indeed acceptable in practice.

8.4.2. ***Case 1: sweep where time Δt spent in each interval Δf is constant for all values of f_0***

Here, since

$$\Delta t = \mu T = \frac{\mu Q}{\pi f} \qquad [8.24]$$

it is necessary that $\mu = \gamma f$ the constant γ has the dimension of time, and

$$\Delta t = \frac{\gamma Q}{\pi} \qquad [8.25]$$

$$\dot{f} = \pm \frac{\pi f^2}{\mu Q^2} = \pm \frac{\pi f}{\gamma Q^2} = \pm \frac{f}{T_1} \qquad [8.26]$$

if we set $T_1 = \dfrac{\gamma Q^2}{\pi}$.

Sweeping at frequency increasing between f_1 and f_2

We deduce from [8.26]

$$f = f_1 \, e^{\frac{t}{T_1}} \qquad [8.27]$$

The constant T_1 is such that, for $t = t_s$ (t_s = sweep duration), $f = f_2$:

$$T_1 = \frac{t_s}{\ln f_2/f_1} \qquad [8.28]$$

where T_1 is the time needed to sweep the interval between two frequencies whose ratio is e. Relations [8.24] and [8.25] lead to

$$T_1 = Q \, \Delta t \qquad [8.29]$$

NOTE.– *Equation [8.27] can also be written as:*

$$f = f_1 \left(\frac{f_2}{f_1} \right)^{\frac{t}{t_s}} \qquad [8.30]$$

Sweep at decreasing frequency

$$f = f_2 \; e^{-\frac{t}{T_1}} \qquad [8.31]$$

the constant T_1 having the same definition as previously.

Expression for E(t)

Increasing frequency:

$$E(t) = 2\,\pi \int_0^t f_1 \; e^{t/T_1} \, dt \qquad [8.32]$$

i.e. [HAW 64], [SUN 75]:

$$E(t) = 2\,\pi\,T_1\,f_1 \left(e^{t/T_1} - 1 \right) = 2\,\pi\,T_1 \left(f - f_1 \right) \qquad [8.33]$$

Decreasing frequency:

$$E(t) = 2\,\pi \int_0^t f_2 \; e^{-t/T_1} \, dt \qquad [8.34]$$

$$E(t) = -2\,\pi\,T_1\,f_2 \left(e^{-t/T_1} - 1 \right) = -2\,\pi\,T_1 \left(f - f_2 \right) \qquad [8.35]$$

Later in this section, and apart from a specific particular case, we will only consider sweepings at increasing frequency, the relations for the other case being either identical or very easy to rewrite.

We assumed above that f_1 is always, whatever the sweep direction, the lowest frequency, and f_2 always the highest frequency. Under this assumption, certain relations depend on the sweep direction. If, on the contrary, it is simply supposed that f_1 is the initial frequency of sweep and f_2 the final frequency, whatever the direction, we obtain the same relations independent of the direction; relations, in addition, identical to those established above, and in what follows in the case of an increasing frequency.

Time t can be expressed versus the frequency f according to:

$$t = T_1 \ln \frac{f}{f_1} \tag{8.36}$$

In spite of the form of relations [8.27] and [8.31], the sweep is known as *logarithmic,* by referring to expression [8.36].

The time necessary to go from frequency f_1 to frequency f_2 is given by:

$$t_s = T_1 \ln \frac{f_2}{f_1} \tag{8.37}$$

which can still be written:

$$t_s = Q\, \Delta t \ln \frac{f_2}{f_1} \tag{8.38}$$

The number of cycles carried out during time t is given by:

$$N = \int_0^t f(t)\, dt = \int_0^t f_1\, e^{\frac{t}{T_1}}\, dt \tag{8.39}$$

$$N = f_1\; T_1 \left(e^{t/T_1} - 1 \right) \tag{8.40}$$

i.e. according to [8.27]:

$$N = T_1 \left(f - f_1 \right) \tag{8.41}$$

The number of cycles between f_1 and f_2 is:

$$N_s = T_1 \left(f_2 - f_1 \right) \tag{8.42}$$

which can be also written, taking into account [8.37],

$$N_s = \frac{t_s \left(f_2 - f_1\right)}{\ln \frac{f_2}{f_1}} \qquad [8.43]$$

The mean frequency (or *average frequency* or *expected frequency*) is equal to:

$$f_m = \frac{N_s}{t_s} = \frac{f_2 - f_1}{\ln f_2/f_1} \qquad [8.44]$$

The number of cycles ΔN performed in the band Δf between the half-power points (during time Δt) is written [8.42]:

$$\Delta N = T_1 \left[f_0 \left(1 + \frac{1}{2Q}\right) - f_0 \left(1 - \frac{1}{2Q}\right)\right]$$

i.e.

$$\Delta N = f_0 \frac{T_1}{Q}$$

$$\Delta N = f_0 \;\Delta t \qquad [8.45]$$

ΔN thus varies like f_0 yielding

$$t_s = \frac{Q \,\Delta N}{f_0} \ln \frac{f_2}{f_1} \qquad [8.46]$$

Also starting from [8.42]:

$$\Delta N = \frac{f_0 \; N_s}{Q\left(f_2 - f_1\right)} \qquad [8.47]$$

The time Δt spent in an interval Δf is:

$$\Delta t = \frac{T_1}{Q}$$

Time Δt is constant regardless of the frequency of f_0.

Example 8.1.

If $Q = 5$, the width of the interval is equal to 20 Hz when $f_0 = 100$ Hz and to 100 Hz when $f_0 = 500$ Hz (Figure 8.6).

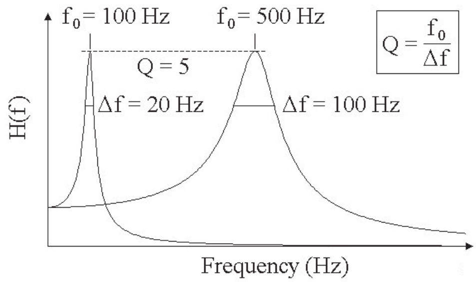


Figure 8.6. *Interval width between two half-power points for Q = 5 and for two values of the natural frequency*

$$\Delta t = \frac{t_s}{Q \ln \frac{f_2}{f_1}} \qquad [8.48]$$

yielding another expression of ΔN :

$$\Delta N = f_0 \; \Delta t$$

$$\Delta N = \frac{f_0 \; t_s}{Q \ln \frac{f_2}{f_1}} \qquad [8.49]$$

The number of cycles N_1 necessary to go from frequency f_1 to a resonance frequency f_0 is:

$$N_1 = T_1 \left(f_0 - f_1 \right) \qquad [8.50]$$

$$N_1 = Q\,\Delta t\left(f_0 - f_1\right) \qquad [8.51]$$

$$N_1 = \frac{Q\,\Delta N}{f_0}\left(f_0 - f_1\right) \qquad [8.52]$$

or

$$N_1 = \frac{t_s\left(f_0 - f_1\right)}{\ln\frac{f_2}{f_1}} \qquad [8.53]$$

This number of cycles is carried out in time:

$$t_1 = T_1 \ln\frac{f_0}{f_1} \qquad [8.54]$$

$$t_1 = \frac{Q\ \Delta N}{f_0}\ln\frac{f_0}{f_1} \qquad [8.55]$$

or

$$t_1 = Q\,\Delta t\,\ln\frac{f_0}{f_1} = \frac{t_s\ \ln\frac{f_0}{f_1}}{\ln\frac{f_2}{f_1}} \qquad [8.56]$$

If the initial frequency f_1 is zero, we have $N_1 = N_0$ given by:

$$N_0 = f_0\ T_1 \qquad [8.57]$$

or

$$N_0 = Q\ \Delta t\, f_0 = Q\,\Delta N \qquad [8.58]$$

It is not possible, in this case, to calculate the time t_0 necessary to go from 0 to f_0.

Sweep rate

According to the sweep direction, we have:

$$\frac{df}{dt} = \begin{Bmatrix} \dfrac{f_1}{T_1} e^{t/T_1} \\ \dfrac{f_2}{T_1} e^{-t/T_1} \end{Bmatrix} = \frac{f}{T_1} \qquad [8.59]$$

The sweep rate is generally expressed as the number of octaves per minute. The number of octaves between two frequencies f_1 and f_2 is equal to [8.4]:

$$n = \frac{\ln f_2/f_1}{\ln 2}$$

yielding the number of octaves per second:

$$R_{os} = \frac{n}{t_s} = \frac{\ln f_2/f_1}{t_s \ln 2} \qquad [8.60]$$

(t_s being expressed in seconds) and the number of octaves per minute:

$$R_{om} = \frac{60\ n}{t_s} = 60\ R_{os} \qquad [8.61]$$

$$t_s = \frac{60}{R_{om} \ln 2} \ln \frac{f_2}{f_1}$$

If we set $f = f_2$ in [8.36] for $t = t_s$, we obtain:

$$t_s = T_1 \ln \frac{f_2}{f_1} = \frac{60}{R_{om}} \frac{\ln f_2 / f_1}{\ln 2} \qquad [8.62]$$

From [8.60] and [8.62]:

$$\ln \frac{f_2}{f_1} = \frac{t_s}{T_1} = R_{os}\ t_s \ln 2$$

$$R_{os} = \frac{1}{T_1 \ln 2}$$ [8.63]

yielding another expression of the sweep law:

$$f = f_1 \, 2^{R_{os} t}$$ [8.64]

or

$$f = f_1 \, 2^{R_{om} t/60}$$ [8.65]

according to the definition of R.

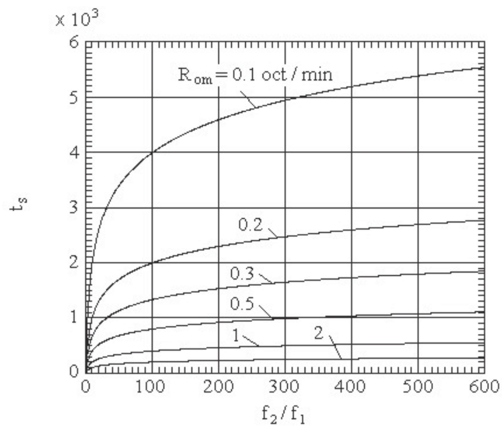


Figure 8.7. *Sweep duration*

Figure 8.7 shows the variations of t_s versus the ratio f_2/f_1, for R_{om} equal to 0.1 – 0.2 – 0.3 – 0.5 – 1 and 2 oct/min (relation [8.61]). In addition we deduce from [8.59] and [8.63] the relation:

$$\frac{df}{dt} = \frac{f}{T_1} = f \, R_{os} \ln 2$$ [8.66]

Case where the sweep rate R_{dm} is expressed in decades per minute

By definition and according to [8.6]

$$R_{dm} = \frac{n_d}{t_s / 60} = \frac{60 \ln f_2/f_1}{t_s \ln 10} \qquad [8.67]$$

$$R_{dm} = \frac{60 \log f_2/f_1}{t_s} \qquad [8.68]$$

or

$$R_{dm} = R_{om} \frac{\ln 2}{\ln 10} \approx \frac{R_{om}}{3.3219\ldots} \qquad [8.69]$$

Time spent between two arbitrary frequencies in the swept interval f_1, f_2

Let us set f_A and f_B ($> f_A$) as the limits of a frequency interval located in (f_1, f_2). The time $t_B - t_A$ spent between f_A and f_B is calculated directly, starting from [8.36] and [8.37] for example:

$$t_B - t_A = t_s \frac{\ln f_B/f_A}{\ln \frac{f_2}{f_1}} \qquad [8.70]$$

Time spent between the half-power points of a linear one-degree-of-freedom system

Let us calculate the time Δt^* between the half-power points using the relations established for small ξ.

The half-power points have as abscissae $f_0 - \frac{\Delta f}{2}$ and $f_0 + \frac{\Delta f}{2}$ respectively, i.e. $f_0 \left(1 - \frac{1}{2Q}\right)$ and $f_0 \left(1 + \frac{1}{2Q}\right)$, yielding, starting from [8.70], the time Δt^* spent between these points:

$$\Delta t^* = t_s \frac{\ln \frac{1 + 1/2\,Q}{1 - 1/2\,Q}}{\ln f_2/f_1} \qquad [8.71]$$

This relation can be written:

$$\Delta t^* = T_1 \ln \frac{1+\xi}{1-\xi}$$

i.e. since $\Delta t = T_1 / Q$

$$\frac{\Delta t^*}{\Delta t} = \frac{1}{2\,\xi} \ln \frac{1+\xi}{1-\xi} \qquad [8.72]$$

Figure 8.8 shows the variations of $\frac{\Delta t^*}{\Delta t}$ versus ξ. It should be noted that, for $\xi < 0.2$, $\frac{\Delta t^*}{\Delta t}$ is very close to 1.

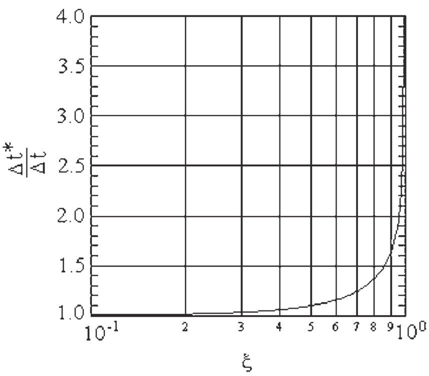


Figure 8.8. *Validity of approximate expression of the time spent between the half-power points*

$$\Delta t^* = \frac{60}{\ln 2} \frac{\ln\left(\frac{1+1/2\,Q}{1-1/2\,Q}\right)}{R_{om}} \qquad [8.73]$$

(where Δt^* is expressed in seconds) [SPE 61], [SPE 62], [STE 73]. The number of cycles in this interval is equal to

$$\Delta N^* = f_0\ \Delta t^*$$

$$\Delta N^* = \frac{60}{\ln 2}\ \frac{f_0}{R_{om}}\ \ln\left(\frac{1+1/2\,Q}{1-1/2\,Q}\right) \qquad [8.74]$$

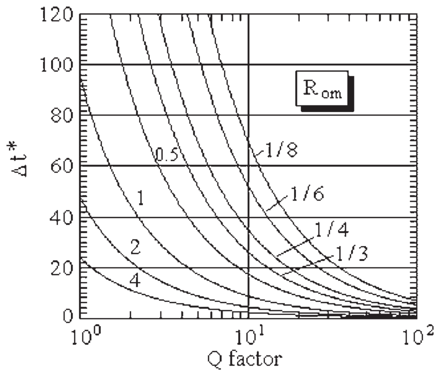


Figure 8.9. *Time spent between the half-power points*

It should be noted that Δt^* given by [8.71] tends towards the value given by [8.48] as Q increases. Figure 8.9 shows the variations of Δt^* versus Q for R_{om} (oct/min) equal to 4, 2, 1, 1/2, 1/3, 1/4, 1/6 and 1/8 respectively.

