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THEORETICAL MECHANICS

LECTURE NOTES
AND SAMPLE PROBLEMS
PART TWO

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Chapter 9. Kinematics of the rigid body

9.1.Introduction

In this chapter we shall deal with the study of the motion of the non deformable bodies, namely of the rigid bodies. We have seen in the previously chapters that in the study of the motion we have to answer to two questions (which will remain the same for the bodies) namely: which is the position of the body in any instant of the motion and how is performed its motion?

The rigid body may be considered (we have seen this propriety in the first part of this mechanics) as a non deformable and continuous system of particles. At the limit the number of these particles tends to infinity. This fact makes as the study of the motion of a body to solve in two ways:

-In the first case are considered the points of the body and are studied the motion of them using the relations find in the previously chapters (in the kinematics of the particle). This way makes as the study of the motion of a body to reduce to the study of the motions of a very large number of points (particles). This way will be used when we want to find the motion of some points from the body.

-The second way to study the motion of a rigid body considers the propriety of the body that to be a continuous and non deformable system. In this case we shall determine the elements of the motion (trajectory, law of motion, velocity, acceleration) for a few points from the body (one point eventually) and we shall find laws of variation of these elements function of the positions of the points from the body.

We specify that due to the finality of the civil engineer's works we are more interested to the way of the motion of the bodies and less to the positions of them. This result from the fact that a well- designed structure acted by the dynamic loads has displacements with small amplitudes in the neighborhood of the static equilibrium position. This will be the reason for *which we shall study the velocities and accelerations and not the position of the body.*

Because we shall use two kinds of variations: one in time and one function of the positions of the points in the rigid body, these last variations will be named as ***distributions****. Therefore we will have* ***distribution of the velocities*** *representing* ***the law of variation of the velocities in the body function of the positions of the points*** *and* ***distribution of the accelerations*** *representing* ***the law of variation of the accelerations in the body function of the positions of the points****.*

In the study of the motion we shall use two reference systems: one fixed system with respect to which is performed the motion of the body, marked $O_1x_1y_1z_1$*, and the other a moving system, joined to the body (and performing the same motion as it) with respect to which we shall define the positions of the points from the body, marked Oxyz.*

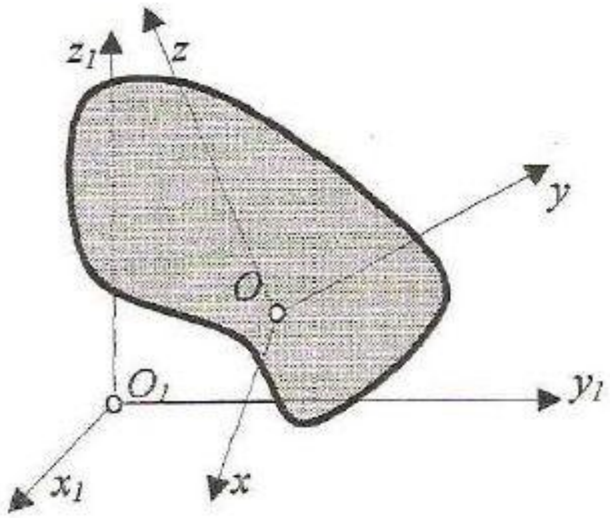


Fig.1.

Because the motion of this last system the unit vectors of the axes are functions of time:

$$\bar{i} = \bar{i}(t)\ ;\ \bar{j} = \bar{j}(t)\ ;\ \bar{k} = \bar{k}(t)$$

In the study of the velocities and accelerations we shall use the derivatives of these unit vectors, consequently in the next section we shall calculate the derivatives with respect to time of these vectors.

9.2. Derivatives of the unit vectors of the moving axes.

For to find the expressions of the derivatives of the moving unit vectors we shall start from the following six well known relations:

$$\bar{i}\cdot\bar{i}=1\ ;\ \bar{j}\cdot\bar{j}=1\ ;\ \bar{k}\cdot\bar{k}=1\ ;\ \bar{i}\cdot\bar{j}=0\ ;\ \bar{j}\cdot\bar{k}=0\ ;\ \bar{k}\cdot\bar{i}=0$$

where the first three relations express the co linearity and the last three the orthogonality of them.

We know that the projection of a vector on an axis is the scalar product of that vector and the unit vector of the axis, so we can write:

$$\begin{aligned}
\dot{\bar{i}} &= (\dot{\bar{i}}\cdot\bar{i})\,\bar{i} + (\dot{\bar{i}}\cdot\bar{j})\,\bar{j} + (\dot{\bar{i}}\cdot\bar{k})\,\bar{k}\\
\dot{\bar{j}} &= (\dot{\bar{j}}\cdot\bar{i})\,\bar{i} + (\dot{\bar{j}}\cdot\bar{j})\,\bar{j} + (\dot{\bar{j}}\cdot\bar{k})\,\bar{k}\\
\dot{\bar{k}} &= (\dot{\bar{k}}\cdot\bar{i})\,\bar{i} + (\dot{\bar{k}}\cdot\bar{j})\,\bar{j} + (\dot{\bar{k}}\cdot\bar{k})\,\bar{k}
\end{aligned}$$

Calculating the derivatives of the first six relations we have:

$$\begin{aligned}
&\dot{\bar{i}}\cdot\bar{i}=0\ ;\ \dot{\bar{j}}\cdot\bar{j}=0\ ;\ \dot{\bar{k}}\cdot\bar{k}=0\ ;\ \dot{\bar{i}}\cdot\bar{j}=-\dot{\bar{j}}\cdot\bar{i}=\omega_z\ ;\\
&\dot{\bar{j}}\cdot\bar{k}=-\dot{\bar{k}}\cdot\bar{j}=\omega_x\ ;\ \dot{\bar{k}}\cdot\bar{i}=-\dot{\bar{i}}\cdot\bar{k}=\omega_y
\end{aligned}$$

where we marked the last derivatives (of the scalar products), that are scalar quantities, with the ω_x, ω_y and ω_z (the index corresponds to the missing direction in the scalar product).

Replacing in the expression of the derivatives of the unit vectors we have:

$$\dot{\bar{i}} = \omega_z \bar{j} - \omega_y \bar{k} \; ; \; \dot{\bar{j}} = -\omega_z \bar{i} + \omega_x \bar{k} \; ; \; \dot{\bar{k}} = \omega_y \bar{i} - \omega_x \bar{j}$$

We can see easy that these relations may we written in the following way:

$$\dot{\bar{i}} = \bar{\omega} \times \bar{i} \; ; \; \dot{\bar{j}} = \bar{\omega} \times \bar{j} \; ; \; \dot{\bar{k}} = \bar{\omega} \times \bar{k}$$

where we have marked with ω the vector having the projections on the axes:

$$\bar{\omega} = \omega_x \bar{i} + \omega_y \bar{j} + \omega_z \bar{k}$$

*This vector is called **angular velocity** and their significance will be seen in the future sections. But we can see this vector is the same no matter where is the origin of the moving system of reference in the body, consequently it is a free vector for the body in motion.*

9.3. Distribution of velocities

*By definition, **the distribution of velocities** is the law of variation of the velocities in the body function of the positions of the points from the body.*

It is enough to obtain a relation between the velocities of two any points from the body for to find the distribution of the velocities. Consequently we shall consider one any point P from the body and the origin O of the moving system of reference. Between the two points we can write the vector relation:

$$\bar{r}_1 = \bar{r}_O + \bar{r}$$

in which $\mathbf{r_1}$ *and* $\mathbf{r_O}$ *are the position vectors, with respect to the origin of the fixed system, of the two points P and O, and* ***r*** *is the position vector of the point P with respect to the point O:*

$$\overline{r} = x\,\overline{i} + y\,\overline{j} + z\,\overline{k}$$

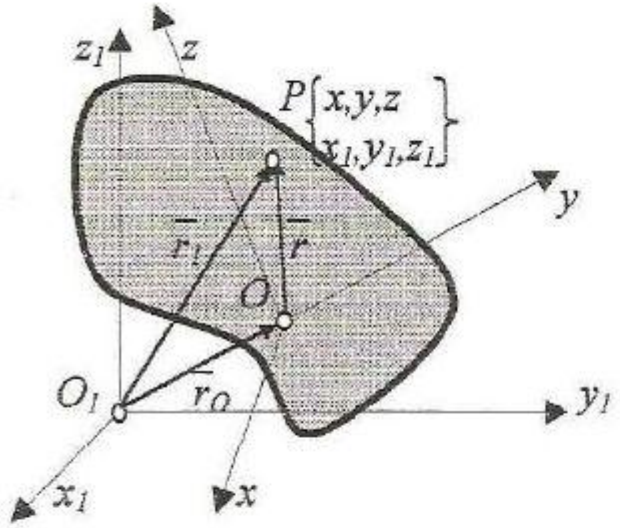


Fig.2.

*In this relation **x**, **y** and **z** are functions of time (the body being non deformable the distance between the two points is unchangeable and the moving system Oxyz is joined to the body).*

We derivate the relation:

$$\dot{\overline{r}}_1 = \dot{\overline{r}}_O + \dot{\overline{r}}$$

in which:

$$\dot{\overline{r}}_1 = \overline{v}_P \,;\, \dot{\overline{r}}_O = \overline{v}_O$$

namely they are the absolute velocities of the two points P and O (being the first derivatives with respect to time of the position vectors with respect to a fixed point).

The derivative of the position vector of the point P with respect to the origin of the moving reference system will be:

$$\dot{\bar{r}} = x\,\dot{\bar{i}} + y\,\dot{\bar{j}} + z\,\dot{\bar{k}} = x\,(\bar{\omega} \times \bar{i}) + y\,(\bar{\omega} \times \bar{j}) + z\,(\bar{\omega} \times \bar{k}) =$$
$$= \bar{\omega} \times (x\,\bar{i} + y\,\bar{j} + z\,\bar{k}) = \bar{\omega} \times \bar{r}$$

We remark that the derivative with respect to time of the vector r has the same expression as the derivatives of the unit vectors because it has constant magnitude.

Removing in the previous relations we shall find finally:

$$\bar{v}_P = \bar{v}_O + \bar{\omega} \times \bar{r}$$

that represents the distribution of velocities (or the law of changing of the velocities of the body's points function of the mutual positions of them).

We note that the distribution of the velocities in a rigid body is defined by two vectors: v_O and ω namely the velocity of an any point and the angular velocity (or by the six scalar parameters: the projections on the axes of an any reference system of these two vectors).

9.4. Distribution of the accelerations

By definition the distribution of the accelerations is the law of changing of the accelerations of the points from a body function to their mutual positions.

For to find this distribution we shall start from the relation between the velocities of two points of the body (namely from the distribution of the velocities) and we shall derivate, with respect to time, the relation:

$$\dot{\bar{v}}_P = \dot{\bar{v}}_O + \dot{\bar{\omega}} \times \bar{r} + \bar{\omega} \times \dot{\bar{r}}$$

In this relation the first two terms are the accelerations of the two points:

$$\dot{\bar{v}}_P = \bar{a}_P \,;\, \dot{\bar{v}}_O = \bar{a}_O$$

The first derivative with respect to time of the angular velocity is marked ε and is called angular acceleration:

$$\dot{\bar{\omega}} = \bar{\varepsilon}$$

The derivative of the position vector r is known:

$$\dot{\bar{r}} = \bar{\omega} \times \bar{r}$$

Replacing in the relation we obtain finally:

$$\bar{a}_P = \bar{a}_O + \bar{\varepsilon} \times \bar{r} + \bar{\omega} \times (\bar{\omega} \times \bar{r})$$

that represents the distribution of the accelerations in a rigid body.

We remark also in this case that if we know the two vectors v_O and ω we know in fact the distribution of the accelerations in the body, namely we know the way of motion of the rigid body.

9.5. Particular motions of a rigid body

We have seen that the way of motion of a body is complete defined if we know two vectors: $\bar{v}_O$ and ω. It is obviously that the particular values of these two vectors will generate the particular motions of the body.

We distinguish the following particular motions:

- *1) $\bar{v}_O = 0$, $\bar{\omega} = 0$. In this case using the two laws (distribution of the velocity and accelerations) results that all the points of the body will have the same velocity (and acceleration):*

$$\bar{v}_P = \bar{v}_O = 0$$

*Results that: the body is not in motion so it is in **REST**.*

- *2) $\bar{v}_O = 0$, $\bar{\omega} = 0$. We shall say that the body is in **TRANSLATION** motion. If the velocity $\bar{v}_O$ has constant direction the motion is called **rectilinear translation motion**, and if it has variable direction then the motion is called **curvilinear translation motion**.*

- *3)* $\overline{v}_O = 0,\ \overline{\omega} = 0$. *In this case the motion is called* **ROTATION** *motion. If the angular velocity* $\overline{\omega}$ *has constant direction then the motion is called* **rotation motion about a fixed axis**, *and if it has variable direction then it is called* **rotation motion about a fixed point**.
- *4)* $\overline{v}_O = 0,\ \overline{\omega} = 0$. *In this case function of the mutual directions of the two vectors we distinguish three motions:*
  - $\overline{v}_O \cdot \overline{\omega} = 0$ *namely the two vectors are perpendicular* ($\overline{v}_O \perp \overline{\omega}$). *If the direction of the angular velocity is constant then the motion is called* **PLANE** *motion.*
  - $\overline{v}_O \times \overline{\omega} = 0$ *namely the two vectors are collinear* ($\overline{v}_O \parallel \overline{\omega}$). *The motion in this case is called* **HELICAL** *motion.*
  - *Finally if the directions of the two vectors are arbitrary then the motion is the* **GENERAL** *motion of the body.*

Systematizing in the table T1 we have the particular motions of a rigid body.

T1

	$\overline{v}_O$	$\overline{\omega}$	*Characteristics*	*Motion*	
1	= 0	= 0	-	*Rest*	
2	$\neq 0$	= 0	$\overline{v}_O$ *has constant direction*	*Translation*	*Rectilinear translation*
			$\overline{v}_O$ *has variable direction*		*Curvilinear translation*
3	= 0	$\neq 0$	$\overline{\omega}$ *has constant direction*	*Rotation*	*Rotation about a fixed axis*
			$\overline{\omega}$ *has variable direction*		*Rotation about a fixed point*
4	$\neq 0$	$\neq 0$	$\overline{v}_O \perp \overline{\omega}$ *and* ω *with constant direction*	*Plan motion*	
			$\overline{v}_O \parallel \overline{\omega}$	*Helical motion*	
			$\overline{v}_O$ *and* $\overline{\omega}$ *with any directions*	*General motion*	

We remark that there is an analogy between the particular motions of the rigid body (subjected to the relationship $\overline{v}_P = \overline{v}_O + \overline{\omega} \times \overline{r}$) and the cases of reduction of the systems of forces (subjected to the relationship $\overline{M}_P = \overline{M}_O + \overline{OP} \times \overline{R}$).

From educational reasons in this chapter we shall study only the motions that can be performed in plane (in two dimensions) namely: the ***translation motion****,* ***rotation motion about a fixed axis*** *and the* ***plane motion****. Also we shall treat the proprieties of the distribution of velocities and accelerations in general motion of a body.*

We make the remark that for each motion we give two definitions: one geometric and one kinematic. The kinematic definition is in fact given in the table T1, namely uses the kinematic elements of the motion. The geometric definition describes the geometric proprieties of the motion (we remind that kinematics is in fact the geometry of the motion). But we shall see that the two definitions are equivalent.

9.6. Translation motion

Kinematic definition*: the translation motion of a rigid body is that motion in which the angular velocity is equal to zero and the velocity of one any point is different to zero:*

$$\overline{v}_O \neq 0\,;\ \overline{\omega} = 0\,.$$

Geometric definition*: the translation motion is that motion of a rigid body in which any straight line belonging to the body remains parallel to itself in the time of motion.*

We shall consider the body in translation with respect to the fixed system of reference $O_1x_1y_1z_1$ for to study the motion.

We shall choose the moving reference system (Oxyz) with the origin in one any point of the body and in the instant when we take this system with the axes parallel to the axes of the fixed system. From the geometric definition results that these axes remain parallel with the fixed axes in the time of motion. Conclusion is that the unit vectors of the moving axes are not functions of time, namely the derivatives with respect to time of them are equal to zero and we have:

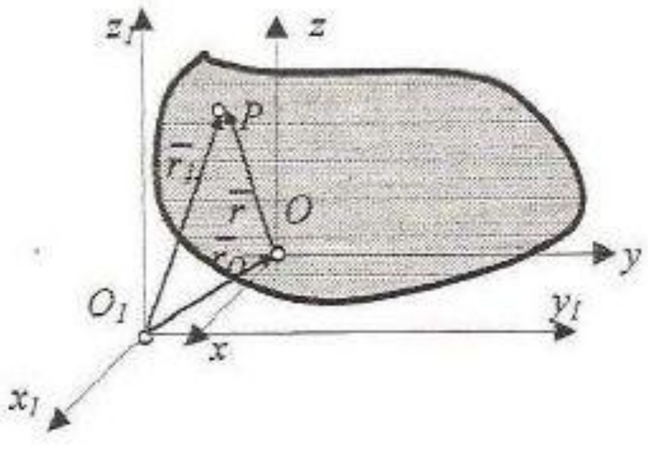


Fig.3.

$$\overline{\omega} = 0 .$$

Because the body is in motion at least one point has velocity different to zero. If this point is the point O then we have:

$$\overline{v}_O \neq 0$$

that represents the kinematic definition of the translation motion.

Between the position vectors of the point P we have the relation:

$$\overline{r}_1 = \overline{r}_O + \overline{r}$$

in which the last vector (the position vector of the point P with respect to the point O) is a constant vector in magnitude (the distance between two points of a rigid body) and in direction (in translation motion the direction of a straight line remains unchangeable). Results that the position vectors of the two points, O and P, differ by a constant, namely the trajectories of the two points are identical but they are translated with the constant vector $\overline{r}$ *(in the case of the rectilinear translation motion the trajectories are parallel lines). Exemplify this fact through the motion of the car body and the motion of the cab of a giant wheel from a park.*

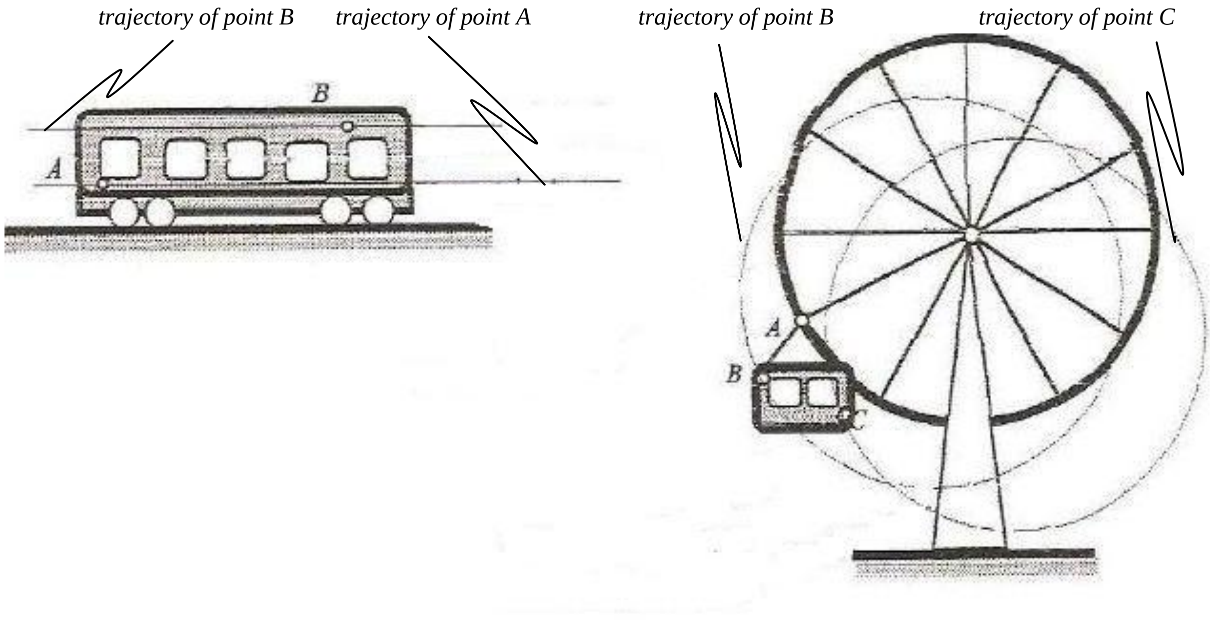


Fig.4.

The distribution of the velocities will be governed by the relation (obtained replacing $\overline{\omega} = 0$ in the expression of the distribution of the velocities):

$$\overline{v}_P = \overline{v}_O$$

meaning that all points of the body have the same velocity.

Removing in the expression of the distribution of accelerations we obtain:

$$\overline{a}_P = \overline{a}_O$$

namely all points will have the same acceleration.

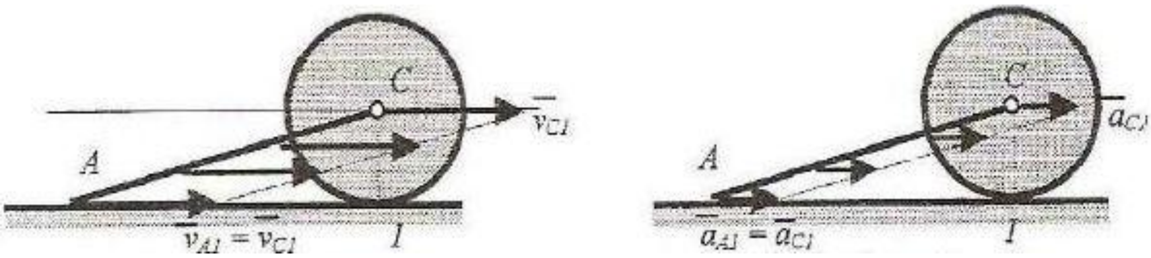
Finally we may state the following propriety: ***in the absence of other conditions, in translation motion it is enough to study the motion of one point from the body.*** *This propriety make that one body in translation motion will be considered as a particle.*

9.7. Sample problems

Problem 1. *A wheel performs a rolling motion on a horizontal surface. By the center of the wheel is hinged a rectilinear rod CA. The end A of the rod is moving on the horizontal surface. Knowing that the velocity of the center of wheel has the expression:*

$$v_C = 3t^2 \ (cm/s)$$

and the radius of the wheel is R = 50 cm represent the distribution of velocities and accelerations on the rod AC at the instant t_1 = 3s from the start of the motion.



Distribution of the velocities *distribution of the accelerations*

Fig.5.

Solution. *The rod AC performs a translation motion because being a straight line it remains parallel to itself in the time of motion. For this we can see that the triangle AIC remains unchangeable in the time of motion (IC is the radius of the wheel and it is the side of the triangle and AC is the hypotenuse o0f the right angle triangle). This translation is rectilinear because the center C of the disc (and the end A of the rod) performs o rectilinear motion on a horizontal straight line parallel with the surface on which the disc rolls.*

The instantaneous velocity of the center C, at the instant t_1, is:

$$v_{C1} = 3.3^2 = 27 \ cm/s$$

The velocities of all points of the rod AC will have the same magnitudes, directions and results the distribution represented in the figure 5.

The instantaneous acceleration of the point C, at the same instant, will be:

$$a_{C1} = \dot{v}_C(t_1) = 6 \,.\, t_1 = 18 \ cm/s^2$$

having the same direction and sense as the velocity of the point C. All points of the rod will have the same acceleration and the distribution of the accelerations will be uniformly as in the figure 5.

Problem 2. *A rectangular plane plate ABCD with the dimensions l_{AB} = 3l and l_{BC} = 30 cm = l is joined to two fixed points O_1 and O_2 with two straight rods hinged at their ends. Knowing that the point E performs an uniformly circular motion with the velocity v_E = 0,6m/s and that at the initial instant the rods are vertically, determine and represent the velocities and accelerations of the points A,B,C, and D at the instant t_1 = 0,5s from the beginning of the motion and also represent* *the distributions of the velocities and accelerations on the BC side of the rectangle at the same instant. Are known: $O_1O_2 = EF = 2l$, $O_1E = O_2F = l$.*

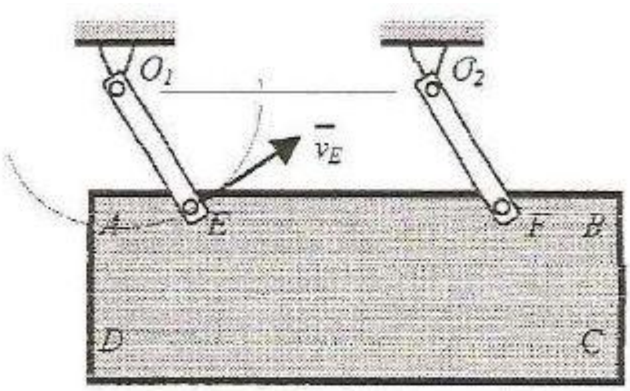


Fig.6.

Solution. *For the beginning we shall define the kind of motion of the rectangle. For this we remark that O_1O_2FE is a parallelogram indifferent to the instant of the motion, so the side AB of the rectangle is all the time parallel to the horizontal direction O_1O_2, namely it remains parallel to itself. Results that the rectangle performs a translation motion and because the point E describes a circular motion the translation is a circular translation motion (all the points of the rectangle will perform circular motions on circles having the same radii). Being a translation motion it is enough to study the motion of one point of the plate and move the elements of the motion in the other points of the rectangle.*

First we shall determine the position of the rectangle at the given instant t_1. For this it is enough to determine the position of the point E on the circle that represents its trajectory. In the circular motion the point E has the angular velocity:

$$\omega_E = \frac{v_E}{l} = \frac{0,6}{0,3} = 2\ rad/s$$

and because the circular motion is uniformly we have the law of variation of the angle in the center in the circular motion (with respect to the vertical position of the rod O_1E that is the initial position):

$$\theta = \omega_E \cdot t$$

At the instant t_1 the angle made by the rod O_1E with the vertical direction is:

$$\theta_1 = \theta(t_1) = 2 \cdot 0,5 = 1\ rad = 57,32^\circ$$

The velocities of all points are equal (in magnitudes, directions and senses) to the velocity of the point E:

$$v_A = v_B = v_C = v_D = v_E = v_F = 0,6 m/s$$

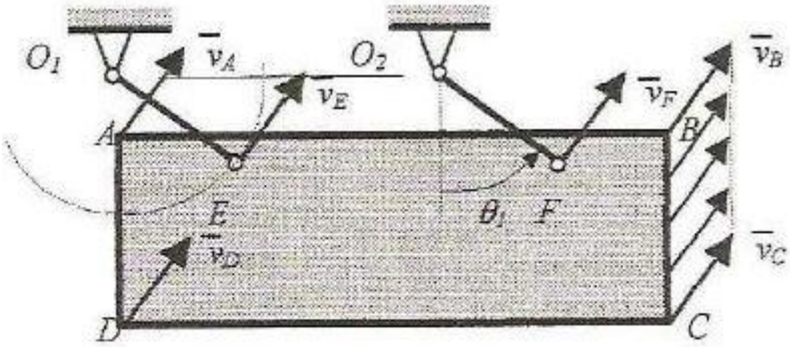


Fig.7.

Distribution of the velocities on the side BC is uniformly and it is represented in the figure 7.

The acceleration of the point E (in circular motion) has two components:

$$\bar{a}_E = \bar{a}_{E\tau} + \bar{a}_{E\nu}$$

where:

$$a_{E\tau} = \dot{v}_E = 0;$$

$$a_{E\nu} = \frac{v_E^2}{O_1E} = \frac{0{,}6^2}{0{,}3} = 1{,}2\ m/s^2.$$

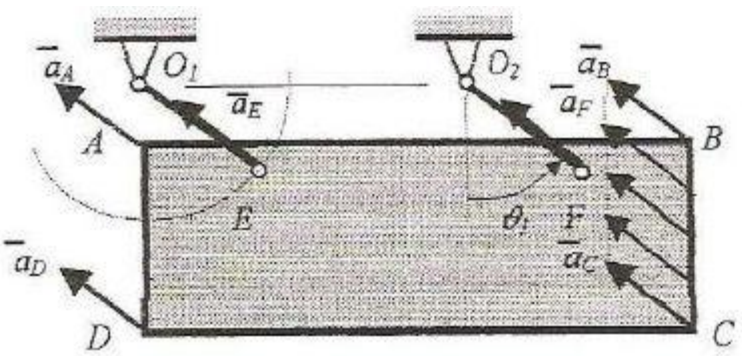


Fig.8.

The normal component has the direction of the radius of the circle with the sense toward to the center of the circle. All the points of the rectangle will have the same accelerations and the distribution of the accelerations on the side BC is uniformly as in the figure 8:

$$a_A = a_B = a_C = a_D = a_E = a_F = 1{,}2 m/s^2.$$

Problem 3. *A right angle triangular prism glides on a horizontal surface as in the figure 9. Knowing that the point A has constant acceleration $a_A = 30\ cm/s^2$ calculate and represent the velocities and accelerations of the three points A, B and C at the instant $t_1 = 1\ s$ from the beginning of the motion (at the initial instant the prism is in rest). Represent at the same instant the distributions of the velocities and accelerations at the given instant.*

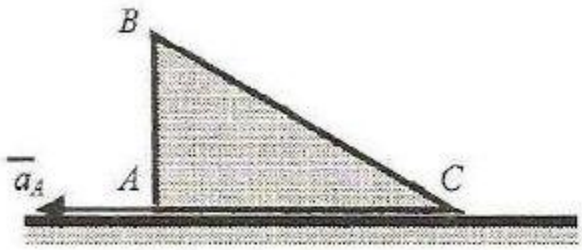


Fig.9.

Problem 4. *The cabin of a rope way moves on the supporting cable by the shape of a parabola. Knowing the suspension points of the cable A and B, the vertical distance in the middle of the distance between the two points and the fact that in the time of motion the cable keeps its shape calculate and represent the velocities and accelerations of the four points of the cabin: C, D, F and E at the instant $t_1 = 3\ s$ from the beginning of the motion. The cabin starts the motion in the point A and has the horizontal velocity constant: $v_h = 1\ m/s$.*

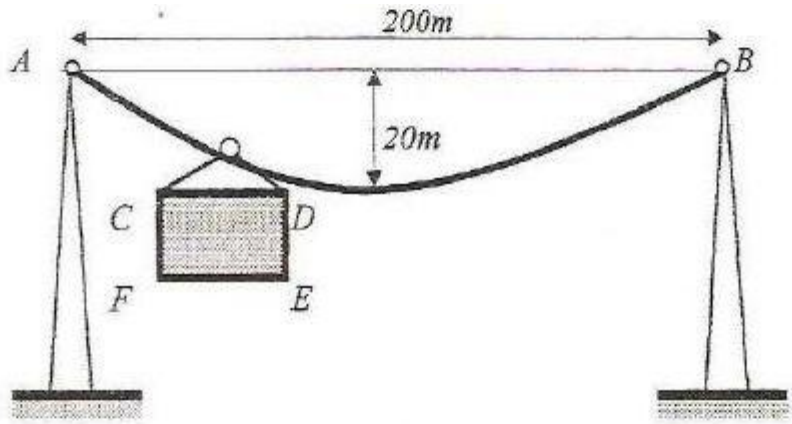


Fig.10.

9.8. Rotation motion about a fixed axis

Kinematic definition: *the rotation motion about a fixed axis is that motion of a rigid body in which one point has zero velocity and the angular velocity is different to zero but has constant direction.*

Geometric definition: *The rotation motion about a fixed axis is the motion of a rigid body in which two points of the body are fixed.*

For the study of this motion we shall suppose that the body is in rotation motion about a fixed axis, namely it has two fixed points O and A. We shall choose the fixed reference system having the origin O_1 in the point O and the axis O_1z_1 to pass through the other fixed point. The moving system, joined to the body, is taken with the origin in one any point of the body, for example in point O and with the Oz axis passing through the point A, namely is the same as the axis O_1z_1. The other two axes are moving axes, they move in the fixed plane $O_1x_1y_1$.

The positions of the moving axes with respect to the fixed axes will be defined by the parameter:

$$\theta = \theta(t).$$

The axis O_1z_1 that is the same as the axis Oz is called ***rotation axis.***

The position vectors of one any point P with respect to the both reference systems are the same and we may write:

$$\overline{r}_1 = \overline{r}$$

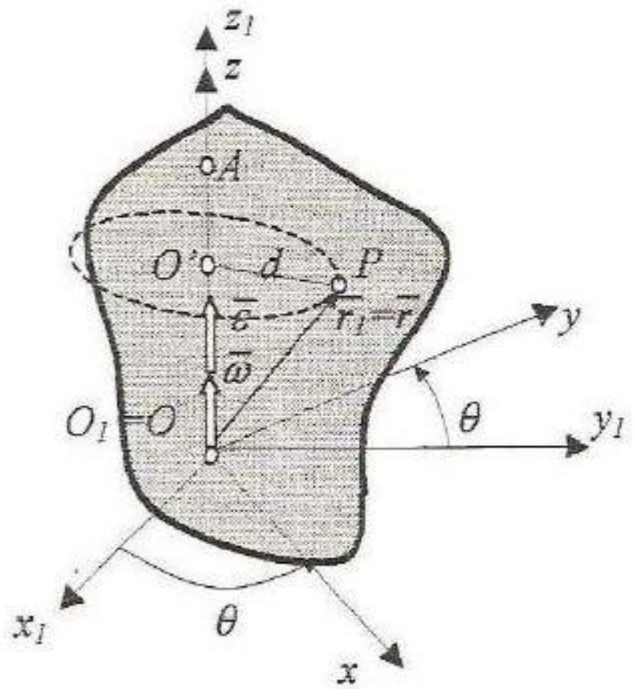


Fig.11.

Expressing the equality of the magnitudes of these two vectors we may write:

$$x_1^2 + y_1^2 + z_1^2 = x^2 + y^2 + z^2$$

But as the two axes O_1z_1 and Oz are the same we have also the equality:

$$z_1 = z$$

Simplifying the previously relation with these coordinates, and knowing that the magnitude of the vector from the right part is constant (definition of the rigid body) and finally marking:

$$x^2 + y^2 = d^2$$

results:

$$x_1^2 + y_1^2 = d^2$$

that is the equation of a circle. Results that ***the trajectories of the points of a body in rotation motion about a fixed axis are circles with the centers on the rotation axis and located in perpendicular plan on that axis.***

The main characteristic of the moving reference system is that the unit vectors of Ox and Oy axes are functions of time, namely:

$$\overline{i} = \overline{i}(t)\ ;\ \overline{j} = \overline{j}(t)$$

and the unit vector of Oz axis is constant:

$$\dot{\overline{k}} = 0$$

From this last relation results (using the relations obtained for the derivatives of the unit vectors):

$$\omega_x = \omega_y = 0$$

namely:

$$\overline{\omega} = \omega_z \overline{k}.$$

For to determine the magnitude of the angular velocity we shall express the variable unit vectors using the projections on the fixed axes (like in the case of the cylindrical reference system) and we obtain:

$$\dot{\overline{i}} = \dot{\theta}\overline{j}\ ;\ \dot{\overline{j}} = -\dot{\theta}\overline{i}$$

from which results finally:

$$\omega = \omega_z = \dot{\theta}$$

This relation justifies the name of this vector as ***angular velocity.***

With this the distribution of the velocities will be:

$$\overline{v_P} = \overline{\omega} \times \overline{r} = \omega \overline{k} \times \overline{r} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = \\ = -\omega y \overline{i} + \omega x \overline{j}$$

The magnitude of the velocity of any point will be:

$$v_P = \omega \sqrt{x^2 + y^2} = \omega . d$$

From the last relations result that: the velocities are linear functions by the distance from the points to the rotation axis.

Suppose now a straight line belonging to the body that passes through the point P and parallel with the rotation axis. All the points of this line, located at the same distances to the rotation axis, will perform circular motions *on identical circles with the same velocities. This means that the considered line performs a circular translation motion about the rotation axis. Consequently for this line it is enough to study one point. We shall choose this point as the intersection point of this line with the Oxy plane and results that it is enough to study the rotation motion only in the fixed plane $O_1x_1y_1$. The intersection point between the rotation axis and this plane (the fixed point $O=O_1$) is called* **rotation center.** *In this way we shall study the rotation motion as a plane motion.*

Consider a body in plane having a fixed point (the rotation center) performing a rotation motion about this point with the angular velocity ω. Representation of this angular velocity is made using only the rotation sense (as in the case of the moment of a couple in statics).

The velocity of one any point P can be obtained determining a time the three characteristics of the velocity:

- *magnitude:*

$$v_P = OP \cdot \omega$$

namely the magnitude of the velocity is equal to the product between the distance from the rotation center to the point and the angular velocity;

- *direction is perpendicular on the radius OP from the rotation center to the point:*

$$\bar{v}_P \perp \overline{OP}$$

- *the sense of the velocity is in the sense of rotation (in the sense of motion) or in the sense of rotation of the angular velocity:*

$$\bar{v}_P \circlearrowleft \bar{\omega}$$

We shall state two proprieties of the distribution of velocities on straight lines in a body in rotation motion:

- *On a straight line passing through the rotation center the distribution of velocities is linear, the velocities being perpendicular on the line;*

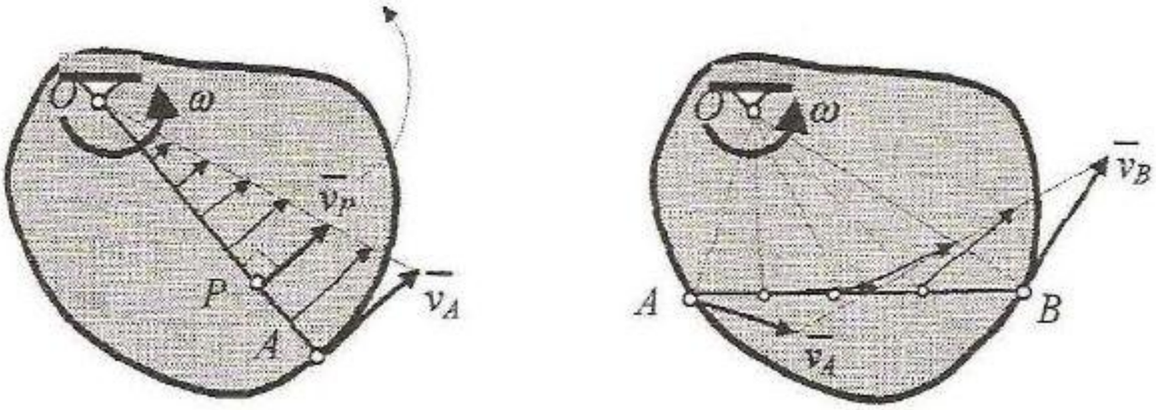


Fig.12.

- *On an arbitrary straight line belonging to a body in rotation motion the ends of the velocities of points are collinear.*

Using these proprieties results that for to represent the distribution of velocities on a straight line passing through the rotation center it is enough to calculate and represent one velocity of one any point from the line and on an arbitrary straight line is necessary to calculate and represent two velocities.

For the distribution of the accelerations we remark first that:

$$\bar a_O = \dot{\bar v}_O = 0;$$
$$\bar\varepsilon = \dot{\bar\omega} = \dot\omega\,\bar k = \ddot\theta\,\bar k = \varepsilon\,\bar k$$

with which we shall find:

$$\bar a_P = \varepsilon\,\bar k \times \bar r + \omega\,\bar k \times (\,\omega\,\bar k \times \bar r\,) = (-\varepsilon y - \omega^2 x)\,\bar i + (\,\varepsilon x - \omega^2 y\,)\,\bar j$$

The magnitude of the acceleration of the point P will be:

$$a_P = \sqrt{(x^2 + y^2)(\varepsilon^2 + \omega^4)} = d\,.\,\sqrt{\varepsilon^2 + \omega^4}$$

We remark that because the point P performs a circular motion we have obviously the equalities:

$$\bar{a}_{P\tau} = \bar{\varepsilon} \times \bar{r}\,;\ \bar{a}_{P\nu} = \omega \bar{k} \times (\,\omega \bar{k} \times \bar{r}\,)$$

Using these relations we can determine the acceleration of a point in two ways: or we calculate directly the magnitude direction of the acceleration, or we calculate the two components of it.

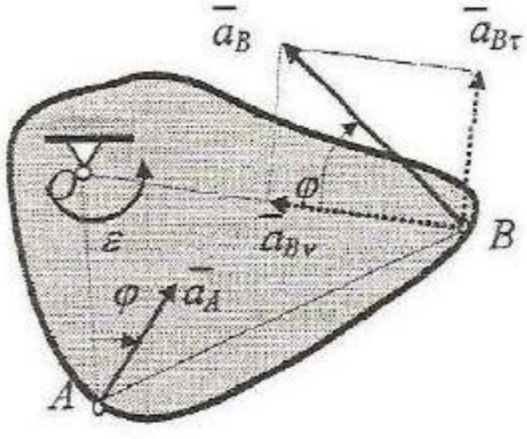


Fig.13.

The direct calculation of the acceleration of a point means that to calculate

- *Magnitude that is:*

$$a_A = OA \cdot \sqrt{\varepsilon^2 + \omega^4}\,;$$

- *Direction makes the angle φ with the radius from the rotation center to the point:*

$$\angle \bar{OA}, \bar{a}_A = \varphi\,;\ tg\varphi = \frac{\varepsilon}{\omega^2}\,;$$

- *The sense of the acceleration is in the same sense of rotation as the sense of the angular acceleration about the rotation center.*

$$\bar{a}_A \curvearrowright \varepsilon$$

Calculation of the components of the acceleration on the two directions:

- *The magnitudes are:*

$$a_{B\tau} = OB \cdot \varepsilon;\ a_{B\nu} = OB \cdot \omega^2$$

- *The direction of the tangent component is perpendicular on the radius from the rotation center to the point and the normal component is collinear with the radius:*

$$\bar{a}_{B\tau} \perp \bar{OB};\ \ \bar{a}_{B\nu} \| \bar{OB};$$

- *The sense of the tangent component is in the rotation sense of the of the angular acceleration about the rotation center and the sense of the normal component is directed to the rotation center:*

$$\bar{a}_{B\tau} \circlearrowright \varepsilon\ ;\ \bar{a}_{B\nu} \rightarrow O$$

Finally the magnitude and direction of the acceleration will be:

$$a_B = \sqrt{a_{B\tau}^2 + a_{B\nu}^2}\ ;\ tg\varphi = \frac{a_{B\tau}}{a_{B\nu}}.$$

9.9. Sample problems

Problem 1. *One disc having the radius R = 40 cm performs a rotation motion about its fixed center with constant angular velocity $\omega = 0,5\ s^{-1}$. Determine and represent, at a given instant, the velocities and accelerations of ends of two perpendicular diameters and finally represent the distribution of the velocities on the two diameters.*

Solution. *Because the disc performs a rotation motion about its center O (fixed point) the magnitudes of the velocities of points will be calculated with the relations:*

$$v_A = OA \cdot \omega = 40 \cdot 0,5 = 20\ cm/s;\ v_B = OB \cdot \omega = 40 \cdot 0,5 = 20\ cm/s;$$
$$v_C = OC \cdot \omega = 40 \cdot 0,5 = 20\ cm/s;\ v_D = OD \cdot \omega = 40 \cdot 0,5 = 20\ cm/s;$$

As we can see all points will have velocities with the same magnitudes because they are at the same distances about the rotation center.

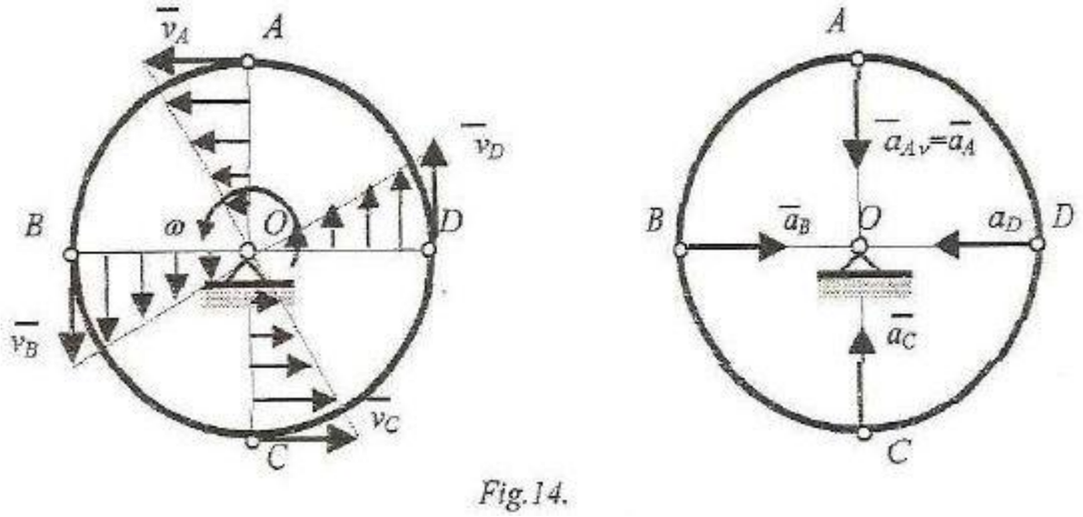


Fig.14.

The directions of these velocities are perpendicular on the radii from point O to the given points:

$$\bar{v}_A \perp \overline{OA}\,;\ \bar{v}_B \perp \overline{OB}\,;\ \bar{v}_C \perp \overline{OC}\,;\ \bar{v}_D \perp \overline{OD}$$

and the senses are in the same sense of rotation, about the rotation center, as the sense of the angular velocity, namely in trigonometric sense.

Distribution of velocities on the two perpendicular diameters is linear having zero value in the rotation center O.

The acceleration of a point in the rotation motion may be determined calculating two components: one tangent and the other normal. For the point A the tangent component has the magnitude:

$$a_{A\tau} = OA\,.\,\varepsilon = OA\,.\,\dot{\omega} = 0$$

because the angular velocity is constant. The second component, normal component, of the acceleration will be:

$$a_{A\nu} = OA\,.\,\omega^2 = 40\,.\,0{,}5^2 = 10\ cm/s^2$$

This component has the direction of the radius of the circle with the sense directed to the center O. Because all the points from the periphery of the circle are at the same distance from the center of the disc (the center of rotation) we will have the equality of the magnitudes of the accelerations of the points from the periphery:

$$a_B = a_C = a_D$$

As we have seen the accelerations may be computed directly, calculating the magnitude and directions of them:

$$a_A = OA \cdot \sqrt{\varepsilon^2 + \omega^4} = 10\ cm/s^2 ;\ tg\ \varphi = \frac{\varepsilon}{\omega^2} = 0 ;\ \rightarrow \varphi = 0 .$$

Problem 6. *The rectangular plate OABC from the figure 15 performs a rotation motion about the point O in the plane of the plate. Knowing that the angle θ has the law of variation : θ(t) = sin t (rad) and the sides of the rectangle are: OA = 40 cm, AB = 30 cm, calculate and represent the velocities and accelerations of the points A, B, and C and represent the distribution of the velocities on the OA and AB sides of the rectangle at the instant t_1 = 1s from the beginning of the motion.*

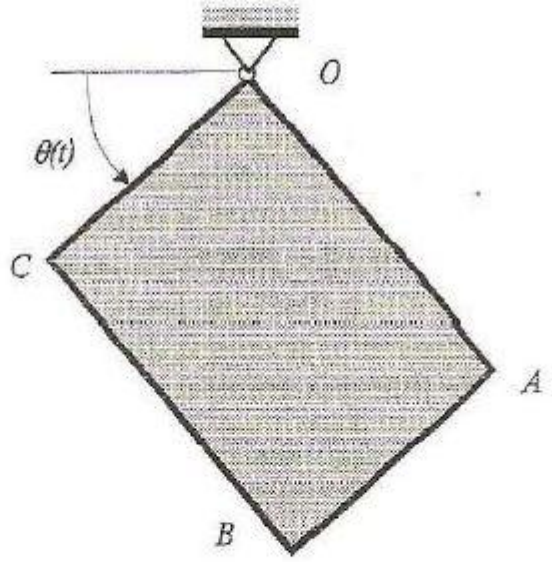


Fig.15.

Solution. *First we shall determine the position of the rectangle at the given instant:*

$$\theta_1 = \theta(t_1) = \sin(1\ rad) = 0{,}841\ rad \quad sau: \quad \theta_1 = 48{,}23^{\circ}$$

For to calculate the velocities we shall calculate first the angular velocity at the instant t_1:

$$\omega_1 = \dot{\theta}(t_1) = \cos t_1 = 0{,}54\ s^{-1}$$

The sense of rotation of the angular velocity will be the sense of the increasing sense of the angle θ (because through derivation we have not changed the sign of it).

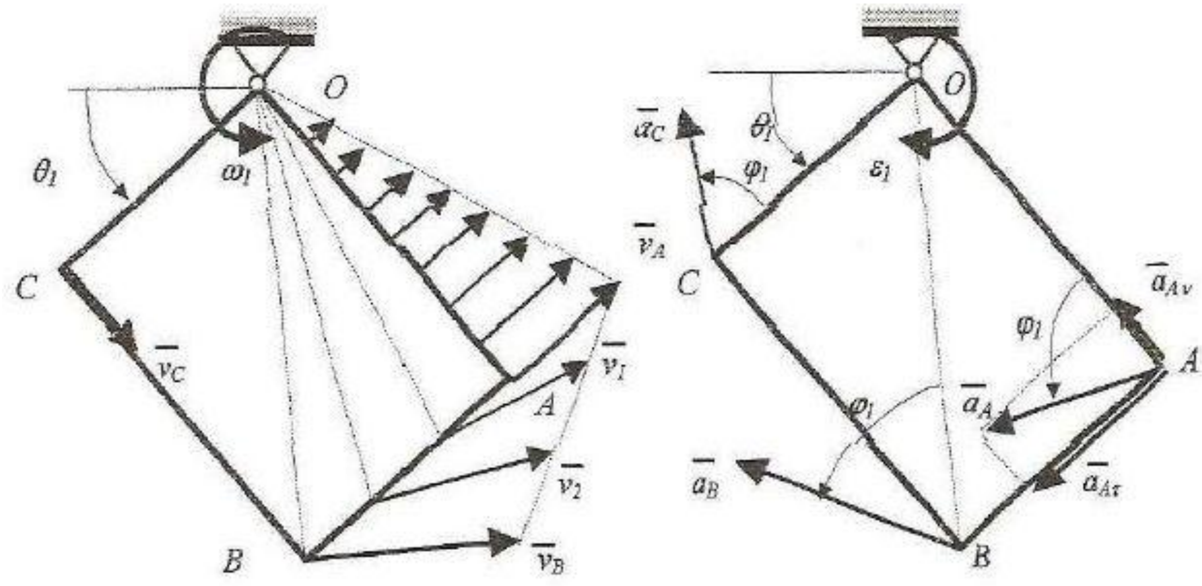


Fig.16.

The velocity of the point A has the magnitude:

$$v_A = OA \cdot \omega_1 = 40 \cdot 0{,}54 = 21{,}6cm/s$$

with the direction perpendicular on the radius OA from the rotation center to the point A and the sense, with respect to the center of rotation, the same sense of rotation as of the angular velocity.

For the point B we have:

$$v_B = OB \cdot \omega_1 = \sqrt{OA^2 + AB^2} \cdot \omega_1 = 50 \cdot 0{,}54 = 27\ cm/s$$

with perpendicular direction on the radius OB and the same rotation sense about the rotation center O as the rotation sense of the angular velocity.

In the same way we shall mare for the point C:

$$v_C = OC \cdot \omega_1 = 30 \cdot 0{,}54 = 16{,}2\ cm/s$$

This velocity will have perpendicular direction on the radius OC.

The distribution of velocities on the OA side of the rectangle is linear because this side is a straight line passing through the rotation center O. For the distribution of velocities on the AB side we shall make in the following way: First we need to find two velocities from this side (but we have the velocities in the two points A and B), next we join the tops of them and we obtain one straight line on that will be located all the tops of the velocities of the points from then side of the rectangle. If we want to represent the distribution of the velocities we choose a few points from the line AB (for example points 1 and 2). Now we draw the radii O1 and O2 and the perpendicular lines on them the intersections of these lines with the line of the tops give us the representation of the velocities of the points 1 and 2.

For to determine the accelerations we shall calculate the acceleration of the point AA using the components and for the other two points we shall calculate directly the accelerations as magnitudes and directions.

Firs we shall calculate the angular acceleration:

$$\varepsilon_1 = \dot{\omega}(t_1) = -\sin t_1 = -0{,}841\ s^{-2}$$

The sign minus shows us that the instantaneous angular acceleration, in this instant, has opposite sense as the angular velocity, so it will have clockwise sense.

The tangent component of the acceleration of the point A will be:

$$a_{A\tau} = OA\ .\ \varepsilon_1 = 40\ .\ 0{,}841 = 33{,}64\ cm/s^2$$

being perpendicular on the radius OA from the rotation center to the point A and with the sense of rotation the same sense as the rotation sense of the angular acceleration about the rotation center.

The normal component is:

$$a_{A\nu} = OA\ .\ \omega_1^2 = 40\ .\ 0{,}54^2 = 11{,}64\ cm/s^2$$

being collinear with the radius OA and directed to the rotation center O.

Now having these two components we can calculate the magnitude and direction of the acceleration of the point A:

$$a_A = \sqrt{a_{A\tau}^2 + a_{A\nu}^2} = \sqrt{33{,}64^2 + 11{,}64^2} = 35{,}59\ cm/s^2;$$

$$tg\varphi_1 = \frac{a_{A\tau}}{a_{A\nu}} = \frac{33{,}64}{11{,}64} = 2{,}89 \longrightarrow \varphi_1 = 70{,}91^\circ$$

For the point B we shall calculate directly the magnitude of the acceleration:

$$a_B = OB\ .\ \sqrt{\varepsilon_1^2 + \omega_1^4} = 50\ .\ \sqrt{0{,}841^2 + 0{,}54^4} = 50\ .\ 0{,}89 = 44{,}5 cm/s^2.$$

The direction is given by the relation:

$$tg\varphi_1 = \frac{\varepsilon_1}{\omega_1^2} = \frac{0{,}841}{0{,}54^2} = 2{,}89$$

We remark that for all points of this body this angle (between the radius from the rotation center to the point of the body and the direction of the acceleration of the point) is the same. Here the angle is measured from the radius OB, around the point B, in the opposite sense of the rotation sense of the angular acceleration. The sense of the acceleration of the point is in the rotation sense of the angular acceleration about the rotation center O.

For the point C we have the magnitude of the acceleration:

$$a_C = OC \cdot \sqrt{\varepsilon_1^2 + \omega_1^4} = 30 \cdot 0{,}89 = 26{,}7\ cm/s^2.$$

and with the same sense of rotation about the rotation center as the angular acceleration or as the same rotation sense as the other accelerations.

Problem 7. *A square having the side $l = 0.5$ m rotates, in the plane of the square, around of a corner with the constant angular acceleration $\varepsilon = 1\ rad/s^2$. Knowing that at the instant $t_o = 0$ the plate has the side OA horizontal and that the velocity of the point B is, in this instant, $v_{Bo} = 1$ m/s, calculate and represent the velocities and accelerations of the tops of the square and represent distribution of velocities on the two diagonals of the square at the instant $t_1 = 1$s from the start of the motion.*

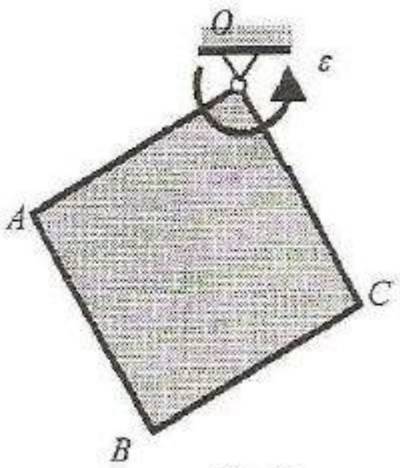


Fig.17.

9.10. The plane motion

Kinematic definition. *The plane motion is that motion of the rigid body in which the direction of the angular velocity is constant and perpendicular on the direction of the velocity of one any point of the body.*

Geometric definition. *The plane motion of a rigid body is that motion in which three points of the body move in the same fixed plane.*

For the study of the motion we shall choose the plane Oxy of the body moving in the fixed plane $O_1x_1y_1$.

The axis Oz, perpendicular on the plane Oxy is perpendicular, all the time of the motion, on the fixed plane $O_1x_1y_1$, so it will have constant direction and we shall have:

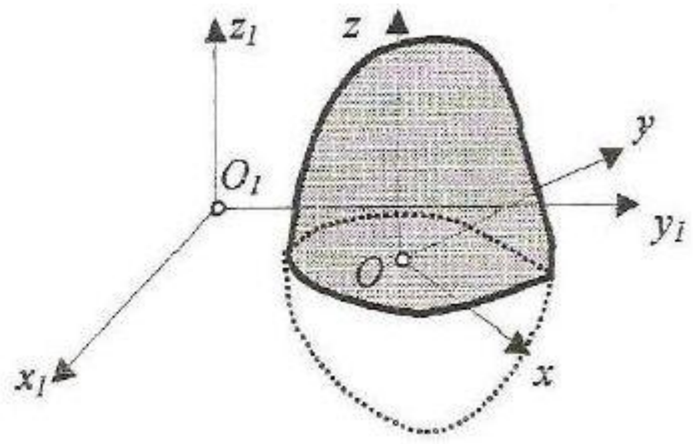


Fig.18.

$$\dot{\overline{k}} = 0; \qquad \omega_x = \omega_y = 0; \qquad \omega = \omega_z$$

Because any straight line, belonging to the body, parallel with Oz remains parallel to itself in the time of motion means that this straight line performs a translation motion. Namely it is enough to study the motion of one single point from this line. Considering the point of intersection of this line and the fixed plane results that for the study of the motion it is enough to study the motion in the considered fixed plane.

Consider, consequently, the fixed plane $O_1x_1y_1$ and we shall study the motion of a plane body in this plane. We shall mark the angle between the axis Ox and O_1x_1 with $\theta(t)$. The unit vectors of the moving axes, expressed with respect to the fixed axes, have the expressions:

$$\overline{i} = \cos\theta\,\overline{i_1} + \sin\theta\,\overline{j_1}$$
$$\overline{j} = \sin\theta\,\overline{i_1} + \cos\theta\,\overline{j_1}$$

These relations are the same as those for the rotation motion about a fixed axis. In conclusion we have:

$$\overline{\omega} = \omega\overline{k} = \dot{\theta}\overline{k} = \dot{\theta}\overline{k_1}$$

For the distribution of velocities we know that the point O has a motion in the fixed plane and results:

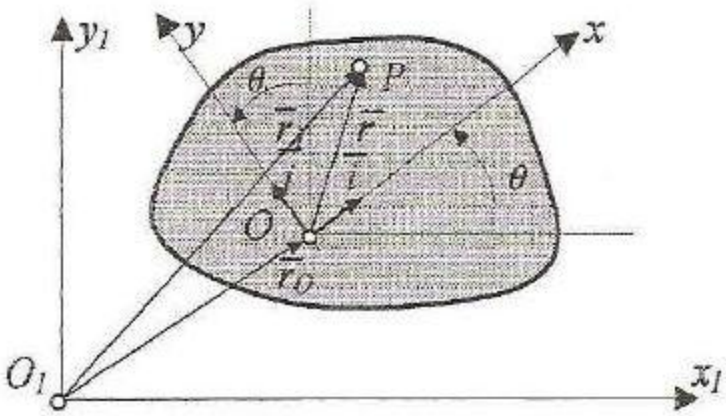


Fig.19.

$$\overline{v}_O = v_{Ox1}\,\overline{i}_1 + v_{Oy1}\,\overline{j}_1 = v_{Ox}\,\overline{i} + v_{Oy}\,\overline{j}$$

We remove in the expression of the distribution of velocities and we shall find:

$$\overline{v}_P = \overline{v}_O + \overline{\omega} \times \overline{r} = v_{Ox}\,\overline{i} + v_{Oy}\,\overline{j} + \omega\,\overline{k} \times (x\,\overline{i} + y\,\overline{j}) =$$
$$= \overline{i}\,(v_{Ox} - \omega y) + \overline{j}\,(v_{Oy} + \omega x)$$

Results that the projections on the moving axes, of the velocity of one any point, are:

$$v_{Px} = v_{Ox} - \omega y; \qquad v_{Py} = v_{Oy} + \omega x$$

We shall investigate, now, if there are points with zero velocity. Supposing that there are these kind of points we shall mark their coordinates, with respect to the moving reference system, with ξ *and* η*. Removing these coordinates in the projections of the velocity of one any point and considering the condition these to be equal to zero we have:*

$$\xi = -\frac{v_{Oy}}{\omega}; \; \eta = \frac{v_{Ox}}{\omega}$$

From the analysis of the equations results that at an instant there is one single point with this propriety (the velocity to be zero). This point *is called* ***instantaneous center of rotation*** *(ICR) and it is marked with I. Because the velocity of the point O and the angular velocity are functions of time results that the two coordinates of the ICR are also functions of time, namely the ICR changes its position with respect to moving reference system and obviously with respect to the fixed reference system.*

The locus of the positions of the ICR with respect to the moving reference system is a plane, moving, curved line called ***moving centrode****, and the locus of this center with respect to the fixed reference system is a plane fixed curved line called* ***fixed centrode.*** *We shall state (without to prove) one propriety of these two curved lines: the moving centrode performs a rolling motion without sliding on the fixed centrode.*

The instantaneous center of rotation has an important propriety used for to find the distribution of velocities on a body in plane motion, that is: ***the instantaneous distribution of velocities in plane motion is the same as in a rotation motion about the instantaneous center of rotation.*** *For to prove this propriety we shall consider that: in the instant when we make the distribution of velocities we choose the origin of the moving reference system in the point I (ICR). This means that:*

$$\overline{v}_O = \overline{v}_I = 0$$

from which results:

$$\overline{v}_P = \overline{\omega} \times \overline{IP}$$

that is the distribution of velocities in rotation motion about the point I.

In this way, if we know the position of the instantaneous center of rotation all the proprieties of the distribution of velocities in rotation motion are used. For to find the position of ICR we have two ways: one analytical and one geometric way.

In the analytical way we shall use the two coordinates of this center. If the origin of the moving reference system is taken in the point with known velocity and the direction of the Ox axis is taken collinear with the known velocity the coordinates of the ICR are:

$$\xi = 0 \; ; \; \eta = \frac{v_O}{\omega}$$

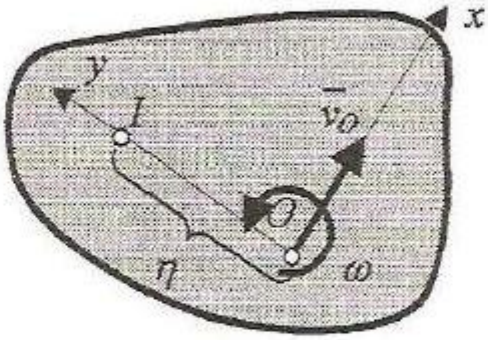


Fig.20.

namely the ICR is located on the Oy axis (perpendicular on the direction of the known velocity).

In geometric way the determination of the position of ICR is based on the proprieties of the distribution of velocities in rotation motion namely on the fact that the velocities are perpendicular on the radii from the rotation center to the points of the body and that the distribution is linear with respect to the rotation center.

In the figure 21 are represented a few cases of geometric determination of the instantaneous center of rotation (ICR) namely:

- *If are known two velocities then will be intersected the perpendicular straight lines on the support lines of the two velocities;*
- *If the two velocities are parallel then the perpendicular lines in the points are collinear, and the ICR is find at the intersection between the common perpendicular line and the straight line that joins the tops of the two velocities;*
- *If are known two simple supports of the body then knowing that the directions of the velocities in the support points are tangent to the continuous line representing the edges of the bodies in contact, the ICR will be located at the intersection point of the two perpendicular lines on the two tangent lines.*

Obviously can be combinations of these few situations when we shall use the proprieties of the distribution of the velocities from the rotation motion.

For the distribution of the accelerations defined by the following relation:

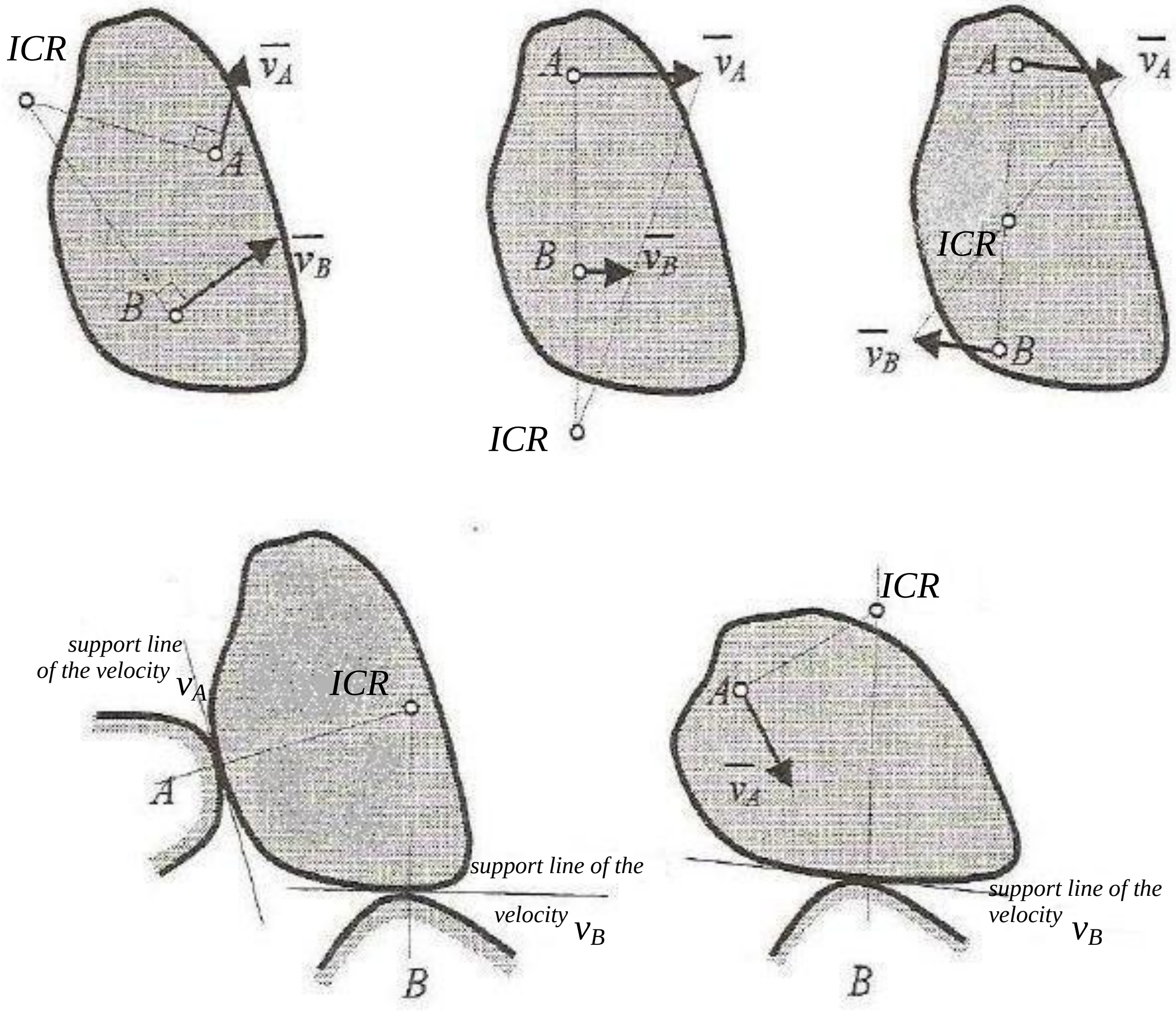


Fig.21.

$$\overline{a}_P = \overline{a}_O + \overline{\varepsilon} \times \overline{r} + \overline{\omega} \times (\overline{\omega} \times \overline{r})$$

we shall remove the particular values of the angular velocity and acceleration and the acceleration of the point O:

$$\overline{a}_O = a_{Ox}\overline{i} + a_{Oy}\overline{j}\ ;\quad \overline{\varepsilon} = \varepsilon\overline{k};\quad \overline{\omega} = \omega\overline{k},\ \overline{r} = x\overline{i} + y\overline{j}$$

Finally results:

$$\overline{a_P} = \overline{i}\,(a_{Ox} - \varepsilon y - \omega^2 x) + \overline{j}\,(a_{Oy} + \varepsilon x - \omega^2 y)$$

*Can show that in the case of the distribution of the accelerations there is one point, and only one, called **instantaneous center of the accelerations**, having the proprieties that with respect to this point the instantaneous distribution of the acceleration is as in a rotation motion about this point and in this point the acceleration is zero. This center is not the same as the ICR and it is also a point that changes its position with respect to the two reference systems. If the two points: instantaneous center of rotation and of accelerations, match then this point is fixed and the motion is a rotation about the fixed point.*

In the following, we shall use this instantaneous center of the accelerations only if this point is given or it can be obtained easy (from the determination of the point with zero acceleration). In the other situations we shall use the methods resulted from the proprieties of the distribution of accelerations in general motion which will be studied in the following section.

9.11. Sample problems

Problem 8. *One straight rod having its length l_{AB} = 50 cm performs a motion in vertical plane so that the two ends A and B move on the two perpendicular straight lines, one horizontal and one vertical. Knowing that the point A moves on the horizontal line with the constant velocity $v_A = v_o$ = 0.8 m/s and that at the initial instant the rod was in rest in vertical position determine: a) distribution of velocities on the rod at the instant t_1 =0.2 s from the start of the motion; b) accelerations of the points A and B at the same instant of the motion.*

Solution. *a) The motion of the rod is a plane motion because it has not a fixed point (so it is not rotation) and in the time of motion the direction of the rod does not remains parallel with itself (so it is not a translation).*

First we shall determine the position of the body at the given instant. This is made remarking that the point A performs a rectilinear motion with given constant velocity. Using the relations from the kinematics of the particle and we work in Cartesian coordinates we can write:

$$\dot{x}_A = v_A\,; \qquad \dot{y}_A = 0 \longrightarrow y_A = 0$$

Integrating the first relation results:

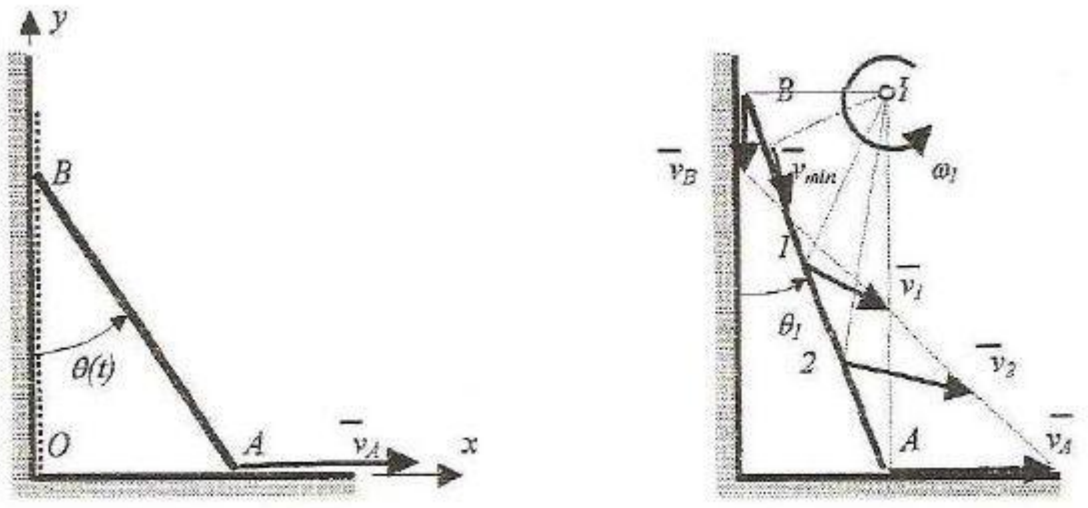


Fig.22.

$$x_A(t) = v_A \cdot t + C$$

where the integration constant C is obtained from the condition that at the initial instant of the motion the point A to be located in the point O. From this condition we find:

$$C = 0 \; ; \quad x_A(t) = v_A \cdot t$$

For the instant t_1 the position of the point A will be:

$$x_A(t_1) = 0{,}8 \cdot 0{,}2 = 0{,}16 \; m$$

From the right angle triangle OAB is obtained the angle made by the rod with the vertical direction in that instant:

$$\sin \theta_1 = \frac{OA}{AB} = \frac{0{,}16}{0{,}5} = 0{,}32 \qquad \theta_1 = 18{,}66^o$$

For the distribution of the velocities we shall determine first the position of the instantaneous center of rotation. Because the rod has two simple supports (in the points A and B) we shall raise perpendicular lines in the two points on the tangent lines to the continuous lines (horizontal in point A and vertical in point B) representing the directions of the velocities in these points. This means that in point A we raise a perpendicular on the horizontal direction and in point B on vertical direction. At the intersection of these two perpendicular lines is located ICR corresponding to the instant t_1. We mark this point with I. The distribution of the velocities will be realized like the body is in rotation motion about the point I.