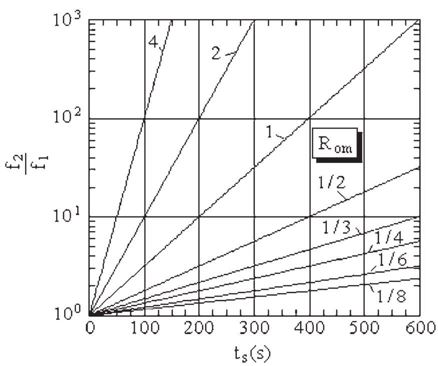


Figure 8.10. *Sweep duration between two frequencies*

Figure 8.10 gives the sweep time necessary to go from f_1 to f_2 for sweep rates R_{om} (oct/min) equal to 4, 2, 1, 1/2, 1/3, 1/4, 1/6 and 1/8.

Number of cycles per octave

If f_A and f_B are two frequencies separated by one octave:

$$f_B = 2\, f_A$$

The number of cycles in this octave is equal to [8.42]

$$N_2 = T_1\, f_A \tag{8.75}$$

i.e. according to [8.43],

$$N_2 = \frac{f_A \;\; f_b}{\ln 2} \tag{8.76}$$

$$N_2 = Q\, \Delta N \frac{f_A}{f_0} \tag{8.77}$$

Time to sweep one octave

Let us set t_2 for this duration

$$t_2 = T_1 \ln 2 \tag{8.78}$$

$$t_2 = Q\, \Delta t \ln 2 \tag{8.79}$$

$$t_2 = \frac{Q\, \Delta N}{f_0} \ln 2 \tag{8.80}$$

Time to sweep 1/nth octave

$$t_n = T_1 \;\ln 2^{1/n} = \frac{T_1}{n} \ln 2 \tag{8.81}$$

$$t_n = \frac{Q\, \Delta t}{n} \ln 2 \tag{8.82}$$

$$t_n = \frac{Q\,\Delta N}{f_0\, n} \ln 2 \qquad [8.83]$$

8.4.3. *Case 2: sweep with constant rate*

If we wish to carry out a sweep with a constant rate, it is necessary that $df/dt =$ constant, i.e., since $\dot{f} = \pm \frac{\pi}{\mu\, Q^2} f^2$

$$\mu = \delta\, f^2 \qquad [8.84]$$

where δ is a constant with dimension of time squared.

$$\Delta t = \delta\, f_0^2 \frac{Q}{f_0}$$

$$\Delta t = \frac{\delta\, f_0\, Q}{\pi} \qquad [8.85]$$

The time spent in the band Δf delimited by the half-power points varies in the same way as the natural frequency f_0

$$\frac{df}{dt} = \pm \frac{\pi}{Q^2} \frac{f^2}{\delta\, f^2} = \pm \frac{\pi}{Q^2 \delta} = \pm\, \alpha \qquad [8.86]$$

where α is a constant.

Increasing frequency sweep

$$f = \alpha\, t + f_1 \qquad [8.87]$$

The constant α is such that $f = f_2$ when $t = t_s$, yielding [BRO 75], [HOK 48], [LEW 32], [PIM 62], [TUR 54], [WHI 72], [WHI 82]:

$$\alpha = \frac{f_2 - f_1}{t_b} \qquad [8.88]$$

This sweep is said to be *linear.*

Decreasing frequency sweep

$$f = -\alpha\ t + f_2 \qquad [8.89]$$

$$\alpha = \frac{\pi}{Q^2\ \delta} = \frac{f_2 - f_1}{t_s} \qquad [8.90]$$

Calculation of the function E(t) [SUN 75]

Increasing frequency:

$$E(t) = 2\ \pi \int_0^t \left(\alpha\ t + f_1\right) dt \qquad [8.91]$$

$$E(t) = 2\ \pi\ t \left(\frac{\alpha\ t}{2} + f_1\right) \qquad [8.92]$$

Decreasing frequency:

$$E(t) = 2\ \pi \int_0^t \left(-\alpha\ t + f_2\right) dt \qquad [8.93]$$

$$E(t) = 2\ \pi\ t \left(-\frac{\alpha\ t}{2} + f_2\right) \qquad [8.94]$$

Sweep rate

This is equal, depending on the direction of sweep, to

$$\frac{df}{dt} = \pm\alpha = \pm \frac{f_2 - f_1}{t_s} \qquad [8.95]$$

8.4.4. *Case 3: sweep ensuring a number of identical cycles ΔN in all intervals Δf (delimited by the half-power points) for all values of f_0*

With this assumption, since the quantity

$$\Delta N = f\,\Delta t = f\,\frac{\mu\,Q}{\pi\,f} = \mu\,\frac{Q}{\pi} \qquad [8.96]$$

must be constant, parameter β must itself be constant, yielding:

$$\dot{f} = \pm\frac{\pi\,f^2}{\mu\,Q^2} = \pm\,a\,f^2 \qquad [8.97]$$

where $a = \dfrac{\pi}{\mu\,Q^2}$. The sweep rate varies as the square of the instantaneous frequency. This expression is written [BIC 70, PAR 61]:

$$\frac{df}{f^2} = \pm\,a\,t \qquad [8.98]$$

Increasing frequency sweep between f_1 and f_2

By integration,

$$\frac{1}{f_1} - \frac{1}{f} = a\,t \qquad [8.99]$$

(at $t = 0$, we assume that $f = f_1$, the starting sweep frequency), i.e. [PAR 61]:

$$f = \frac{f_1}{1 - a\,f_1\,t} \qquad [8.100]$$

or, since, for $t = t_s$, $f = f_2$:

$$a = \frac{f_2 - f_1}{f_1\,f_2\,t_s} \qquad [8.101]$$

In this case, little used in practice, the sweep is known as *hyperbolic* [BRO 75] (also called *parabolic sweep,* undoubtedly because of the form of the relation [8.97], and *log–log sweep* [ELD 61], [PAR 61]).

NOTE.–

In spite of the form of the denominator of expression [8.100], frequency f cannot be negative. For that, it would be necessary that $1 - a\, f_1\, t < 0$*, i.e.*

$$t > \frac{1}{a\, f_1} = \frac{t_s\, f_2}{f_2 - f_1}$$

i.e. that $t > t_s$.

Decreasing frequency sweeping between f_2 *and* f_1

$$\frac{df}{f_2} = -a\, dt \qquad [8.102]$$

$$\frac{1}{f_2} - \frac{1}{f} = -a\, t \qquad [8.103]$$

$$f = \frac{f_2}{1 + a\, f_2\, t} \qquad [8.104]$$

For $t = t_s$ we have $f = f_1$, yielding:

$$a = \frac{f_2 - f_1}{f_1\, f_2\, t_s} \qquad [8.105]$$

Expression for E(t)

Function $E(t)$ in the sine term can be calculated from expression [8.2] of $f(t)$:

$$E(t) = \int_0^t 2\pi\, f(t)\, dt \qquad [8.106]$$

Increasing frequency sweep [CRU 70], [PAR 61]

$$E(t) = 2\,\pi \int_0^t \frac{f_1\, dt}{1 - a\, f_1\, t} = \frac{2\,\pi}{a} \int_0^{a\, f_1\, t} \frac{d(a\, f_1\, t)}{1 - a\, f_1\, t} \quad [8.107]$$

$$E(t) = -\frac{2\,\pi}{a} \ln(1 - a\, f_1\, t)$$

$$E(t) = \frac{2\,\pi}{a} \ln\left(\frac{1}{1 - a\, f_1\, t}\right) \quad [8.108]$$

i.e. taking into account [8.100]

$$E(t) = \frac{2\,\pi}{a} \ln \frac{f}{f_1} \quad [8.109]$$

Decreasing frequency sweep

We have in the same way:

$$E(t) = 2\,\pi \int_0^t \frac{f_2\, dt}{1 - a\, f_2\, t} = \frac{2\,\pi}{a} \int_0^{a\, f_2\, t} \frac{d(a\, f_2\, t)}{1 - a\, f_2\, t}$$

$$E(t) = \frac{2\,\pi}{a} \ln(1 + a\, f_2\, t) \quad [8.110]$$

Sweep rate

Increasing frequency:

$$\frac{df}{dt} = a\, f^2 = \frac{f_2 - f_1}{f_1\, f_2\, t_s}\, f^2 \quad [8.111]$$

Decreasing frequency:

$$\frac{df}{dt} = -a\, f^2 \quad [8.112]$$

Tables 9.2 to 9.8 at the end of Chapter 9 summarize the relations calculated for the three sweep laws (logarithmic, linear and hyperbolic).

Chapter 9

Response of a Linear One-Degree-of-Freedom System to a Swept Sine Vibration

9.1. Influence of sweep rate

An extremely slow sweep rate makes it possible to measure and plot the transfer function of the one-degree-of-freedom system without distortion, and to obtain correct values for the resonance frequency and Q factor.

When the sweep rate increases, it is seen that the transfer function obtained increasingly differs from the real transfer function. The deformations of the transfer function result in (Figure 9.1):

– a reduction of the maximum δH;

– a displacement of the abscissa of the maximum δf_r;

– a displacement δf_m of the median axis of the curve (which loses its symmetry);

– an increase in the bandwidth Δf (interval between the half-power points).

When the sweep rate increases:

– beats caused by interference between the free response of the mechanical system, relatively important after resonance, and the excitation "swept sine" imposed on the system, are observed appearing on the signal of the response according to time [BAR 48], [PIM 62]. The number and importance of these beats are weaker since the damping is greater;

– then, as if the system were subjected to a shock, the sweep duration decreases. The largest peak of the response occurs for $t > t_b$ (residual "response" observed when the duration of the shock is small compared to the natural period of the system). We will see an example of this in section 9.2.3.

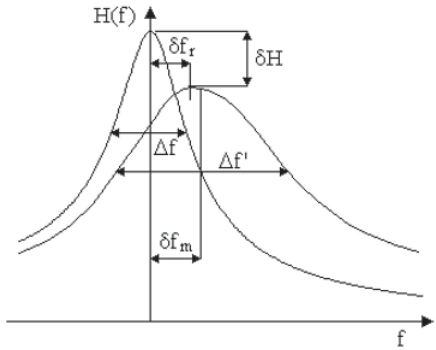


Figure 9.1. *Deformation of the transfer function related to a large sweep rate (according to [REE 60])*

Example 9.1.

Figure 9.2 shows the transfer function of a one-degree-of-freedom system which is measured with a slow swept sine vibration and a too fast swept sine vibration with increasing frequencies. The peak is shifted to the right. Conversely, if the swept sine vibration is carried out with decreasing frequencies, the peak shifts to the left (Figure 9.3).

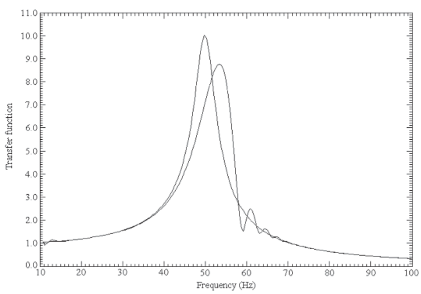


Figure 9.2. *Swept sine vibration with increasing frequencies*

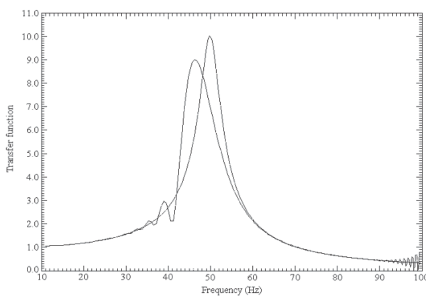


Figure 9.3. *Swept sine vibration with decreasing frequencies*

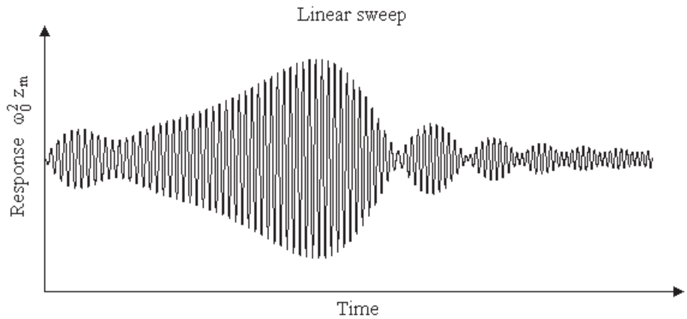


Figure 9.4. *Sweep rate influence on the response of a one-degree-of-freedom system*

9.2. Response of a linear one-degree-of-freedom system to a swept sine excitation

9.2.1. *Methods used for obtaining response*

The calculation of the response of a linear one-degree-of-freedom system cannot be carried out entirely analytically because of the complexity of the equations (except in certain particular cases). Various methods have been proposed to solve the differential movement equation (analog [BAR 48], [MOR 65], [REE 60], numerical [HAW 64]), using the Fourier transformation [WHI 72], the Laplace transformation [HOK 48], the convolution integral [BAR 48], [LEW 32], [MOR 65], [PAR 61], [SUN 75], [SUN 80], a series [BAR 48], [MOR 65], [PAR 61], [SUN 75], Fresnel integrals [DIM 61], [HOK 48], [LEW 32], [WHI 72], asymptotic developments [KEV 71], parameter variation techniques [SUZ 78a], [SUZ 78b], [SUZ 79], numerical integration, etc.

In general, the transient period of the beginning of the sweep, which relates to only a low number of cycles compared to the total number of cycles of sweep, is neglected. However, it is better to choose the initial frequency of sweep at least an octave below the first resonance frequency of the material, to ensure that this has no effect [SUN 80].

9.2.2. *Convolution integral (or Duhamel's integral)*

We will see in the following sections that the choice of the initial and final frequencies of sweep influence the amplitude of the response, which is all the more sensitive since the sweep rates are larger.

If the excitation is an acceleration, the differential equation of the movement of a linear one-degree-of-freedom system is written:

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{x}(t) \qquad [9.1]$$

$$\ddot{z} + 2\xi\omega_0\,\dot{z} + \omega_0^2\,z = -\ddot{x}(t) \qquad [9.2]$$

The solution can be expressed in the form of Duhamel's integral:

$$\omega_0^2\,z(t) = -\frac{\omega_0}{\sqrt{1-\xi^2}}\int_0^t \ddot{x}(\lambda)\,e^{-\xi\,\omega_0\,(t-\lambda)}\sin\omega_0\sqrt{1-\xi^2}\,(t-\lambda)\,d\lambda \qquad [9.3]$$

if $z(0) = \dot{z}(0) = 0$ (λ = variable of integration). The excitation $\ddot{x}(t)$ is given by:

$$\ddot{x}(t) = \ddot{x}_m\,\sin\,E(t)$$

where $E(t)$ is given, according to the case, by [8.92] for a linear sweep, by [8.33] for a logarithmic sweep or by [8.108] for a hyperbolic sweep (increasing frequency). If we set

$$h = \frac{f}{f_0}\left(= \frac{\Omega}{\omega_0}\right)$$

and

$$\theta = \omega_0\,t$$

these expressions can be written respectively in reduced form:

$$E(\theta) = \theta\left[\frac{h_2 - h_1}{2\,\theta_s}\,\theta + h_1\right] \qquad [9.4]$$

$$E(\theta) = \theta_1\left(h - h_1\right) \qquad [9.5]$$

$$E(\theta) = -\frac{h_1\, h_2\, \theta_s}{h_2 - h_1} \ln\left[1 - \frac{(h_2 - h_1)\theta}{h_2\, \theta_s}\right] \qquad [9.6]$$

where

$$\theta_1 = \theta_s / \ln(h_2 / h_1)$$

and

$$\theta_s = \omega_0\, t_s$$

This yields, λ being a variable of integration:

$$q(\theta) = \frac{\omega_0^2\, z_m}{(-\ddot{x}_m)} = \frac{1}{\sqrt{1-\xi^2}} \int_0^\theta \sin\left[E(\lambda)\right] e^{-\xi(\theta-\lambda)} \sin\left[\sqrt{1-\xi^2}\,(\theta-\lambda)\right] d\lambda \qquad [9.7]$$

It should be noted that the reduced response $q(\theta)$ is a function of the parameters ξ, θ_s, h_1 and h_2 only, and is independent of the natural frequency f_0.

Numerical calculation of Duhamel's integral

Direct calculation of $q(\theta)$ from numerical integration of [9.7] is possible, but it:

– requires a number of points of integration that become larger as the sweep rate becomes smaller;

– sometimes introduces, for the weak rates, singular points in the plot of $q(\theta)$, which do not necessarily disappear on increasing the number of points of integration (or changing the X-coordinate).

The results given in the following sections were obtained in this way. Integration was carried out by Simpson's method.

NOTE.– *If the response is characterized by the absolute acceleration of the mass of the one-degree-of-freedom system, we have, from [4.71]:*

$$\ddot{y}(t) = \omega_0 \int_0^t \ddot{x}(\lambda)\, e^{-\xi\, \omega_0\, (t-\lambda)} \left[\frac{1-2\xi^2}{\sqrt{1-\xi^2}} \sin \omega_0 \sqrt{1-\xi^2}\,(t-\lambda) + 2\,\xi \cos \omega_0 \sqrt{1-\xi^2}\,(t-\lambda)\right] d\lambda$$

[9.8]

yielding

$$q(\theta)=\frac{\ddot{y}_m}{\ddot{x}_m}=\int_0^{\theta} \sin\left[E(\lambda)\right] e^{-\xi(\theta-\lambda)}\left[\frac{1-2\xi^2}{\sqrt{1-\xi^2}}\sin\left[\sqrt{1-\xi^2}\,(\theta-\lambda)\right]+2\,\xi\cos\left[\sqrt{1-\xi^2}\,(\theta-\lambda)\right]\right]d\lambda \qquad [9.9]$$

9.2.3. *Response of a linear one-degree-of freedom system to a linear swept sine excitation*

The numerical integration of expression [9.7] was carried out for various values of h_1 and h_2, for $\xi = 0.1$, with between 400 and 600 points of calculation (according to the sweep rate) and, according to the sweep direction, $E(\theta)$ being given by [9.4] if the sweep is at an increasing frequency or by

$$E(\theta) = \theta\left[-\frac{h_2 - h_1}{2\,\theta_s}\,\theta + h_2\right] \qquad [9.10]$$

if the frequency is decreasing. On each curve response $q(\theta)$, we have to note:

– the highest maximum;

– the lowest minimum (it was noted that these two peaks always follow each other);

– the frequency of the excitation at the moment when these two peaks occur;

– the frequency of the response around these peaks starting from the relation $f_R = \frac{1}{2\,\Delta\theta}$, with $\Delta\theta$ being the interval of time separating these two consecutive peaks.

The results are presented in the form of curves in reduced coordinates with:

– on the abscissae, parameter η defined by:

$$\eta = \frac{Q^2}{f_0^2}\left(\frac{df}{dt}\right)_{f=f_0} \qquad [9.11]$$

We will see that this is used by the majority of authors [BAR 48], [BRO 75], [CRO 56], [CRO 68], [GER 61], [KHA 57], [PIM 62], [SPE 61], [TRU 70], [TRU 95], [TUR 54], in this form or a very close form ($\frac{7}{\pi}\eta$, $\frac{2}{\pi}\eta$, ...).

Since, for a linear sweep and according to the direction of sweep,

$$f = \frac{f_2 - f_1}{t_s} t + f_1 \tag{9.12}$$

or

$$f = -\frac{f_2 - f_1}{t_s} t + f_2$$

we have:

$$|\eta| = \frac{Q^2}{f_0^2} \frac{f_2 - f_1}{t_s} \tag{9.13}$$

If frequency and time are themselves expressed in reduced form, η can be written:

$$\eta = 2\pi Q^2 \left(\frac{dh}{d\theta}\right)_{h=1} \tag{9.14}$$

with, for linear sweep with increasing frequency:

$$h = \frac{dE}{d\theta} = \frac{h_2 - h_1}{\theta_s} \theta + h_1 \tag{9.15}$$

and with decreasing frequency:

$$h = -\frac{h_2 - h_1}{\theta_s} \theta + h_2 \tag{9.16}$$

yielding:

$$\frac{dh}{d\theta} = \pm \frac{h_2 - h_1}{\theta_s} \qquad [9.17]$$

and

$$|\eta| = 2\,\pi\,Q^2\,\frac{h_2 - h_1}{\theta_s} \qquad [9.18]$$

– On the ordinates the ratio G of the largest positive or negative peak (in absolute value) of the response $q(\theta)$ to the largest peak which would be obtained in steady state mode ($Q / \sqrt{1-\xi^2}$).

Calculations were carried out for sweeps at increasing and decreasing frequency. These showed that:

– for given η results differ according to the values of the limits h_1 and h_2 of the sweep; there is a couple h_1, h_2 for which the peaks of the response are largest. This phenomenon is all the more sensitive since η is larger ($\eta \geq 5$);

– this peak is sometimes positive, sometimes negative;

– for given η, sweep at decreasing frequency leads to responses larger than sweep at increasing frequency.

Figure 9.5 shows the curves $G(\eta)$ thus obtained. These curves are envelopes of all the possible results.

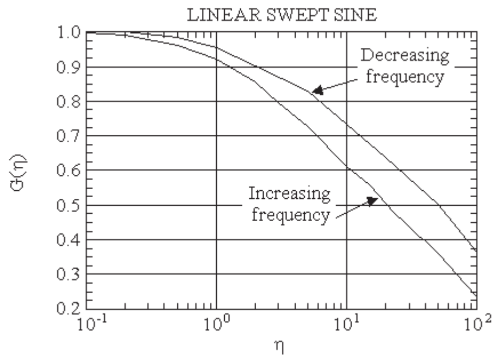


Figure 9.5. *Attenuation of the peak versus the reduced sweep rate*

In addition, Figure 9.6 shows the variations with η of the quantity

$$\frac{Q\,\delta f}{f_R} = \frac{\delta f}{\Delta f}$$

where $\delta f = f_p - f_R$, difference between the peak frequency of the transfer function measured with a fast sweep and resonance frequency f_R $\left(= f_0 \sqrt{1 - 2\,\xi^2}\right)$ measured with a very slow sweep.

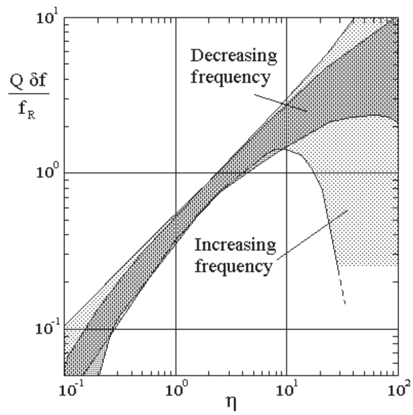


Figure 9.6. *Shift in the resonance frequency (linear and logarithmic sweep)*

The frequency f_p is that of the excitation at the moment when the response passes through the highest peak (absolute value). Δf is the width of the resonance peak measured between the half-power points (with a very slow sweep).

The values of the frequencies selected to plot this curve are those of the peaks (positive or negative) selected to plot the curve $G(\eta)$ of Figure 9.5 (for sweeps at increasing frequency). Following the initial and final frequencies, the speed and direction of sweep, $\frac{Q\,\delta f}{f_R}$ can vary within a certain range.

NOTE.– *These curves have been plotted for η varying between 0.1 and 100. This is a very important domain. To be convincing, it is enough to calculate for various values of η the number of cycles N_b carried out between h_1 and h_2 for given Q. This number of cycles is given by:*

$$N_s = \frac{f_1 + f_2}{2} t_s = \frac{f_1/f_0 + f_2/f_0}{2} f_0 t_s$$

$$N_s = \frac{h_1 + h_2}{2} \frac{\theta_s}{2\pi} \tag{9.19}$$

In addition, we showed in [9.18] that

$$\eta = 2\pi Q^2 \left(\frac{h_2 - h_1}{\theta_s}\right)$$

$$\eta = Q^2 \frac{f_2 - f_1}{f_0^2 t_s}$$

$$t_s = Q^2 \frac{f_2 - f_1}{f_0^2 \eta}$$

yielding, since $N_s = \frac{f_1 + f_2}{2} t_s$

$$N_s = \frac{\left(h_2^2 - h_1^2\right) Q^2}{2\eta} \tag{9.20}$$

Example 9.2.

$h_1 = 0.5$
$Q = 5$
$h_2 = 1.5$

If $\eta = 0.1$ there are $N_s = 250$ cycles and if $\eta = 10$ there are $N_s = 2.5$ cycles.

For the higher values of η and for certain couples h_1, h_2, it can happen that the largest peak occurs after the end of sweep ($t > t_s$). There is, in this case, a "residual" response, the system responding to its natural frequency after an excitation of short duration compared to its natural period ("impulse response"). The swept sine can be considered as a shock.

Example 9.3.

$\eta = 60$
$f_1 = 10$ Hz
$f_0 = 20$ Hz
$f_2 = 30$ Hz
$Q = 5$

With these data, the duration t_b is equal to 20.83 ms.

Figure 9.7 shows the swept sine and the response obtained (velocity: $\dot{f} = 960$ Hz/s).

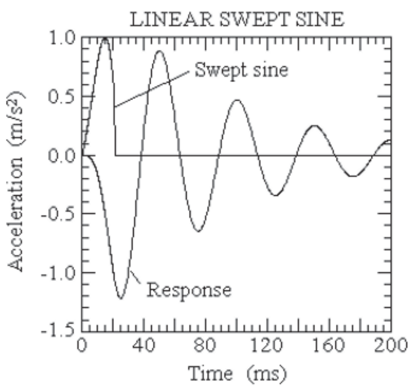


Figure 9.7. *Example of response to a fast swept sine*

It should be noted that, for this rate, the excitation resembles a half-sine shock of duration t_s and amplitude 1.

On the shock response spectrum of this half-sine (Figure 9.8), we would read on the Y-axis (for $f_0 = 20$ Hz on the X-axis) an amplitude of the response of the one-degree-of-freedom system ($f_0 = 20$ Hz, $Q = 5$) equal to 1.22 m/s^2, a value which is that read above on the curve in Figure 9.7.

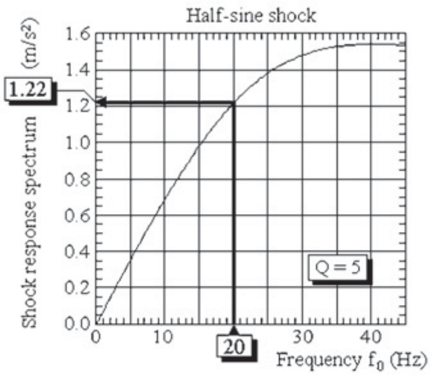


Figure 9.8. *Shock response spectrum of a half-sine shock*

For the same value of η, and with the same mechanical system, we can obtain, by taking $f_1 = 1$ Hz and $f_2 = 43.8$ Hz, an extreme response equal to 1.65 m/s^2 (Figure 9.9).

In this case the duration has as a value of 44.58 ms.