We shall calculate the angular velocity in the given instant knowing the velocity of the point A and the distance $IA = AB \cdot \cos \theta_1$,from the ICR to the point A:

$$\omega_1 = \frac{v_A}{IA} = \frac{0,8}{0,5 \times 0,947} = 1,688 \ rad/s$$

This velocity has the same sense of rotation about the ICR as the rotation sense of the velocity v_A about the same point.

The velocity of the point B will be in magnitude:

$$v_B = IB \, . \, \omega_1 = AB \, . \, \sin\theta_1 \, . \, \omega_1 = 0,5 \, . \, 0,32 \, . \, 1,688 = 0,27 \ m/s$$

with the direction perpendicular on the radius IB (namely vertical direction, this direction being the trajectory of the point B) and with trigonometric sense of rotation about the point I.

Distribution of velocities will result join the tops of the two velocities and drawing other few velocities. For example we draw the perpendicular line from I on the AB. This point is with minimum velocity from the rod AB because it is at the minimum distance from ICR. This velocity is collinear with the direction of the rod AB. For the other points, as the points 1 and 2 we shall make in the same way, namely we draw the radii from the point I to the points of the rod and then perpendiculars on these radii. The tops of the velocities are collinear, namely they are located on the line joining the tops of the velocities from A and B.

b)For to calculate the accelerations we shall remark that the point A performs a rectilinear motion with constant velocity, so the acceleration of this point is zero:

$$a_A = 0$$

Consequently the point A is the instantaneous center of the accelerations and the accelerations of the other points (distribution of the accelerations) can be calculated as the body should perform a rotation motion about this point. We shall mark this point A with Q (dedicated notation for the instantaneous center of the accelerations). The accelerations can be calculated only is known the angular acceleration. So we shall calculate first this acceleration. By definition the angular acceleration is the derivative of the angular velocity, but this velocity has to be calculated for an any instant, namely it has to be function of time.

We shall consider the rod making the angle $\theta(t)$ and we have the relation:

$$\omega(t) = \frac{v_A}{IA} = \frac{v_o}{l.\cos\theta(t)}$$

from which results the angular acceleration at the given instant:

$$\varepsilon_1 = \varepsilon(t_1) = \dot{\omega}(t_1) = \frac{0,8.\dot{\theta}(t_1).\sin\theta_1}{0,5.\cos^2\theta_1} = \frac{0,8.\omega_1.\sin\theta_1}{0,5.\cos^2\theta_1} = 0,96 \ rad/s^2$$

This angular acceleration has the same rotation sense as the angular velocity because through the derivation does not change the sign.

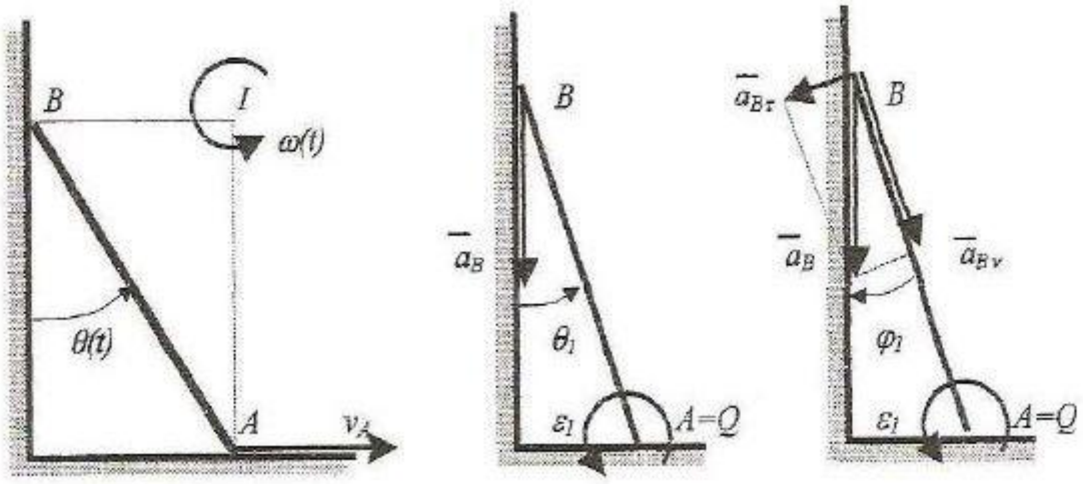


Fig.23.

Now we shall calculate the acceleration of the point B in two ways. Firs we shall calculate this acceleration directly as magnitude:

$$a_B = QB \cdot \sqrt{\varepsilon_1^2 + \omega_1^4} = 0{,}5 \cdot 3 = 1{,}5\ m/s^2$$

with the direction defined by the angle φ_1 resulting from the relation:

$$tg\ \varphi_1 = \frac{\varepsilon_1}{\omega_1^2} = 0{,}337; \qquad \varphi_1 = 18{,}66^\circ = \theta_1$$

and having the sense of rotation the same as the sense of the angular acceleration ε_1 around the instantaneous center of the accelerations Q.

Second time we shall calculate the acceleration of the point B determining the components of it, namely:

-the tangent component with the magnitude:

$$a_{B\tau} = AB \cdot \varepsilon_1 = 0{,}5 \cdot 0{,}96 = 0.48\ m/s^2$$

and having the direction perpendicular on the AB and with the same sense of rotation about the point A as the angular acceleration ε_1.

-the normal component will have the magnitude:

$$a_{B\nu} = AB \cdot \omega_1^2 = 0{,}5 \cdot 1{,}688^2 = 1{,}424\ m/s^2$$

and it is collinear with AB and with the sense directed to the point A.

The acceleration of the point B will be calculated in magnitude with the relation:

$$a_B = \sqrt{a_{B\tau}^2 + a_{B\nu}^2} = \sqrt{0,48^2 + 1,424^2} = 1,5\ m/s^2$$

with the direction making the angle φ_1 with the direction of AB:

$$tg\ \varphi_1 = \frac{a_{B\tau}}{a_{B\nu}} = \frac{0,48}{1,424} = 0,337 \longrightarrow \varphi_1 = 18,66^\circ.$$

We remark that the two methods of determination of the accelerations of the points in a rigid body are equivalent. More, here the acceleration of the point B could be calculated from the rectilinear motion of the point using the knowledge from the motion in Cartesian system writing the coordinate y_B and deriving it twice.

Problem 9. *The bar AB performs a plane motion in vertical plane so that the extremity A describe with constant velocity $v_A = v_o = 0,5\ m/s$ the horizontal, fixed straight line, and it resting in the fixed point C. Calculate and represent the velocity and acceleration of the point B, the minimum velocity on the rod AB at the instant $t_1 = 1s$ from the start of the motion and represent the distribution of velocities on the rod at the same instant. It is known that in the initial instant the rod is in rest in vertical position and has the length $l_{AB} = 1,5m$.*

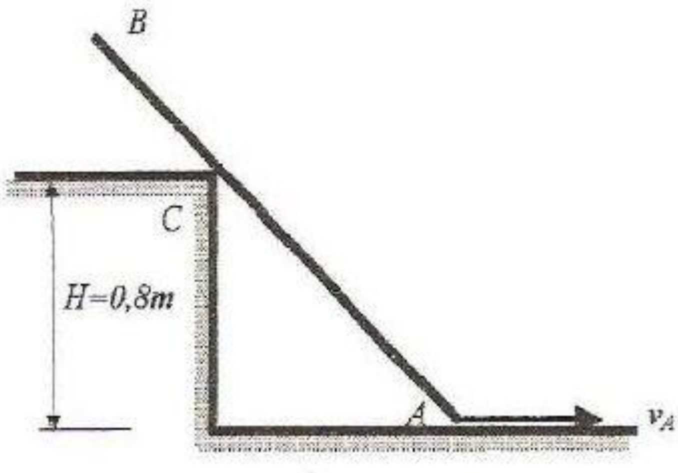


Fig.24.

9.12. Proprieties of the distribution of velocities and accelerations in general motion of a rigid body

We will show a few proprieties of the distribution of velocities and accelerations in the general motion of the rigid body highlighting *those proprieties which can be used in the plane motion of the body (the general motion ob the body in two dimensions).*

The proprieties of the distribution of velocities result from the relation:

$$\bar{v}_P = \bar{v}_O + \bar{\omega} \times \bar{OP}$$

These proprieties are:

- *1)The projections of the velocities of the points from a straight line belonging to a body, on that line, are equal.*
-

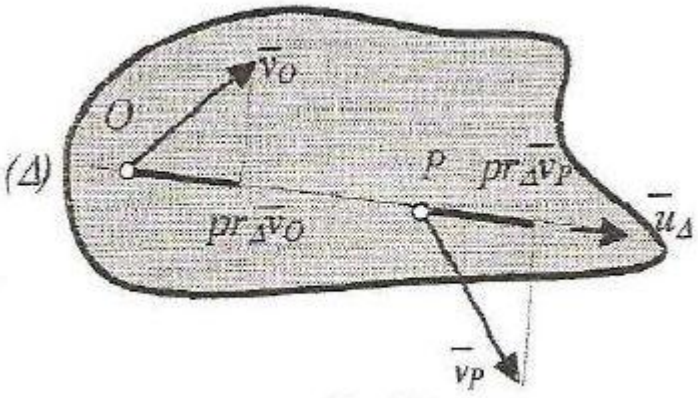


Fig.25.

The demonstration of this propriety is made calculating the scalar product of the relation representing the distribution of velocities with the unit vector of the direction of the straight line:

$$\bar{v}_P \cdot \bar{u}_\Delta = \bar{v}_O \cdot \bar{u}_\Delta + (\bar{\omega} \times \bar{OP}) \cdot \bar{u}_\Delta$$

The third term of the relation is a mixed product of two collinear vectors (OP and u_Δ) and consequently it is equal to zero and the first two terms represent even the projections of the two velocities on the straight line passing through the two points.

- *2) The projections of the velocities of all points of a rigid body on the direction of the angular velocity are equal. This propriety is proved as the previous one namely calculating the scalar product between the relation representing the distribution of velocities and the unit vector of the direction of the angular velocity. This propriety is automatically checked in two* *dimensions because the direction of the angular velocity is perpendicular on the plane containing the velocities of the points.*

- *3) The difference of the velocities of two any points from a body is a perpendicular vector on the straight line passing through the two points and it represents the velocity of one point, from the two, in rotation about the other point.*

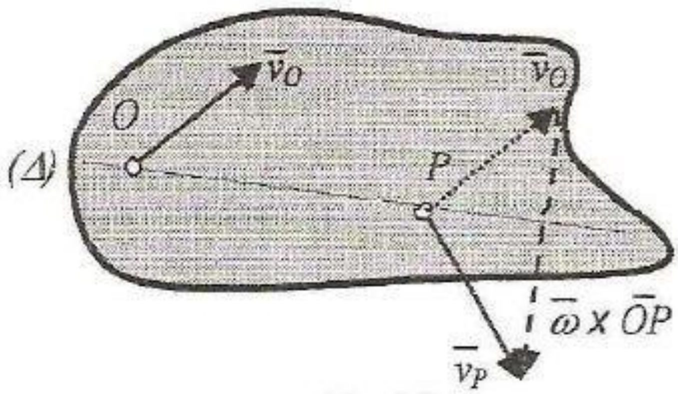


Fig.26.

This propriety is proved bringing in the left side of the relation representing the distribution of velocities the first term from the right part of the relation. In the right part remains one vector product perpendicular on OP and representing the velocity of the point P in rotation about the point O.

- *4) The distribution of velocities can considered as a sum of two distributions: the first from one translation motion with the velocity of a point from the body and the second from a rotation motion about the point considered in the first distribution.*

This propriety results remarking that the relation representing the distribution of velocities has two terms corresponding to the two distributions. If we mark:

$$\bar{v}_O = \bar{v}_P^{tr/O} \quad ; \quad \bar{\omega} \times \bar{OP} = \bar{v}_P^{rot/O}$$

then we have obviously:

$$\bar{v}_P = \bar{v}_P^{tr/O} + \bar{v}_P^{rot/O}$$

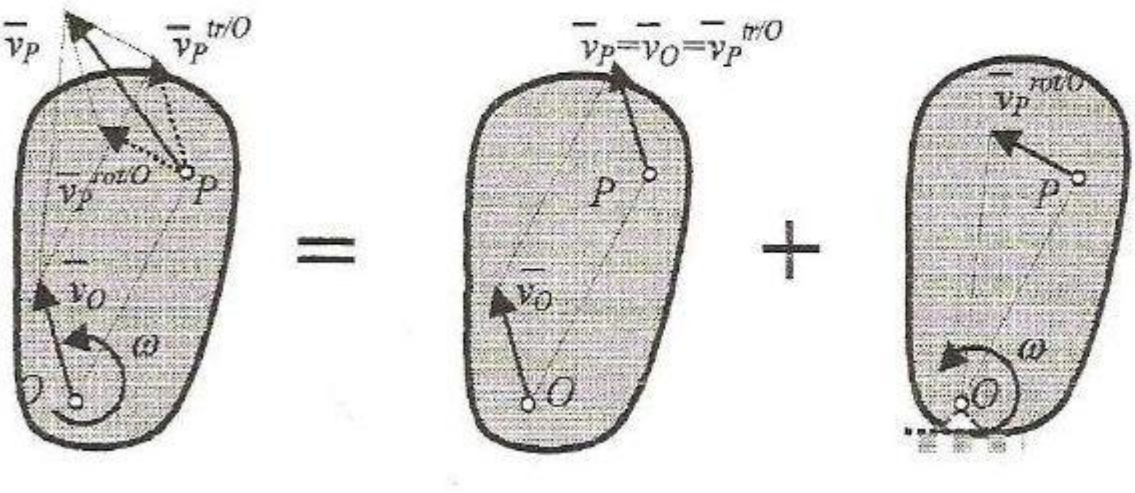


Fig.27.

that is even the stated propriety. We have to remark that the rotation motion is made as the point considered in the translation motion is a fixed point.

- 5) On an arbitrary straight line from the body the tops of the velocities of the points of the line are collinear. This propriety is the consequently of the previous propriety in which we remark that if we make the vector addition of the two distribution from which one is uniformly and the second is linear.
- 6) In the general motion, in space, no points with zero velocity.
- 7) The distribution of velocities has always two invariants: ω the angular velocity and $\omega \cdot v_O$ the scalar product of the angular velocity with the velocity of one any point from the body.
- 8) In general motion, in space, the distribution of velocities is identical with a distribution from a helical motion with respect to an instantaneous axis of the helical motion.

For the distribution of the acceleration we shall state the following proprieties, resulted from the relation defining the distribution of the acceleration:

$$\bar{a}_P = \bar{a}_O + \bar{\varepsilon} \times \overline{OP} + \bar{\omega} \times (\bar{\omega} \times \overline{OP})$$

- 1) The distribution of the accelerations can be considered as a sum of two distributions: one from a translation motion with the *acceleration of a point from the body and the second from a rotation motion about the point considered at the first distribution.*

This propriety results directly if we mark:

$$\bar{a}_O = \bar{a}_P^{tr/O}; \quad \bar{\varepsilon} \times \bar{OP} + \bar{\omega} \times (\bar{\omega} \times \bar{OP}) = \bar{a}_P^{rot/O}$$

with that the distribution of the acceleration is expressed:

$$\bar{a}_P = \bar{a}_P^{tr/O} + \bar{a}_P^{rot/O}$$

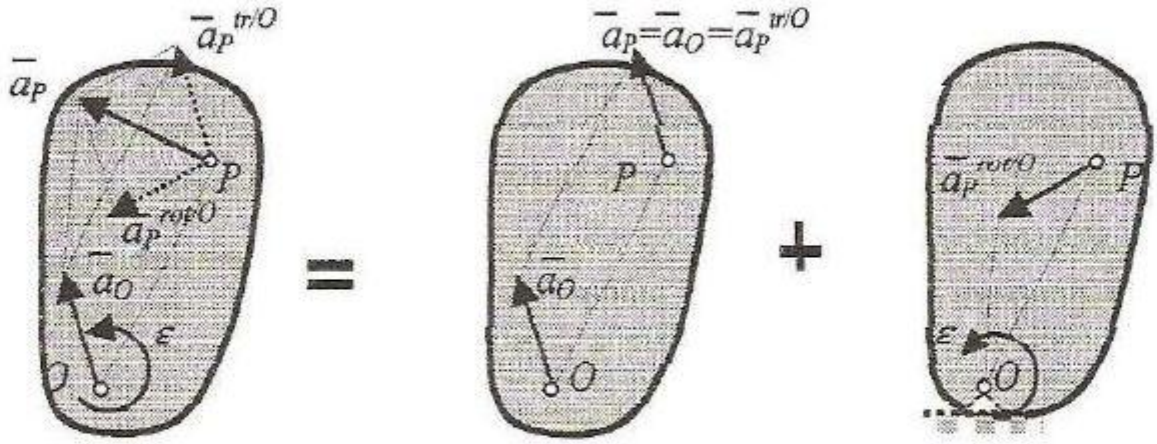


Fig.28.

- *2) On an arbitrary straight line from the body the tops of the accelerations of the points of the line are collinear. This propriety is the consequently of the previous propriety in which we remark that if we make the vector addition of the two distribution from which one is uniformly and the second is linear.*
- *3) In general motion there is always one point, and only one, in which the acceleration is equal to zero. This point is called instantaneous center of the accelerations.*
- *4) In general motion the distribution of the accelerations is identical with a distribution from a rotation about the instantaneous center of the accelerations.*

9.13. Sample problems

Problem 10. *A rod AB by the length l_{AB} = 1,2 m performs a motion so that the extremity A describes, uniformly, one circle with the center in O and the radius R = 0,5 m and rests in the fixed point C (the extremity of the horizontal diameter). Knowing the velocity of the point A, v_A = 1 m/s calculate, using the proprieties of the distribution of the velocities and accelerations in general motion, the velocities of the points C and B and the accelerations of the points A and B in the instant when the extremity A is located on the vertical line passing through the center O of the circle.*

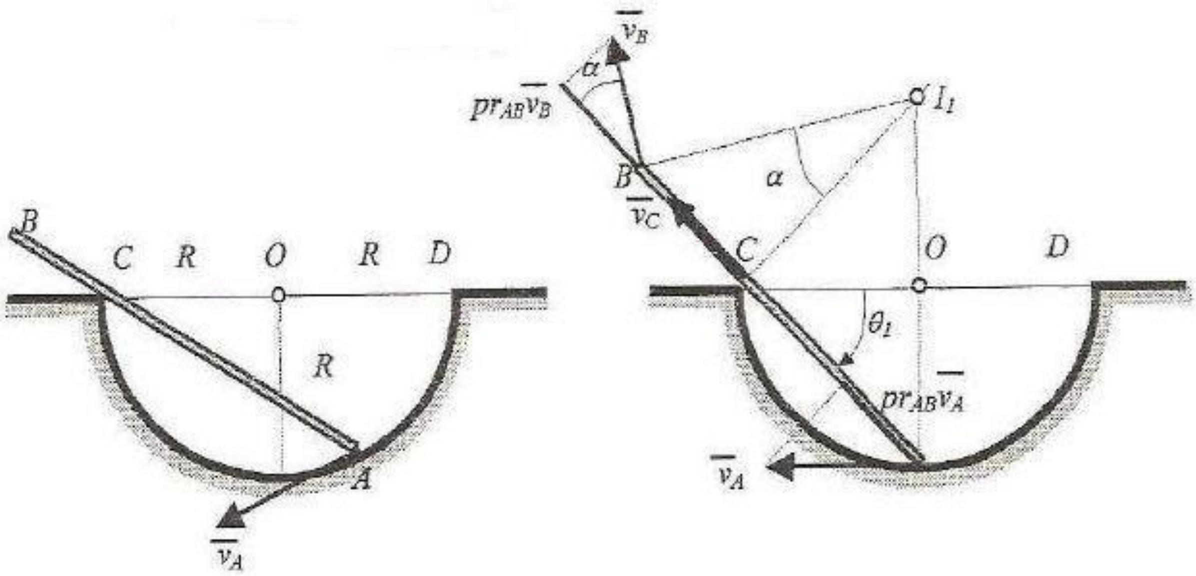


Fig.29.

Solution. *We shall represent the rod AB in the required position and we remark that it makes an 45° angle with the horizontal (or vertical) direction. for to calculate the velocities we shall use the first propriety of the distribution of velocities, namely that the projections of the velocities of the points of the rod on the direction of the rod are equal. But for to know the directions of the velocities we need to know the position of the instantaneous center of rotation. This center will be found at the intersection of the perpendicular direction on the velocity v_A in the point A (Straight line that passes through the point O) with the perpendicular direction on the rod AB in point C (that is inclined with 45° about the horizontal direction).*

We know that the projections of the velocities of the points of the rod AB on the direction AB are equals and equal with:

$$pr_{AB}v_A = v_A \cdot cos45^o = 1 \cdot \frac{\sqrt{2}}{2} = 0,707\ m/s$$

The point C is the point in which is coming the perpendicular line from point I to the direction AB, results that the velocity of the point C is collinear with the direction AB, namely it is the point with minimum velocity on the rod AB and it is equal in magnitude with the projections of all velocities on that direction:

$$v_C = 0,707 m/s$$

For the point B the velocity is perpendicular on the radius from I to B we have:

$$v_B \,.\, \cos\alpha = pr_{AB}\, v_B$$

where the direction results from the triangle I_1CB:

$$\cos\alpha = \frac{I_1C}{I_1B} = \frac{I_1C}{\sqrt{I_1C^2 + (AB - AC)^2}} =$$

$$= \frac{R / \cos 45^\circ}{\sqrt{(R/\cos 45^\circ)^2 + (l_{AB} - 2R/\cos 45^\circ)^2}} =$$

$$= \frac{0,5/0,707}{\sqrt{(0,5/0,707)^2 + (1,2 - 2.0,5.0,707)^2}} = 0,82$$

Results the magnitude of the velocity in point B:

$$v_B = \frac{0,707}{0,82} = 0,862 m/s$$

An other way for to determine the velocities is that in which knowing that the velocity of the point C is collinear with the direction of the rod AB we can write:

$$\overline{v_A} = \overline{v_A}^{tr/C} + \overline{v_A}^{rot/C}$$

where the magnitudes of the two components are:

$$v_A^{tr/C} = v_C = 0,707 m/s;$$
$$v_A^{rot/C} = CA \,.\, \omega_1 \longrightarrow \omega_1 = 1\ rad/s$$

because the first component the component of the velocity v_A on the direction of the rod AB and the second component is the component of the velocity v_A perpendicular on the direction of the rod AB. The sense of the angular velocity is clockwise.

Now, using the same relation we shall have for the point B:

$$\overline{v_B} = \overline{v_B}^{tr/A} + \overline{v_B}^{rot/A}$$

where we have the following components:

$$v_B^{tr/A} = v_A = 1 m/s$$

with the same direction and sense as in the point A, and:

$$v_B^{rot/A} = AB \,.\, \omega_1 = 1,2 \,.\, 1 = 1,2 \ m/s$$

This component is perpendicular on the rod AB and with the same sense of rotation about the point A as the angular velocity ω_1.

The angle between the two components is 135^o and consequently results the magnitude:

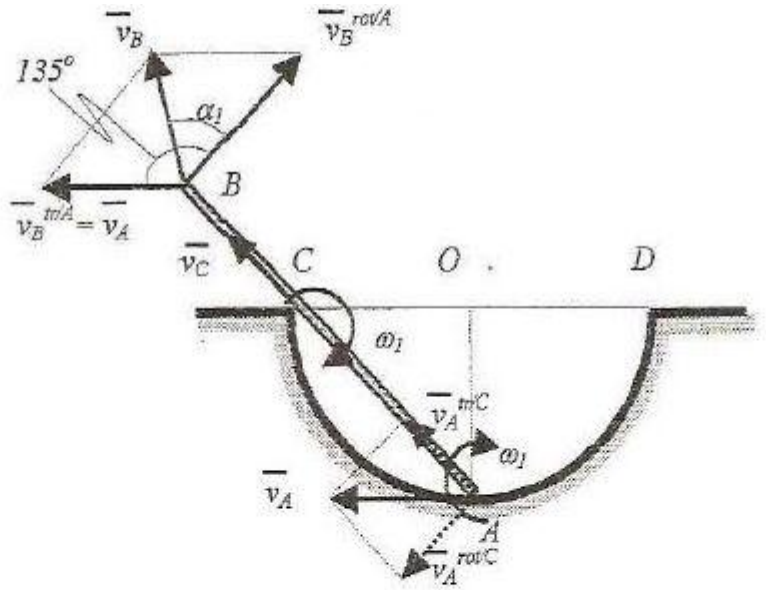


Fig.30.

$$v_B = \sqrt{(v_B^{tr/A})^2 + (v_B^{rot/A})^2 + 2 v_B^{tr/A} . v_B^{rot/A} . \cos 135^o} =$$
$$= \sqrt{1 + 1,2^2 - 2.1.1,2.0,707} = 0,862 m/s$$

and the direction:

$$\sin\alpha_1 = \frac{v_B^{tr/A}}{v_B} \sin 135^o = \frac{1}{0,862} 0.707 = 0,82 \longrightarrow \alpha_1 = 55,12^o$$

For the calculation of the acceleration of the point A we know that this point performs a circular motion (on the circle with the center in O and the radius R) with constant velocity. Results the two components of the acceleration of this point:

$$a_{A\tau} = \dot{v}_A = 0$$
$$a_{A\nu} = \frac{v_A^2}{R} = 2 \ m/s^2 = a_A$$

This last component has the direction of AO and the sense directed to the center of the circle O.

For to determine the acceleration of the point B we have need to determine, first, the angular velocity at an any instant of the motion. For this we shall consider the rod in an any position (making the angle θ(t) with the horizontal direction) and after that we obtain the position of the instantaneous center of rotation I we calculate the angular velocity. We remark that the distance IA is constant because the right angle C is with its tip on the circle so IA is the diameter of the circle. Because the velocity of the point A is constant too results that the angular velocity is also constant:

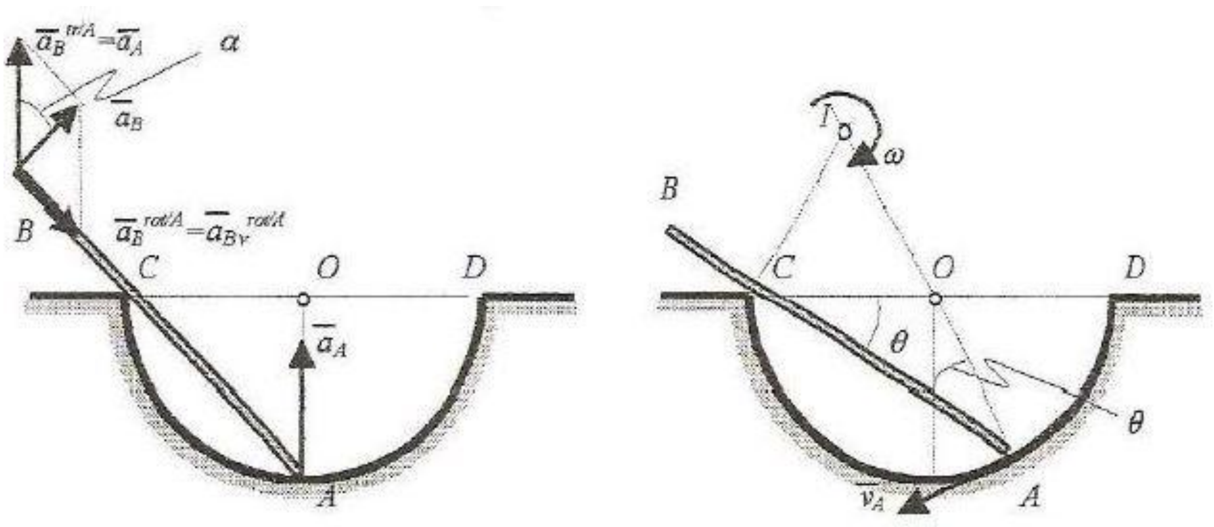


Fig.31.

$$\omega = \frac{v_A}{IA} = 1\ m/s$$

Results, obviously that the angular acceleration is equal to zero:

$$\varepsilon = \dot{\omega} = 0$$

Having the angular acceleration of the rod and knowing the relation:

$$\overline{a_B} = \overline{a_b}^{tr/A} + \overline{a_B}^{rot/A}$$

where the first component is equal in magnitude, direction and sense with the acceleration of the point A:

$$a_b^{tr/A} = a_A$$

we can determine the acceleration of the point B without to known the position of the instantaneous center of the accelerations.

The component corresponding to the rotation about the point A hat also two components, from which the firs is the tangent component with the magnitude:

$$a_{B\tau}^{rot/A} = AB \,.\, \varepsilon_1 = 0.$$

The second component is the normal one and has the magnitude:

$$a_{B\nu}^{rot/A} = AB \,.\, \omega_1^2 = 1{,}2 \,.\, 1^2 = 1{,}2 \; m/s^2,$$

with the direction of the rod and the sense directed to the point A (the center of rotation corresponding to this component).

The magnitude of the acceleration of the point B will be (the composition of two components):

$$a_B = \sqrt{(a_B^{tr/A})^2 + (a_B^{rot/A})^2 + 2.a_B^{tr/A}.a_B^{rot/A}.\cos 135^o} =$$
$$= \sqrt{2^2 + 1{,}2^2 - 2.2.1{,}2.0{,}707} = 1{,}43 \; m/s^2$$

Its direction with respect to the vertical direction will be obtained with the relation:

$$\sin\alpha = \frac{a_B^{rot/A}}{a_B}\sin 135^o = \frac{1{,}2}{1{,}43}0{,}707 = 0{,}593 \longrightarrow \alpha = 36{,}39^o$$

Problem 11. *A plate having square shape with the side $l = 0{,}8$ m moves in the plane of the plate. Knowing the velocities of two tops and the accelerations of two tops (as in the figure 32) at an instant of the motion calculate and represent the velocities and accelerations of the other tops of the plate and also the velocity and acceleration of the center of the square in the same instant. Are known: $v_A = v_C = 2$ m/s, $a_B = a_C$. Represent the distribution of velocities on the four sides of the square in the given instant.*

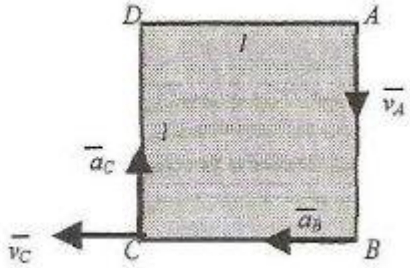


Fig.32.

Chapter 10. Relative motion of the particle

10.1. Introduction

In the previously two chapters we have studied the absolute motions of the particle and the rigid body, namely the motions performed with respect to a fixed reference system. But as we know, in nature, there are not fixed reference systems. In most of the engineering works the reference systems joined to the Earth are considered as fixed systems. But there are problems in which this simplification does not satisfy the requirements of those problems and we need to consider the motions with respect to the reference systems in motion too.

In this chapter we shall deal with the relative motion of the particle considering that the knowledge from this chapter are enough for to explain all those engineering problems in which are these motions.

We shall use two reference systems and namely: one moving system marked Oxyz, about which is performed the relative motion of the particle, and a fixed system marked $O_1x_1y_1z_1$*, about which is performed the motion of the moving reference system.*

In the time of the relative motion we shall distinguish three kinds of motions:

-relative motion of the particle *that is the motion of the particle about the moving reference system;*

-absolute motion of the particle *that is the motion of the particle about the fixed reference system;*

-transport motion *that is the motion of the moving reference system about the fixed system.*

In the following we shall use the derivatives with respect to time in the process of the calculation of the velocities and accelerations. A part of these time functions are expressed with respect to moving reference system so is necessary, first, to define the derivatives of the vectors function of time expressed about the moving reference system.

10.2. Absolute and relative derivatives.

We shall name ***absolute derivative*** *the derivative with respect to time that will consider the variation in time of the derivative quantities and also of the reference system about it is expressed. We shall name* ***relative derivative*** *that derivative which will consider only the variation in time of the vector without to consider the motion of the moving reference system. This derivative is called also* ***local derivative.***

Let be a vector, function of time and expressed function a moving reference system:

$$\overline{V} = V_x(t) \cdot \overline{i(t)} + V_y(t) \cdot \overline{j(t)} + V_z(t) \cdot \overline{k(t)}$$

The derivative with respect to time will be:

$$\dot{\overline{V}} = (\dot{V}_x \cdot \overline{i} + \dot{V}_y \cdot \overline{j} + \dot{V}_z \cdot \overline{k}) + (V_x \cdot \dot{\overline{i}} + V_y \cdot \dot{\overline{j}} + V_z \cdot \dot{\overline{k}})$$

The first parenthesis is the derivative of the vector without to consider the motion of the moving reference system, namely it is the local derivative of the vector and it is marked:

$$\frac{\partial \overline{V}}{\partial t} = \dot{V}_x \cdot \overline{i} + \dot{V}_y \cdot \overline{j} + \dot{V}_z \cdot \overline{k}$$

We remark that this derivative is obtained considering the moving reference system as a fixed one, or as the moving system is stopped from its motion in the time of this derivative.

The second parenthesis contains only the derivatives of the unit vectors of the moving axes, derivatives which were solved in the previous chapter and we have:

$$V_x \cdot \mathring{\bar{i}} + V_y \cdot \mathring{\bar{j}} + V_z \cdot \mathring{\bar{k}} = \bar{\omega} \times \bar{V}$$

Here we remark that this derivative is obtained as the vector would have constant magnitude and direction with respect to the moving reference system or as this vector is joined to the moving system.

Finally we may write:

$$\mathring{\bar{V}} = \frac{\partial \bar{V}}{\partial t} + \bar{\omega} \times \bar{V}$$

From this relation results that generally the absolute derivative does not coincide with the relative derivative. However we remark that the absolute derivative of the angular velocity of the moving reference system is always equal to the relative derivative of it.

10.3. Composition of the velocities in the relative motion of the particle.

Consider a particle P in motion about a reference system Oxyz, system in motion too about a fixed reference system $O_1x_1y_1z_1$.

Among the position vectors we have the relation:

$$\bar{r}_1 = \bar{r}_O + \bar{r}$$

If we calculate the derivative of this results:

$$\mathring{\bar{r}}_1 = \mathring{\bar{r}}_O + \mathring{\bar{r}}$$

where the first two terms are the absolute velocities of the two points P and O:

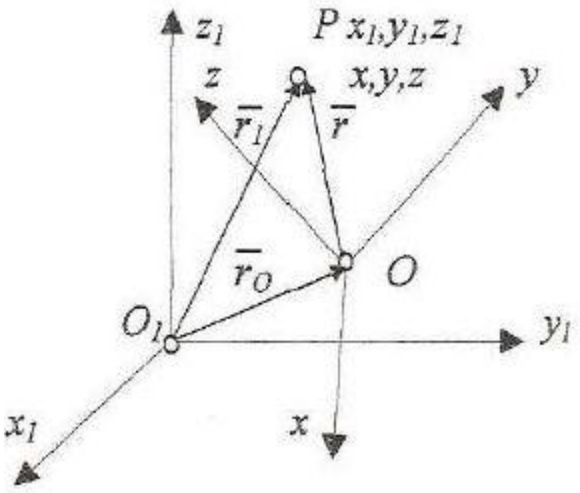


Fig.1.

$$\dot{\overline{r}}_1 = \overline{v}_a \ ; \ \dot{\overline{r}}_O = \overline{v}_O$$

The derivative of the position vector of the point P with respect to the origin of the moving reference system is made as we have seen before and we have:

$$\dot{\overline{r}} = \frac{\partial \overline{r}}{\partial t} + \overline{\omega} \times \overline{r}$$

Replacing in the initial relation we will have:

$$\overline{v}_a = \overline{v}_O + \overline{\omega} \times \overline{r} + \frac{\partial \overline{r}}{\partial t}$$

The last term is a velocity being the first derivative with respect to time but it considers the motion of the particle about the moving reference system only, namely it consider only the relative motion of the particle. We shall name this as ***relative velocity*** *and we shall mark:*

$$\overline{v}_r = \frac{\partial \overline{r}}{\partial t}$$

As we have presented before this velocity is calculated as the moving reference system is a fixed one.