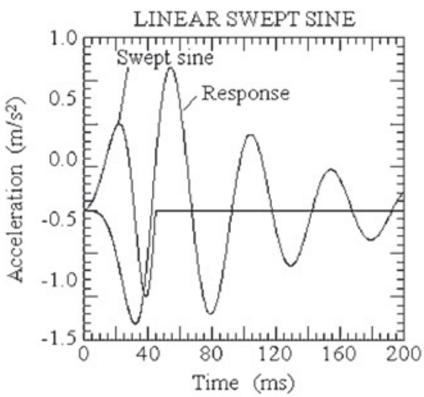


Figure 9.9. *Response for the same η value and for other limit frequencies of the swept domain*

Note on parameter η

As defined by [9.11], this parameter η is none other than the quantity π/μ from relation [8.22]. If we calculate the number of cycles ΔN according to η carried out in the band Δf delimited by the half-power points, we obtain, according to the sweep mode:

– linear sweep

$$\Delta N = \frac{f_0^2}{Q} \frac{t_s}{f_2 - f_1}$$

$$\eta = \frac{Q^2}{f_0^2} \frac{f_2 - f_1}{t_s}$$

yielding

$$\Delta N = \frac{Q}{\eta} \qquad [9.21]$$

– logarithmic sweep

$$\eta = \frac{Q^2}{f_0 \, T_1}$$

$$\Delta N = \frac{f_0}{Q} \frac{t_s}{\ln f_2/f_1} = \frac{f_0 \, T_1}{Q}$$

$$\Delta N = \frac{Q}{\eta} = \frac{Q^2}{f_0 \, t_s} \ln \frac{f_0}{f_1} \qquad [9.22]$$

– hyperbolic sweep

$$\eta = Q^2 \frac{f_2 - f_1}{f_1 \, f_2 \, t_s}$$

$$\Delta N = \frac{f_1 \, f_2 \, t_s}{Q(f_2 - f_1)}$$

$$\Delta N = \frac{Q}{\eta} \qquad [9.23]$$

For given Q and η, the number of cycles carried out in the band Δf is thus identical. As a consequence, the time Δt spent in Δf is, whatever the sweep mode, for η and Q constant

$$\Delta t = \frac{Q}{f_0 \, \eta} \qquad [9.24]$$

The expressions of the parameters considered in Chapter 8 expressed in terms of η are given in Tables 9.2 to 9.7 at the end of this chapter.

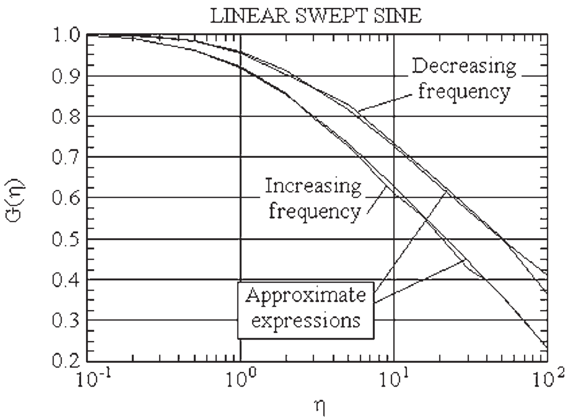


Figure 9.10. *Validity of approximate expressions for attenuation G(η)*

A good approximation of the curve at *increasing frequency* can be obtained by considering the empirical relation (Figure 9.10):

$$G(\eta) = 1 - \exp\left[-2.55\ \eta^{-0.39}\right] - 0.003\ \eta^{0.79} \qquad [9.25]$$

$(0 \le \eta \le 100)$. To represent the curve $G(\eta)$ relating to sweeps at *decreasing frequency*, we can use in the same interval the relation:

$$G(\eta) = 1 - \exp\left(-3.18\ \eta^{-0.39}\right) \qquad [9.26]$$

When damping tends towards zero, the time necessary for the establishment of the response tends towards infinity. When the sweep rate is weak, F.M. Lewis [LEW 32] and D.L. Cronin [CRO 68] stated the response of an undamped system as:

$$u_m = 3.67 \sqrt{\frac{f_0^2}{\left|\frac{df}{dt}\right|_{f = f_0}}} \quad [9.27]$$

i.e. if sweep is linear, by:

$$u_m = 3.67 \; f_0 \sqrt{\frac{t_s}{f_2 - f_1}} \quad [9.28]$$

NOTE.–

For the response of a simple system having its resonance frequency f_0 outside the swept frequency interval (f_1, f_2) in steady state mode, or for an extremely slow sweep, the maximum generalized response is given:

– for $f_0 < f_1$, by

$$u_m = \frac{\ell_m}{\sqrt{\left[1 - \left(\frac{f_1}{f_0}\right)^2\right]^2 + \frac{f_1^2}{Q^2 \, f_0^2}}} \quad [9.29]$$

– for $f_0 > f_2$, by

$$u_m = \frac{\ell_m}{\sqrt{\left[1 - \left(\frac{f_2}{f_0}\right)^2\right]^2 + \frac{f_2^2}{Q^2 \, f_0^2}}} \quad [9.30]$$

When the sweep rate is faster it is possible to obtain an approximate value of the response by successively combining [9.25] and [9.29], [9.25] and [9.30]:

– $f_0 < f_1$

$$u_m = \frac{\ell_m \left\{ 1 - exp\left[-2.55 \ \eta^{-0.39} \right] - 0.003 \ \eta^{0.79} \right\}}{\sqrt{\left[1 - \left(\frac{f_1}{f_0} \right)^2 \right]^2 + \frac{f_1^2}{f_0^2 \ Q^2}}} \qquad [9.31]$$

– $f_0 > f_2$

$$u_m = \frac{\ell_m \left\{ 1 - exp\left[-2.55 \ \eta^{-0.39} \right] - 0.003 \ \eta^{0.79} \right\}}{\sqrt{\left[1 - \left(\frac{f_2}{f_0} \right)^2 \right]^2 + \frac{f_2^2}{f_0^2 \ Q^2}}} \qquad [9.32]$$

where η is given by [9.13].

9.2.4. *Response of a linear one-degree-of-freedom system to a logarithmic swept sine*

The calculation of Duhamel's integral [9.7] was carried out under the same conditions as in the case of linear sweep, with:

$$E(\theta) = \theta_1 \left(h - h_1 \right) \qquad [9.33]$$

or

$$E(\theta) = \theta_1 \left(h_2 - h \right) \qquad [9.34]$$

according to the direction of sweep, with $\xi = 0.1$, for various values of the sweep rate, the limits h_1 and h_2 being those which, for each value of η, lead to the largest response (in absolute value). The curves $G(\eta)$ thus obtained were plotted on Figure 9.11, η being equal to:

$$|\eta| = \frac{Q^2}{f_0^2} \left(\frac{df}{dt} \right)_{f=f_0}$$

$$|\eta| = \frac{Q^2}{f_0^2}\left(\frac{f}{T_1}\right)_{f=f_0} = \frac{Q^2}{f_0\ T_1}$$

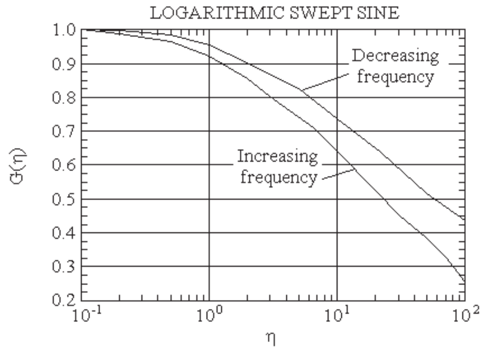


Figure 9.11. *Attenuation versus reduced sweep rate*

$$|\eta| = \frac{2\,\pi\,Q^2}{\theta_1} = \frac{2\,\pi\,Q^2}{\theta_s} \ln \frac{h_2}{h_1} \tag{9.35}$$

where

$$\theta_1 = 2\,\pi\,f_0\ T_1 \tag{9.36}$$

These curves can be represented by the following empirical relations (for $0 \le \eta \le 100$):

– *for increasing frequencies:*

$$G(\eta) = 1 - \exp\left[-\,2.55\ \eta^{-0.39}\right] - 0.0025\ \eta^{0.79} \tag{9.37}$$

– *for decreasing frequencies:*

$$G(\eta) = 1 - \exp\left[-\,3.18\ \eta^{-0.38}\right] \tag{9.38}$$

Figure 9.12 shows the calculated curves and those corresponding to these relations.

The remarks relating to the curves $G(\eta)$ for the linear sweep case apply completely to the case of logarithmic sweep.

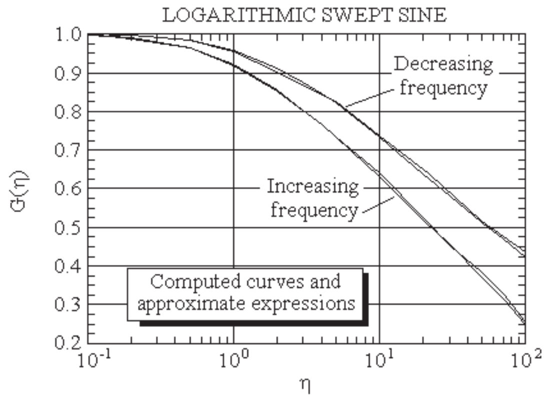


Figure 9.12. *Validity of the approximate expressions for attenuation G(η)*

NOTE.– *These curves are envelopes of the points obtained for various values of parameters f_1, f_2, f_0 and Q (for each value of η).*

The number of cycles between h_1 and h_2 is given here by:

$$N_s = \frac{f_2 - f_1}{\ln f_2/f_1} t_s$$

$$N_s = \frac{h_2 - h_1}{\ln h_2/h_1} \frac{\theta_s}{2\pi} \tag{9.39}$$

yielding, starting from [9.35]:

$$N_s = \frac{Q^2}{\eta}\left(h_2 - h_1\right) \tag{9.40}$$

$$\left.\begin{aligned} \theta_s &= \frac{2\pi Q^2}{\eta} \ln \frac{h_2}{h_1} \\ t_s &= \frac{Q^2}{\eta f_0} \ln f_2/f_1 \end{aligned}\right] \tag{9.41}$$

Example 9.4.

If $f_1 = 10$ Hz

$Q = 5$

$f_2 = 30$ Hz

$f_0 = 20$ Hz

η	N_s	t_s (s)
0.1	250	137.33
10	2.5	0.1373
60	0.417	0.02289
100	0.25	0.01373

Table 9.1. *Examples of sweep durations for given values of* η

Figure 9.13 shows the swept sine (log) for increasing frequency and the response calculated with these data for $\eta = 60$.

It is possible to find other limits of the swept range (f_1, f_2) leading to a larger response.

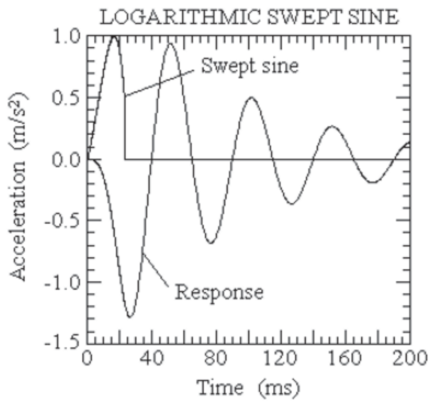


Figure 9.13. *Example of response to a fast swept sine*

The curves $G(\eta)$ obtained in the case of linear and logarithmic sweeps at increasing and decreasing frequencies are superimposed in Figure 9.14. We obtain very similar curves (for a given sweep direction) with these two types of sweeps.

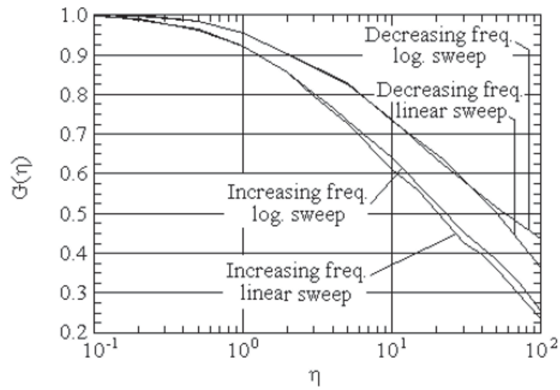


Figure 9.14. *Comparison of the attenuation of linear and logarithmic sweeps*

9.3. Choice of duration of swept sine test

In this section, the duration of the tests intended to simulate a certain particular swept sine real environment will not be considered.

During an identification test intended to measure the transfer function of a mechanical system, it is important to sweep slowly so that the system responds at its resonance with an amplitude very close to the permanent response, whilst adjusting the duration of sweep to avoid prohibitive test times.

It has been seen that a good approximation to the measure of the resonance peak could be obtained if η was sufficiently small; J.T. Broch [BRO 75] advised, for example, that $\eta \le 0.1$, which ensures an error lower than $1-G=1\%$. For a given $1-G_0$ error, the curve $G(\eta)$ makes it possible to obtain the limit value η_0 which η should not exceed:

$$\eta = \frac{\left(\frac{df}{dt}\right)_{f=f_0} Q^2}{f_0^2} \le \eta_0$$

Linear sweep	$\dfrac{f_2 - f_1}{t_s} \dfrac{Q^2}{f_0^2} \le \eta_0$	$t_s \ge \dfrac{f_2 - f_1}{\eta_0\, f_0^2} Q^2$
Logarithmic sweep	$\dfrac{Q^2}{f_0} \dfrac{\ln\left(f_2/f_1\right)}{t_s} \le \eta_0$	$t_s \ge \dfrac{Q^2}{f_0\, \eta_0} \ln \dfrac{f_2}{f_1}$
		$R_{os} \le \dfrac{f_0\, \eta_0}{Q^2 \ln 2}$
		$R_{om} \le 60 \dfrac{f_0\, \eta_0}{Q^2 \ln 2}$
Hyperbolic sweep	$Q^2 \dfrac{f_2 - f_1}{f_1\, f_2\, t_s} \le \eta_0$	$t_s \ge Q^2 \dfrac{f_2 - f_1}{f_1\, f_2\, \eta_0}$

Table 9.2. *Minimal sweeping duration versus sweeping mode*

It should be noted that, in this last case, t_s is independent of f_0. In both other cases, f_0 being in general unknown, f_0 will be selected as equal to the value of the swept frequency range which leads to the largest duration t_s.