If we consider now only the first two terms of the right part of the relation these represent the velocity of the particle P as this point moves together with the moving reference system (as it is joined to the moving system). But this system performs the transport motion consequently it will be named ***transport motion of the particle****:*

$$\overline{v}_t = \overline{v}_O + \overline{\omega} \times \overline{r}$$

We can write finally:

$$\overline{v}_a = \overline{v}_t + \overline{v}_r$$

namely the absolute velocity is the sum of the transport and relative velocities of the particle.

This relation can be used to determine one velocity if we know the other two.

10.4. Composition of the acceleration in relative motion of the particle.

Let to calculate the derivative of the expression of the absolute velocity:

$$\overset{\circ}{\overline{v}}_a = \overset{\circ}{\overline{v}}_O + \overset{\circ}{\overline{\omega}} \times \overline{r} + \overline{\omega} \times \overset{\circ}{\overline{r}} + \frac{d}{dt}\left(\frac{\partial \overline{r}}{\partial t}\right)$$

The first two terms are the absolute derivatives of the absolute velocities of two any points, so they represent the absolute accelerations of the two points:

$$\overset{\circ}{\overline{v}}_a = \overline{a}_a ; \quad \overset{\circ}{\overline{v}}_O = \overline{a}_O$$

The derivative of the angular velocity is the angular acceleration. As we have seen the absolute derivative of the angular velocity is equal to the relative (local) derivative of it:

$$\overset{\circ}{\overline{\omega}} = \overline{\varepsilon}$$

The derivative of the position vector with respect to the origin of the moving reference system is as we have find:

$$\overset{\circ}{\overline{r}} = \frac{\partial \overline{r}}{\partial t} + \overline{\omega} X \overline{r}$$

The absolute derivative of the relative derivative (that is a function of time expressed with respect to the moving reference system) is made as the derivative of the position vector r:

$$\frac{d}{dt}\left(\frac{\partial \overline{r}}{\partial t}\right) = \frac{\partial^2 \overline{r}}{\partial t^2} + \overline{\omega} X \frac{\partial \overline{r}}{\partial t}$$

Removing in the first relation we have:

$$\overline{a}_a = \overline{a}_O + \overline{\varepsilon} X \overline{r} + \overline{\omega} X \left(\frac{\partial \overline{r}}{\partial t} + \overline{\omega} X \overline{r}\right) + \frac{\partial^2 \overline{r}}{\partial t^2} + \overline{\omega} X \frac{\partial \overline{r}}{\partial t} =$$
$$= \overline{a}_O + \overline{\varepsilon} X \overline{r} + \overline{\omega} X (\overline{\omega} X \overline{r}) + \frac{\partial^2 \overline{r}}{\partial t^2} + 2\overline{\omega} X \frac{\partial \overline{r}}{\partial t}$$

*We remark that the fourth term is the local derivative by the second order with respect to time of the position vector (this derivative considers only the variations about the moving reference system without to consider the motion of this reference system), consequently it is the **relative acceleration** of the particle and we shall mark:*

$$\frac{\partial^2 \overline{r}}{\partial t^2} = \overline{a}_r$$

The first three terms represent the acceleration of the particle as the particle is connected to the moving reference system performing the motion together with this system, namely the particle performs the transport motion. We shall name this acceleration as the ***transport acceleration*** *of the particle:*

$$\overline{a}_O + \overline{\varepsilon} \times \overline{r} + \overline{\omega} \times (\overline{\omega} \times \overline{r}) = \overline{a}_t$$

The last term does not belong to the relative motion and the transport motion, or better, it has one component from the transport motion (the angular velocity) and one component from the relative motion (the relative velocity). This component of the acceleration is called ***Coriolis acceleration*** *ant it keep in mind that the two motions (transport and relative motions) are in interaction:*

$$2\overline{\omega} \times \frac{\partial \overline{r}}{\partial t} = 2\overline{\omega}_t \times \overline{v}_r = \overline{a}_c$$

With these we may write:

$$\overline{a}_a = \overline{a}_t + \overline{a}_r + \overline{a}_c$$

10.5. Method of the stopped motions

If we consider the two relations corresponding to the composition of the velocities and accelerations:

$$\overline{v}_a = \overline{v}_t + \overline{v}_r$$
$$\overline{a}_a = \overline{a}_t + \overline{a}_r + \overline{a}_c$$

results that we can determine the absolute velocity and acceleration of a particle in a composite motion using a method that allows to calculate the velocities and accelerations separately in the two motions: transport and relative motions. This method is called ***method of the stopped motions*** *and it is used in the following way:*

1) Firs we shall define the two motions: the transport motion and the relative motion. Generally these motions are obtained from the fact that the particle is in motion about a body that it is in motion too. In this case the body performs the transport motion and the particle the relative motion about the body.

2) We consider the transport motion stopped. In the case when the transport motion is performed by a body we shall stop the motion of it. We shall consider only the motion of the particle about the body in rest, namely the relative motion of the particle. Studying this motion we determine the relative velocity and the relative acceleration:

$$\overline{v}_r\, ,\ \overline{a}_r$$

3) This time we shall stop the particle on the body (we stop the relative motion) and we study the motion of the particle together with the body that performs the transport motion. For the particle results the transport velocity and transport acceleration:

$$\overline{v}_t\, ,\ \overline{a}_t$$

4) Having the relative velocity and the angular velocity of the transport motion we calculate the Coriolis acceleration. In two dimensions for to calculate can be considered a reference system with the origin in the particle and the Px axis on the direction and sense of the relative velocity. In this case we have:

$$\overline{v}_r = v_r\,\overline{i}$$

The axis Py is taken so that the system to be an right hand system. Results for the angular velocity of the transport motion:

$$\overline{\omega}_t = \pm\,\omega_t\,\overline{k}$$

The (+) sign corresponds to the counterclockwise sense of rotation of the angular velocity. The Coriolis acceleration will result:

$$\bar{a}_c = 2\,(\pm\,\omega_t\,\bar{k})\times v_r\,\bar{i} = \pm 2\omega_t v_r\,\bar{j}$$

namely this acceleration has the direction of the axis Py (perpendicular on the direction of the relative velocity). We may remark that if the angular velocity is positive then the Coriolis acceleration results also positive.

5) Using the two relations we add in vector way the velocities and accelerations resulting the absolute velocity and absolute acceleration of the particle. If for the velocities, because we have only two components, we may use the rule of the parallelogram, for the accelerations the simplest way to calculate is that to choose a reference system and projecting all the components on the aces we may use the theorem of projections to calculate the projections on the two axes of the absolute acceleration.

10.6. Sample problems.

Problem 1. *One triangular prism performs a horizontal rectilinear translation motion. In the same time on the slope side of the prism slides a particle P. Knowing that at an instant the velocity and acceleration of the translation motion of the prism are v_A = 2m/s and a_A = 1 m/s^2 and also that in the same instant the particle is located in the position B and has the relative velocity and acceleration (about the prism) v_P = 2 m/s and a_P = 2 m/s^2 determine the absolute velocity and acceleration of the particle in the same instant.*

Solution. *The slipping motion of the prism on the horizontal surface is the transport motion, and the sliding motion of the particle on the inclined surface AB of the prism is the relative motion of the particle.*

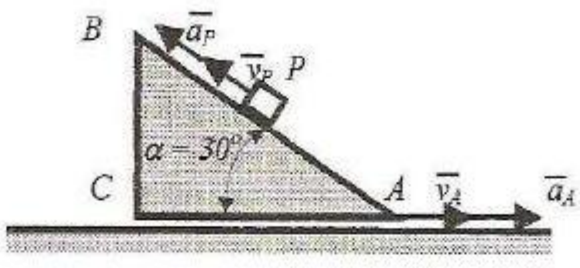


Fig.2.

We shall consider the transport motion stopped, namely we shall stop the slipping motion of the prism on the horizontal surface. It remains only the slipping motion of the particle on the sloped surface of the prism that is a rectilinear motion. Results that the relative *velocity and acceleration of the particle at the given instant are known and they have the direction of the straight line on which is performed the motion:*

$$v_r = v_P = 2m/s \; ; \; a_r = a_P = 2m/s^2$$

This velocity and acceleration are represented in the figure 3.

The transport motion is the rectilinear translation motion of the prism and consequently the transport velocity and acceleration of the particle P are even the velocity and acceleration of the point A (moved in the point P) because in translation motion all the points of the body (the point P is considered belonging to the prism) have the same velocity and acceleration:

$$\overline{v_t} = \overline{v_A} \; ; \; \overline{a_t} = \overline{a_A}$$

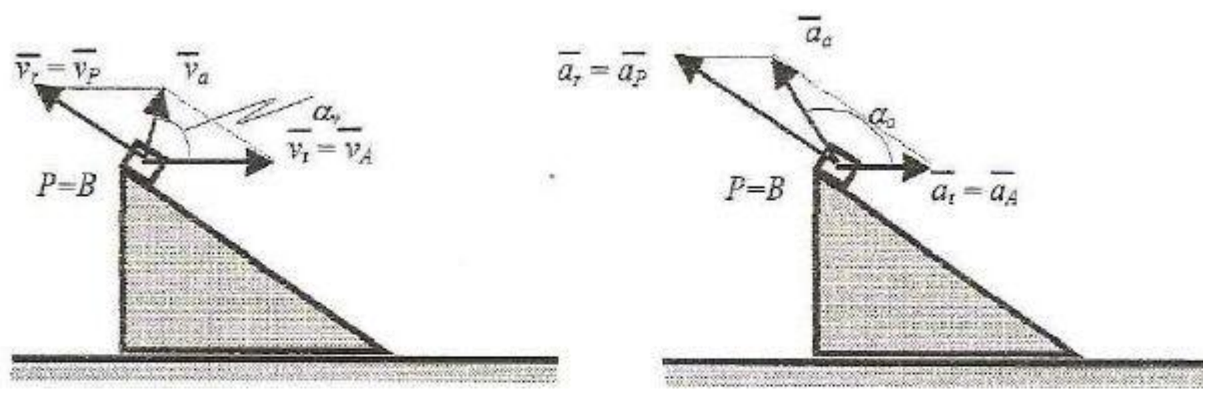


Fig.3.

Because the transport motion is a translation we have obviously:

$$\overline{\omega_t} = 0$$

and consequently the Coriolis acceleration is equal to zero:

$$\overline{a_c} = 2\overline{\omega_t} \times \overline{v_r} = 0$$

The absolute velocity results summing the two velocities and it has the magnitude:

$$v_a = \sqrt{v_t^2 + v_r^2 + 2v_t v_r \cos 150^\circ} = \sqrt{2^2 + 2^2 - 2.2.2.0,865} = 1,035m/s$$

and the direction defined, with respect to the horizontal direction, by the angle α_v :

$$\sin\alpha_v = \frac{v_r}{v_a}\sin 30^\circ = \frac{2}{1,035}0,5 = 0,966 \longrightarrow \alpha_v = 75^\circ$$

Because the Coriolis acceleration is equal to zero we have only two components of accelerations and we shall use for summing the parallelogram rule. The magnitude of the absolute acceleration will be:

$$a_a = \sqrt{a_t^2 + a_r^2 + 2a_t a_r \cos 150^\circ} = \sqrt{1^2 + 2^2 - 2.1.2.0{,}865} = 1{,}239 m/s^2$$

with the direction defined about the horizontal direction:

$$\sin\alpha_a = \frac{a_r}{a_a}\sin 30^0 = \frac{2}{1{,}239}0{,}5 = 0{,}807\ ; \quad \alpha_a = 180^0\text{-}53{,}81^o\text{=}126{,}18^o$$

Problem 2. *A bar having the length l = 1 m performs a rotation motion about a fixed point O with the angular velocity $\omega = \pi t/3$ (s^{-1}). In the same time on the bar slides a small collar P with the law of motion (about the bar) $s(t) = OP = 12{,}5\ t^2$ (cm). Knowing that the motion of the bar stars from horizontal determine the absolute velocity and acceleration of the collar when it arrives in the middle of the bar.*

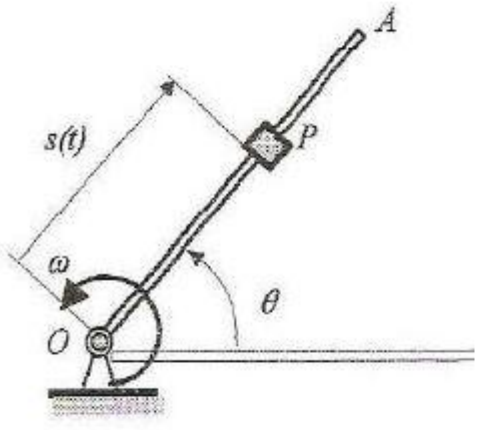


Fig.4.

Solution. *The rotation motion of the bar is the transport motion, and for the particle (that is considered stopped from its sliding motion on the bar) P results a circular motion with the center O and the radius OP and having as law of motion the rotation motion of the bar, namely with the angular velocity ω. If we stop the bar in an any position, the collar P slides along the bar, consequently the relative motion is a rectilinear motion on the bar OA with the law of motion s(t).*

Now we shall determine the position of the bar when the collar arrives in the middle of the bar. For that we shall suppose that this position is obtained at the instant t_1 and results:

$$s(t_1) = l/2$$

or:

$$12{,}5 \cdot t_1^2 = 50 \longrightarrow t_1 = 2\ s$$

In this instant the angle made by the bar with the horizontal (the initial position) is $\theta(t_1)$, where the angle θ results from the definition of the angular velocity:

$$\overset{\circ}{\theta} = \omega \longrightarrow \theta(t) = \int \omega dt + C \longrightarrow \theta(t) = \frac{\pi t^2}{6} + C$$

but because the initial position is the horizontal the constant C is equal to zero and the position of the bar at the given instant results:

$$\theta_1 = \theta(t_1) = \frac{2\pi}{3}$$

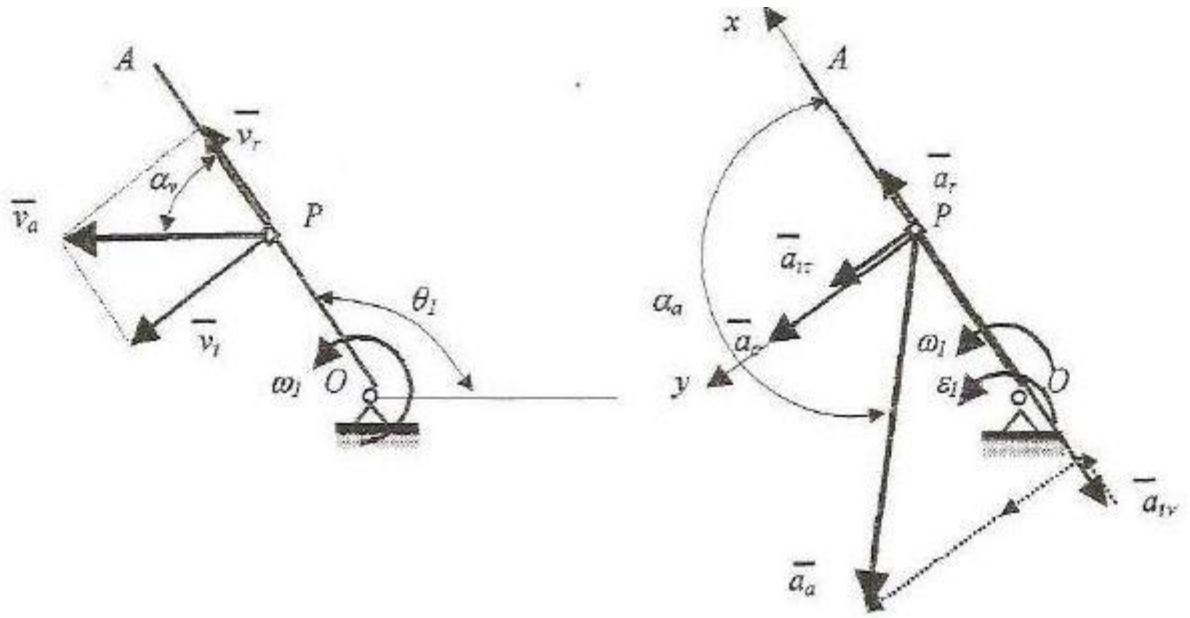


Fig.5.

We shall consider the bar stopped in this position and from the rectilinear motion of the particle P along the bar result the relative velocity and acceleration:

$$v_r = \overset{\circ}{s}(t_1) = 25t_1 = 50 \ cm/s$$
$$a_r = \overset{\circ\circ}{s}(t_1) = \overset{\circ}{v}(t_1) = 25 \ cm/s^2$$

having the direction of the straight line OA along that is made the sliding motion of the collar P.

The transport motion of the particle results if the collar is stopped on the bar OA (in the given position, namely at the half of the bar) and performs the motion considering joined to the bar (like it is a point of the bar). In this way the transport velocity will have the magnitude:

$$v_t = OP(t_1) \cdot \omega(t_1) = \frac{l}{2} \cdot \frac{\pi t_1}{3} = 104,7 \ cm/s$$

the direction being perpendicular on the OP and with the sense so that to be in the rotation sense of the angular velocity about the rotation center O.

The transport acceleration resulted from the circular motion of the point P will have two components. The tangent component has the magnitude:

$$a_{t\tau} = OP(t_1)\,.\,\varepsilon_1 = OP(t_1)\,.\,\dot{\omega}(t_1) = \frac{l}{2}\,.\,\frac{\pi}{3} = 52{,}34\ cm/s^2$$

the direction being perpendicular on the OP and the sense in the rotation sense of the angular acceleration about the rotation center.

The normal component has the magnitude:

$$a_{t\nu} = OP(t_1)\,.\,\omega(t_1)^2 = \frac{l}{2}\,.\,\frac{\pi^2 t_1^2}{9} = 219{,}1\ cm/s^2$$

being collinear with OP and directed toward the rotation center O.

For the calculation of Coriolis acceleration we shall choose a reference system (used only for the vector computation) with the origin in point P, with the Px axis on the direction and the sense of the relative velocity (here on the direction OA directed toward the end A of the bar) and with the Py axis perpendicular on the bar (on the relative velocity) and with the sense so that to form a right hand system. With respect to this system we have:

$$\overline{v_r} = v_r\,\overline{i} = 50\,\overline{i}\,;\ \overline{\omega_{t1}} = \omega_t(t_1)\,\overline{k} = 2{,}09\,\overline{k}$$

With this The Coriolis acceleration will be:

$$\overline{a_c} = 2\ \overline{\omega_t}(t_1)\ \times\ \overline{v_r}(t_1) = 2\,.\,2{,}09\overline{k} \times 50\,\overline{i} = 209{,}4\,\overline{j}$$

The absolute velocity of the particle will be obtained with the rule of the parallelogram:

$$v_a = \sqrt{v_t^2 + v_r^2} = \sqrt{104{,}7^2 + 50^2} = 116{,}02\ cm/s$$

The direction with respect to the direction of the bar will be defined by the angle:

$$tg\ \alpha_v = \frac{v_t}{v_r} = \frac{104{,}7}{50} = 2{,}09 \qquad \alpha_v = 64{,}47^\circ$$

For the absolute acceleration, that is the resultant vector of minimum three components:

$$\overline{a}_a = \overline{a}_t + \overline{a}_r + \overline{a}_c$$

the rule of the parallelogram is difficult to use. In this way we shall determine this acceleration using the theorem of projections. Consequently we need a system of reference and here we shall use the same reference system as the calculation of the Coriolis acceleration. The projections of the components on the two axes will be:

$$a_{ax} = a_r - a_{tv} = 25 - 219{,}1 = -194{,}1 \ cm/s^2;$$
$$a_{ay} = a_{t\tau} + a_c = 52{,}34 + 209{,}4 = 261{,}74 \ cm/s^2$$

We obtain the magnitude of the absolute acceleration:

$$a_a = \sqrt{a_{ax}^2 + a_{ay}^2} = \sqrt{194{,}1^2 + 261{,}74^2} = 325{,}82 \ cm/s^2$$

having the direction defined with respect to the Px axis:

$$\cos\alpha_a = \frac{a_{ax}}{a_a} = -0{,}595 \longrightarrow \alpha_a = 180^o - 53{,}44^o = 126{,}56^o$$

Problem 3. *One disc performs a translation motion along its horizontal diameter, on a horizontal fixed bar, with the law of motion OA = at^2. On the periphery of the disc a particle moves with the law of motion BP = s(t) = $R\pi t^2/12$. Knowing that a = 10 cm/s^2 and R = 25 cm, determine the absolute velocity and acceleration of the particle P at the instant t_1 = 2s from the start of the motion.*

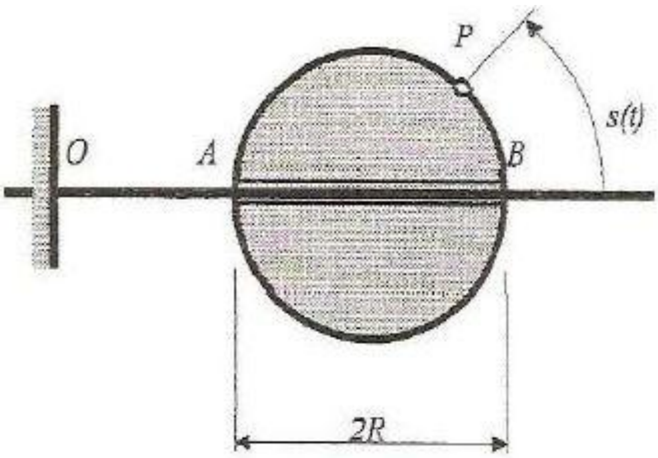


Fig.6.

Problem 4. *One circular frame by the radius R = 30 cm rotates in its plane around the fixed point O with the constant angular velocity* ω_o *= 1 rad/s. In the same time on the frame moves, starting from point O, with constant velocity* v_o *= 25 cm/s with respect to the frame, one small ring P. Calculate the absolute velocity and acceleration of the particle P when the diameter of the frame becomes horizontal and knowing that at the initial instant it is vertical.*

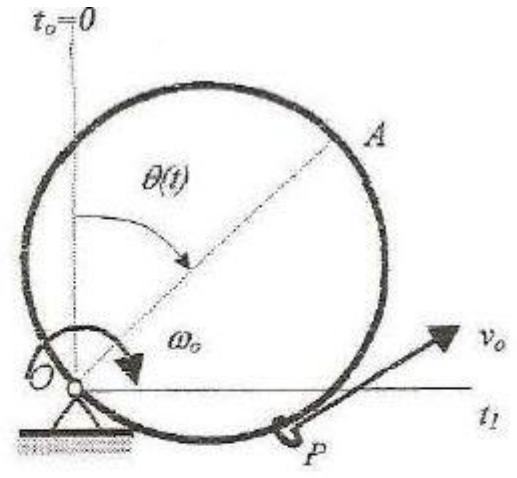


Fig.7.

Chapter 11. Plane mechanism with one degree of freedom

11.1. Introduction

This chapter will have as object the study of the distribution of velocities for the systems of bodies named simple mechanism.

*One **mechanism is a system of bodies that has possibilities of motion**, namely having degrees of freedom. If the system is made from plane bodies, located in the same plane and performing motion in the same plane then we say that it is a **plane mechanism**. If the motion of the system may be expressed function of one kinematic parameter then it said that the **mechanism has one degree of freedom**. For this kind of mechanism we have the relation:*

$$N_{df} = 3N_b - (3N_{fs} + 2N_{sh} + N_{ss}) = 1$$

where we have marked: N_{df} the number of the degrees of freedom, N_b the number of the bodies, N_{fs} the number of the fixed supports, N_{sh} the number of the simple hinges and N_{ss} the number of the simple supports.

11.2. Centers of rotation

As we have shown, in this chapter we shall study only the distribution of velocities for plane mechanisms. For to develop an efficient method, to obtain the distribution of velocities we shall make some remarks about characteristic elements of the distribution of velocities.

In plane motions a body may perform only three kinds of motions namely: translation motion, rotation motion and plane motion. If we consider only the distribution of velocities then we can remark the following:

- *-In rotation motion there is one point (the fixed point called rotation center), and only one, that has zero velocity and with respect it the distribution of velocities is linear;*
- *-Because we study the instantaneous distribution of velocities in plane motion there is a point, and only one (the instantaneous center of rotation), that has zero velocity and with respect to it the instantaneous distribution of velocities is linear;*
- *In translation motion there are not points with zero velocity, but we can think in the following way: the plane motion in two dimensions is the general motion, so it contains the particular motions namely the translation motion too. Therefore the translation motion is a particular plane motion and the instantaneous center of rotation will be located at the intersection of the perpendicular lines on the velocities of two points.*

ICR∞

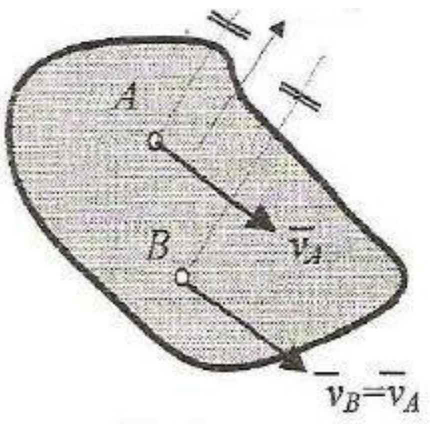


Fig.1.

But the velocities are the same (in magnitude, direction and sense) in the two points resulting that the two perpendiculars intersect at infinity distance. We can say the in translation motion there is an instantaneous center of rotation but it is located at infinity.

*Finally we can remark that: all the three motions, no matter the kind of them, have each a rotation center with zero velocity and about that the instantaneous distribution of velocities is linear. These centers will be named **absolute centers of rotation**.*

Results obviously that the number of the absolute centers of rotation is equal to the number of bodies from the mechanism:

$$N_{ACR} = N_B$$

Let to consider now two bodies of the mechanism. There is always one point in which the velocities determined on the two bodies are equal. For example if the two bodies have one internal hinge, indifferent to the kind of motions performed by the two bodies, the point representing the internal hinge has only one velocity indifferent on which body is considered this point.

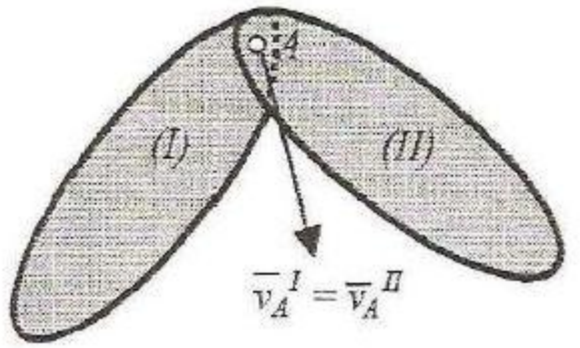


Fig.2.

We can remark in this example that if the two bodies should have two points in which the velocities are equal, the two bodies should perform the same motion and we can consider one body made from the two. Results consequently that for each pair of two bodies there is one this kind of point. These points are called ***relative centers of rotation****. The name of these centers are coming from the fact that if one body (from two) stops then this point becomes the rotation center of the other body, namely this point is the rotation center of the relative rotation motion of one body with respect to the other.*

The number of this kind of centers is equal:

$$N_{RRC} = C^2_{Nb} = N_b(N_b - 1)/2$$

where C^2_{Nb} is the number of combinations from the number of bodies taken two by two.

The importance of the two kinds of center is that: the instantaneous distribution of velocities is made in relation to the absolute center *of rotation of each body, and the relative centers of rotation make the connections between the distributions on the bodies of the mechanism.*

The centers of rotation are determined or using the constraints and connections of the mechanism or using the proprieties of the distribution of velocities.

If do not consider the fixed support (because if a body has this kind of constraint it is fixed so it has not motion and we can eliminate this kind of body from the mechanism) the other constraints and connections highlight the following proprieties:

- ***One hinged support*** *(fixed hinge) is always* ***the absolute rotation center*** *of the body on which it is located. This is because the hinged support fixes a point of the body.*

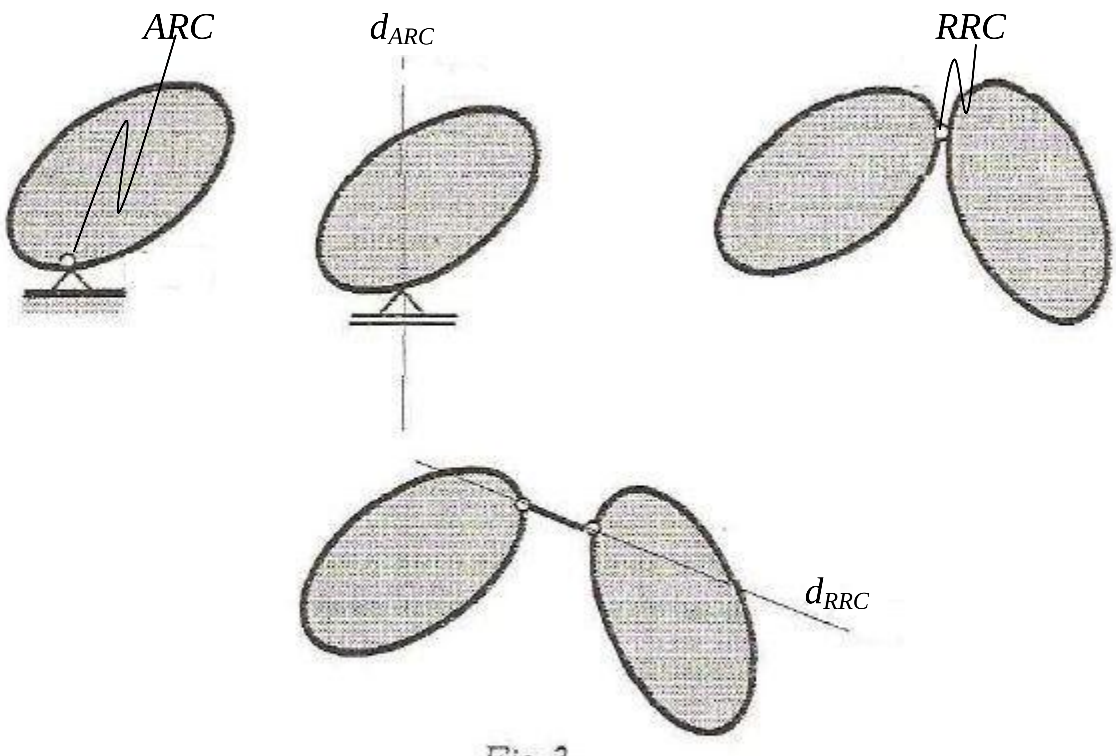


Fig.3.

- ***One simple support*** *defines a straight line, the direction of the simple support, on which is the* ***absolute rotation center*** *of the body. This propriety results from the fact that this constraint allows the sliding motion of the body on the boundary surface, consequently the velocity of the contact point is tangent to this surface and the instantaneous rotation center is located on the perpendicular line to the surface in this point, namely on the* *direction of the simple support. We shall mark this line with* $\boldsymbol{d_{ARC}}$*, namely the line on which is located the absolute rotation center.*

- ***One internal hinge*** *is always the* ***relative rotation center*** *of the two bodies joined by the hinge.*
- *If two bodies are connected among them with* ***one simple internal connection*** *then the* ***relative rotation center*** *is located on de direction of this connection. This propriety results from the fact that this connection refers to the relative motions of the bodies and consequently if we stop one body (from two) then the rotation center of the other is located on the direction of the simple connection. We shall mark the line of this direction with* $\boldsymbol{d_{RRC}}$.

For to know all the rotation centers of a mechanism we have to use the proprieties of the distribution of velocities in motion in plane also (in two dimensions). This is made with two ***collinearity theorems****.*

11.3. Collinearity theorems

We shall study two colliniarity theorems of the rotation centers, theorems with which we can determine the positions of all rotation centers of a plane mechanism with one degree of freedom.

I st ***theorem****. In a plane mechanism with one degree of freedom the absolute rotation centers of two any bodies and the relative rotation center of the two bodies are collinear.*

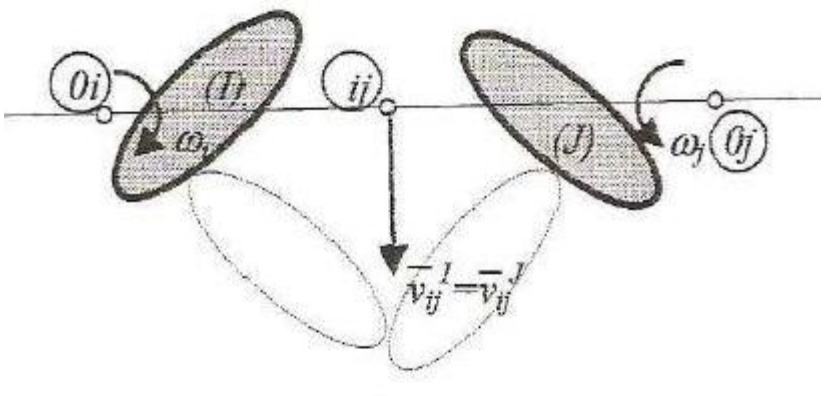


Fig.4.

Consider two any bodies from a plane mechanism with one degree of freedom marked (I) and (J). The absolute rotation centers of the two *bodies will be marked (0i) and (0j) and the relative rotation center corresponding to the two bodies will be marked (ij). The theorem says that the three centers of rotation (0i), (0j) and (ij) are located on the same straight line.*

For to demonstrate this theorem it is enough to consider the velocity of the relative rotation center calculated from the body (I) and from the body (J). Because the velocity of the point (ij) is only one it has to be perpendicular on the radius from the absolute rotation center (0i) to (ij) and also on the radius from (0j) to (ij) resulting that the two radii have to be collinear.

***IInd theorem**. The relative rotation centers of three bodies, taken two by two, from a plane mechanism with one degree of freedom are collinear.*

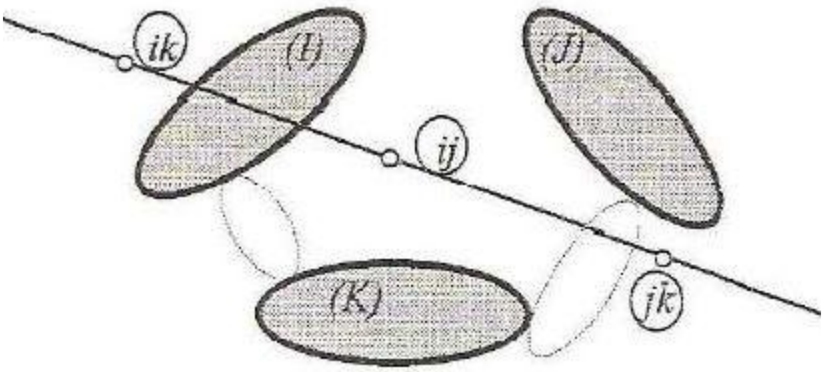


Fig.5.

We shall mark the three bodies (I), (J) and (K) and the relative rotation centers corresponding to them (ij), (ik) and (jk). The theorem says that the three relative rotation centers are located on the same straight line.

Being relative rotation centers we shall consider that one body from the three performs transport motion and the other two relative motions about the first. Consequently if we stop the body (K) will remain only the relative motions of the two bodies (I) and (J) and the two relative rotation centers (ik) and (jk) will become the two absolute rotation centers of the two bodies (I) and (J). But corresponding to first theorem the two absolute rotation centers have to be collinear with the relative rotation center (ij). We can see that in this demonstration we have not changed the position of the three points but only their nature.

For to use efficiently these two theorems I propose to use them as a rule with the name ***rule of the indexes****.*

Rule of the indexes*. If two centers have one common index then on the straight line passing through the two centers is located the center of rotation corresponding to the non-common indexes of the two centers.*

We may remark that this rule contains the two theorems (reason for which we have marked the absolute centers of rotation with two indexes from which one is 0).