Example 9.5.

The required sweep rate for a logarithmic sweep between 5 Hz and 2,000 Hz.

Let us assume that a resonance with a Q factor having possibly the value 50, can be found on the studied structure (Q factors are in general weaker).

Then the reduced sweep rate $\eta = \dfrac{Q^2}{f_0\; t_b} \ln \dfrac{f_2}{f_1}$ has to be lower than 0.1, or

$$t_b \ge \frac{Q^2}{0.1\, f_0} \ln \frac{f_2}{f_1}$$

The natural frequency being unknown (the purpose of the test is to measure it) we consider the most penalizing case for the calculation of the duration by supposing that f_0 is equal to the lowest frequency of the swept frequency range, i.e. in this example, to 5 Hz. This yields

$$t_b \geq \frac{50^2}{0.15} \ln \frac{2,000}{5}$$

$$t_b \geq 2,9957 \text{ s}$$

i.e. $t_b \geq 499.29 \text{ min}$. The number of octaves between 5 Hz and 2,000 Hz equals

$$n = \frac{\ln(f_2 / f_1)}{\ln 2} = \frac{\ln(2,000/5)}{\ln 2} \approx 8.64$$

The sweep rate expressed in octaves per minute must thus be equal to

$$\frac{8.64}{499.29} \approx 0.017$$

For a Q factor equal to 10, the sweep rate would be equal to 19.97 min (i.e. 0.43 octave/min).

This example shows the limits of the rule which specify a sweep rate equal to 1 octave per minute. This rule does not apply for low natural frequencies or high Q factors.

Figure 9.15 shows the required sweep rate as a function of the natural frequency, for three Q factor values (5, 10 and 50). It can be seen that the sweep rate has to be lower than 1 octave per minute if f_0 is approximately less than 2.5 Hz for Q = 5, 11.5 Hz for Q = 10 and 280 Hz for Q = 50 (Figure 9.16).

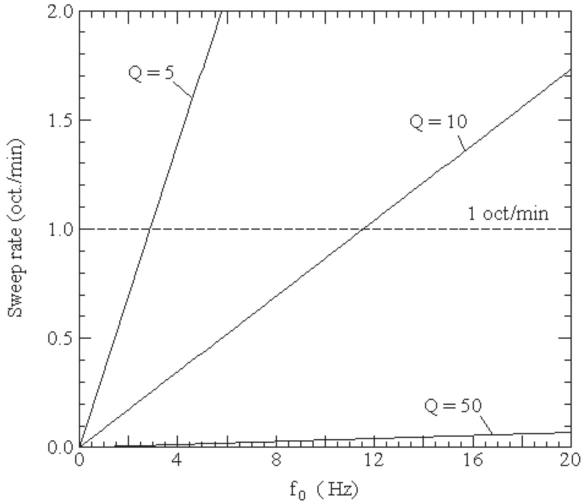


Figure 9.15. *The required sweep rate with respect to the natural frequency for a Q factor equal to 5, 10 and 50*

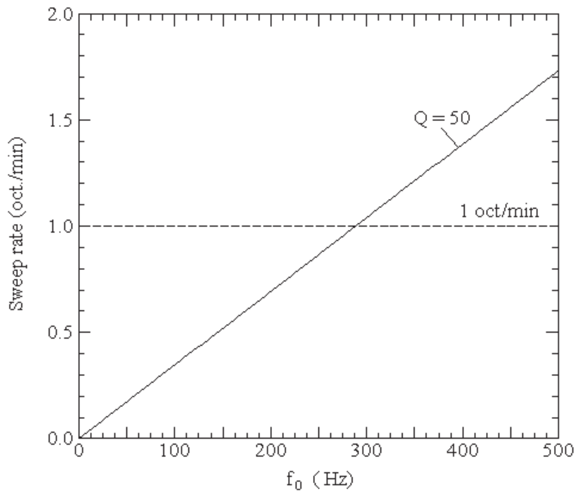


Figure 9.16. *The required sweep rate with respect to the natural frequency for a Q factor equal to 50*

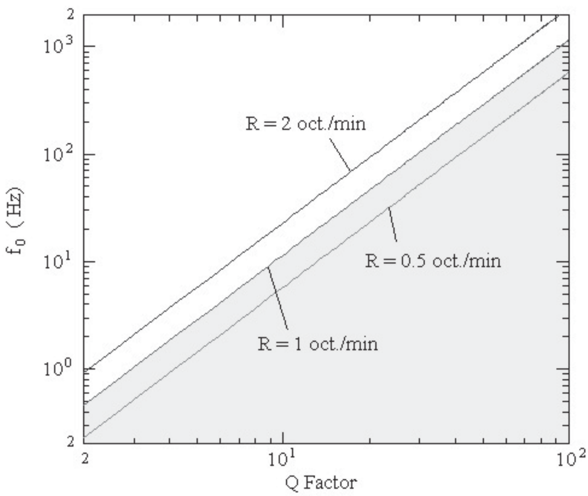


Figure 9.17. *Lowest natural frequency which can be to be correctly measured versus Q factor, for various values of the sweep rate*

The couples, natural frequency – Q factor which can be measured with a negligible error ($\eta < 0.1$) with a sweep rate equal to 1 octave/minute locate at the top of the curve corresponding to this value in Figure 9.17. For example, a Q factor Q = 30 at 20 Hz will not be correctly measured if the sweep rate is equal to 1 octave/minute.

9.4. Choice of amplitude

To search for resonance frequency the amplitude of the excitation must be:

– sufficiently high to correctly "reveal" the peaks of resonance in the response. If the structure is linear, the values of Q measured are independent of the sweep level;

– sufficiently weak not to damage the specimen (by exceeding an instantaneous stress level or by fatigue). The choice must thus be made by taking the levels of the real vibratory environment into account.

If the structure is not linear, the value of Q measured depends on the level of the excitation. Generally, Q decreases when the level of the excitation increases. If we wish to use an experimental transfer function in calculations, we will have to measure this function using an excitation in which the level is close to those of the real environment that the structure will exist in. Two levels (or more) often have to be chosen.

9.5. Choice of sweep mode

The more common use of swept sine is for the determination of the dynamic properties of a structure or of a material (natural frequencies, Q factors). For this type of test, the sweep rate must be sufficiently slow so that the response reaches a strong percentage of the response in steady state excitation (however, it will be seen (Volume 3) that there are methods using very fast sweeps).

The relations allowing the determination of the test duration are based on the calculations carried out in the case of a linear one-degree-of-freedom system. It is admitted then that if this condition is obeyed, the swept sine thus defined will also create in a several degrees-of-freedom system responses very close to those which one would obtain in steady state mode; this assumption can be criticized for structures having modes with close frequencies.

For this use, it may be worthwhile choosing a sweep mode similar to that which leads to the lowest test duration for the same percentage G of steady state response. According to the relations given in section 9.3, it unfortunately appears that the mode determined according to this criterion is a function of the natural frequency f_0 to be measured in the swept frequency interval. Generally a logarithmic sweep is the preferred choice in practice.

The hyperbolic sweep, little used to our knowledge, presents the interesting property of carrying out a constant number of cycles in each interval Δf delimited by the half-power points of a linear one-degree-of-freedom system, whatever the frequency of resonance (or a constant number of cycles of amplitude higher than P% of the Q factor [CRE 54]).

This property can be used to simulate the effects of a shock (the free response of a system is made with the same number of cycles whatever the resonance frequency f_0, provided that the Q factor is constant) or to carry out fatigue tests. In this case, it must be noted that, if the number of cycles carried out at each resonance is the same, the damage created by fatigue will be the same as if the excitation produces a maximum response displacement z_m (i.e. a stress) identical to all the frequencies.

Gertel [GER 61] advises this test procedure for materials for which the lifespan is extremely long (equipment installed on various means of transport, such as road vehicles or aircraft).

The sweep duration t_s can be defined *a priori* or by imposing a given number ΔN of cycles around each resonance, the duration t_s then being calculated from:

$$t_s = Q \, \Delta N \, \frac{f_2 - f_1}{f_1 \, f_2} \qquad [9.42]$$

by introducing the value of the highest Q factor measured during the identification tests (search for resonance) or a value considered representative in the absence of such tests. Limits f_1 and f_2 delimit a frequency range which must include the principal resonance frequencies of the material. This type of sweep is sometimes used in certain spectral analyzers for the study of experimental signals whose frequency varies with time [BIC 70].

To simulate an environment of short duration Δt, such as the propulsion of a missile, on a material of which the resonance frequencies are little known, it is preferable to carry out a test where each resonance is excited during this time Δt (logarithmic sweep) [PIM 62]. The total duration of sweep is then determined by relation [8.38]:

$$t_s = Q \, \Delta t \, \ln \frac{f_2}{f_1} \qquad [9.43]$$

NOTE.–

The test duration t_s thus calculated can sometimes be relatively long, particularly as it is generally necessary to subject the specimen to the vibrations on each of its three axes. Thus, for example, if $Q = 10$, $\Delta t = 20$ s, $f_1 = 10$ Hz and $f_2 = 2{,}000$ Hz:

$$t_s \approx 1{,}060 \text{ s},$$

yielding a total test duration of 3 x 1,060 s = 3,180 s (53 min).

C.E. Crede and E.J. Lunney [CRE 56] recommend that a sweep is carried out in several frequency bands simultaneously to save time. The method consists of cutting out the signal (swept sine) which would be held between times t_1 and t_2 in several intervals taken between t_1 and t_a : t_a and t_b, ... , t_n and t_2, and applying the sum of these signals to the specimen.

Knowing that the material is especially sensitive to the vibrations whose frequency is located between the half-power points, it can be considered that only the swept component which has a frequency near the frequency of resonance will act significantly on the behavior of the material, the others having little effect. In addition, if the specimen has several resonant frequencies, all will be excited simultaneously as in the real environment.

Another possibility consists of sweeping the frequency range quickly and in reducing the sweep rate in the frequency bands where the dynamic response is important to measure the peaks correctly.

C.F. Lorenzo [LOR 70] proposes a control technique based on this principle, usable for linear and logarithmic sweeps, making it possible to reduce with equal precision the test duration by a factor of about 7.5 (for linear sweep).

The justification for a test with linear sweep clearly does not appear, unless we accept the assumption that the Q factor is not constant whatever the natural frequency f_0. If Q can vary according to a law $Q = \text{constant} \times f_0$ (the Q factor often being an increasing function of the natural frequency), it can be shown that the best mode of sweep is the linear sweep.

Type of sweep		Hyperbolic	Logarithmic	Linear
Sweep rate		$\dot{f} = \pm \frac{\pi}{\mu\, Q^2} f^2 = \pm a\, f^2$	$\dot{f} = \pm \frac{\pi}{\gamma\, Q^2} f = \pm \frac{f}{T_1}$	$\dot{f} = \pm \frac{\pi}{\delta\, Q^2} = \pm \alpha$
Constant η		$\eta = \frac{\pi}{\mu} = a\, Q^2$	$\eta = \frac{\pi}{\gamma\, f_0} = \frac{Q^2}{T_1\, f_0}$	$\eta = \frac{\pi}{\delta\, f_0^2} = \frac{\alpha\, Q^2}{f_0^2}$
		$\eta = Q^2\, \frac{f_2 - f_1}{f_1\, f_2\, t_s}$	$\eta = \frac{Q^2}{f_0\, T_1}$	$\eta = \frac{Q^2}{f_0^2}\, \frac{f_2 - f_1}{t_s}$
Law f(t)	f↑	$f = \frac{f_1}{1 - a\, f_1\, t}$	$f = f_1\, e^{t/T_1}$	$f = \alpha\, t + f_1$
	f↓	$f = \frac{f_2}{1 + a\, f_2\, t}$	$f = f_2\, e^{-t/T_1}$	$f = -\alpha\, t + f_2$
	f↑	$E(t) = -\frac{2\,\pi}{a} \ln\left(1 - a\, f_1\, t\right)$	$E = 2\,\pi\, T_1\left(f - f_1\right)$	$E = 2\,\pi\, t\left(\frac{\alpha\, t}{2} + f_1\right)$
	f↓	$E(t) = \frac{2\,\pi}{a} \ln\left(1 + a\, f_2\, t\right)$	$E = 2\pi T_1\left(f_2 - f\right)$	$E = 2\,\pi\, t\left(-\frac{\alpha\, t}{2} + f_2\right)$
	Cst	$a = \frac{f_2 - f_1}{t_s\, f_1\, f_2} = \frac{\pi}{\mu\, Q^2}$	$T_1 = \frac{t_b}{\ln\left(f_2/f_1\right)} = \frac{\gamma\, Q^2}{\pi}$	$\alpha = \frac{\pi}{Q^2\, \delta} = \frac{f_2 - f_1}{t_s}$
		$a = \frac{1}{Q\, \Delta N} = \frac{\eta}{Q^2}$	$T_1 = \frac{Q^2}{\eta\, f_0}$	/
η		$Q^2\, \frac{f_2 - f_1}{f_1\, f_2\, t_s}$	$\frac{Q^2\, \ln\left(f_2/f_1\right)}{f_0\, t_s}$	$\frac{Q^2}{f_0^2}\, \frac{f_2 - f_1}{t_s}$