11.4. Sample problems

Problem 1. *For the plane mechanism from the figure 6 determine all rotation centers.*

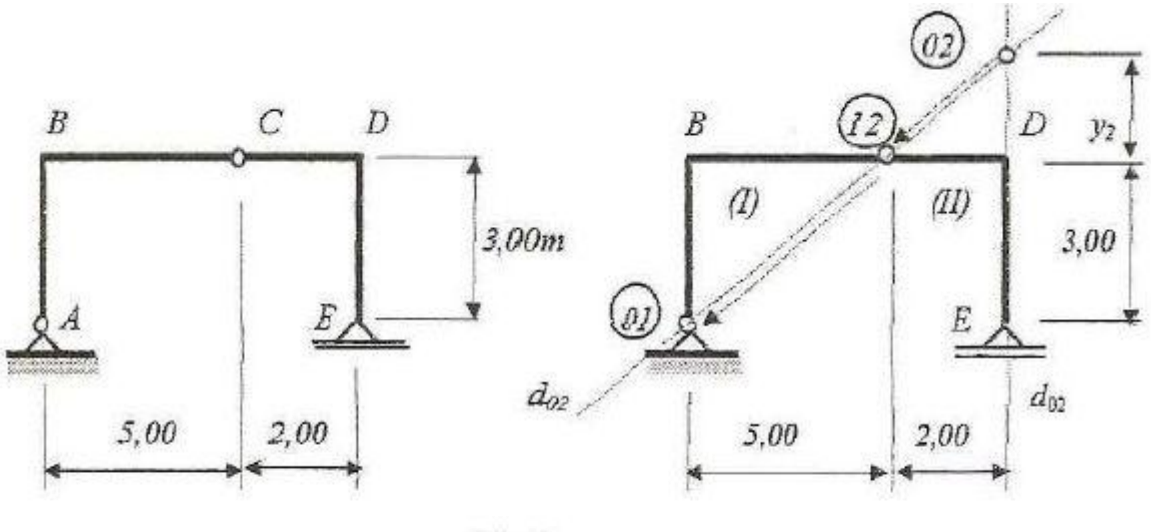


Fig.6.

Solution. *The given mechanism has two bodies: ABC and CDE which we note (I) and (II). The hinged support from the body (I) is the absolute rotation center of this body and we shall mark (01). The internal hinge from the point C is the relative rotation center of the two bodies and it will be marked (12). The vertical simple support from E defines a straight line (the vertical line passing through point E) on which is located the absolute rotation center of the body (II), namely this line is* d_{02}.

Remains to determine the absolute rotation center of the body (II). This is made in the following way: we know that a center (one point) can be determined at the intersection of two straight lines. But we know one line d_{02} *on which is located the search center so it is enough to determine one other line* d_{02}*. For this we shall use the collinearity theorems (or the rule of the indexes). Here the rule of the indexes shows us: the two known centers (01) and (12) have one common index (figure 1), so on the straight line passing through the two centers is located the rotation center corresponding to the non-common indexes (02). In this way we have the second line* d_{02} *that* *intersected with the first (the vertical line through point E) give us the position of the absolute rotation center (02).*

The position of the absolute rotation center (02) is determined with respect to the known points in the following way: if one line is vertically then we have to determine the position only in vertical direction from a known point to that center. Let to note this distance with y_2. Starting from the center the find center (02) on the inclined straight line to the first known center (here the relative rotation center (12)0 we make one right angle triangle (here (02)(12)D). From the known center where we have arrived (here (12)) we go on the inclined straight line until the second known center (here the absolute rotation center (01)) and we make another right angle triangle (here the triangle (12)(01)B). The two triangles are like and we can write the likeness relation:

$$\frac{D(12)}{D(02)} = \frac{(12)B}{B(01)};$$

from which results the position of the absolute rotation center (02):

$$\frac{2}{y_2} = \frac{5}{3} \longrightarrow y_2 = 1{,}20m$$

Problem 2. *Determine the rotation center for the mechanism from the figure 7.*

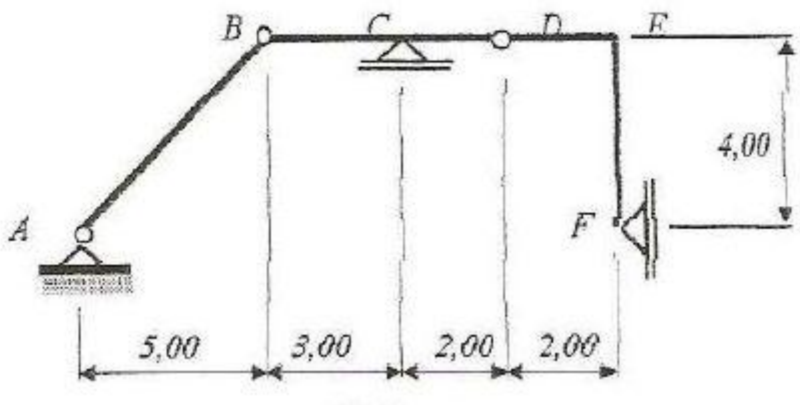


Fig.7.

11.5. Method of the diagrams of velocities' projections

For to determine the distribution of velocities on a mechanism there are a lot of methods. The most of them solve this problem on straight lines from the body. The best method to obtain the distribution of velocities in a plane mechanism with one degree of freedom is ***the method of the diagrams of velocities' projections****.*

For to obtain this method we shall consider one body of the mechanism and we shall suppose known: the position of the absolute rotation center and the instantaneous angular velocity of the body. Because we want to achieve the instantaneous distribution of velocities we can consider the mechanism stopped at that instant in the considered instantaneous position.

For to study the motion of the body we shall consider a reference system with its origin in the absolute rotation center and the two axes in the convenient directions namely the horizontal and vertical directions.

The relation that defines the distribution of velocities of the considered body is:

$$\bar{v}_P = \bar{\omega} \times \overline{AP}$$

where AP is the position vector of the any point P with respect to the absolute rotation center.

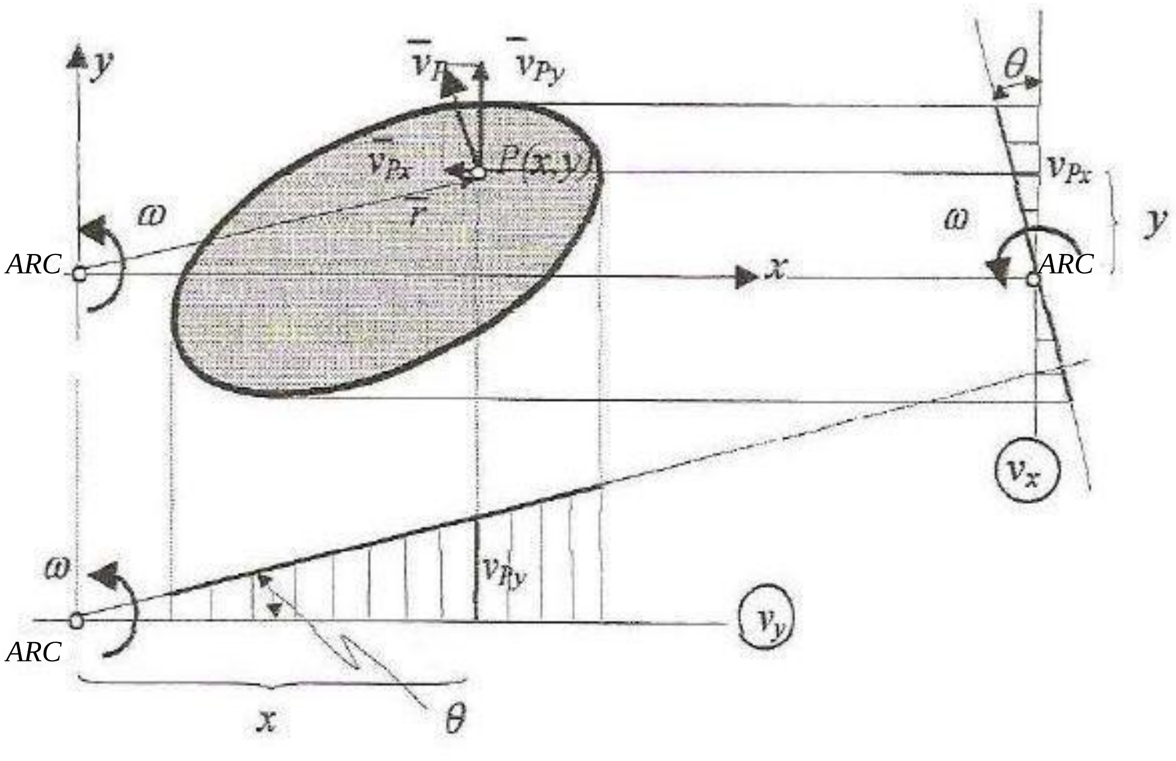


Fig.8.

The projections of the velocity of the point P on the two axes are:

$$v_{Px} = -\omega . y; \quad v_{Py} = \omega . x$$

From these last two relations result the following proprieties:

- *With respect to the absolute rotation center the variation of the projections of velocities is linear being zero in the right of the absolute rotation center.*
- *All the points of the body which have the same x coordinate will have the same projection of the velocities on the axis Oy, and all points with the same y coordinate will have the same projection of the velocities on the axis Ox.*

*If we consider two reference lines parallel to the two axes and we represent perpendicular on them the two projections and we join the top of the represented projections with the absolute rotation center (projected on the reference lines) then we obtain straight lines which represent the variations of the velocities' projections on the direction of the corresponding axes. These straight lines are called **diagrams of velocities' projections**. Results that:*

- *The diagrams of velocities projections are straight lines passing through the absolute rotation centers (projected on the reference lines).*
- *The slope of the diagram is the angular velocity of the body:*

$$tg\theta = \frac{v_{Py}}{x} = \frac{|v_{Px}|}{y} = \omega$$

- *The diagrams in two orthogonal projections are perpendicular.*

Knowing these we can draw the diagrams of the velocities' projections for a mechanism with one degree of freedom in two dimensions.

11.6. The steps of the determining of velocities using the method of the diagrams of velocities' projections.

In the following we shall give the steps which we have to pass for to calculate the angular velocities and velocities of points of a plane mechanism with one degree of freedom. These steps are:

- *1) First we check that the mechanism to have one degree of freedom. After we determine the fixed bodies and they are removed from the mechanism. We remind that the fixed bodies are in the following four cases (or in the cases which can be reduced to these four cases): one body with one fixed support, one body with a fixed hinge (an internal hinge can be a fixed hinge having the same behavior as a hinged support if it is in contact with a fixed body) and a simple support (the direction of the simple support does not pass through the hinge), one body with three simple supports (the directions of the supports are not all the three parallel or concurrent in the same point) and finally two bodies each of them with a fixed hinge and an internal hinge between them (the three hinges are not collinear). We denote (number) the bodies of the mechanism.*
- *2) We determine the centers of rotation and the directions on which they are resulted from the constraints and connections of the mechanism.*
- *3) We determine all the absolute rotation centers using the theorems of collinearity. An absolute rotation center will be determined at the intersection of two straight lines. These lines are obtained or from the constraints, or using the collinearity theorems. For to determine an absolute rotation center we shall start from another known absolute rotation center.*
- *4) We draw the diagrams of the velocities' projections starting from the body that has given motion. The successions of the following operations are: we draw the two reference lines, we project on these lines the absolute rotation centers of all bodies. It is drawn the diagram of the first body (the body with given motion) that is a straight line that passes through the absolute rotation center of the body projected on the reference line, diagram*

that is rotated in the rotation sense of the angular velocity. It is projected on the find diagram the relative rotation center with the following body and is drawn the diagram of the following body as a straight line passing through this relative rotation center and the absolute rotation center of this body. Step by step we draw the diagrams of all bodies from the mechanism,

- *5) We calculate the angular velocities of the bodies and then the projections of the velocities of points from the bodies of the mechanism.*

11.7. Sample problems

Problem 3. *For the mechanism from the figure 9. calculate the velocities of the marked points using the method of diagrams of velocities' projections.*

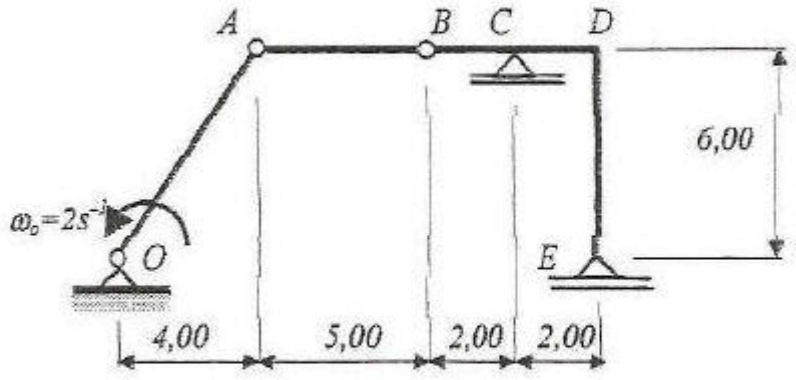


Fig.9.

Solution. *The mechanism has one degree of freedom because we have:*

$N_{df} = 3 \cdot 3 - (2 \cdot 3 + 2) = 1$

namely it is made from three bodies and has three simple hinges and two simple supports.

In this mechanism we have not fixed bodies and consequently we shall solve this mechanism with three bodies. We shall mark these bodies as: (I), (II) and (III).

The fixed hinge from the point O (the hinged support) is the absolute rotation center of the first body and it is marked (01), the internal simple hinges from A and B are the relative rotation centers of the bodies joined by them namely (12) and (23). The two vertical simple supports located on the third body (III) define each of them one vertical straight line (the directions of the simple supports) on which is located the absolute rotation center (03). We shall mark these lines with d_{03}.

Because we have two lines d_{03} *at the intersection point of them have to by located the absolute rotation center (03). But because the two lines are parallel their intersection is* *located at infinity on the vertical direction. This means that any vertical straight line will passes through this absolute rotation center (03).*

For the distribution of velocities of a plane mechanism with one degree of freedom generally is enough to determine all the absolute rotation centers because the existing relative rotation centers generally are enough for the distribution. This thing is because the transmission of the motion can be made from a body to the other for the neighbor bodies, generally is not necessary to transmit the motion from example from the first body to the last.

Taking in consideration all of these, for the distribution of velocities in this problem is necessary to determine the absolute rotation center (02). For this center we have need to know two straight line on which is located this center. In this problem we have not any this kind of lines, but we can determine them using the collinearity theorems (or the rule of indexes). Because we have only relative rotation center between neighbor bodies and the first theorem uses for to determine one absolute rotation center also relative rotation centers, it is obviously that we have to start always from a neighbor body (of the body for which we determine the absolute rotation center) with known absolute rotation center. Here we shall start from the body (I) that has the known absolute rotation center and it is neighbor body to the body (II) so we know also the relative rotation center between them. Consequently remarking that the two centers (01) and (12) have a common index results that on the straight line passing through these two points is located the absolute rotation center (02). In this way we have find the first line d_{02}. But we have need to find another line d_{02}. Another neighbor body to the body (II) with known absolute rotation center is the body (III). We remark that the absolute rotation center (03) and the relative rotation center (23) have the common index 3 and in this way on the line passing through the two centers id located the absolute rotation center (02) corresponding to the non-common indexes. But the absolute rotation center (03) is located at infinity distance on vertical direction, this d_{02} is also vertical. Intersecting the two lines d_{02} results the center (02). The position of this center will be obtained if we determine the distance y_2 on vertical direction. As we have seen in the previous problem we obtain two like triangles in which we can write the likeness relation:

$$\frac{5}{y_2} = \frac{4}{6} ; \longrightarrow y_2 = 7{,}50.$$

In this problem is not necessary for the solution to find the relative rotation center (13) but we shall determine also this center for to show how we find a relative rotation center in a mechanism.

One relative rotation center is find, as all the other rotation center, at the intersection of two straight lines. These lines can be obtained or from the first theorem when we know the absolute rotation centers of the corresponding two bodies (as in this problem), or using the second theorem when in the combination enter only relative rotation centers. In this case the first line is found considering the absolute rotation centers of the two bodies (I) and (III) and we remark that for these two centers the index 0 is common. It is obtained the d_{13} line in vertical direction. Because we have only the two relative rotation centers :(12) and (23) with the common index 2 results passing through these two points another straight line d_{13}. Intersecting the two lines we obtain the position of the relative rotation center (13).

Having all the rotation centers we can draw the diagrams of the velocities' projections. We shall draw firs the two reference lines on orthogonal directions (v_x) and (v_y) and we project on these two lines the absolute rotation centers of the mechanism. If at the first two centers we have not any problem to project on the reference lines at the absolute rotation center (03) we make the remark that on the vertical reference line this center is located at the infinity, and on the horizontal reference line any point of the reference line should be the projection of this center because any vertical straight line passes through this center.*

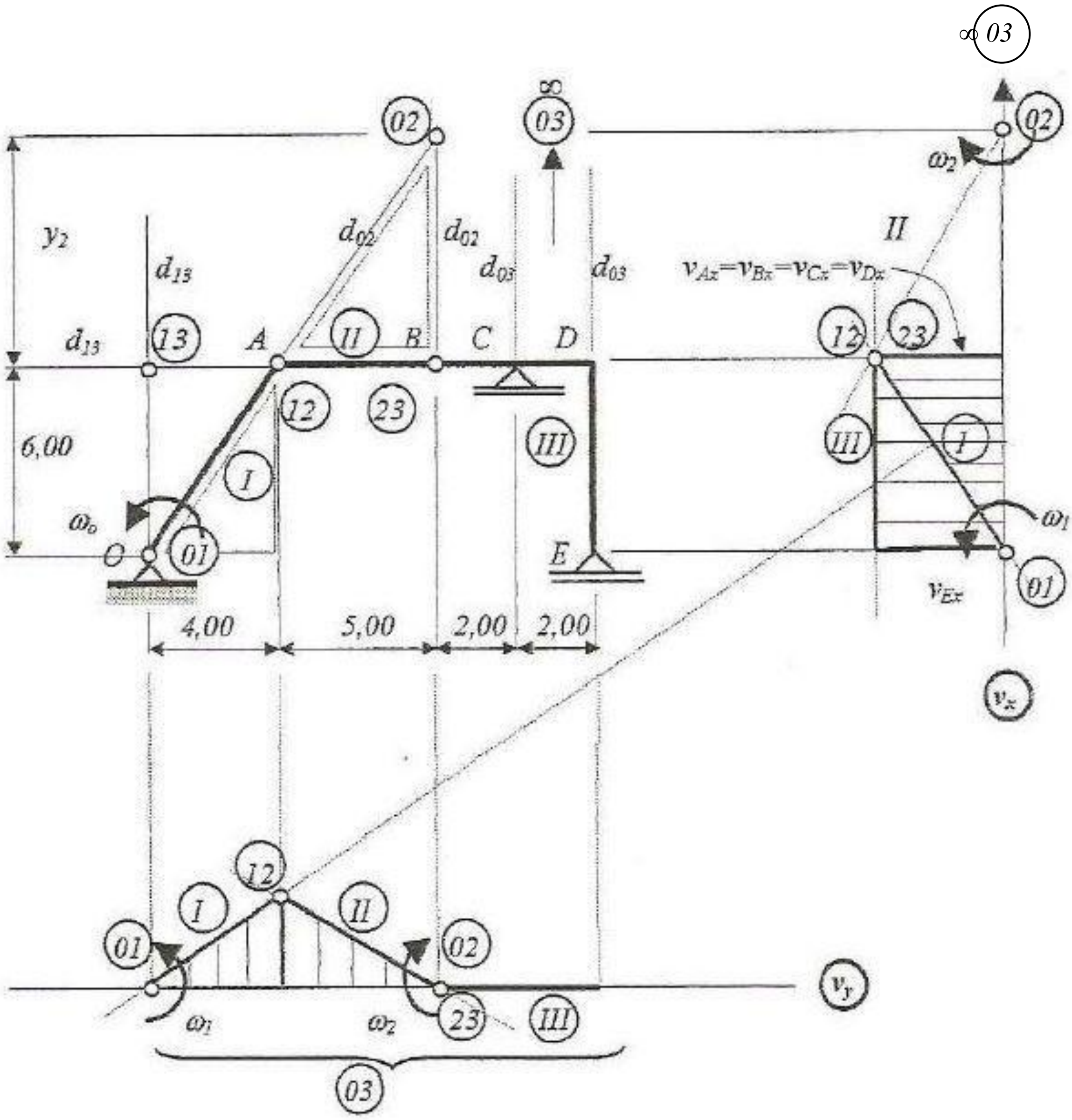


Fig.10.

We begin to draw the diagrams of velocities with the body (I) at which we know the angular velocity. The diagram of this body is a straight line passing through the absolute rotation center (01) from the reference line and it is rotated, with respect to the reference line, in the rotation sense of the angular velocity of the body. In the both projections we shall have the same rotation of the diagrams of the body (I) and in this way the two diagrams of the same body (I) are perpendicular. From the line representing the diagram of the body we shall cut the portion corresponding to the body (in the picture we marked with bolded line this portion).

On the line representing the diagram of the first body we shall project the relative rotation center with the following body, here the center (12). We join this center with the absolute rotation center (02), from the reference line, and we obtain the diagram if the body (II). We project on these diagrams the relative rotation center with the next body, here the body (III), namely the center (23) and joining with the absolute rotation center of the body (III) is obtained the diagrams of this body. In this way we have obtained, in two perpendicular projections, the diagrams of the velocities' projections.

We shall begin now to calculate the velocities. First we determine the angular velocities of the bodies starting from the known angular velocity:

$$\omega_1 = \omega_o = 2\ rad/s$$

The angular velocity of the next body is obtained from the propriety of the relative rotation center that in this point the velocities from the two bodies are equal. We can write, for example from the v_y diagram the following relation:

$$\omega_1 \,.\, 4 = \omega_2 \,.\, 5$$

namely the velocity v_{y12} is equal to the angular velocity multiplied with the distance from the absolute center of the body to that point (12) on the both bodies. Results for the second body:

$$\omega_2 = 1{,}6\ rad/s$$

The body (III) having the absolute rotation center at infinity has a translation motion and consequently it has zero angular velocity:

$$\omega_3 = 0.$$

Now, for to calculate the velocities of the points it is enough to determine the projections of the velocities on the two directions. For each projection we have to calculate the product between the angular velocity of the body and the distance from the absolute rotation center of the body to the point. We have for the marked points:

$$\overline{v_A} \begin{cases} v_{Ax} = \omega_1 \,.\, 6 = 2 \,.\, 6 = 12\ m/s\ ; \\ v_{Ay} = \omega_1 \,.\, 4 = 2 \,.\, 4 = 8\ m/s; \end{cases}$$

$$\overline{v_B} \begin{cases} v_{Bx} = \omega_1 \,.\, 6 = 2 \,.\, 6 = 12\ m/s\ ; \\ v_{By} = 0; \end{cases}$$

The points B, C, D and E are points on a body in translation motion so they will have equal velocities.

In this way we have solved the problem entirely.

Problem 4. *For the plane mechanism represented in the figure 11 determine the angular velocities of the bodies and the velocities of the marked points using the method of the diagrams of the velocities' projections. We know that the velocity of the point D is vertically and has the magnitude v = 10 m/s.*

Solution. *The mechanism has one degree of freedom because it has three bodies, one simple fixed hinge (hinged support), one internal double hinge that is equivalent to two simple hinges and two simple supports:*

$N_{df} = 3 \cdot 3 - (2 \cdot 3 + 2) = 1$

As we can see the mechanism has not fixed bodies. We shall mark the bodies: the body AB with (I), the body BCD with (II) and the body BEF as (III).

The fixed hinge from A is the absolute rotation center (01) of the body (I), the internal double hinge from B is in the same time the relative rotation centers (12), (13) and (23). We remark that the number of the relative rotation centers from a multiple internal hinge is not equal to the number of the equivalent simple hinges. On the horizontal straight line passing through the horizontal simple support from D is located the absolute rotation center (02) so this line is d_{02}, and on the vertical straight line that passes through the point F is located the absolute rotation center (03) so this line is d_{03} .

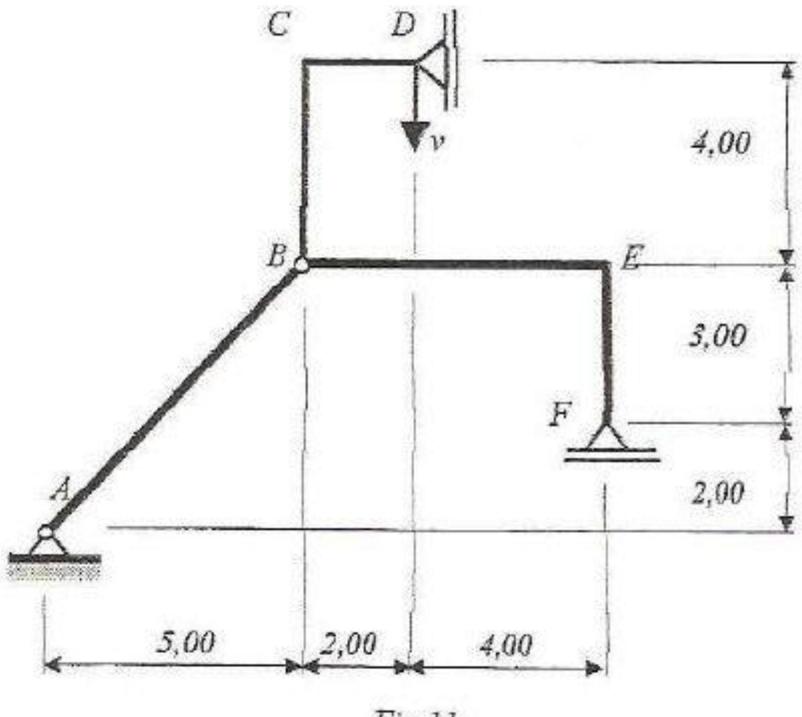


Fig.11.

The absolute rotation centers (02) and (03) will be obtained on the straight line passing through the centers (01) and (12) – the common index is 1 – line that is the same as the line passing through (01) and (13) – common index 1 - , consequently this straight line is in the same time the line d_{02} and d_{03}. Intersecting with the two lines corresponding to the directions of the simple supports we obtain the absolute rotation centers (02) and (03). For to determine the positions of these *centers we shall write the likeness relations in the triangles (02)CB and ABG and also (03)EB and ABG:*

$$\frac{2+x_2}{4}=\frac{5}{5}\ ;\longrightarrow x_2=2;$$

$$\frac{6}{y_3+4}=\frac{5}{5}\ ;\longrightarrow y_3=2.$$

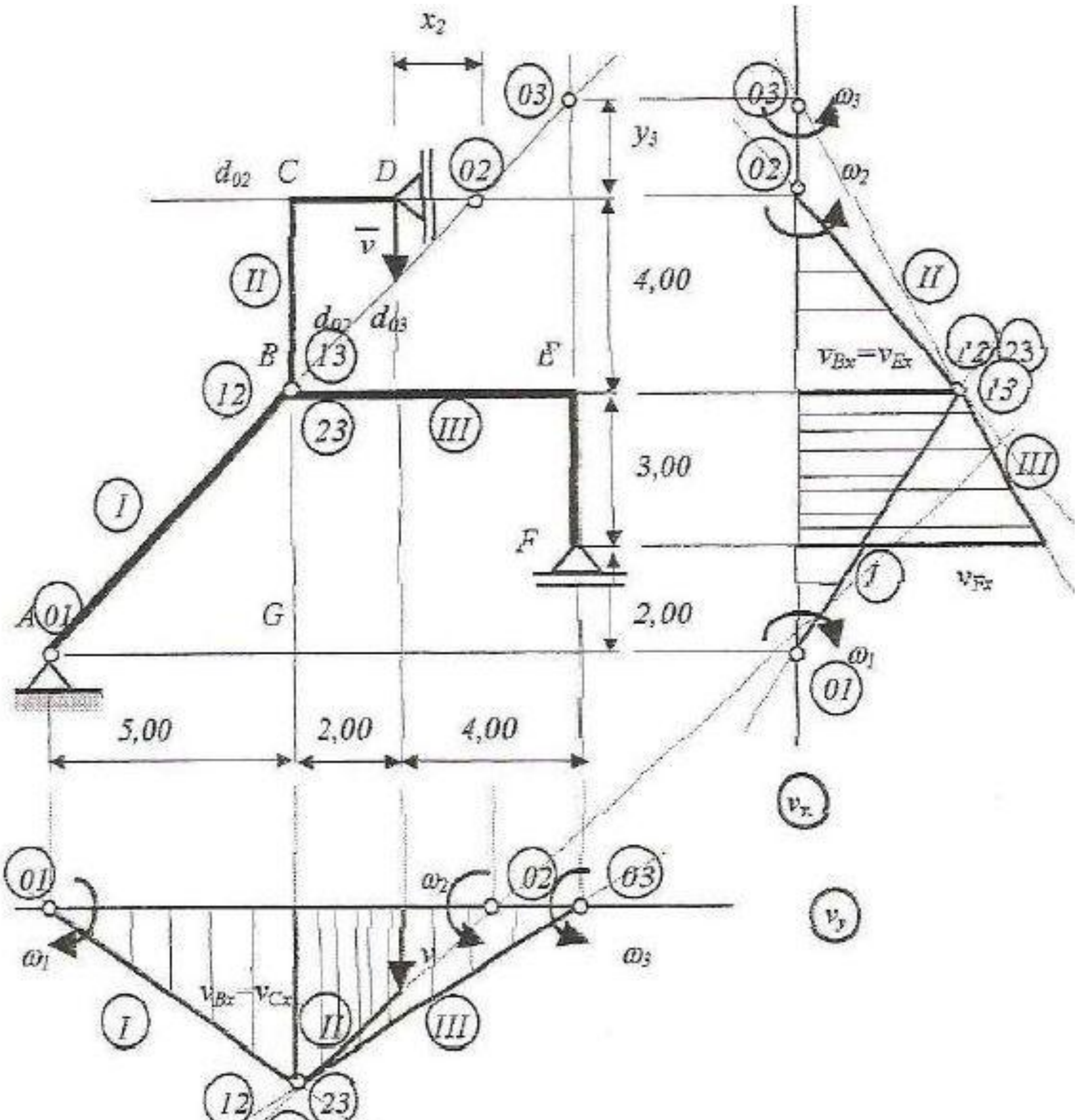


Fig.12.

To draw the diagrams of the velocities' projections we shall start from the vertical projection v_y of the body (II) because is known the vertical velocity of the point D.

The angular velocity of the body (II) is obtained from the given velocity of the point D with the relation:

$$2 \cdot \omega_2 = 10 \longrightarrow \omega_2 = 5 \ rad/s$$

The angular velocities of the other bodies of the mechanism result from the equalities:

$$5 \cdot \omega_1 = 4 \cdot \omega_2 \longrightarrow \omega_1 = 4 \ rad/s$$
$$6 \cdot \omega_3 = 4 \cdot \omega_2 \longrightarrow \omega_3 = 3{,}33 \ rad/s$$

The velocities' projections of the marked points will be:

$$\overline{v_B} \begin{cases} v_{Bx} = 5 \cdot \omega_1 = 20 \ m/s \\ v_{By} = 5 \cdot \omega_1 = 20 \ m/s \end{cases}$$

$$\overline{v_C} \begin{cases} v_{Cx} = 0 \\ v_{Cy} = 4 \cdot \omega_2 = 20 \ m/s \end{cases}$$

$$\overline{v_E} \begin{cases} v_{Ex} = 6 \cdot \omega_3 = 20 \ m/s \\ v_{Ey} = 0 \end{cases}$$

$$\overline{v_F} \begin{cases} v_{Fx} = 9 \cdot \omega_3 = 30 \ m/s \\ v_{Fy} = 0 \end{cases}$$

Problem 5,6. *Using the method of the diagrams of velocities' projections determine the angular velocities of the bodies and the velocities of the marked points for the mechanisms from the figures 13 and 14.*

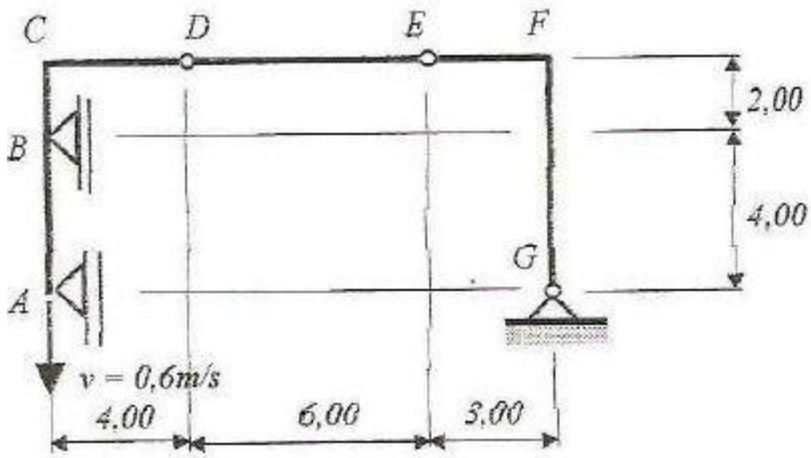


Fig.13.

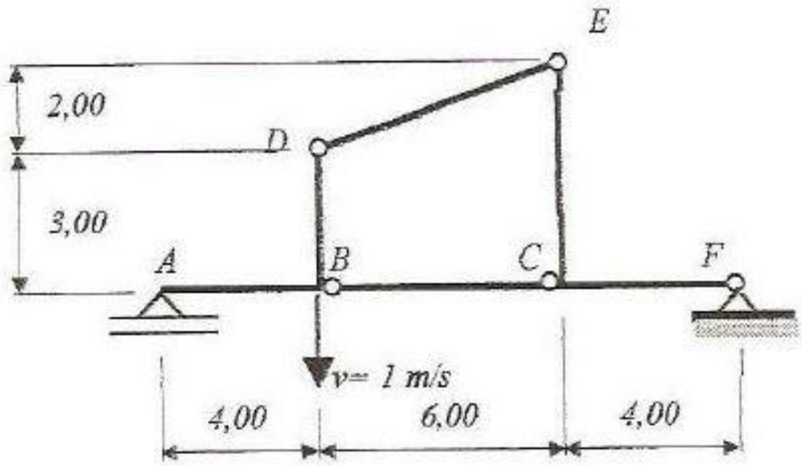


Fig.14.

DYNAMICS

Chapter 12. Introduction.

As we have presented in the introduction in the mechanics, ***dynamics*** *is the part of the theoretical mechanics that deals with the study of the mechanical motion of the bodies considering the masses of them and the forces which act about them. Therefore in dynamics we shall use all the notions from statics and kinematics.*

As the other parts of the mechanics, in dynamics we use some basic notions specific to this part of the mechanics. These notions are: ***linear momentum, angular momentum, kinetic energy, work, potential energy, mechanical energy, power*** *and also other notions but without importance in this course. The first three notions are* ***notions of the state*** *named in this way because they define the mechanical state at an instant of the motion for the bodies. The other notions are called* ***derived notions*** *because they are derived or from the first three notions or from the other notions from statics and kinematics.*

Besides of these two kinds of notions in the dynamics we shall use ***additional notions*** *which are used for to define some of the previous notions. These additional notions are* ***moments of inertia****.*

In this part we shall make in the following way: first we shall define all the specific basic notions of the dynamics, after we shall state three theorems called ***general theorems****, and finally we shall study the motions of the particle, rigid body and systems of particles and rigid bodies.*

We make the remark that the principal problem of the dynamics is to determine the join (the relations) between the ***cause*** *and* ***effect****,* *namely between the forces and the motion produced by them or vice versa between the motion and the forces induced by it in the mechanical system.*

We shall start to study the notions from the additional notions namely from the moment of inertia because these will be used in the definitions of the other specific basic notions.

We will make in the following way (as for all the other notions in dynamics): first we define the notion for a particle, after for a system of particles and finally for the rigid body.

Chapter 13. Moments of inertia.

13.1. Definitions.

We can remark , from a simple experience, that the inertia of the bodies in rotation motion does not depend only by their masses but also the distribution of the masses with respect to the rotation axis or rotation point. This kind of inertia is emphasized by the notion called ***moment of inertia****.*

Suppose a particle P by mess **m** *and a fixed reference system, for example one axis (Δ).*

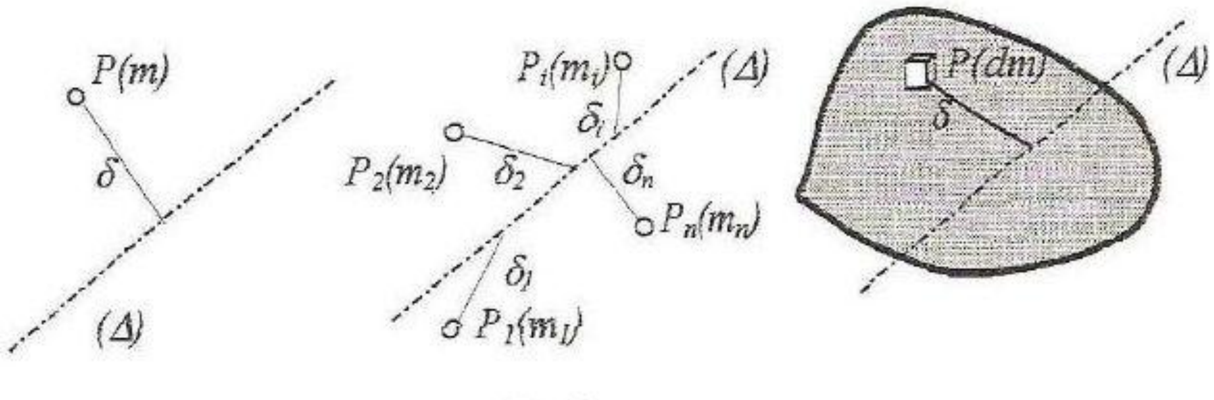


Fig.1.

By definition we call ***moments of inertia of the particle P with respect to the axis (Δ), marked $J_Δ$, the scalar quantity equal to the product between the mass of the particle and the distance from the particle to the axis at the power two:***

$$J_Δ = m \cdot δ^2$$

For a system of particles P_i having the masses $\mathbf{m_i}$*, by definition* ***the moment of inertia of the system with respect to the axis (Δ) is the scalar quantity equal to the sum of the moments of inertia of the particles with respect to the axis (Δ)****. This moment of inertia is marked also J_Δ and it is:*

$$J_\Delta = \Sigma J_{\Delta i} = \Sigma m_i \,.\, \delta_i^2$$

If we have a rigid body, then we can consider it as a continuous and non deformable system of particles by the elementary masses **dm***. In this way results the definition of the moment of inertia of a rigid body about the axis (Δ):*

$$J_\Delta = \int_V dJ_\Delta = \int_V dm \,.\, \delta^2$$

where we have marked $\mathbf{dJ_\Delta}$ *the moment of inertia of the elementary mass* **dm** *about the same axis.*

In the following we shall use only the moments of inertia of the bodies therefore we shall give the following definitions only for the rigid body.

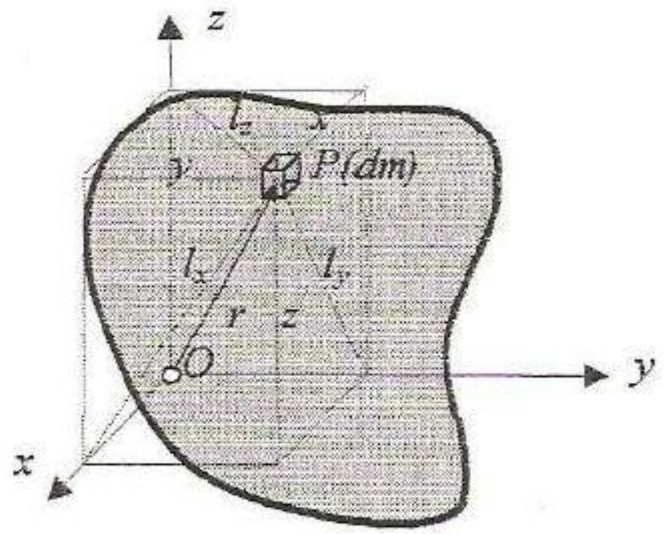


Fig.2.

*Consider a rigid body by mass **M** and a Cartesian reference system.*

With respect to the axes, reference planes and the origin of the system we define the following moments of inertia:

- ***Moment of inertia about a plane (planar moment of inertia)**. In the reference system we have:*

$$J_{xOy} = \int_V z^2 dm ; \quad J_{xOz} = \int_V y^2 dm ; \quad J_{yOz} = \int_V x^2 dm$$

- ***Moment of inertia about an axis (axial moment of inertia)**:*

$$J_x = \int_V l_x^2 dm = \int_V (y^2 + z^2) dm ;$$
$$J_y = \int_V l_y^2 dm = \int_V (x^2 + z^2) dm ;$$
$$J_z = \int_V l_z^2 dm = \int_V (x^2 + y^2) dm$$

- ***Moment of inertia about a point (polar moment of inertia)**:*

$$J_O = \int_V r^2 dm = \int_V (x^2 + y^2 + z^2) dm$$

We make the remark that these moments of inertia can be calculated about any plane, axis or point from space.

*In technical problems we have also a scalar quantity having the same dimension as the moment of inertia called **inertia product**. These kinds of moments of inertia are defined about a Cartesian reference system in space in the following way:*

$$J_{xy} = \int_V xy dm ; \quad J_{xz} = \int_V xz dm ; \quad J_{yz} = \int_V yz dm$$

13.2. Proprieties of the moments of inertia.

From the definition relations of the moments of inertia result some proprieties:

- *The moment of inertia about an axis is equal to the sum of the moments of inertia about two perpendicular planes having as intersection line the given axis:*

$$J_x = J_{xOy} + J_{xOz}\,; \quad J_y = J_{xOy} + J_{yOz}\,; \quad J_z = J_{yOz} + J_{xOz}$$

- *The moment of inertia about a point is equal to the sum of the moments of inertia about three perpendicular axes, two by two, axes which pass through the given point:*

$$J_O = J_{xOy} + J_{xOz} + J_{yOy}$$

- *The moment of inertia about a point is equal to the semi sum of the moments of inertia about three perpendicular planes two by two which pass through the given point:*

$$J_O = \frac{1}{2}(J_x + J_y + J_z)$$

- *The moments of inertia about axes, planes and points can by only positive or equal to zero, but the inertia products can be positive, negative or zero.*
- *The planar moments of inertia are zero for plane plates about the plane of the plate.*
- *The axial moments of inertia are zero for rectilinear bars about the axes of the bars.*
- *The polar moments of inertia are zero for the particle about itself.*

- *The inertia product is zero if one reference plane, from the two, about which is calculated, is symmetry plane.*

13.3. Moment of inertia's variation with respect to parallel axes.

To consider a rigid body by mass M and two reference systems with parallel axes marked Oxyz and $O_1x_1y_1z_1$. We shall mark with a, b and c the coordinates of the point O_1 with respect to the system Oxyz. We consider also one any point P of the body having the coordinates x,y,z with respect to the system Oxyz and the coordinates x_1, y_1 and z_1 with respect to the system $O_1x_1y_1z_1$. We have obviously the relations between the coordinates:

$$x = x_1 + a;\ y = y_1 + b;\ \ z = z_1 + c$$

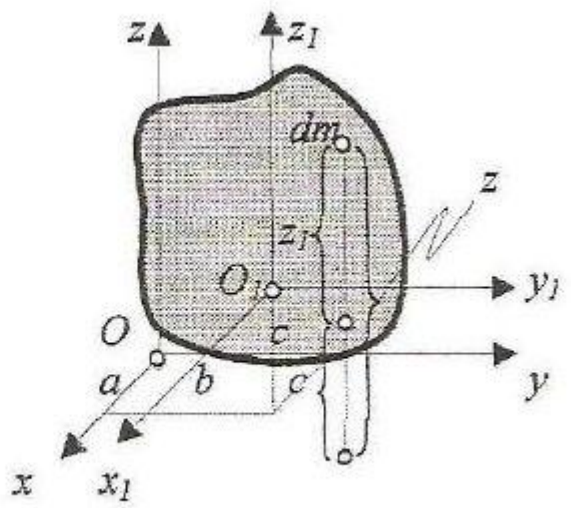


Fig.3.

We shall find a relation between the moments of inertia about the two reference systems. The demonstration is made for only one moment of inertia, for all the other the relations will be obtained in the same way.

We consider the moment of inertia about the reference plane xOy and we replace in the definition relation the expression of the coordinate with respect to the system Oxyz function of the corresponding coordinate with respect to the other reference system:

$$J_{xOy} = \int_V z^2 dm = \int_V (z_1 + c)^2 dm = \int_V (z_1^2 + 2cz_1 + c^2) dm =$$

$$= \int_V z_1^2 dm + 2c \int_V z_1 dm + c^2 \int_V dm$$

We remark that the first integral is the moment of inertia of the body about the plane $O_1x_1y_1$, the second integral is the statically moment of the body about the same plane and the last integral is the mass of the body. Using the corresponding notation we find the relation:

$$J_{xOy} = J_{x_1O_1y_1} + 2c S_{x_1O_1y_1} + c^2 M$$

In the particular case when the origin of the $O_1x_1y_1z_1$ reference system is the center of gravity (center of mass) of the body, the distance c is the coordinate z_C of the mass center and the statically moment about the reference plane that passes through the mass center is equal to zero. In this case the relation is transformed in:

$$J_{xOy} = J_{x_1C_1y_1} + z_C^2 . M$$

where $J_{x_1C_1y_1}$ is the central planar moment of inertia (moment of inertia about the plane passing through the mass center).

Making in the same way for all the other moments of inertia we shall find the relations:

$$J_{xOz} = J_{x_1C_1z_1} + x_C^2 . M; \quad J_{yOz} = J_{y_1C_1z_1} + y_C^2 . M$$

$$J_x = J_{x_1} + (y_C^2 + z_C^2).M; \qquad J_y = J_{y_1} + (x_C^2 + z_C^2).M;$$

$$J_z = J_{z_1} + (x_C^2 + y_C^2).M;$$

$$J_O = J_C + M.OC^2;$$

$$J_{xy} = J_{x_1y_1} + M.x_C.y_C; \; J_{yz} = J_{y_1z_1} + M.y_C.z_C;$$

$$J_{xz} = J_{x_1z_1} + M.x_C.z_C.$$

For the planar, axial and polar moments of inertia we can state the law of variation called ***parallel axes theorem****:* ***the moment of inertia of a body about a plane, or an axis, or a point is equal to the sum between the moment corresponding central moment of inertia (about the plane, or axis passing through the center of mass, or about the center of mass) having parallel axes to the given reference system and the product of the entire mass of the body and the squared distance from the center of mass and the plane, axis or the given point.*** *It is obviously that for the inertia products we can state the same law of variation with respect to parallel reference systems. This theorem is known as the* ***Steiner's theorem*** *also.*

13.4. Moment of inertia's variation about the rotation of the axes.

We have seen that if we change the reference system but we keep the directions of the axes the moments of inertia change. It is obviously that if we change the directions of the axes we shall obtain variations of moments of inertia. In this section we shall determine the way in which is modified the moments of inertia when we rotate the axes of the reference system. Because in problems is important to know the variation of the axial moments of inertia in this section we shall study only the variation of the axial moment of inertia about a rotated axis.

Suppose one body and a reference system about which we know all the moments of inertia, namely the moments of inertia about the three axes and the three inertia products, and also an any axis (Δ) passing through the origin of the system and making with the three axes the angles α, β and γ, namely the axis (Δ) is rotated with respect to the three axes with these three angles.

By definition the moment of inertia of the body about the axis (Δ) is:

$$J_{\Delta} = \int_V \delta^2 dm$$

where δ is the distance from the any point P by the elementary mass dm to the axis (Δ). This distance can be expressed from the right angle triangle OPP' in the following way:

$$\delta = r \cdot \sin\theta$$

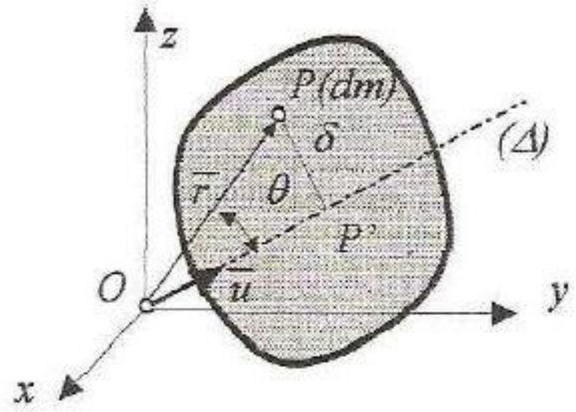


Fig.4.

distance that can be expressed also using the vector product:

$$\delta^2 = (\overline{r} \times \overline{u})^2$$

where the expressions of the two vectors are:

$$\overline{r} = x\overline{i} + y\overline{j} + z\overline{k};$$
$$\overline{u} = \cos\alpha\,\overline{i} + \cos\beta\,\overline{j} + \cos\gamma\,\overline{k}$$

Performing the vector product and calculating the second power results:

$$\delta^2 = (\cos\alpha . y - \cos\beta . x)^2 + (\cos\beta . z - \cos\gamma . y)^2 + \\ + (\cos\gamma . x - \cos\alpha . z)^2$$

Replacing in the expression of the moment of inertia and calculating the second power results finally:

$$J_\Delta = J_x.\cos^2\alpha + J_y.\cos^2\beta + J_z.\cos^2\gamma - 2J_{xy}.\cos\alpha.\cos\beta -$$
$$-2J_{xz}.\cos\alpha.\cos\gamma - 2J_{yz}.\cos\beta.\cos\gamma$$

As we can see this moment of inertia is a function of the angles with respect to the three axes:

$$J_\Delta = J_\Delta(\alpha,\beta,\gamma)$$

namely when are modified these angles is modified the moment of inertia also. The extreme values of this function are called ***principal moments of inertia*** *and the axes with respect to which are obtained these moments of inertia are called* ***principal axes of inertia****. Without to prove these statements we shall set out the followings:*

- *The principal axes of inertia form a three orthogonal reference system resulting three principal moments of inertia.*
- *With respect to this principal reference system the inertia products are equal to zero. results that if with respect to a system of reference the inertia products are zero that system is a principal system of reference in that point. We remark that the symmetry axes are principal axes of inertia.*

13.5. Moments of inertia in two dimensions.

Supposing that the body is a plane body (plane plate) and the study of the motion is made in the same plane the moments of inertia are interested only in that plane namely in two dimensions. In this way considering a plane body and the corresponding reference system Oxy in plane are the following kinds of moments of inertia:

- *Axial moments of inertia:*

$$J_x = \int_V y^2 dm; \quad J_y = \int_V x^2 dm;$$

- *Polar moment of inertia:*

$$J_O = \int_V r^2 dm = \int_V (x^2 + y^2)dm$$

- *Inertia product:*

$$J_{xy} = \int_V xydm$$

Between the axial moments of inertia and the polar moment of inertia we can write the relation:

$$J_O = J_x + J_y$$

The relations of variations of the moments of inertia with respect to parallel axes will be:

$$J_x = J_{Cx} + M\,.\,y_C^2;\ J_y = J_{Cy} + M\,.\,x_C^2;\ J_O = J_C + M\,.\,OC^2,$$
$$J_{xy} = J_{Cxy} + M\,.\,x_C\,.\,y_C$$

where the moments of inertia marked with the index C are central moments of inertia.

With respect to one any axis but that passes through the origin of the reference system the moment of inertia will have the following expression:

$$J_\Delta = J_x\,.cos^2\alpha + J_y\,.cos^2\beta - 2J_{xy}\,.\,cos\alpha\,.\,cos\beta$$

13.6. Moments of inertia for simple usual homogeneous bodies.

In this section we shall calculate the central moments of inertia for a few bodies often used in problems in two dimensions. For each *body we shall calculate the four central moments of inertia: two axial moments, one polar moment of inertia and one inertia product. We shall see that having these four moments of inertia in fact we have the moments of inertia about any other reference system.*

- ***Rectilinear bar.*** *Consider a rectilinear bar by the mass M and length l and a reference system with the origin in the mass center and having the axes: one collinear to the axis of the bar and the other perpendicular on it.*

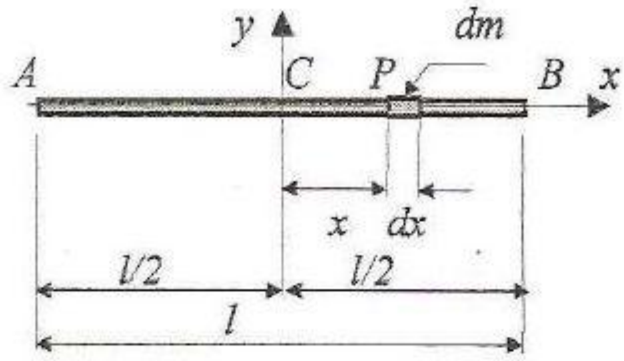


Fig.5.

Because the axis Cx is the axis of the bar the moment of inertia with respect to this axis is equal to zero:

$$J_x = 0$$

We shall calculate only the central moment of inertia about the axis perpendicular to the bar. For calculation we shall choose, in one any point P by coordinate x, one infinitesimal element of mass **dm** *and length* **dx***. The bar being homogeneous we have the relation that expresses this propriety:*

$$\mu = \frac{M}{l} = \frac{dm}{dx}$$

With this the moment of inertia about the Cy axis is:

$$J_y = \int_{AB} x^2 dm = \int_{AB} x^2 \mu dx = \frac{M}{l} \int_{-l/2}^{l/2} x^2 dx = \frac{Ml^2}{12}$$

Because the central polar moment of inertia is the sum of the axial moments of inertia results obviously:

$$J_C = \frac{Ml^2}{12}$$

Because this shape is symmetrical about the both axes the central inertia product is equal to zero:

$$J_{xy} = 0$$

- ***Rectangular plate.*** *Consider now a rectangular plane plate by the sides **b** and **h** and the mass **M**. Also we shall consider a reference system with its origin in the mass center and with the axes parallel to the sides of the rectangle.*

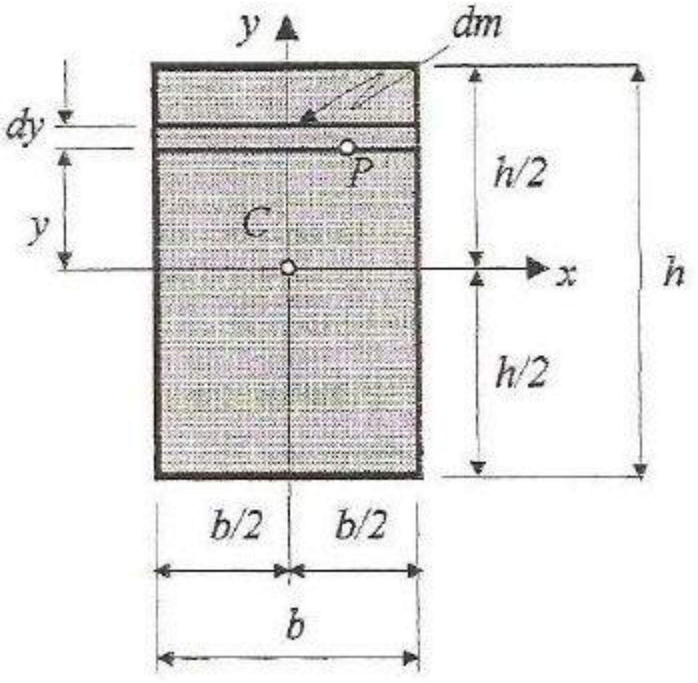


Fig.6.

Also in this case we shall calculate one central axial moment of inertia because the rectangle has the same position about the both axes (one side is parallel to one axis and the other side is perpendicular to that axis) and finally we shall change the index x with y and b with h resulting the other central axial moment of inertia about the other axis.

At the other hand knowing that the central polar moment of inertia is the sum of the two central axial moments of inertia and also the shape is symmetrical therefore the inertia product is equal to zero:

$$J_{xy} = 0;$$

so it is enough to calculate that single central axial moment of inertia.

We shall choose an infinitesimal element of mass for to calculate the moment of inertia about the axis Cx in the any point P (having the coordinate y). Here, in this case, we can make an artifice for to make the calculation easier. We know that the element of mass has to be a punctual element. The conditions for to be a punctual element are: to be with infinitesimal mass (but this condition can be obtained by one element with one infinitesimal dimension) and to have the central axial moment of inertia about its central axis parallel to the given axis (about which we calculate the moment of inertia of the body) equal to zero. This last condition can be obtained also for an element having the shape as a bar parallel to the given axis. Results that we may choose the mass element as a rectangle with infinitesimal thickness dy.

The orthogonality condition of the plate will be:

$$\mu = \frac{M}{A} = \frac{dm}{dA}$$

where the element of the area is:

$$dA = b \cdot dy$$

The moment of inertia with respect to the Cx axis will be:

$$J_x = \int_A y^2 dm = \mu \int_A y^2 dA = \frac{M}{bh} \int_{-h/2}^{h/2} y^2 b dy = \frac{Mh^2}{12}$$

With respect to the Cy axis we will have:

$$J_y = \frac{Mb^2}{12};$$

Finally we have also the central polar moment of inertia:

$$J_C = J_x + J_y = \frac{M}{12}(b^2 + h^2)$$

- ***Right-angle triangular plate.*** *We shall consider a right-angle triangular homogeneous plate having the two perpendicular sides b and h and the mass equal to M and also a central reference system with the axes parallel to the two perpendicular sides. For to make easier the calculation of the moments of inertia we shall calculate first the moments of inertia about another reference system namely a reference system with the axis collinear with the two perpendicular sides of the triangle. After using the relations of variation with respect to parallel axes we will obtain the central moments of inertia. Consequently we shall choose $O_1x_1y_1$ as reference system.*

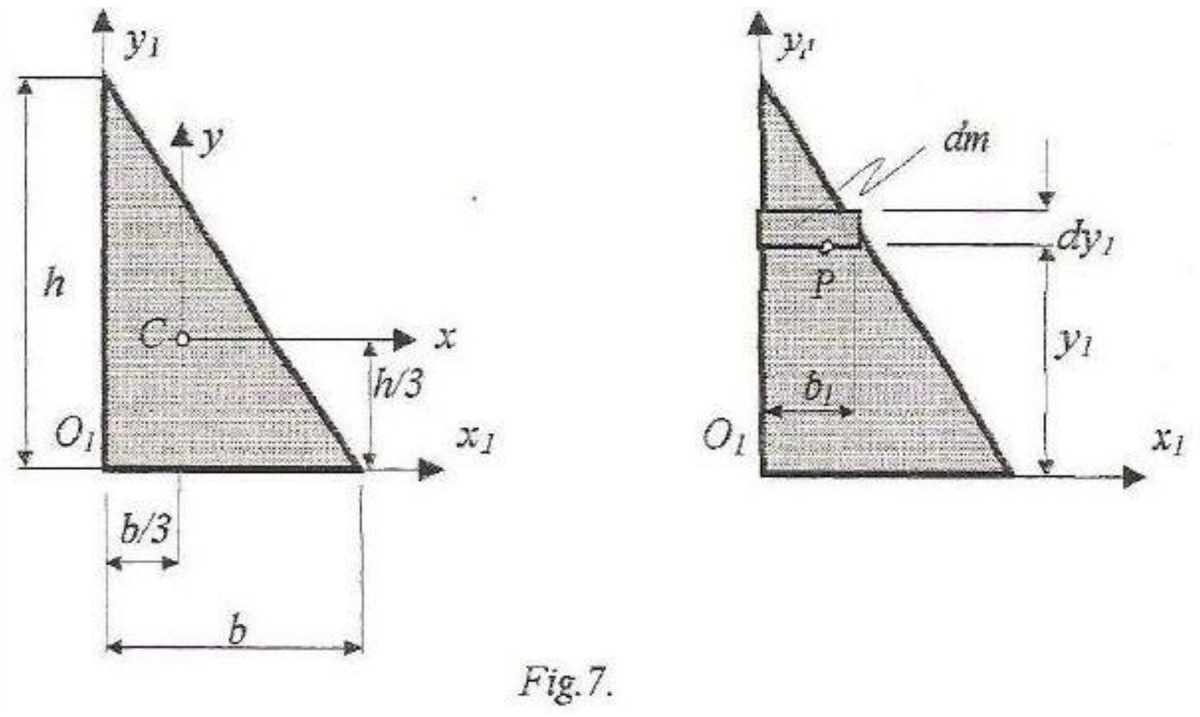


Fig.7.

The element of mass is taken in the point P by coordinate y_1 and having the shape of a bar (small rectangle as for the previous plate). The area of this element is:

$$dA = b_1 \cdot dy_1$$

where b_1 will be obtained from the similarity relation of two triangles (one the given triangle and the other the triangle with the base b_1 and the height (h-y)). The similarity relation is:

$$\frac{b_1}{b} = \frac{h-y}{h} \longrightarrow b_1 = \frac{b}{h}(h-y_1)$$

The moment of inertia of the triangle with respect to the O_1x_1 axis is:

$$J_{x_1} = \int_A y_1^2 dm = \frac{2M}{h^2}\int_0^h y_1^2(h-y_1)dy_1 = \frac{Mh^2}{6}$$

and with respect to the O_1y_1 axis is equal (because the triangle has the same position about the both axes – one side is collinear and the other is perpendicular to the axis O_1x_1 or O_1y_1 – so we change x_1 with y_1 and h with b):

$$J_{y_1} = \frac{Mb^2}{6}$$

This time the inertia product is not equal to zero. it will be calculated with the relation:

$$J_{x_1y_1} = \int_A x_1 y_1 dm$$

in which x_1 and y_1 are the coordinates of the centroid of the infinitesimal element of mass considered so that to comply with the conditions of the particle. Because the infinitesimal element having the shape of a bar meets these *conditions we shall use the same element of mass for this inertia product where we will have:*

$$x_1 = \frac{b_1}{2}$$

With all these we calculate:

$$J_{x_1 y_1} = \frac{Mb}{h^3}\int_0^h y_1(h-y_1)^2\,dy_1 = \frac{Mbh}{12}$$

Now we shall use the theorems of variation of the moments of inertia with respect to parallel axes and we obtain:

$$J_x = J_{x_1} - M \cdot y_C^2 = \frac{Mh^2}{18};$$
$$J_y = J_{y_1} - M \cdot x_C^2 = \frac{Mb^2}{18};$$
$$J_{xy} = J_{x_1 y_1} - M \cdot x_C \cdot y_C = -\frac{Mbh}{36};$$

In the calculation of the inertia product we remark that function of the position of the triangle with respect to the reference system we obtain different signs. In the figure 8 we present the way in which results the sign function of the position of the triangle with respect to the reference system. The rule may be expressed in the following way: if the right angle is situated in an odd frame then the sign of the inertia product is negative and is it is situated in an even frame then the sign is positive.

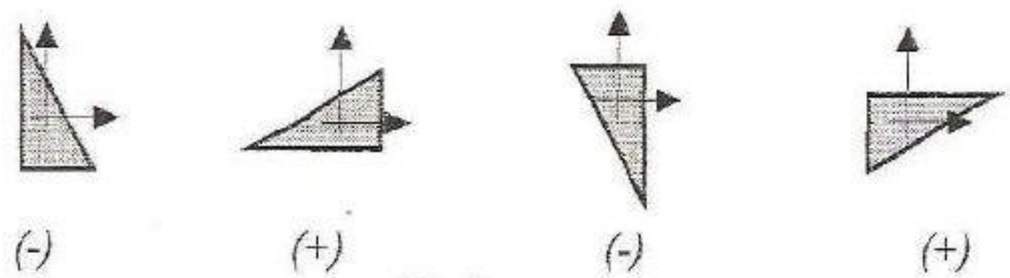


Fig.8.

The central polar moment of inertia results summing the two central axial moments of inertia:

$$J_C = J_x + J_y = \frac{M}{18}(b^2 + h^2)$$

- **Circular disc.** *Suppose a circular plate by radius R and mass M and the reference system with the origin in the center of the circle that is in the same time the center of mass of it. Because this shape is symmetrical with respect to any diameter we have the proprieties: the inertia product is equal to zero:*

$$J_{xy} = 0,$$

and the two axial moments of inertia are equal and equal to half from the central polar moment of inertia:

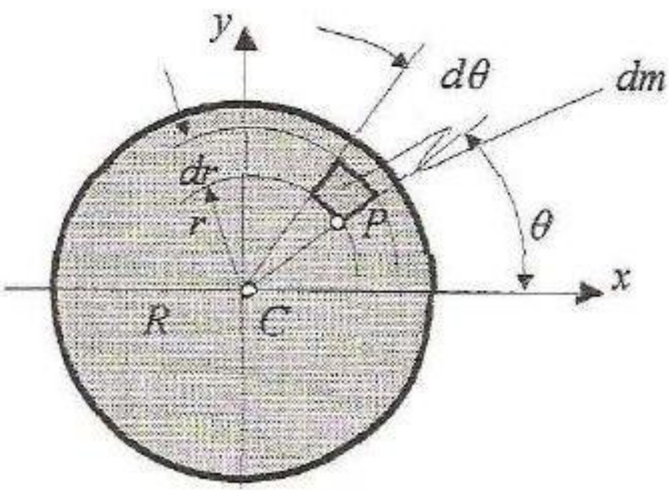


Fig.9.

$$J_x = J_y = \frac{J_C}{2}$$

This means that will be enough to calculate the central polar moment of inertia.

The homogeneous propriety will be expressed with the relation:

$$\mu = \frac{M}{A} = \frac{dm}{dA}$$

For to calculate the central polar moment of inertia we shall consider, in the any point P, the elementary mass in polar coordinates (r,θ) and we shall have:

$$dm = \mu . dA = \frac{M}{\pi R^2} r.d\theta.dr$$

The polar moment of inertia will be:

$$J_C = \int_A r^2 dm = \frac{M}{\pi R^2} \iint_A r^3 dr d\theta = \frac{M}{\pi R^2} \int_0^R r^3 dr \int_0^{2\pi} d\theta = \frac{MR^2}{2}$$

and the central axial moments of inertia will result

$$J_x = J_y = \frac{MR^2}{4}$$

13.7. Sample problems

Problem 1. *Calculate the polar moment of inertia about the point O for the body represented in the figure 10. Are known:* $M_{bar} = 10$ *kg,* $M_{plate} = 69$ *kg and* $r = a = 10$ *cm.*

Solution. *The body may be divided in three simple bodies namely: one rectilinear bar (1), one rectangular plate (2) and a circular plate (3) that will be subtracted from the sum of the first two. The relation:*

the given body = (1) + (2) – (3)

is kept also for the moments of inertia, which by definition is a sum.
The masses of the three bodies are:

$$M_1 = M_b = 10\ kg\ ;\ M_2 = \mu A_2\ ;\ M_3 = \mu A_3\ .$$

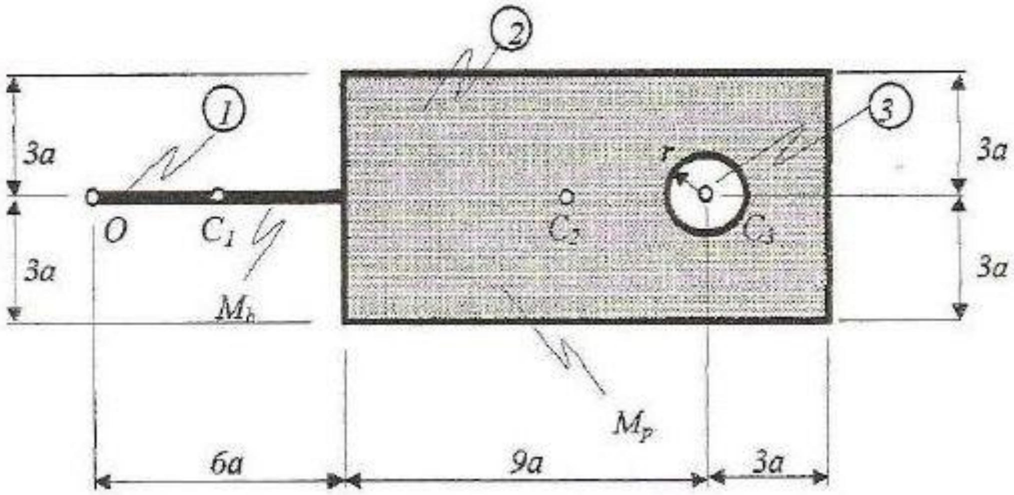


Fig.10.

where μ is the specific mass of the plate:

$$\mu = \frac{M_p}{A_p} = \frac{M_p}{A_2 - A_3} = \frac{M_p}{6a \cdot 9a - \pi a^2} = \frac{69}{0,69} = 100\ kg/m^2$$

We have:

$M_2 = 72\ kg$; $M_3 = 3\ kg$.

The polar moment of inertia with respect to the point O will be:

$$J_O = \Sigma(J_{C_i} + M_i \cdot OC_i^2) = (\frac{M_1 l^2}{12} + M_1 \cdot OC_1^2) + [\frac{M_2}{12}(b^2 + h^2) + $$
$$+ M_2 \cdot OC_2^2] - (\frac{M_3 r^2}{2} + M_3 \cdot OC_3^2)$$

or replacing with the corresponding values we obtain the polar moment of inertia about the point O:

$$J_O = \frac{10 \cdot 60^2}{12} + 10 \cdot 30^2 + \frac{72}{12}(120^2 + 60^2) + 72 \cdot 120^2 -$$
$$-(\frac{3 \cdot 10^2}{2} + 3 \cdot 150^2) = 1.089.200\ kgcm^2$$

Problem 2. *Calculate the moment of inertia about the axis (Δ) for the homogeneous plane plate from the figure 11. Are known: M = 87 kg, r = a = 0,2 m.*

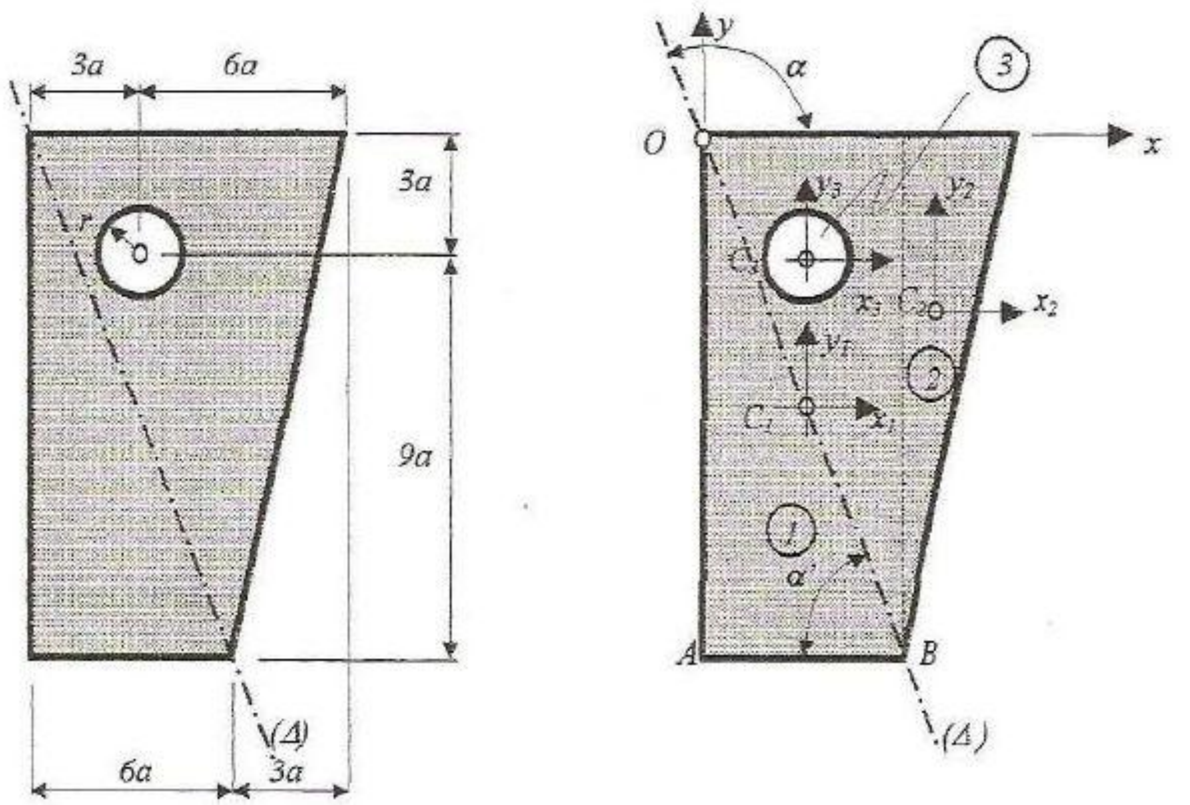


Fig.11.