Table 9.3. *Summary of sweep expressions*

Type of sweep	Hyperbolic	Logarithmic	Linear
Number of cycles carried out during t_s between the frequencies f_1 and f_2	$N_s = \frac{1}{a} \ln \frac{f_2}{f_1}$	$N_s = T_1 \left(f_2 - f_1\right)$	$N_s = \frac{1}{2\,\alpha} \left(f_2^2 - f_1^2\right)$
	$N_s = \frac{f_1\, f_2}{f_2 - f_1} t_s \ln \frac{f_2}{f_1}$	$N_s = \frac{f_2 - f_1}{\ln \frac{f_2}{f_1}} t_s$	$N_s = \frac{f_1 + f_2}{2} t_s$
	$N_s = Q\, \Delta N \ln \frac{f_2}{f_1}$	$N_s = Q\, \Delta N \frac{f_2 - f_1}{f_0}$	$N_s = \frac{Q\, \Delta N}{2\, f_0^2} \left(f_2^2 - f_1^2\right)$
	$N_s = \frac{Q^2}{\eta} \ln \frac{f_2}{f_1}$	$N_s = \frac{Q^2}{\eta\, f_0} \left(f_2 - f_1\right)$	$N_s = \frac{Q^2}{2\, \eta\, f_0^2} \left(f_2^2 - f_1^2\right)$
Sweep duration between the frequencies f_1 and f_2	$t_s = \frac{1}{a} \frac{f_2 - f_1}{f_1\, f_2}$	$t_s = T_1 \ln \frac{f_2}{f_1}$	$t_s = \frac{1}{\alpha} \left(f_2 - f_1\right)$
	$t_s = Q\, f_0\, \Delta t \frac{f_2 - f_1}{f_1\, f_2}$	$t_s = Q\, \Delta t \ln \frac{f_2}{f_1}$	$t_s = \frac{Q\, \Delta t}{f_0} \left(f_2 - f_1\right)$
	$t_s = Q\, \Delta N \frac{f_2 - f_1}{f_1\, f_2}$	$t_s = \frac{Q\, \Delta N}{f_0} \ln \frac{f_2}{f_1}$	$t_s = \frac{Q\, \Delta N}{f_0^2} \left(f_2 - f_1\right)$
	$t_s = \frac{Q^2}{\eta} \frac{f_2 - f_1}{f_1\, f_2}$	$t_s = \frac{Q^2}{\eta\, f_0} \ln \frac{f_2}{f_1}$	$t_s = \frac{Q^2}{\eta\, f_0^2} \left(f_2 - f_1\right)$
Interval of time spent in the band Δf	$\Delta t = \frac{1}{a\, Q\, f_0}$	$\Delta t = \frac{T_1}{Q} = \text{constant}$	$\Delta t = \frac{f_0}{\alpha\, Q}$
	$\Delta t = \frac{f_1\, f_2\, t_s}{f_0\, Q \left(f_2 - f_1\right)}$	$\Delta t = \frac{t_s}{Q \ln f_2/f_1}$	$\Delta t = \frac{f_0\, t_s}{Q \left(f_2 - f_1\right)}$
	$\Delta t = \frac{Q}{\eta\, f_0}$	$\Delta t = \frac{Q}{\eta\, f_0}$	$\Delta t = \frac{Q}{\eta\, f_0}$

Table 9.4. *Summary of sweep expressions*

Type of sweep	**Hyperbolic**	**Logarithmic curve**	**Linear**
Number of cycles carried out in the interval Δf (between the half-power points) of a one-degree-of-freedom system	$\Delta N = \frac{\pi}{a\,Q}$ =constant	$\Delta N = \frac{f_0\ T_1}{Q}$	$\Delta N = \frac{f_0^2}{\alpha\,Q}$
	$\Delta N = \frac{N_s}{\ln \frac{f_2}{f_1}}$	$\Delta N = \frac{f_0\ N_s}{Q\left(f_2 - f_1\right)}$	$\Delta N = \frac{2\,f_0^2\ N_s}{Q\left(f_2^2 - f_1^2\right)}$
	$\Delta N = \frac{f_1\ f_2\ t_s}{Q\left(f_2 - f_1\right)}$	$\Delta N = \frac{f_0\ t_s}{Q \ln f_2/f_1}$	$\Delta N = \frac{f_0^2\ t_s}{Q\left(f_2 - f_1\right)}$
	$\Delta N = \frac{Q}{\eta}$	$\Delta N = \frac{Q}{\eta}$	$\Delta N = \frac{Q}{\eta}$
Number of cycles to be carried out between f_1 and f_0 (resonance frequency)	$N_1 = \frac{1}{a} \ln \frac{f_0}{f_1}$	$N_1 = T_1\left(f_0 - f_1\right)$	$N_1 = \frac{1}{2\,\alpha}\left(f_0^2 - f_1^2\right)$
	$N_1 = Q\ \ \Delta N \ln \frac{f_0}{f_1}$	$N_1 = \frac{Q\,\Delta N}{f_0}\left(f_0 - f_1\right)$	$N_1 = \frac{Q\,\Delta N}{2\,f_0^2}\left(f_0^2 - f_1^2\right)$
	$N_1 = \frac{f_1\ f_2\ t_s}{f_2 - f_1} \ln \frac{f_0}{f_1}$	$N_1 = t_s\,\frac{f_0 - f_1}{\ln f_2/f_1}$	$N_1 = \frac{t_s}{2}\,\frac{f_0^2 - f_1^2}{f_2 - f_1}$
	$N_1 = \frac{Q^2}{\eta} \ln \frac{f_0}{f_1}$	$N_1 = \frac{Q^2}{\eta\,f_0}\left(f_0 - f_1\right)$	$N_1 = \frac{Q^2}{2\,\eta\,f_0^2}\left(f_0^2 - f_1^2\right)$
Time t_1 between f_1 and f_0	$t_1 = \frac{1}{a}\,\frac{f_0 - f_1}{f_0\ \ f_1}$	$t_1 = T_1 \ln \frac{f_0}{f_1}$	$t_1 = \frac{1}{\alpha}\left(f_0 - f_1\right)$
	$t_1 = Q\ \ \Delta N\ \ \frac{f_0 - f_1}{f_0\,f_1}$	$t_1 = \frac{Q\,\Delta N}{f_0} \ln \frac{f_0}{f_1}$	$t_1 = \frac{Q\,\Delta N}{f_0^2}\left(f_0 - f_1\right)$

Table 9.5. *Summary of sweep expressions*

Type of sweep	Hyperbolic	Logarithmic curve	Linear
Time t_1 between f_1 and f_0	$t_1 = t_s \frac{f_2}{f_0} \frac{f_0 - f_1}{f_2 - f_1}$	$t_1 = t_s \frac{\ln f_0/f_1}{\ln f_2/f_1}$	$t_1 = t_s \frac{f_0 - f_1}{f_2 - f_1}$
	$t_1 = \frac{Q^2}{\eta} \frac{f_0 - f_1}{f_0 f_1}$	$t_1 = \frac{Q^2}{\eta f_0} \ln \frac{f_0}{f_1}$	$t_1 = \frac{Q^2}{\eta f_0^2} \left(f_0 - f_1\right)$
Number of cycles to carry out between $f_1 = 0$ and f_0	/	$N_0 = T_1 f_0$	$N_0 = \frac{1}{2\alpha} f_0^2$
	/	$N_0 = Q \Delta N$	$N_0 = \frac{Q \Delta N}{2}$
	/	/	$N_0 = \frac{f_0^2}{2 f_2} t_s$
	/	/	$N_0 = \frac{Q^2}{2 \eta}$
Time t_0 between $f_1 = 0$ and f_0	/	/	$t_0 = \frac{f_0}{\alpha}$
	/	/	$t_0 = \frac{Q \Delta N}{f_0}$
	/	/	$t_0 = \frac{f_0}{f_2} t_s$
	/	/	$t_0 = \frac{Q^2}{\eta f_0}$
Mean frequency	$f_m = \frac{f_1 f_2}{f_2 - f_1} \ln \frac{f_2}{f_1}$	$f_m = \frac{f_2 - f_1}{\ln f_2/f_1}$	$f_m = \frac{f_1 + f_2}{2}$
Time spent between f_a and $f_c \in (f_1, f_2)$	$t_c - t_a = \left[\frac{1}{f_a} - \frac{1}{f_c}\right] \frac{f_1 f_2}{f_2 - f_1} t_s$	$t_c - t_a = t_s \frac{\ln f_c/f_a}{\ln f_2/f_1}$	$t_c - t_a = t_s \frac{f_c - f_a}{f_2 - f_1}$
	$t_c - t_a = \frac{Q^2}{\eta} \left(\frac{1}{f_a} - \frac{1}{f_c}\right)$	$t_c - t_a = \frac{Q^2}{\eta f_0} \ln \frac{f_c}{f_a}$	$t_c - t_a = \frac{Q^2}{\eta f_0^2} \left(f_c - f_a\right)$

Table 9.6. *Summary of sweep expressions*

Type of sweep	**Hyperbolic**	**Logarithmic curve**	**Linear**
Numbers of cycles per octave (f_A = lower frequency of the octave)	$N_2 = \dfrac{\ln 2}{a}$	$N_2 = T_1\ f_A$	$N_2 = \dfrac{3}{2\ \alpha} f_A^2$
	$N_2 = 2\ f_A\ t_s\ \ln 2$	$N_2 = \dfrac{f_A\ t_s}{\ln 2}$	$N_2 = \dfrac{3\ f_A\ t_s}{2}$
	$N_2 = Q\,\Delta N \ln 2$	$N_2 = Q\,\Delta N \dfrac{f_A}{f_0}$	$N_2 = \dfrac{3\,Q\,\Delta N}{2\,f_0^2} f_A^2$
	$N_2 = \dfrac{Q^2}{\eta} \ln 2$	$N_2 = \dfrac{3\,Q^2\,f_A^2}{2\,\eta\,f_0^2}$	$N_2 = \dfrac{Q^2\,f_A^2}{\eta\,f_0}$
Time necessary to sweep an octave	$t_2 = \dfrac{1}{2\,a\ f_A}$	$t_2 = T_1 \ln 2$	$t_2 = \dfrac{f_A}{\alpha}$
	$t_2 = \dfrac{Q\,f_0\,\Delta t}{2\,f_A}$	$t_2 = Q\,\Delta t \ln 2$	$t_2 = \dfrac{Q\,\Delta t}{f_0} f_A$
	$t_2 = \dfrac{Q\,\Delta N}{2\,f_A}$	$t_2 = \dfrac{Q\,\Delta N}{f_0} \ln 2$	$t_2 = \dfrac{Q\,\Delta N}{f_0^2} f_A$
	‼ EMBED	$t_2 = \dfrac{Q^2\ \ln 2}{\eta\,f_0}$	$t_2 = \dfrac{Q^2\,f_A}{\eta\,f_0^2}$

Table 9.7. *Summary of sweep expressions*

Type of sweep	**Hyperbolic**	**Logarithmic curve**	**Linear**
Time necessary to sweep $1/n^{th}$ octave	$t_n = \frac{1}{a\, f_A} \frac{2^{1/n} - 1}{2^{1/n}}$	$t_n = \frac{T_1}{n} \ln 2$	$t_n = \frac{f_A}{\alpha}\left(2^{1/n} - 1\right)$
	$t_n = \frac{Q\, \Delta N}{f_A} \frac{2^{1/n} - 1}{2^{1/n}}$	$t_n = \frac{Q\, \Delta N}{f_0\, n} \ln 2$	$t_n = \frac{Q\, \Delta N}{f_0} f_A \left(2^{1/n} - 1\right)$
	$t_n = \frac{Q\, f_0\, \Delta t}{f_A} \frac{2^{1/n} - 1}{2^{1/n}}$	$t_n = \frac{Q\, \Delta t}{n} \ln 2$	$t_n = \frac{Q\, \Delta t}{f_0} f_A \left(2^{1/n} - 1\right)$
	$t_n = \frac{Q^2}{\eta\, f_A} \frac{2^{1/n} - 1}{2^{1/n}}$	$t_n = \frac{Q^2 \ln 2}{\eta\, f_0\, n}$	$t_n = \frac{Q^2}{\eta\, f_0} f_A \left(2^{1/n} - 1\right)$
Sweep rate	/	$R_{om} = \frac{60 \ln f_2/f_1}{t_s \ln 2}$	$R = 60 \frac{f_2 - f_1}{t_s}$
	/	$R_{om} = \frac{60}{T_1 \ln 2}$	/
	/	$R_{om} = \frac{60\, \eta\, f_0}{Q^2 \ln 2}$	$R = 60 \frac{\eta\, f_0^2}{Q^2}$

Table 9.8. *Summary of sweep expressions*

Appendix

Laplace Transformations

A.1. Definition

Consider a real continuous function $f(t)$ of the real definite variable t for all $t \geq 0$ and set

$$F(p) \equiv L[f(t)] = \int_0^{\infty} e^{-p\,t}\, f(t)\, dt \qquad [A.1]$$

(provided that the integral converges). The function $f(t)$ is known as "*origina*" or "*object*", the function $F(p)$ as "*image*" or "*transform*".

Example A.1.

Consider a step function applied to $t = 0$ and of amplitude f_m. Integral [A.1] gives simply

$$F(p) = \int_0^{\infty} e^{-p\,t}\, f_m\, dt = f_m \left[-\frac{e^{-p\,t}}{p} \right]_0^{\infty} \qquad [A.2]$$

$$F(p) = \frac{f_m}{p} \qquad [A.3]$$

A.2. Properties

In this section some useful properties of this transformation are given, without examples.

A.2.1. *Linearity*

$$L\left[f_1(t) + f_2(t)\right] = L\left[f_1(t)\right] + L\left[f_2(t)\right] \qquad [A.4]$$

$$L\left[c\ f(t)\right] = c\ L\left[f(t)\right] \qquad [A.5]$$

A.2.2. *Shifting theorem (or time displacement theorem)*

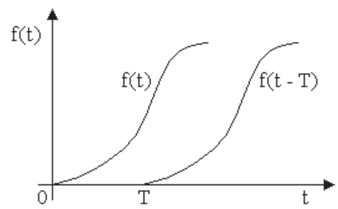


Figure A.1. *Shifting of a curve with respect to the variable t*

Consider $f(t)$, a transformable function, and operate a translation parallel to the axis 0t, of amplitude T ($T \geq 0$). If $F(p)$ is the transform of $f(t)$, the transform of $f(t - T)$ is equal to

$$\phi(p) = e^{-p\,T}\ F(p) \qquad [A.6]$$

(formula of the translation on the right or *shifting theorem*).

Application

A square shock can be considered as being created by the superposition of two levels, one of amplitude f_m applied at time $t = 0$ (transform: $\dfrac{f_m}{p}$, see preceding *example), the other of amplitude* $-f_m$ *applied at time* $t=\tau$, *of transform* $-\dfrac{f_m}{p}e^{-p\tau}$ *yielding the expression of the transform*

$$L(p)=\frac{f_m}{p}\left(1-e^{-p\tau}\right) \qquad [A.7]$$

A.2.3. *Complex translation*

$$L\left[f(t)\,e^{-a\,t}\right]=F(p+a) \qquad [A.8]$$

This result makes it possible to write the transform of $f(t)\,e^{-a\,t}$ directly when that of $f(t)$ is known.

A.2.4. *Laplace transform of the derivative of f(t) with respect to time*

The transform of the derivative $f'(t)$ of $f(t)$ with respect to t is equal to

$$L[f'(t)]=p\,F(p)-f\left(0^+\right) \qquad [A.9]$$

where $F(p)$ is the Laplace transform of $f(t)$ and $f\left(0^+\right)$ is the value of the first derivative of $f(t)$ for $t=0$ (as t approaches zero from the positive side).

In a more general way, the transform of the n[th] derivative of $f(t)$ is given by

$$\int L\left[\frac{d^n f}{dt^n}\right]=p^n F(p)-p^{n-1}f\left(0^+\right)-p^{n-2}f'\left(0^+\right)-\cdots-pf^{(n-2)}\left(0^+\right)-f^{(n-1)}\left(0^+\right)$$

[A.10]

where $f^{(n-1)}\left(0^+\right)$, $f^{(n-2)}\left(0^+\right)$,..., $f'\left(0^+\right)$ are the successive derivatives of $f(t)$ for $t=0$ (as t approaches zero from the positive side).

A.2.5. *Derivative in the p domain*

The n[th] derivative of the transform $F(p)$ of a function $f(t)$ with respect to the variable p is given by

$$\frac{d^n F}{dp^n} = (-1)^n \ L\left[t^n \ f(t)\right] \tag{A.11}$$

A.2.6. *Laplace transform of the integral of a function f(t) with respect to time*

If $\lim \int_0^\varepsilon f(t)\ dt = 0$ when $\varepsilon \to 0$,

$$L\left[\int_0^t f(t)\ dt\right] = \frac{F(p)}{p} \tag{A.12}$$

and to the order n

$$L\left[\int_0^t dt \int_0^t dt \cdots \int_0^t f(t)\ dt\right] = \frac{F(p)}{p^n} \tag{A.13}$$

A.2.7. *Integral of the transform $F(p)$*

The inverse transform of the integral of $F(p)$ between p and infinity is equal to:

$$\int_p^\infty F(p)\ dp = L\left[\frac{f(t)}{t}\right] \tag{A.14}$$

When integrating n times between the same limits, it becomes

$$\int_p^\infty dp \int_p^\infty dp \cdots \int_p^\infty F(p)\ dp = L\left[\frac{f(t)}{t^n}\right] \tag{A.15}$$

A.2.8. *Scaling theorem*

If a is a constant,

$$L\left[f\left(\frac{t}{a}\right)\right] = a\ F(a\ p) \qquad [A.16]$$

$$L\left[f(a\ t)\right] = \frac{1}{a}\ F\left(\frac{p}{a}\right) \qquad [A.17]$$

A.2.9. *Theorem of damping or rule of attenuation*

$$F(p + a) = \int_0^{\infty} e^{-p\ t}\ e^{-a\ t}\ f(t)\ dt \qquad [A.18]$$

The inverse transform of $F(p + a)$ is thus $e^{-a\ t}\ f(t)$. It is said that the function $e^{-a\ t}$ damps the function $f(t)$ when a is a positive real constant.

A.3. Application of Laplace transformation to the resolution of linear differential equations

For the principal use of the Laplace transformation, the interest resides in the property relating to derivatives and integrals which become, after transformation, products or quotients of the transform $F(p)$ of $f(t)$ by p or its powers.

Let us consider, for example, the second order differential equation

$$\frac{d^2q(t)}{dt^2} + a\ \frac{dq(t)}{dt} + b\ q(t) = f(t) \qquad [A.19]$$

where a and b are constants. Let us make $Q(p)$ and $F(p)$ the Laplace transforms of $q(t)$ and $f(t)$ respectively. From the relationships in section A.2.4, we have

$$L[\ddot{q}(t)] = p^2\ Q(p) - p\ q(0) - \dot{q}(0) \qquad [A.20]$$

$$L[\dot{q}(t)] = p\ Q(p) - q(0) \qquad [A.21]$$

where $q(0)$ and $\dot{q}(0)$ are the values of $q(t)$ and its derivative for $t = 0$. Due to the linearity of the Laplace transformation, it is possible to transform each member of differential equation [A.19] term by term:

$$L[\ddot{q}(t)] + a\ L[\dot{q}(t)] + b\ L[q(t)] = L[f(t)] \qquad [A.22]$$

By replacing each transform with its expression, this becomes

$$p^2\ Q(p) - p\ q(0) - \dot{q}(0) + a\ [p\ Q(p) - q(0)] + b\ Q(p) = F(p) \qquad [A.23]$$

$$Q(p) = \frac{F(p) + p\ q(0) + a\ q(0) + \dot{q}(0)}{p^2 + a\ p + b} \qquad [A.24]$$

Let us expand the rational fraction $\frac{F(p) + p\,q(0) + a\,q(0) + \dot{q}(0)}{p^2 + a\,p + b}$ into partial fractions; while noting that p_1 and p_2 are the roots of the denominator $p^2 + a\,p + b$, we have

$$\frac{A\ p + B}{p^2 + a\ p + b} = \frac{C}{p - p_1} + \frac{D}{p - p_2} \qquad [A.25]$$

with

$$A = q(0) \qquad C = \frac{A\ p_1 + B}{p_1 - p_2}$$

$$B = a\ q(0) + \dot{q}(0) \qquad D = -\frac{A\ p_2 + B}{p_1 - p_2}$$

yielding

$$Q(p) = \frac{F(p)}{p^2 + a\ p + b} + \frac{1}{p_1 - p_2}\left[\frac{A\ p_1 + B}{p - p_1} - \frac{A\ p_2 + B}{p - p_2}\right] \qquad [A.26]$$

i.e.

$$Q(p) = \frac{F(p)}{p_1 - p_2}\left[\frac{1}{p - p_1} - \frac{1}{p - p_2}\right]$$

$$+\frac{1}{p_1-p_2}\left[\frac{q(0)\ p_1+a\ q(0)+\dot{q}(0)}{p-p_1}-\frac{q(0)\ p_2+a\ q(0)+\dot{q}(0)}{p-p_2}\right] \quad \text{[A.27]}$$

$$Q(p)=\int_0^t \frac{f(\lambda)}{p_1-p_2}\left[e^{p_1\ (t-\lambda)}-e^{p_2\ (t-\lambda)}\right]d\lambda$$

$$+\frac{1}{p_1-p_2}\left\{\left[q(0)\ p_1+a\ q(0)+\dot{q}(0)\right]e^{p_1\ t}-\left[q(0)\ p_2+a\ q(0)+\dot{q}(0)\right]e^{p_2\ t}\right\}$$

[A.28]

where λ is a variable of integration. In the case of a system initially at rest, $q(0)=\dot{q}(0)=0$ and

$$Q(p)=\int_0^t \frac{f(\lambda)}{p_1-p_2}\left[e^{p_1\ (t-\lambda)}-e^{p_2\ (t-\lambda)}\right]d\lambda \quad \text{[A.29]}$$

A.4. Calculation of inverse transform: Mellin–Fourier integral or Bromwich transform

Once the calculations are carried out in the domain of p, where they are easier, it is necessary to return to the time domain and to express the output variables as a function of t.

We saw that the Laplace transform $F(p)$ of a function $f(t)$ is given by [A.1]

$$F(p)\equiv L[f(t)]=\int_0^\infty e^{-p\ t}\ f(t)\ dt \quad \text{[A.30]}$$

The inverse transformation is defined by the integral known as the Mellin–Fourier integral

$$L^{-1}[F(p)]\equiv f(t)=\frac{1}{2\ \pi\ i}\int_{C-i\ \infty}^{C+i\ \infty}F(p)\ e^{p\ t}\ dp \quad \text{[A.31]}$$

and calculated, for example, on a Bromwich contour composed of a straight line parallel to the imaginary axis, of positive abscissae C [ANG 61], C being such that all the singularities of the function $F(p)\, e^{p\,t}$ are on the left of the line [BRO 53]; this contour thus goes from $C - i\,\infty$ to $C + i\,\infty$.

If the function $F(p)\, e^{p\,t}$ only has poles, then the integral is equal to the sum of the corresponding residues, multiplied by $2\,\pi\,i$. If this function has singularities other than poles, it is necessary to find, in each case, a equivalent contour to the Bromwich contour allowing calculation of the integral [BRO 53] [QUE 65].

The two integrals [A.1] and [A.31] establish a one-to-one relationship between the functions of t and those of p.

These calculations can in practice be rather complex and, where possible, it is preferred to use tables of inverse transform which directly provide the transforms of the most common functions [ANG 61, DIT 67, HLA 69, SAL 71]. The inverse transformation is also performed using these tables after having expressed results as a function of p in a form revealing transforms whose inverse transform appears in Table A.1.

Example A.2.

Let us consider the expression of the response of a one-degree-of-freedom damped system subjected to a rectangular shock of amplitude one of the form $f(t) = 1$ and of duration τ. For this length of time τ, i.e. for $t \le \tau$, the Laplace transform is given by (Table A.1):

$$F(p) = \frac{1}{p} \qquad \text{[A.32]}$$

NOTE.– *After the end of the shock, it would be necessary to use the relation* $F(p) = \dfrac{1 - e^{-p\,\tau}}{p}$.

Equation [A.24] applies with a = 2 ξ and b = 1, yielding

$$q(t) = L^{-1}\left[\frac{\dfrac{1}{p} + p\,q_0 + 2\,\xi\,q_0 + \dot{q}_0}{\left(p^2 + 2\,\xi\,p + 1\right)}\right] \qquad \text{[A.33]}$$

$$q(t) = L^{-1}\left[\frac{1}{p\left(p^2+2\xi p+1\right)}\right] + q_0 \; L^{-1}\left[\frac{p}{p^2+2\xi p+1}\right]$$

$$+ \left(2\xi q_0 + \dot{q}_0\right) L^{-1}\left[\frac{1}{p^2+2\xi p+1}\right] \qquad [A.34]$$

yielding, using Table A.1 ($\xi \neq 1$):

$$q(t) = 1 - \frac{e^{-\xi t}}{\sqrt{1-\xi^2}}\left(\xi \sin\sqrt{1-\xi^2}\; t + \sqrt{1-\xi^2}\; \cos\sqrt{1-\xi^2}\; t\right)$$

$$+ q_0 \frac{e^{-\xi t}}{\sqrt{1-\xi^2}}\left(\sqrt{1-\xi^2}\; \cos\sqrt{1-\xi^2}\; t - \xi \sin\sqrt{1-\xi^2}\; t\right)$$

$$+\left(2\xi q_0 + \dot{q}_0\right) \frac{e^{-\xi t}}{\sqrt{1-\xi^2}} \sin\sqrt{1-\xi^2}\, t \qquad [A.35]$$

$$q(t) = 1 + \frac{e^{-\xi t}}{\sqrt{1-\xi^2}}\left\{\sqrt{1-\xi^2}\; \left(q_0 - 1\right) \cos\sqrt{1-\xi^2}\; t - \left[\xi\left(1-q_0\right) - \dot{q}_0\right] \sin\sqrt{1-\xi^2}\; t\right\}$$

[A.36]

A.5. Laplace transforms

Function $f(t)$	**Transform** $L[f(t)] = F(p)$
1	$\frac{1}{p}$
t	$\frac{1}{p^2}$
e^{at}	$\frac{1}{p-a}$
$\sin a\, t$	$\frac{a}{p^2+a^2}$

$\cos a t$	$\dfrac{p}{p^2 + a^2}$
$\text{sh } a t$	$\dfrac{a}{p^2 - a^2}$
$\text{ch } a t$	$\dfrac{p}{p^2 - a^2}$
t^2	$\dfrac{2}{p^3}$
t^n	$\dfrac{n!}{p^{n+1}}$ (n integer ≥ 0)
$\sin^2 t$	$\dfrac{2}{p\left(p^2 + 4\right)}$
$\cos^2 t$	$\dfrac{p^2 + 2}{p\left(p^2 + 4\right)}$
$a t - \sin a t$	$\dfrac{a^3}{p^2\left(p^2 + a^2\right)}$
$\sin a t - a t \cos a t$	$\dfrac{2 a^3}{\left(p^2 + a^2\right)^2}$
$t \sin a t$	$\dfrac{2 a p}{\left(p^2 + a^2\right)^2}$
$\sin a t + a t \cos a t$	$\dfrac{2 a p^2}{\left(p^2 + a^2\right)^2}$
$t \cos a t$	$\dfrac{p^2 - a^2}{\left(p^2 + a^2\right)^2}$
$a \sin b t - b \sin a t$	$\dfrac{a b\left(a^2 - b^2\right)}{\left(p^2 + a^2\right)\left(p^2 + b^2\right)}$