Solution. *The homogeneous plate is divided in three simple plates: one rectangle, one triangle and one circular plate that is subtracted from the sum of the first two. The specific mass of the body (considering π = 3) will be:*

$$\mu = \frac{M}{A} = \frac{M}{A_1 + A_2 - A_3} = \frac{87}{72a^2 + 18a^2 - 3a^2} = 25\ kg/m^2$$

with which are calculated the masses of the three simple bodies:

$$M_1 = \mu \,.\, A_1 = 72\ kg\ ;\ M_2 = \mu \,.\, A_2 = 18\ kg\ ;\ M_3 = \mu \,.\, A_3 = 3\ kg$$

We shall choose one reference system that has to be with its origin on the axis about which is calculated the moment of inertia, here the point O (fig. 11.).

The three moments of inertia about this system will be calculated with the relations:

$$J_x = \Sigma\,(J_{xi} + M_i \,.\, y_i^2)\ ;\ J_y = \Sigma\,(J_{yi} + M_i \,.\, x_i^2)\ ;\ J_{xy} = \Sigma\,(J_{xyi} + M_i \,.\, x_i \,.\, y_i)$$

where J_{xi} ,J_{yi} and J_{xyi} are the central moments of inertia of the component simple bodies, and x_i and y_i are the coordinates of the centers of mass of these simple bodies. We shall have:

$$J_x = [\frac{M_1 h_1^2}{12} + M_1.(6a)^2] + [\frac{M_2 h_2^2}{18} + M_2.(4a)^2] - [\frac{M_3 r^2}{4} +$$

$$+ M_3.(3a)^2] = \frac{72 \cdot 2{,}4^2}{12} + 72 \,.\, 1{,}2^2 + \frac{18 \cdot 2{,}4^2}{18} + 18 \,.\, 0{,}8^2 -$$

$$- \frac{3 \cdot 0{,}2^2}{4} - 3 \,.\, 0{,}6^2 = 154{,}41 \text{ kgm}^2;$$

$$J_y = [\frac{M_1 b_1^2}{12} + M_1.(3a)^2] + [\frac{M_2 b_2^2}{18} + M_2.(7a)^2] - [\frac{M_3 r^2}{4} +$$

$$+ M_3.(3a)^2] = \frac{72 \cdot 1{,}2^2}{12} + 72 \,.\, 0{,}6^2 + \frac{18 \cdot 0{,}6^2}{18} + 18 \,.\, 1{,}4^2 -$$

$$- \frac{3 \cdot 0{,}2^2}{4} - 3 \,.\, 0{,}6^2 = 69{,}09 \text{ kgm}^2;$$

$$J_{xy} = [\,0 + M_1.(3a)\,.\,(-6a)] + [-\frac{M_2 b_2 h_2}{36} + M_2.\,(7a)\,.\,(-4a)] -$$

$$- [0 + M_3.(3a)\,.\,(-3a)] = 72 \,.\, 0{,}6 \,.\, (-1{,}2) - \frac{18 \cdot 0{,}6 \cdot 2{,}4}{18} +$$

$$+ 18 \,.\, 1{,}4 \,.(-\,0{,}8) - 3 \,.\, 0{,}6 \,.\, (-0{,}6) = -\,70{,}20 \text{ kgm}^2$$

The moment of inertia about the (Δ) axis will be calculated with the relation:

$$J_\Delta = J_x \,.\, \cos^2\alpha + J_y \,.\, \cos^2\beta - 2\, J_{xy} \,.\, \cos\alpha \,.\, \cos\beta$$

where the angle α results from the right angle triangle ABO. We have:

$$\cos\alpha = -\cos\alpha' = -\frac{6a}{6a\sqrt{5}} = -\,0{,}447;$$

$$\sin\alpha = \sin\alpha' = \frac{12a}{6a\sqrt{5}} = 0{,}894$$

Replacing the previous relation the calculated moments of inertia and the cosines and sinus of the direction of the axis (Δ) we have finally:

$$J_\Delta = 154{,}41 \,.\, (-0{,}447)^2 + 69{,}09 \,.\, 0{,}894^2 - 2 \,.\, (-70{,}2) \,.\, 0{,}894 \,.\, (-0{,}447) =$$
$$= 30 \text{ kgm}^2.$$

Problem 3. *Calculate the polar moment of inertia about the point O for the body represented in the figure 12.*

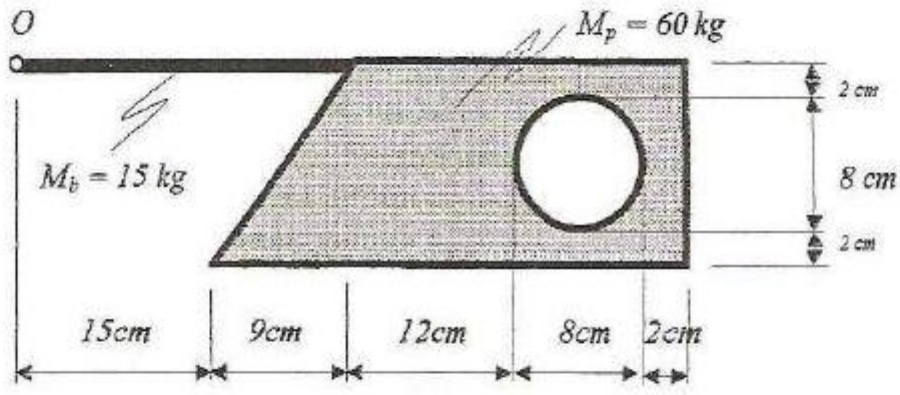


Fig.12.

Problem 4. *Calculate the moment of inertia with respect to the axis (Δ) for the homogeneous plane plate from the figure 13.*

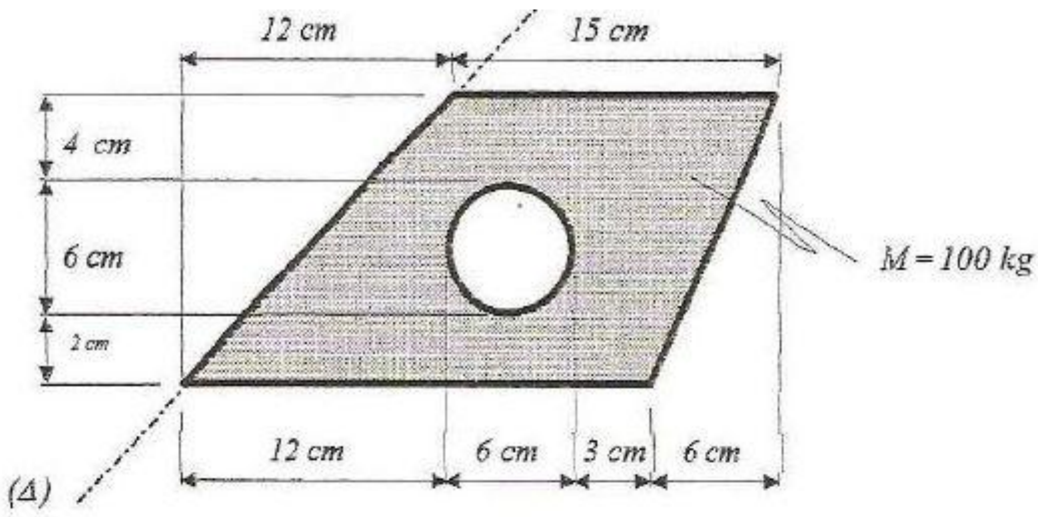


Fig.13.

Chapter 14. Fundamental notions in dynamics.

14.1. Linear momentum.

We consider a particle P by mass ***m*** *that has, at one any instant of the motion the instantaneous velocity* $\overline{\mathbf{v}}$. *By definition* ***the linear momentum of a particle is the vector quantity, marked*** $\overline{H}$, ***and equal to the product between the mass and the instantaneous velocity of the particle:***

$$\overline{H} = m \cdot \overline{v}$$

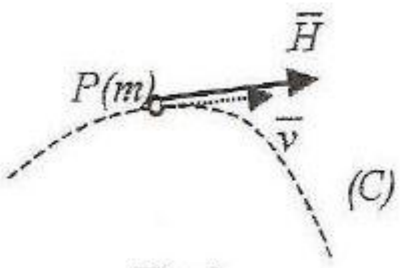


Fig.1.

Finding through a multiplication with one positive scalar quantity (the mass m of the particle), the linear momentum has the same direction and sense as the instantaneous velocity of the particle.

For a system of particles P_i *by messes* m_i *and instantaneous velocities* $\overline{v}_i$ *(Fig.2.), by definition* ***the linear momentum of the system of particles is the vector quantity, marked also*** $\overline{H}$ ***and equal to the sum of the linear momentums of the particles from the system:***

$$\overline{H} = \Sigma \overline{H}_i = \Sigma m_i \cdot \overline{v}_i$$

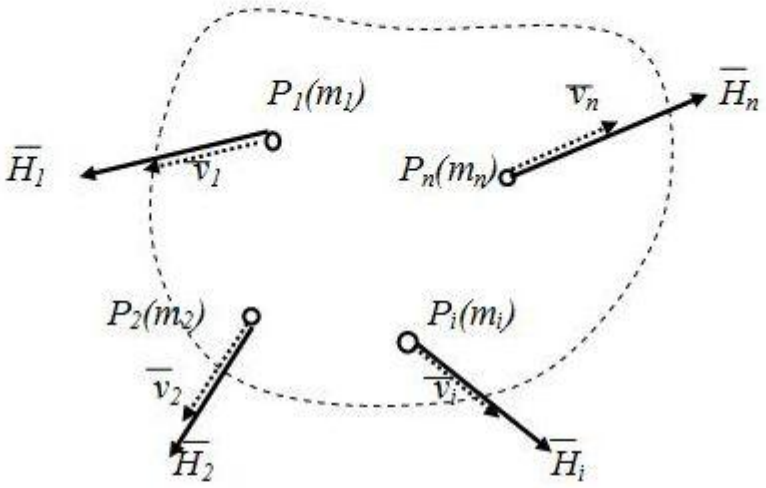


Fig.2.

The magnitude, direction and sense of this linear momentum are known because they are determined using the rule of parallelogram, or other way to calculate a vector sum, but the point of application have to be determined.

If we consider now a rigid body by mass M in motion having an instantaneous distribution of velocities (Fig.3.), then the body may be considered as a continuous and non deformable system of particles P by elementary masses **dm** *and velocity* $\overline{\mathbf{v}}$*. Using the previously relations we can obtain the linear momentum of the rigid body:*

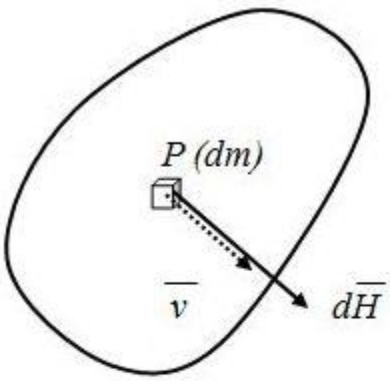


Fig.3.

$$\overline{H} = \int_V d\overline{H} = \int_V \overline{v}dm$$

where we have marked $d\overline{H}$ the linear momentum of the elementary mass dm.

Knowing that the velocity of one any point can be expressed using the first derivative with respect to time of the point position vector:

$$\overline{v} = \frac{d\overline{r}}{dt}$$

and after removing in the definition relation and also remarking that the mass does not depend by time and the integral is made function of a parameter that does not function of time we write:

$$\overline{H} = \int_V \frac{d\overline{r}}{dt} dm = \frac{d}{dt}\left(\int_V \overline{r} dm\right) = \frac{d}{dt}\left(\overline{r_C} \cdot M\right) = M \cdot \overline{v_C}$$

This result is obtained because the parenthesis is the statically moment of the body with respect to the fixed point, and using the theorem of the statically moments it is equal to the statically moment of the mass center where is considered concentrated entire mass of the body. This relation shows as that ***the linear momentum of a body (or of a mechanical system) is equal to the linear momentum of the mass center where is considered concentrated entire mass of the body.***

This result is true for the system of particles or other kind of mechanical systems.

With this we have defined the point of application of the linear momentum of a system or a body, namely it is the mass center of the system of body.

We remark also that the linear momentum of a rigid body does not depend by the distribution of the velocities in the body, but only by the velocity of the mass center.

14.2. Angular momentum.

Consider a particle P by mess ***m*** *in motion with the instantaneous velocity* $\overline{\boldsymbol{v}}$ *and a fixed point O.*

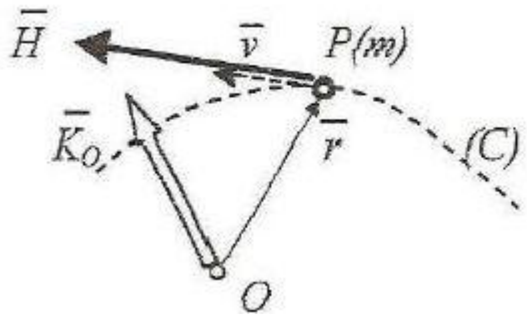


Fig.4.

By definition is called ***angular momentum of the particle P with respect to the point O the vector quantity marked*** $\overline{K}_O$ ***and equal to the moment of the linear momentum of the particle about the point O****:*

$$\overline{K}_O = \overline{M}_O(\overline{H}) = \overline{r} \times \overline{H} = \overline{r} \times m\overline{v}$$

Being one vector product the angular moment is a perpendicular vector on the two vectors: $\overline{\boldsymbol{r}}$ *and* $\overline{\boldsymbol{H}}$ *or* $\overline{\boldsymbol{r}}$ *and* $\overline{\boldsymbol{v}}$*. Results that it is perpendicular on the plane generated by the point O and the tangent to the trajectory in point P.*

For a system of particles P_i *by masses* $\boldsymbol{m}_i$ *and instantaneous velocities* $\overline{\boldsymbol{v}}_i$ *we define as* ***angular momentum of the system with respect to the fixed point O, and it is marked*** $\overline{K}_O$***, the sum of the angular momentums of the particles from the system calculated about the same point O*** *(this is the resultant angular momentum):*

$$\overline{K}_O = \Sigma \overline{K}_{Oi} = \Sigma \overline{r}_i \times \overline{H}_i = \Sigma \overline{r}_i \times m_i \overline{v}_i$$

For a rigid body considered as a continuous and non deformable system of particles by elementary masses ***dm*** *the angular momentum of it with respect to the fixed point O will be:*

$$\overline{K}_O = \int_V d\overline{K}_O = \int_V \overline{r} \times d\overline{H} = \int_V \overline{r} \times \overline{v} dm$$

where we have marked $d\overline{K}_O$ the angular momentum of the elementary mass dm with respect to the point O.

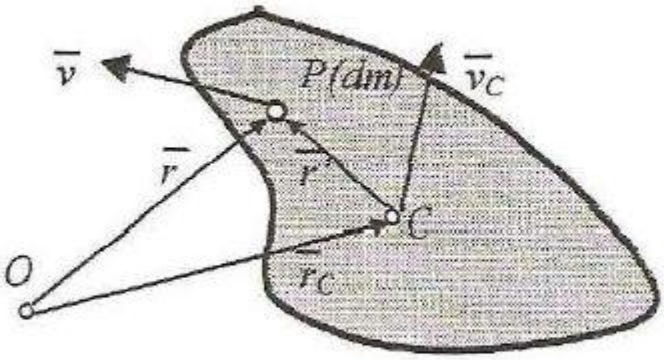


Fig.5.

Express now the position of the any point P of the rigid body using the position of the mass center C:

$$\overline{r} = \overline{r}_C + \overline{r}'$$

and calculate de derivative of this last relation

$$\dot{\overline{r}} = \dot{\overline{r}}_C + \dot{\overline{r}}'$$

The three terms from this last relation are: the (absolute) velocity of the point P, the (absolute) velocity of the point C and the velocity of the point P from the relative motion about the point C:

$$\dot{\overline{r}} = \overline{v}\ ;\ \dot{\overline{r}}_C = \overline{v}_C\ ;\ \dot{\overline{r}}' = \overline{v}'$$

If we shall replace in the expression of the angular momentum with respect to the point O then we shall obtain:

$$\overline{K}_O = \int_V (\overline{r}_C + \overline{r}') \times (\overline{v}_C + \overline{v}')\,dm = \overline{r}_C \times \overline{v}_C \int_V dm + \overline{r}_C \times \int_V \overline{v}'\,dm + \\ + \left(\int_V \overline{r}'\,dm\right) \times \overline{v}_C + \int_V \overline{r}' \times \overline{v}'\,dm$$

The second and the third terms of the development of the integral are equal to zero because in their expressions results the statically moment of the body with respect to the mass center, that it is zero from the theorem of the statically moments:

$$\int_V \overline{r}' \, dm = 0 \, ;$$

$$\int_V \overline{v}' \, dm = \int_V \frac{d}{dt}(\overline{r}' \, dm) = \frac{d}{dt}\int_V \overline{r}' \, dm = 0.$$

The integral from the first term is the entire mass of the body and the forth term of the development is the angular momentum of the body calculated about the mass center considering only the motion of the body about the mass center (that can be considered as a fixed point as in relative motion). Marking these terms we have:

$$\overline{K}_O = \overline{r}_C \times M\overline{v}_C + \overline{K}'_C$$

*This relation is called **Koenig's theorem for the angular momentum** and it state the following propriety: **the angular momentum of a body (or mechanical system) with respect to an any fixed point is equal to the sum between the angular momentum of the body about the mass center considering the motion of the body about this point (as the mass center is a fixed point) and the angular momentum of the mass center where is considered concentrated the entire mass of the body calculated about the given fixed point.***

We can remark that if the mass center is considered as a fixed point then the motion of the body about this point is a rotation motion about the mass center.

14.3. Kinetic energy.

*Consider again a particle P by mass **m** and having the instantaneous velocity* $\overline{v}$. *We define **the kinetic energy of the particle the scalar***

quantity equal to the semi product of the mass and its velocity at the power two*:*

$$E = \frac{mv^2}{2} = \frac{m\overline{v}^2}{2}$$

We remark that the kinetic energy is a positive scalar quantity and also that in its expression the velocity can be considered or scalar or vector quantity.

For a system of particles P_i by messes $\mathbf{m_{i}}$- *and instantaneous velocities* $\overline{\mathbf{v}_i}$ ***the kinetic energy of the system is the scalar quantity marked also E and equal to the sum of the kinetic energies of the particles from the system****:*

$$E = \Sigma E_i = \Sigma \frac{m_i v_i^2}{2} = \Sigma \frac{m_i \overline{v_i}^2}{2}$$

For a rigid body making in the same way as for the angular momentum we have:

$$E = \int_V dE = \frac{1}{2}\int_V v^2 dm = \frac{1}{2}\int_V \overline{v}^2 dm$$

Expressing the velocity of one any point of the body function the velocity of the mass center we obtain:

$$\overline{v} = \overline{v}_C + \overline{v}'$$

that removing in the expression of the kinetic energy we may write:

$$E = \frac{1}{2}\int_V (\overline{v}_C + \overline{v}')^2 dm = \frac{1}{2}\int_V (\overline{v}_C^2 + 2\overline{v}_C \cdot \overline{v}' + \overline{v}'^2)dm =$$
$$\frac{1}{2}v_C^2\int_V dm + \overline{v}_C \cdot \int_V \overline{v}' dm + \frac{1}{2}\int_V v'^2 dm$$

The first integral is the entire mass of the body the second is equal to zero because it can be expressed function the statically moment with *respect to the mass center and the third integral is the kinetic energy of the body considering only the motion about the mess center (as it is a fixed point). We have finally:*

$$E = \frac{1}{2}Mv_C^2 + E'$$

relation that represents ***the Koenig's theorem for the kinetic energy*** *that states:* ***the kinetic energy of a rigid body (or a mechanical system) is equal to the sum between the kinetic energy of the mass center where is considered concentrated the entire mass of the body and the kinetic energy of the body from the motion about the mass center (considered as a fixed point)****.*

14.4. Work (mechanical work).

Consider a force $\overline{F}$ that acts about the particle P. Under the action of this force the particle will has a certain displacement. If the interval of time is very small, ***dt****, then the displacement of the particle (in fact of the point of application of the force) will be very small, infinitesimal. This displacement marked* $\overline{\mathbf{dr}}$ *and that changes the point of application of the force from P in P_1 is called elementary displacement and from mathematically point of view it is the differential of the position vector of the particle P. We can say that at limit the direction of the elementary displacement is tangent to the trajectory in point P.*

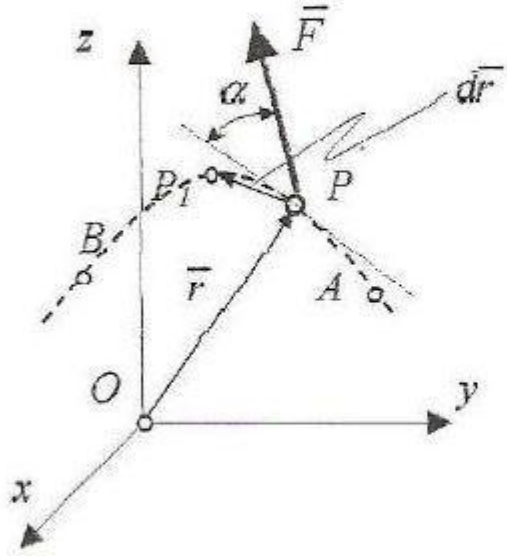


Fig.6.

The scalar quantity, marked dL, called elementary mechanical work by definition is equal to the scalar product between the force and elementary displacement of the point of application of the force in the dt interval of time:

$$dL = \bar{F} \cdot d\bar{r}$$

From the definition of the scalar product is obtained:

$$dL = F \cdot dr \cdot \cos\alpha = F_\tau \cdot dr = F \cdot dr_F$$

where we have marked F_τ the projection of the force $\bar{F}$ on the direction of the tangent to the trajectory in point P and dr_F the projection of the elementary displacement $d\bar{r}$ on the direction of the force.

*Also we remark that if the angle α is less then 90^o the elementary work is **positive** and we shall say that it is **active work**, and if the angle α is bigger than 90^o the elementary work is **negative** and we shall say it is **resistant work**.*

If the force and the displacement are expressed with respect to a fixed reference system then the expression of the elementary work becomes:

$$dL = X.dx + Y.dy + Z.dz$$

Resulted from a scalar product the elementary work will have all the proprieties of this kind of product namely:

- *If the force is the resultant of a system of concurrent forces then we have:*

$$dL = \bar{F} \cdot d\bar{r} = (\Sigma \bar{F}_i) \cdot d\bar{r} = \Sigma dL_i$$

meaning that the elementary work is equal to the sum of the elementary works of the force components passing the same elementary displacement.

- *If the elementary displacement is the resultant of a set of elementary displacements then we have:*

$$dL = \overline{F} \cdot d\overline{r} = \overline{F} \cdot (\Sigma\, d\overline{r}_i) = \Sigma\, dL_i$$

meaning that the elementary work is equal to the sum of the works of the force passing all the component displacements.

Now to consider the force acting about the particle P in the interval of time corresponding to pass from point A to point B on the trajectory. The work produced by the force $\overline{F}$ in this interval of time will be the sum of the works produced by the force $\overline{F}$ on a lot of elementary displacements which summed will give us the work from A to B. This sum at limit is the curvilinear integral and it is the ***total work****:*

$$L_{AB} = \int_{AB} dL = \int_{AB} \overline{F} \cdot d\overline{r}$$

Generally the force is a function of the position, velocity of the particle about it acts and time:

$$\overline{F} = \overline{F}\,(r, v, t)$$

consequently the work depends by the trajectory, velocity and time resulting that the integral is curvilinear.

If the forces act about a rigid body the total work of these forces is the sum of the works produced by all forces acting about the body. If about the body act also concentrated moments (concentrated couple) the work will be calculated with the relation:

$$dL = \Sigma\,(\overline{F}_i \cdot d\overline{r}_i)$$

where $\overline{F_i}$ represent the two forces of the couple which have the sum equal to zero:

$$\Sigma \overline{F}_i = 0$$

Expressing the elementary displacement using the distribution of velocities we have:

$$d\bar{r}_i = \bar{v}_i\, dt = (\bar{v}_O + \bar{\omega} \times \bar{r}_i)dt$$

and the expression of the work becomes:

$$dL = \Sigma [\bar{F}_i \cdot (\bar{v}_O + \bar{\omega} \times \bar{r}_i)dt\,] = \Sigma (\bar{F}_i \cdot \bar{v}_O)dt + \Sigma [\bar{F}_i \cdot (\bar{\omega} \times \bar{r}_i)\,]dt = \bar{v}_O \cdot (\Sigma \bar{F}_i)dt + \bar{\omega} \cdot [\Sigma (\bar{r}_i \times \bar{F}_i)\,]dt = \bar{M} \cdot d\bar{\theta}$$

where $d\bar{\theta}$ is the elementary rotation of the body.

Finally for the actions about a body the elementary work will be calculated with:

$$dL = \Sigma (\bar{F}_i \cdot d\bar{r}_i) + \Sigma (\bar{M}_j \cdot d\bar{\theta})$$

14.5. Conservative forces, force function.

If the projections of the force $\bar{F}$ meet property:

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x};\ \frac{\partial X}{\partial z} = \frac{\partial Z}{\partial x};\ \frac{\partial Z}{\partial y} = \frac{\partial Y}{\partial z}$$

meaning that they meet the Cauchy's conditions then there is a scalar function depending only by the position of the point of application of the force:

$$U = U(x,y,z)$$

so that the projections of the force will be:

$$X = \frac{\partial U}{\partial x}\ ;\ Y = \frac{\partial U}{\partial y}\ ;\ Z = \frac{\partial U}{\partial z}$$

*In this case the force is called **conservative force** and the scalar function U is called **force function**. Because this propriety the expression of the force will be:*

$$\bar{F} = \frac{\partial U}{\partial x}\bar{i} + \frac{\partial U}{\partial y}\bar{j} + \frac{\partial U}{\partial z}\bar{k} = gradient\ (U) = \nabla U$$

*namely it is an **exactly total differential**.*

From this propriety results a very important propriety of these kinds of forces namely the total work produced by a conservative force does not depends by the trajectory of the point of application of the force between the two positions:

$$L_{AB} = \int_{AB} dL = \int_A^B dU = U_B - U_A$$

where we have marked :

$$U_A = U(x_A, y_A, z_A)\ ;\ U_B = U(x_B, y_B, z_B)$$

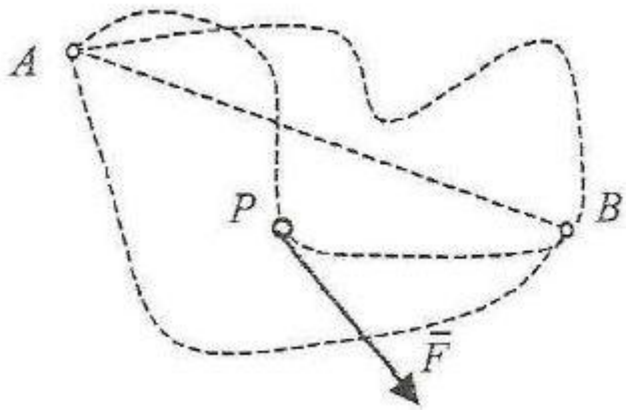


Fig.7.

If about a particle (or a body) act a system of conservative forces coming from the same number of scalar force functions:

$$\bar{F}_i = \nabla U_i$$

then the resultant force of the system:

$$\bar{R} = \Sigma \bar{F}_i = \Sigma \nabla U_i = \nabla (\Sigma U_i) = \nabla U$$

is also a conservative force that comes from a force function equal to the sum of the force functions of the components of the system:

$$\Sigma U_i = U$$

14.6. Potential energy.

Consider a particle acted by conservative forces. We call ***potential energy*** *the capacity of the particle, acted by conservative forces, to produce mechanical work because of its position with respect to a certain reference system.*

We know that the total work produced by the conservative forces between two positions is:

$$L_{AB} = \int_{AB} dL = \int_A^B dU = U_B - U_A$$

By definition the potential energy marked is:

$$V = -L = U_A - U_B$$

that if the point A is taken so that the force function to be zero in this point and the point B is an any point with the coordinates x, y, z results finally for the potential energy:

$$V = -U(x,y,z)$$

14.7. Mechanical energy.

We consider a mechanical system (particle, rigid body or a system of particles of bodies) acted by a system of conservative forces. At an instant of the motion of the system we define as ***mechanical energy*** *of the system the sum between the kinetic energy corresponding to that instant and the potential energy corresponding to that position of the system:*

$$E_m = E + V$$

Chapter 15. General theorems

15.1. Introduction.

In the previous chapters we have presented the fundamental notions in dynamics, notions with which we shall express the characteristics of the motions of the mechanical systems. In this chapter we shall state three theorems called **general theorems** *which make the join between the fundamental notions and the forces acting about the mechanical system, or between the cause and effect.*

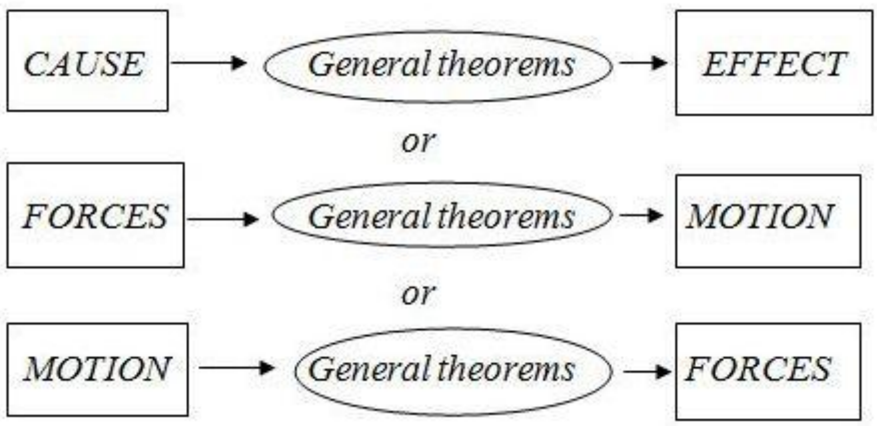


Fig.1.

The motion of a particle can be studied (as we will sown in the following chapter) using only the ***principle of the independent action of the force.*** *For a system of particles we can use the same way to study the motion but because in this case we have also a very large number of unknown internal forces this way is difficult to use. Using the general theorems in different cases* *we can eliminate the internal forces from the study (from the equations) so the number of the unknowns is less.*

The general theorems are: ***theorem of the linear momentum, theorem of the angular momentum*** *and* ***theorem of the kinetic energy.*** *the first two theorems are vector theorems using vector notions therefore they can be projected on the axes of a reference system, but the third theorem is a scalar theorem giving us only one equation.*

In the next sections we will show that the first two theorems applied to the state of rest of the mechanical system are transformed in conditions of equilibrium. Also we shall express conditions in which the fundamental notions are preserved.

The theorems will be enounced for an any mechanical system (meaning or a particle, or a system of particles, or a rigid body , or a system of rigid bodies) but they will be proved only for a system of particles. For a particle in demonstration we eliminate the index and for the rigid body the finite sum is transformed in integral (continuous and infinity sum).

15.2. Theorem of the linear momentum.

The first derivative with respect to time of the linear momentum of a mechanical system is equal to the resultant force of all forces acting about the system:

$$\dot{\overline{H}} = \overline{R}$$

For to prove his theorem we shall start from the definition of the linear momentum for a system of particles:

$$\overline{H} = \Sigma m_i \overline{v}_i$$

Deriving, with respect to time, the linear momentum of the system and considering the second principle of the mechanics results finally:

$$\dot{\overline{H}} = \Sigma m_i \dot{\overline{v}}_i = \Sigma m_i \overline{a}_i = \Sigma \overline{F}_i = \overline{R}$$

We note that for a particle this theorem is the same with the second principle of the mechanics.

Because the linear momentum of a system or a body is in fact the linear momentum of the mass center where we consider concentrated the entire mass of the system or body, the derivative of this linear momentum:

$$\dot{\overline{H}} = M.\dot{\overline{v}}_C = M.\overline{a}_C$$

that if is removed in the expression of the theorem we shall obtain a new form of this theorem called ***theorem of the mass center motion****:*

$$M.\overline{a}_C = \overline{R}$$

and that say: ***the mass center moves as in this point is concentrated the entire mass of the system or of the body and about it are acting all the forces****.*

Generally this theorem is used in scalar form projecting on the axis of a reference system:

-Cartesian reference system:

$$M\ddot{x}_C = \Sigma X_i \; ; M\ddot{y}_C = \Sigma Y_i \; ; M\ddot{z}_C = \Sigma z_i \; ;$$

-Cylindrical reference system:

$$M(\ddot{\rho}_C - \rho_C \cdot \dot{\theta}^2) = \Sigma F_{i\rho} \; ; M(\rho_C \cdot \ddot{\theta} + 2\,\dot{\rho}_C \cdot \dot{\theta}) = \Sigma F_{in} \; ;$$
$$M z_C = \Sigma \ddot{z}_i \; ;$$

-Frenet's system:

$$Ma_{C\tau} = \Sigma F_{i\tau} \; ; Ma_{C\nu} = \Sigma F_{i\nu} \; ; 0 = \Sigma F_{i\beta}$$

If the mechanical system (particle, system of particles or rigid body) is in rest then the velocities and accelerations are zero and this theorem becomes condition of equilibrium (in particular for a system of concurrent forces).

This theorem highlights one propriety of the linear momentum: ***for an isolated mechanical system or acted by a system of forces in equilibrium, the linear momentum is preserved.***

Replacing in the theorem the condition of equilibrium (or the nonexistence of the forces for an isolated system):

$$\bar{R} = 0$$

results:

$$\bar{H} = \bar{C}$$

where C is a vector constant that can be obtained from the initial conditions of the motion. We remark also that the linear momentum can be preserved only on a certain directions:

$$R_{\Delta} = 0\,; \longrightarrow H_{\Delta} = C.$$

15.3. Theorem of the angular momentum.

We suppose one system of particles P_i by masses m_i and one fixed point O. The angular momentum of the system with respect to this point is:

$$\bar{K}_O = \Sigma \bar{r}_i \times m_i \bar{v}_i$$

Deriving with respect to time we obtain:

$$\dot{\bar{K}}_O = \underbrace{\Sigma \dot{\bar{r}}_i \times m_i \bar{v}_i}_{=0} + \Sigma \bar{r}_i \times m_i \dot{\bar{v}}_i = \Sigma \bar{r}_i \times m_i \bar{a}_i = \Sigma \bar{r}_i \times \bar{F}_i = \Sigma \bar{M}_{Oi}$$

namely:

$$\dot{\bar{K}}_O = \bar{M}_O$$

and represents ***the theorem of the angular momentum: the first derivative of the angular momentum of a mechanical system (particle, system of particles or rigid body) calculated about a fixed point is equal to the resultant moment of all forces which act about the mechanical system calculated about the same fixed point****.*

Using the Koenig's theorem for the angular momentum:

$$\bar{K}_O = \bar{r}_C \times M\bar{v}_C + \bar{K}'_C$$

expressing the resultant moment function of the moment about the mass center:

$$\bar{M}_O = \bar{M}_C + \overline{OC} \times \bar{R}$$

and removing in the expression of the theorem we obtain:

$$\underbrace{\dot{\bar{r}}_C \times M\bar{v}_C}_{=0} + \cancel{\bar{r}_C \times M\dot{\bar{v}}_C} + \dot{\bar{K}}'_C = \bar{M}_C + \cancel{\overline{OC} \times \bar{R}}$$

In this last relation the first term is equal to zero being a vector product of two collinear terms and the second term is reduced with the last term of the relation being the same using the theorem of the motion of the mass center. Finally we have:

$$\dot{\bar{K}}'_C = \bar{M}_C$$

representing ***the theorem of the angular momentum about the mass center*** *of the mechanical system. We remark that in this form the mass center is* *considered as a fixed point and also that this is the single moving point about that the theorem keeps its shape.*

If the mechanical system is in rest, the velocities and the accelerations are zero and the theorem becomes a condition of equilibrium for a system of any forces.

Also this theorem gives us an important propriety of preservation of the angular momentum: ***if the mechanical system is isolated or is acted by a system of forces in equilibrium the angular momentum of the system is preserved about any point from space.***

$$\bar{M}_O = 0\ ; \longrightarrow \bar{K}_O = \bar{C}$$

Also in the case of a mechanical system acted by central forces (the supports of these forces are passing through the same point) the angular momentum of the system about the center of the forces is preserved (because the resultant moment about this point is equal to zero).