$\frac{1}{2a^3}\left(\sin at - at\cos at\right)$	$\frac{1}{\left(p^2+a^2\right)^2}$
$\frac{\cos at - \cos bt}{b^2-a^2}$	$\frac{p}{\left(p^2+a^2\right)\left(p^2+b^2\right)}$
$\frac{e^{-at}-e^{-bt}}{b-a}$	$\frac{1}{(p+a)(p+b)}$
$\frac{b\,e^{-bt}-a\,e^{-at}}{b-a}$	$\frac{p}{(p+a)(p+b)}$
$t\,e^{at}$	$\frac{1}{(p-a)^2}$
$t^n\,e^{at}$	$\frac{n!}{(p-a)^{n+1}}(n=1,2,3,...)$
$e^{-at}\cos bt$	$\frac{p+a}{(p+a)^2+b^2}$
$e^{-at}\sin bt$	$\frac{b}{(p+a)^2+b^2}$
$1-\frac{e^{-\frac{at}{2}}}{\sqrt{1-\frac{a^2}{4}}}\left[\frac{a}{2}\sin\sqrt{1-\frac{a^2}{4}}\,t+\sqrt{1-\frac{a^2}{4}}\cos\sqrt{1-\frac{a^2}{4}}\,t\right]$	$\frac{1}{p\left(p^2+ap+1\right)}$
$\frac{e^{-\xi t}}{\sqrt{1-\xi^2}}\left[\sqrt{1-\xi^2}\cos\sqrt{1-\xi^2}\,t-\xi\sin\sqrt{1-\xi^2}\,t\right]$ $(\xi<1)$	$\frac{p}{p^2+2\xi p+1}$
$\frac{e^{-\xi t}}{\sqrt{1-\xi^2}}\sin\sqrt{1-\xi^2}\,t$	$\frac{1}{p^2+2\xi p+1}$
$\frac{e^{-\xi h t}}{h\sqrt{1-\xi^2}}\sin h\sqrt{1-\xi^2}\,t$	$\frac{1}{p^2+2h\xi p+h^2}$

$e^{-\xi h t} \cos h\sqrt{1-\xi^2}\ t - \frac{\xi\, e^{-\xi h t}}{\sqrt{1-\xi^2}} \sin h \sqrt{1-\xi^2}\ t$	$\frac{p}{p^2 + 2 h \xi p + h^2}$
$\frac{e^{-\xi t}\left(\sin\sqrt{1-\xi^2}\ t - t\sqrt{1-\xi^2}\cos\sqrt{1-\xi^2}\ t\right)}{2\left(1-\xi^2\right)^{3/2}}$	$\frac{1}{\left(p^2 + 2\xi p + 1\right)^2}$
$\frac{t\, e^{-\xi t}}{2\sqrt{1-\xi^2}} \sin\sqrt{1-\xi^2}\ t$	$\frac{p}{\left(p^2 + 2\xi p + 1\right)^2}$
$t\, e^{-\xi \Omega t} \sin \Omega t$	$-2\frac{\left(p+\xi\Omega\right)\Omega}{\left[\left(p+\xi\Omega\right)^2 + \Omega^2\right]^2}$
$t\, e^{-\xi \Omega t} \cos \Omega t$	$\frac{p^2 + 2\xi\Omega p + \xi^2\left(\Omega^2 - 1\right)}{\left[\left(p+\xi\Omega\right)^2 + \Omega^2\right]^2}$
$\frac{1+\xi^2}{2} e^{-\xi\Omega t}\left[\frac{\sin\Omega t}{\Omega} - t\cos\left(\Omega t + \phi\right)\right]$	$\frac{p^2}{\left[\left(p+\xi\Omega\right)^2 + \Omega^2\right]^2}$
$\frac{1+\xi^2}{2} e^{-\xi\Omega t}\left[\frac{\sin\Omega t}{\Omega} - t\cos\left(\Omega t + \phi\right)\right]$ $-t\, e^{-\xi\Omega t}\cos\Omega t - \xi\, t\, e^{-\xi\Omega t}\sin\Omega t$	$\frac{\Omega^2\left(1+\xi^2\right)}{\left[\left(p+\xi\Omega\right)^2 + \Omega^2\right]^2}$
$t - 2\xi + e^{-\xi t}\left[2\xi\cos\sqrt{1-\xi^2}\ t + \frac{2\xi^2 - 1}{\sqrt{1-\xi^2}}\sin\sqrt{1-\xi^2}\ t\right]$	$\frac{1}{p^2\left(p^2 + 2\xi p + 1\right)}$

Table A.1. *Laplace transforms*

These transforms can be used to calculate others, for example starting from decompositions in partial fractions such as

$$\frac{1}{p\left(p^2 + a\, p + 1\right)} = \frac{1}{p} - \frac{p + a}{p^2 + a\, p + 1} \qquad [A.37]$$

$$\frac{1}{p^2\left(p^2+a\,p+1\right)}=\frac{1}{p^2}-\frac{a}{p}+\frac{a\,p}{p^2+a\,p+1}+\frac{a^2-1}{p^2+a\,p+1} \qquad [A.38]$$

where, in these relations, there arises $a = 2\,\xi$.

A.6. Generalized impedance – the transfer function

If the initial conditions are zero, equation [A.24] can be written

$$\left(p^2+a\,p+b\right)Q(p)=F(p) \qquad [A.39]$$

i.e. while setting

$$Z(p)=p^2+a\,p+b \qquad [A.40]$$

$$F(p)=Z(p)\,Q(p) \qquad [A.41]$$

By analogy with the equation which links the current $I(\Omega)$ (output variable) and the tension $E(\Omega)$ (input variable) in an electrical supply network in sinusoidal mode

$$E(\Omega)=Z(\Omega)\,I(\Omega) \qquad [A.42]$$

$Z(p)$ is called the *generalized impedance of the system*, and $Z(\Omega)$ is the *transfer impedance* of the circuit. The inverse of $A(p)$ of $Z(p)$, $1/Z(p)$, is the *operational admittance*. The function $A(p)$ is also termed the *transfer function*. It is by its intermediary that the output is expressed versus the input:

$$Q(p)=\frac{1}{Z(p)}\,F(p)=A(p)\,F(p) \qquad [A.43]$$

Vibration Tests: a Brief Historical Background

The first studies on shocks and vibrations were carried out at the beginning of the 1930s to improve the behavior of buildings during earthquakes. With this framework in mind, M.A. Biot defined the shock spectrum to characterize these phenomena and to compare their severity. The term *shock spectrum* has since been changed to *shock response spectrum* (SRS) in order to avoid any confusion and to clearly show that it characterizes the response of a (linear with one-degree-of-freedom) system subjected to the studied shock.

Vibration tests on aircraft were developed from 1940 to verify the resistance of parts and equipment prior to their first use [BRO 67].

Such tests became necessary as a result of:

– the increasing complexity of on-board flight equipment which was more sensitive to vibrations;

– improved performance of aircraft (and, more generally, of vehicles), to the extent that the sources of vibration initially localized in engines became extended substantially outwards to the ambient medium (aerodynamic flows).

The chronology of such developments can be summarized as follows [HUN 99], [PUS 77]:

1940 Measurement of resonance frequencies.
Self-damping tests.
Sine tests (at fixed frequency) corresponding to the frequencies created by engines running at a constant speed.
Combined tests (temperature, humidity, altitude).

The exciters which were used at the time were mechanical and the vibration was created by the rotation of off-centered mass. Shock machines, of standard impact, were developed shortly after. The table, guided by vertical columns, fell into a tub filled with sand. The shape of the shock created during the impact could be selected by fixing pieces of wood of particular form under the table of the machine.

1946 The first electrodynamic exciters were developed [DEV 47], [IMP 47]. Their limited power made it possible to carry out only tests of sinusoidal vibrations.

At this time, the first standards were written and used for the acceptance tests carried out on each material. The measured vibratory environments being in general of random type, the standards quickly evolved towards "swept sine" tests which made it possible to cover a broad range of frequencies in spite of the limitations of the exciters.

1950 Swept sine tests were introduced to simulate variations in engine speed, or to excite all of the resonance frequencies of the test item, regardless of its value.

Test severities resulted from measurements of the real environment taken on a category of carriers. The measured signals were filtered using square filters and the largest peak of the response of the filters was drawn on an amplitude-filter central frequency diagram.

The group of points thus obtained was largely enveloped by straight line segments in order to define a swept sine test, with constant displacement at low frequency, then eventually with constant velocity and finally with constant acceleration.

Thus, the standards proposed swept sine tests, which are often still specified today in certain documents. It was, however, understood that it would be better to apply random vibration tests, and it was attempted to specify swept narrowband random vibration tests, broadband random vibration tests not being possible because of the lack of power of machines. All of these studies were essentially completed for military applications.

Similarly to today, shocks carried out on the shock machines were limited to simple shapes: half-sine wave, square (or trapezoidal) shock and terminal peak sawtooth shock. For convenience, and in order to reduce costs, the possibility of creating shocks directly with an exciter was studied. With the test specimen remaining on the same machine for both shocks and vibrations, it was possible to gain much time.

1953 Specifications and tests with random vibrations (introduction of jet engines, simulation of jet flows and aerodynamic turbulences with continuous spectra). These tests were highly controversial until the 1960s [MOR 53]. To overcome the insufficient power of such installations, attempts were made to promote swept narrowband random variations in the frequency domain of interest [OLS 57].

1955 First publications on acoustic vibrations (development of jet rockets and engines, effect of acoustic vibrations on their structures and equipment).

1957 First acoustic chambers [BAR 57], [COL 59], [FRI 59].

1960 The specification of random vibration became essential and the possibilities of an exciter were sufficient to carry out broad band random vibrations. Studies were carried out to determine equivalences between random and sine vibrations.

Missiles and also space vehicles and satellite launchers use many pyrotechnic devices, which enable them to use very precise time slots during the operation of equipment (separation between propelling stages, firing of an engine for example). These devices contain small amounts of explosives which generate very short, but locally very severe, high frequency shocks, which are propagated in the structures while attenuating and combining with the response of structures. The frequency contents of these "pyrotechnical shocks" thus increases with frequencies closer to those of the equipments. Their amplitude remaining still significant, these shocks can produce important malfunctions.

In the 1960s some publications reported the new interest in these shocks, which were often regarded as not very severe because of the very high frequency. Following incidents, a very large number of works were published in the early 1980s, and this interest has continued until today, both to measure shocks, study the propagation, to attenuate or to filter them mechanically and to take them into account in the softwares used for the dimensioning of parts.

With acceptance tests arriving late in the design/production process, in the event of problems with the behavior of the materials, it was preferred in around 1960 to carry out qualification tests before beginning the production of the products, using standards still defined without reference to the real environment.

1965 J.W. Cooley and J.W. Tukey's algorithm for calculating FFTs [COO 65].

Although the spectrum of shock is still criticized and not used to specify the shocks in the standards, it was very useful for severity comparison of several shocks in the absence of a more powerful tool. Some first attempts were made to try to control the exciters directly starting from a shock response spectrum, in order to be able to simulate shocks for which the SRS is difficult to reproduce starting from a simple shape shock.

1967 Increasing number of publications on acoustic vibrations.

1970 Tri-axial test facility [DEC 70].

Development of digital control systems.

1975 The use of standards that superficially recreate the environment sometimes led to the creation of products which were too large for their environment, or sometimes to imaginary problems – the material being designed more to resist the qualification tests than to resist real conditions of the environment. It was in addition often necessary to reduce the mass of the material to the maximum. It was thus necessary to dimension the material to resist, with a certain margin, its real conditions of use.

This remark was at the origin of the development of a method transforming and epitomizing measurements of the mechanical environment into test specifications expressed in a simple form and with a reduced duration in order to reduce costs. This procedure implies:

– determination of the life profile of the products;

– searching for measurements corresponding to each condition of the identified environment;

– then, the synopsis of all the data collected in order to calculate the simplest possible specification from it, with a small number of tests of reduced duration if the real environment is of long duration;

– finally, the tests thus determined must be organized in order to ensure the best representativeness of the tests with the lowest cost.

1975 Extreme response spectra and fatigue damage spectra developed; useful in writing specifications (a method in four stages starting from the lifecycle profile).

Equivalence necessary during the synopsis is based on two criteria: the reproduction in the tests of the largest stresses created in the product when it is in its real environment (except duration reduction) and the reproduction of the fatigue damage related to a large number of stress cycles undergone by the material. These two criteria are the base of the extreme response spectra and fatigue damage spectra developed around 1975, unifying the methods of shock and vibration analysis. The application of this method supposes the exploitation of many measurements and the realization of calculations, which led to the development of software running under Windows and, associated with databases, under Unix.

1984 Account taken of the tailoring of tests in certain standard documents (MIL-STD-810F [MIL 97], [GAM 92]): development of specifications on the basis of measuring the real environment.

Versions of the standards MIL STD 810 D in the USA and GAM EG 13 in France, then NATO standards, have themselves evolved in this direction in years 1980/1985, requiring the *test tailoring*. However, only standard GAM EG 13 proposes and describes in its technical appendices the method by the equivalence of damages.

At that time the MIL STD 810 standard explicitly authorized the use of the shock response spectrum to specify shocks.

1995 Taking the environment into account in the project management (according to the R.G. Aero 00040 Recommendation).

Test tailoring makes it possible to demonstrate during the qualification tests that the developed product will be resistant to its future real environment. These tests arrive late in the event of failure, since they oblige a resumption of the design of the object. This is why in around 1990 the concept of *tailoring the product to its environment* was introduced, which encourages taking into account the real environment through a step of tailoring at the very beginning of the project.

	1930	**Shock spectrum (earthquakes)**
	1935	
	1940	**High impact shock machines** Measurement of resonance frequencies Self-damping tests Sine tests (on resonance)
First standards **Acceptance tests**	1945	First electrodynamic exciter **Impact shock machines**
	1950	Swept sine tests Swept narrow band random tests
	1955	**Shocks on exciters**
Qualification tests **Severities specified in standards**	1960	Broad band random vibration tests **Pyrotechnic shocks**
	1965	**Exciter control using a SRS**
	1970	Triaxial test facility **Development of exciter control using a SRS**
	1975	MRS and FDS - Test tailoring 4 steps (IRIS 80)
	1980	**Papers on pyroshocks**
Qualification tests **Test tailoring** **(MIL STD 810 D, GAM EG13)** **NATO Standards**	1985	**Reference to SRS in the MIL STD810D Std** Test tailoring softwares on PC
	1990	Test tailoring softwares under UNIX
Taking environement into account **in the project management** **(RG Aero 00040)**	1995	
	2010	Test tailoring softwares on workstation

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