Remarking also that the derivative of the angular momentum is in fact the moment of the derivative of the linear momentum about the same point the two theorems (of the linear momentum and of the angular momentum) can be enounced under one form namely under the form of the ***theorem of the force couple system****:* ***the first derivative with respect to time of the force couple system in an any fixed point of the linear momentum is equal to the force couple system of all forces acting about the mechanical system about the same point****:*

$$\dot{\tau}_O (\bar{H}) = \tau_O (\bar{F}_i).$$

15.4. Theorem of the kinetic energy.

Suppose a mechanical system P_i *of masses* $\boldsymbol{m_i}$ *which move with the velocities* $\bar{v}_i$. *The kinetic energy of the system is:*

$$E = \frac{1}{2} \Sigma m_i \bar{v}_i^2$$

To calculate the variation of the kinetic energy in an infinitesimal interval of time:

$$dE = \frac{1}{2}\Sigma m_i\, d(\bar{v}_i^2) = \frac{1}{2}\Sigma m_i\, 2\,\bar{v}_i \,.\, d\bar{v}_i = \Sigma m_i \frac{d\bar{v}_i}{dt} .\, \bar{v}_i\, dt =$$
$$= \Sigma m_i \bar{a}_i \,.\, d\bar{r}_i = \Sigma \bar{F}_i \,.\, d\bar{r}_i = \Sigma\, dL_i = dL$$

namely:

$$dE = dL$$

that represents ***the theorem of the kinetic energy (the differential form of the theorem): the variation of the kinetic energy in an interval of time is equal to the work produced by all forces which act about the mechanical system going through the displacements corresponding to the same interval of time.***

We remark that in this theorem the work is of the external but also internal forces (this is different to the first two theorems in which the internal forces are not considered because they are pares, equals in magnitude and with opposite senses):

$$dL = dL_{ext} + dL_{int}$$

This is because the internal forces (equals in magnitudes and with opposite senses) have different displacements of their points of application. But if the mechanical system is no-deformable then the relative displacements of the points are zero and the work of the internal forces is equal to zero. This means that for a rigid body we have:

$$dL = dL_{ext}$$

Integrating this theorem (the differential form) between two any positions A and B we obtain:

$$\int_A^B dE = \int_{AB} dL$$

or:

$$E_B – E_A = L_{AB}$$

representing the ***theorem of the kinetic energy under finite form.*** *The statement of the theorem remains unchangeable but it is changed the interval of time.*

If the forces which are acting about the mechanical system (internal and external forces) are conservative forces then we can highlight the property of preservation of the mechanical energy (from this property comes the name of conservative forces). In this way for the conservative forces the elementary work is equal:

$$dL = dU$$

and the total work is:

$$L_{AB} = U_B - U_A$$

where U is the resultant force function (the sum of the force functions of all forces acting about the mechanical system). Removing in the expression of the theorem of the kinetic energy (the finite form) results:

$$E_B – E_A = U_B - U_A$$

and knowing that we have the equality:

$$V = -U$$

results:

$$E_B + V_B = E_A + V_A$$

or finally:

$$E_{mB} = E_{mA}$$

This last relation is the propriety of preservation of the mechanical energy: ***in a conservative mechanical system (acted only by conservative forces) the mechanical energy is preserved***.

Chapter 16. Dynamics of the particle.

16.1. Introduction.

In this chapter we shall study the motion of the particle considering its mass and the actions of the forces. First we shall study the motion of the free particle, and after the constrained particle. We shall see that for to study the motion of the particle it is enough to us only the second principle of mechanics or the theorem of the linear momentum. The other two theorems will be used only in the cases in which they bring some simplifications of the study of motion.

As we have presented before the general theorems make the join between the actions about the particle, generally these are the systems of forces (here concurrent systems of forces) and the motion of the particle.

If we know the forces and is asked to determine the motion then we shall say that we have the ***direct problem of the dynamics of particle,*** *and if the we know the motion and is asked to determine the system of forces that produces the motion then we say we have* ***the reverse problem of the dynamics of particle****.*

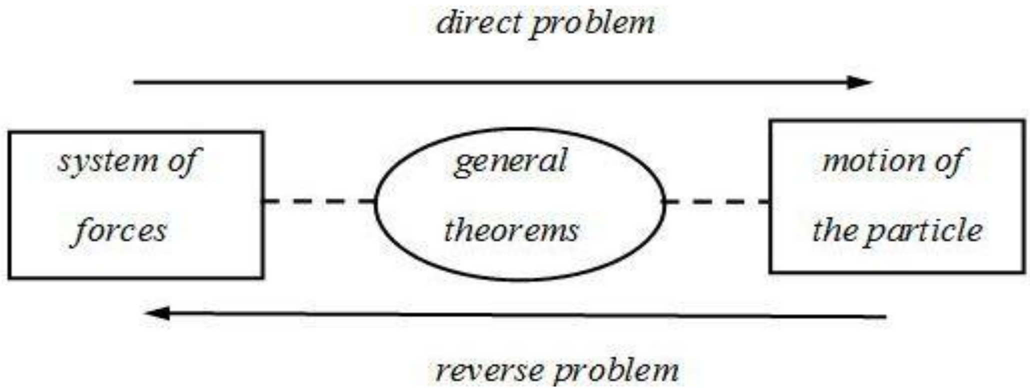


Fig.1.

There are also mixed problems in which we have to solve the problem in the both senses.

16.2. Dynamics of the free particle.

Suppose a free particle P by mass m acted by a system of concurrent forces with the resultant force $\overline{F}$. Generally the forces are functions by the position, velocity of the particle and time:

$$\overline{F} = \overline{F}(\overline{r},\dot{\overline{r}},t)$$

We want to determine the motion of this particle supposing that we know all the forces acting about it.

We know that a free particle in space has three degrees of freedom that means the motion of the particle is defined by three scalar, independent kinematic position parameters (position parameters function of time), parameters which can be, for example, the three coordinates of the particle with respect to an arbitrary reference system. This means that for to determine these three parameters we need to use or three scalar equations or one vector equation. This kind of vector equation (or three scalar equations if we project on the three axes of the reference system) can be obtained using the theorem of the linear momentum (or the second principle of mechanics). This means that the theorem of the linear momentum solves entirely the problem of dynamics of a particle. The vector equation resulted using this theorem will be:

$$m\ddot{\overline{r}}(t) = \overline{F}(\overline{r},\dot{\overline{r}},t)$$

that represents a differential vector equation by second order in $\overline{r}(t)$. The general solution of this equation will be:

$$\overline{r}(t) = \overline{r}(\overline{C}_1,\overline{C}_2,t)$$

where $\overline{C}_1$ and $\overline{C}_2$ are two vector integration constants which will be determined from the initial conditions referring to the initial position and initial velocity of the particle:

$$\overline{r}(t_o) = \overline{r}_o \,;\, \dot{\overline{r}}(t_o) = \overline{v}_o$$

These initial conditions removed in the general solution and its derivative will allows to determine the two integration constants which will be resulted function of these initial conditions:

$$\overline{C}_1 = \overline{C}_1(t_o,\overline{r}_o,\overline{v}_o) \,;\, \overline{C}_2 = \overline{C}_2(t_o,\overline{r}_o,\overline{v}_o)$$

Now if we replace these constants in the general solution is obtained the solution of the problem (of the motion) in the given initial conditions:

$$\overline{r}(t) = \overline{r}(t_o,\overline{r}_o,\overline{v}_o,t)$$

that represents the law of motion of the particle.

The scheme of solving the direct problem of the dynamics of free particle is represented in the figure 2. In this figure is given the solution in vector way but generally in problems we shall use a convenient reference system and we project the equations on the axes of the system. In this case each vector constant will be replaced with three scalar constants in space and with two scalar constants in plane (in two dimensions).

The integration of the differential equation of the motion (or of the system of differential scalar equations) is generally difficult to make and from this reason the integration is made numerical.

For the reverse problem of the dynamics of the free particle the solution is find easier because it will be obtained through derivation of the given law of motion and replacing in the differential equation results the resultant force acting about the particle. We can remark that in this case we may have a lot of solutions because we have a lot of systems of forces with the same resultant force.

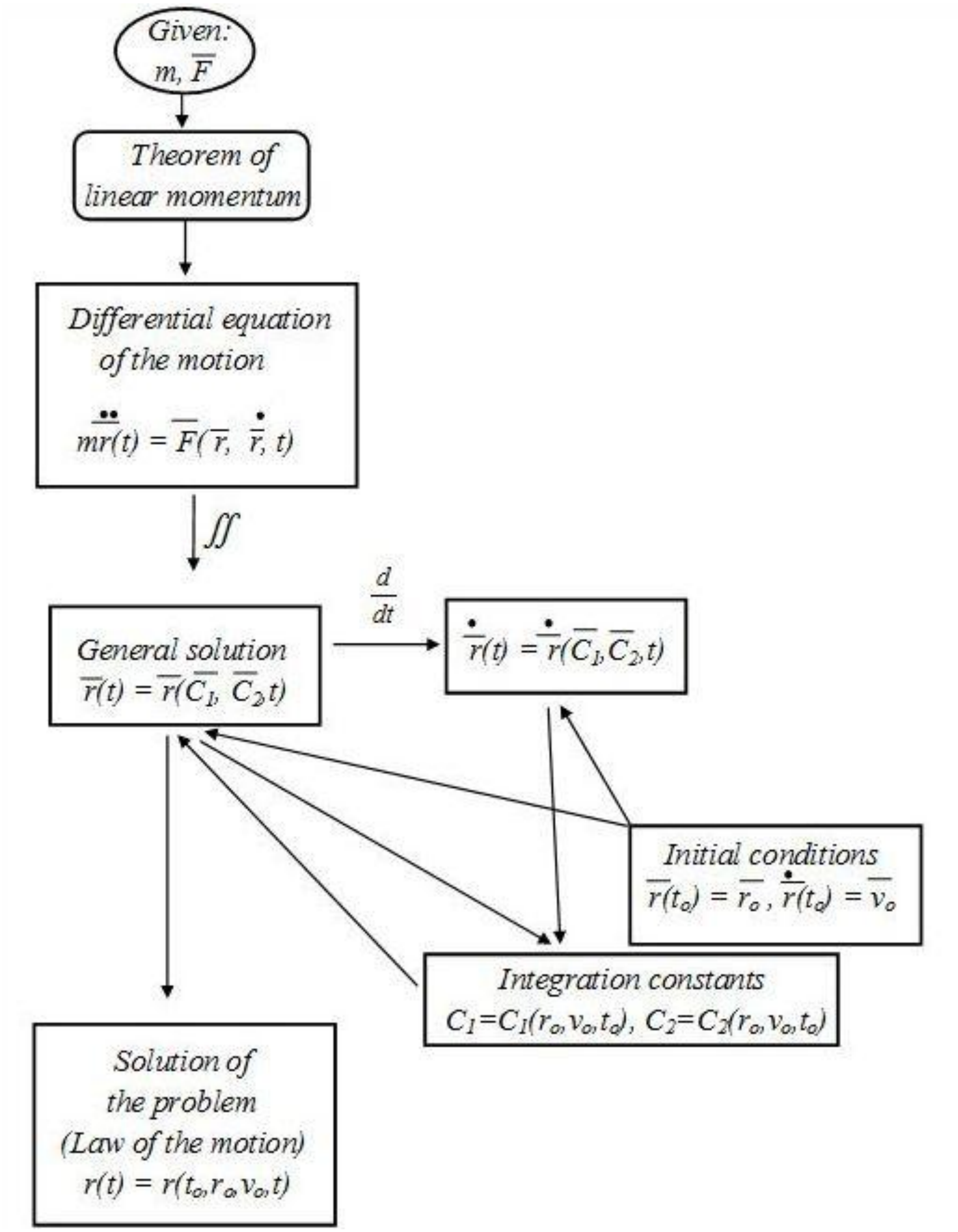


Fig.2.

16.3. Sample problems.

Problem 1. *One bullet P by mass m is launched from the height H with the horizontal initial velocity in the gravitational field. Determine the position in which the bullet arrives on the surface of the ground and its velocity in the same instant knowing that the motion is performed without the resistance of the air.*

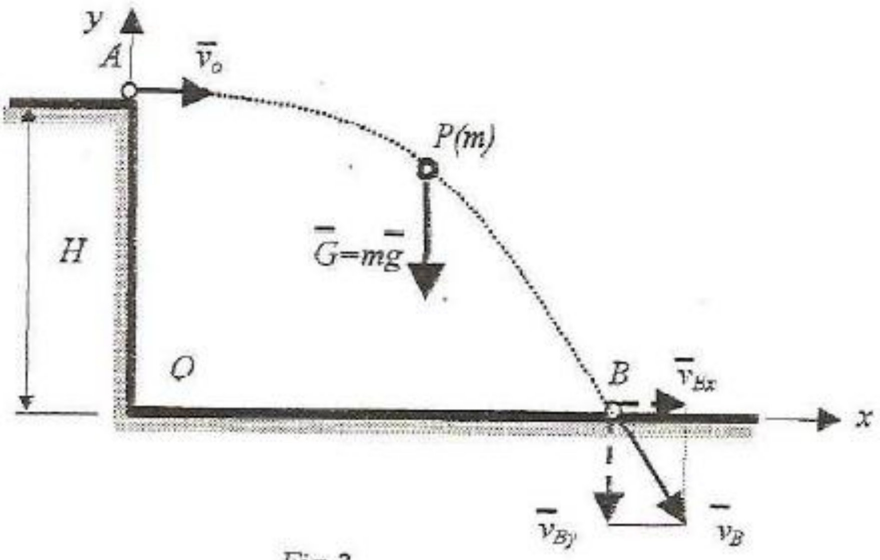


Fig.3.

Solution. *1) As we remark we have in this problem a motion in the vertical plane, so having a plane motion we shall choose a convenient reference system. The reference system is taken so that to highlight the main proprieties of the motion. For example if the motion or the forces are joined to a fixed point then we shall choose a polar reference system with the origin in that point, or if we know the trajectory of the motion then we choose the Frenet's system and if the forces have constant directions then the best choice is the Cartesian fixed system. In this problem the single force acting about the particle (the bullet) in the time of motion is its weight that has constant direction for motion of the particle with small displacement (with respect to the radius of the earth). So in this problem we shall choose the Cartesian fixed system. This system will have the directions: horizontal and vertical directions and with the origin in point O taken so that the initial position of the particle to be on the axis Oy.*

2) Now we shall choose the method of study. We can choose between the three general theorems but because we study a motion of a free particle in two dimensions (in vertical plane, so we shall have two scalar independent kinematic parameters) we choose, as we have seen, the theorem of the linear momentum:

$$m\overline{a} = \Sigma \overline{F_i}$$

3) Because we took a Cartesian reference system the two scalar independent kinematic parameters will be the two coordinates of the particle. The expression of the acceleration in Cartesian coordinates is known.

The single force acting about the particle is the weight that has the magnitude, direction and sense known consequently we can determine its projections on the two axes.

4) We project the resulted equation from the theorem of the linear momentum on the two axes of the reference system:

$$\begin{cases} ma_x = \Sigma X_i; \\ ma_y = \Sigma Y_i; \end{cases}$$

Replacing the expressions of the acceleration in Cartesian coordinates function to the chosen kinematic parameters and the projections of the weight on the two axes we obtain:

$$\begin{cases} m\ddot{x} = 0; \\ m\ddot{y} = -mg \end{cases} \longrightarrow \begin{cases} \ddot{x} = 0; \\ \ddot{y} = -g \end{cases}$$

that represent the system of the differential equations of the particle's motion (after that we have simplified with the mass of the particle). This system is made from two differential equations by second order function of x and y, the two chosen kinematic parameters.

5) Now we shall integrate the differential equations. Because the two equations are independent their integration will be made separately and because the equations are with separable variables we shall integrate step by step:

$$\begin{cases} \dot{x} = C_1; \\ \dot{y} = -gt + C_2 \end{cases}$$

$$\begin{cases} x(t) = C_1.t + C_3 \ ; \\ y(t) = -\dfrac{gt^2}{2} + C_2.t + C_4 \end{cases}$$

This final result is the general solution of the system if differential equations which depends by four scalar integration constants.

6) For to find the four integration constants we shall consider the initial conditions referring to the position and velocity of the particle. These conditions will be projected on the two axes of the reference system and we have:

$$t_o = 0 \ ; \quad x(0) = 0 \ ; \ y(0) = H \ ; \ v_x(0) = v_o \ ; \ v_y(0) = 0.$$

Replacing in the previous four relations are obtained (four algebraic equations) the four integration constants:

$$C_1 = v_o \ ; \ C_2 = 0 \ ; \ C_3 = 0 \ ; \ C_4 = H.$$

7) We shall remove these integration constants in the general solution and we have:

$$\begin{cases} x(t) = v_o t \\ y(t) = -\dfrac{gt^2}{2} + H \end{cases}$$

that represents the solution of the motion in the given initial conditions, namely these are ***the laws of motion of the particle****.*

8) Now having the laws of motion, we can determine any element of the particle's motion. In this way if we consider the condition that the particle to arrive on the surface of the earth we can determine the instant of this fact:

$$y(t_B) = 0\ ; \longrightarrow \quad t_B$$

namely:

$$0 = -\frac{gt_B^2}{2} + H \longrightarrow t_B = \sqrt{\frac{2H}{g}}$$

and this replaced in the x(t) will give us the searched distance:

$$x_B = x(t_B) = v_o \sqrt{\frac{2H}{g}}$$

For to calculate the velocity in point B we replace t_B in the expressions of the two projections of velocity:

$$v_{Bx} = v_x(t_B) = v_o;$$
$$v_{By} = v_y(t_B) = -\sqrt{2gH}\ ;$$

With these we obtain the magnitude of the velocity in this point:

$$v_B = \sqrt{v_o^2 + 2gH}$$

Problem 2. *One missile P by mass m is launched from the surface of the earth with the initial velocity v_o on a direction making the angle α with the horizontal. If the resistance of the air is considered proportional to the velocity of the missile (the proportionality factor is $k_1 = km$) and the motion of the missile is made in vertical plane determine the height at which it will arrives, the position in which it will touches the ground and its velocity in the same instant. Is considered as the motion is made in the gravitational field and it has small scale. (Fig.4).*

Problem 3. *One missile P by mass m is launched from the surface of the earth with the initial velocity v_o on a direction making the angle α with the horizontal. Knowing that the motion is performed in gravitational field without the resistance of the air and it has large scale (it is considered the curvature of the earth) to determine the height that rises the missile with respect to the surface of the earth. It will be considered the radius of the earth equal to R (Fig.5.)*

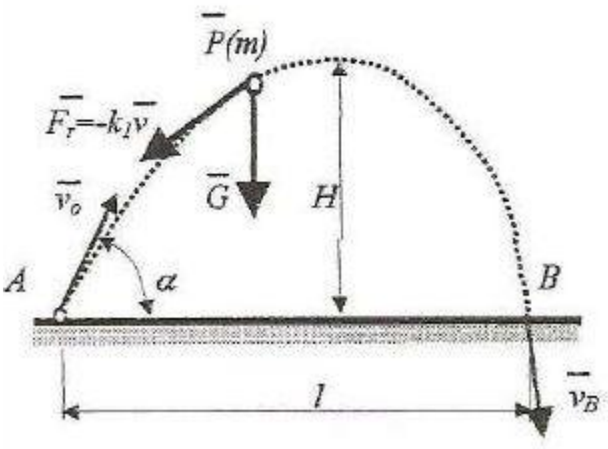


Fig.4.

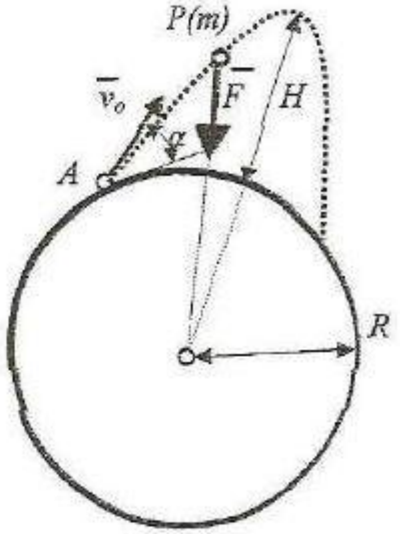


Fig.5.

16.4. Dynamics of the constrained particle.

Suppose a particle P by mass ***m*** *having some ideal constraints, but keeping at least one degree of freedom (to have possibilities of motion). We know that the constraints eliminate degrees of freedom consequently the particle will have the following number of degrees of freedom:*

$$N_{df} = 3 - l$$

where ***l*** *is the number of the simple ideal constraints. The number of the scalar kinematic parameters which define the motion of the particle is the same as the number of the degrees of freedom.*

Using the axiom of the constraints the simple ideal constraints can be removed with the same number of unknown reaction forces, functions of time, and because this fact they are called ***dynamic reaction forces****. Consequently the number of the scalar unknown introduces by the reactions is equal to* ***l****, so the number of the unknowns (unknown kinematic parameters and unknown reactions) is the same as for the free particle.*

At the other hand after which we have removed the constraints with the reaction forces the particle becomes a free particle and we can study its motion using the theorem of the linear momentum. The difference by a free particle is that the unknowns are kinematic parameters and forces as for a mixed problem of the free particle.

The scalar differential equations resulted from the theorem of the linear momentum, projected on the axes of a convenient reference system contain kinematic parameters (which have to be determined for to know the motion of the particle) and also dynamic reactions equivalent to the constraints of the particle. These equations form a system of scalar differential equations with the same number of unknowns.

This system of equations is decomposed in two subsystems (making the corresponding substitutions):

- one subsystem having the same number of equations as the number of the degrees of freedom or the number of the kinematic parameters containing only the kinematic parameters as unknowns (and their derivatives obviously). This is the ***subsystem of the differential equations of the motion****;*

- one subsystem having the same number of equations as the number of the scalar unknown reactions from the constraints and containing besides of the kinematic parameters the unknown scalar reaction forces.

The solving of the differential equations of the motion is made as for the free particle but we have less number of equations and we have in this way the motion of the particle. After which we know the motion of the particle, from the second subsystem of equations we determine the dynamic reactions.

In plane problem (in two dimensions) the constrained particle has always one degree of freedom and one simple constraint.

Besides the theorem of the linear momentum in some cases we can use the theorem of the angular momentum or the theorem of the kinetic energy. In space the theorem of the angular momentum can replace the theorem of the linear momentum for to study entirely the motion of the particle but in two *dimensions (in plane motion) this theorem (theorem of the angular momentum) give us only one scalar differential equation so this theorem can't be used for to study entirely the motion of the particle, but the equation obtained from this theorem may be used for to determine a part of the elements of motion.*

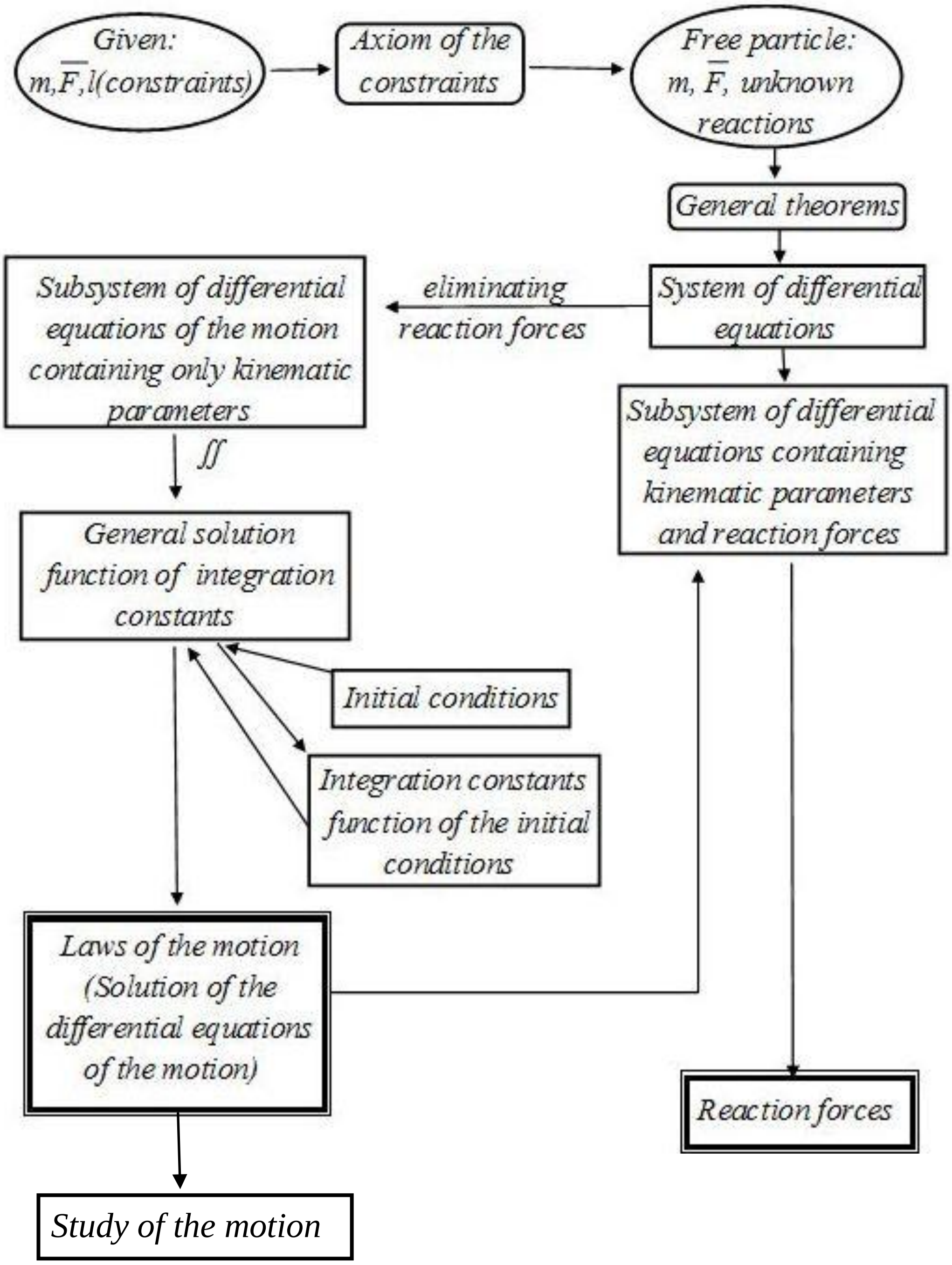


Fig.6.

The theorem of the kinetic energy is a scalar theorem so it will give us only one scalar differential equation. Consequently it will be used if the particle has only one degree of freedom and the constraints of the particle are ideal because in *this case it is obtained directly the differential equation of the motion. The advantage of to use the theorem of the kinetic energy is that we can obtain the first integral of the differential equation of the motion (the finite form of the theorem). The scheme of solving this kind of problem is represented in the figure number 6.*

16.5. Sample problems

Problem 4. *A particle P by mass* **m** *is linked with an ideal wire without mass and having its length* **l** *by a fixed point O. The particle is in rest when the wire is vertically and at the initial instant it will have an initial horizontal velocity v_o. Knowing that the motion of the particle is performed in vertical plane under the action of its weight without any frictions determine the differential equation of the motion of the particle, its velocity and the tension from the wire when this makes with the vertical direction an angle by 60^o. After determine the value of the initial velocity v_o as the particle to describe an entire circle.*

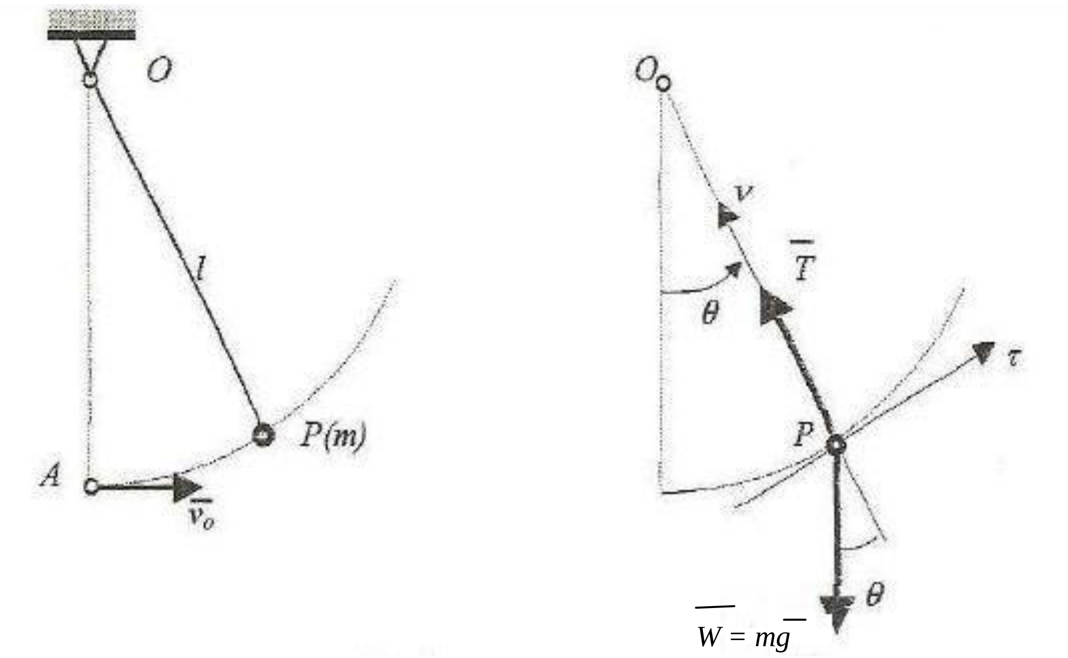


Fig.7.

Solution. *1) The particle has one simple ideal constraint therefore it has one degree of freedom and we shall choose, for to study the motion one single kinematic parameter and this will be the angle θ.*

2) We shall eliminate the constraint of the particle (the ideal wire by length l) and we shall replace with one corresponding reaction force the tension T from the wire. In this way the particle becomes a free particle acted by two forces: the weight W known in magnitude, direction *and sense and the unknown reaction force $\overline{T}$ corresponding to the removed wire. The position of this free particle is defined by the kinematic parameter θ(t).*

3) For to study the motion of the particle we shall choose as reference system the Frenet's system because we know the trajectory of the particle (the circle with the center in point O and radius l). This system has the Pτ axis tangent to the circle representing the trajectory of the particle and the positive sense in the increasing sense of the kinematic parameter and the axis Pν on the direction of the wire (the radius of the circle) and with the sense directed toward the fixed point O.

4) For the study we shall use the theorem of the linear momentum:

$$m\,\overline{a} = \Sigma \overline{F_i}$$

or projecting on the two axes we have:

$$\begin{cases} ma_\tau = \Sigma F_{i\tau} \\ ma_\nu = \Sigma F_{i\nu} \end{cases}$$

Replacing the expression of the acceleration function of the chosen parameter and also the projections of the forces on the two axes we obtain the differential equations:

$$\begin{cases} m\,l\ddot{\theta} = -mg\sin\theta \\ m\,l\dot{\theta}^2 = T - mg\cos\theta \end{cases}$$

*5) The first equation is **the differential equation of the motion of the particle** because it does not contain the dynamic reaction force, equation that can be written also in the following form:*

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

The first integral of this differential equation is:

$$\frac{\dot{\theta}^2}{2} = \frac{g}{l}\cos\theta + C$$

where C is one integration constant that can be determined from the initial conditions:

$$t_o = 0\,; \longrightarrow \theta(0) = 0\,;\ \dot{\theta}(0) = \omega_o = \frac{v_o}{l}$$

Replacing these conditions in the previously relation is obtained the integration constant:

$$C = \frac{v_o^2}{2l^2} - \frac{g}{l}$$

In this instant we have the angular acceleration of the particle and the angular velocity function of the position of the particle on the circle:

$$\ddot{\theta} = -\frac{g}{l} sin\theta ; \quad \frac{\dot{\theta}^2}{2} = \frac{g}{l} cos\theta + \frac{v_o^2}{2l^2} - \frac{g}{l}$$

6) The second equation resulted from the theorem of the linear momentum is the equation that allows to determine the dynamic reaction. If we replace the expressions of the kinematic parameter's derivatives is obtained for the dynamic reaction force the expression:

$$T(\theta) = 3\ mg\ cos\theta + m\frac{v_o^2}{l} - 2mg$$

7) For to determine the elements of the motion in any instants of the motion we shall make particularly the kinematic parameter corresponding to those instants. In this way if we want the velocity of the particle in the instant when the wire makes the angle θ_1 = 60^o with the vertical direction we shall make:

$$v_1 = v(\theta_1) = \dot{\theta}(\theta_1) \,.\, l = \sqrt{v_o^2 - gl}$$

The tension in the same instant will be:

$$T(\theta_1) = m\frac{v_o^2}{l} - 0{,}5mg$$

Finally for to determine the initial velocity corresponding to the complete circular motion of the particle we shall consider the condition as the wire to remain tensioned in the entire time of motion, namely the tension from the wire to remain positive for all values of the angle θ:

$$T(\theta) \geq 0 \quad pentru \ \ 0 \leq \theta \leq 2\pi$$

The magnitude of the tension being a function by the cosines it is obviously that the most unfavorable situation corresponds for:

$$\theta_2 = \pi$$

whence results that the minimum value of the initial velocity have to be:

$$T(\theta_2) = 0 \longrightarrow v_o = \sqrt{5gl}\,.$$

As we have seen if the particle has one degree of freedom the elements of the motion can be determined using the theorem of the kinetic energy and sometimes the theorem of the angular momentum. In this case the differential equation of the motion can be obtained writing under differential form the theorem of the kinetic energy:

$$dE = dL$$

where the kinetic energy at an instant is:

$$E = \frac{1}{2}mv^2 = \frac{1}{2}m(l\dot{\theta})^2.$$

The elementary work is produced by the weight only because the reaction forces from ideal constraints are perpendicular on the elementary displacement of the particle (that is tangent to the trajectory as the velocity of the particle). The elementary work of the weight is calculated projecting the force on the direction of the elementary displacement (on the direction of the tangent to the trajectory) and it will be:

$$dL = -G_{dr}\,.\,dr = -Mg\sin\theta\,.\,l\,d\theta$$

The sign minus is because the elementary displacement is performed in the sense of the parameter θ increasing namely in the positive sense of the axis Pτ and the projection of the weight is in the opposite sense.

Differentiating the kinetic energy and removing in the expression of the theorem of kinetic energy is obtained the differential equation of the motion:

$$\frac{1}{2}m2l^2\,.\dot{\theta}\,.\,d\dot{\theta} = -Mg\,.\sin\theta\,.\,l\,.\,d\theta$$

or dividing by dt we have the same differential equation.

For to calculate the velocity of the particle at the instant t_1 (for $\theta_1 = 60^o$) we shall use the theorem of the kinetic energy under finite form:

$$E_1 - E_o = L_{o1}$$

where the two kinetic energies have the expressions:

$$E_1 = \frac{1}{2}mv_1^2\,;\,E_o = \frac{1}{2}mv_o^2$$

The work of the weight (that is a conservative force) is:

$$L_{oI} = \pm mg\,\Delta z$$

relation in which the sign (+) corresponds to the displacement of the particle in the sense of action (down) of the weight and the sign (-) corresponds to the opposite sense. The Δz is the difference of the quota of the two positions. We obtain:

$$L_{oI} = -mg\,(l - l\cos\theta_1) = -\,0{,}5\,mgl$$

Replacing in the expression of the theorem of kinetic energy we obtain the same value for the velocity of the particle as in the previously calculation.

For to determine the differential equation of the motion we can use also the theorem of the angular momentum about the fixed point O:

$$\dot{K}_O = M_O$$

or knowing that the angular momentum of a particle is the moment of the linear momentum with respect to the point O (the linear momentum has the direction and sense of the velocity of the particle namely tangent to the circle representing the trajectory of the particle or perpendicular to the radius of this circle):

$$\frac{d}{dt}[(ml\dot{\theta}).\,l] = -\,mg\,.\,l\sin\theta$$

The sign (-) from the right side of the relation is due to the fact that we have considered the trigonometrically sense of rotation as the positive sense. Finally after derivation and simplifying this relation we obtain the same differential equation of the motion.

Problem 5. *The particle p by mass m is launched from the point A on an inclined surface with the initial velocity v_o. knowing that the motion on the inclined surface and also on the circular surface is performed without friction determine the velocities of the particle in the positions B, C and D and the reaction forces in the same positions. The motion of the particle is performed due to its weight in vertical plane. Are given: AB = 5 m, R = 1 m , v_o = 4 m/s.*

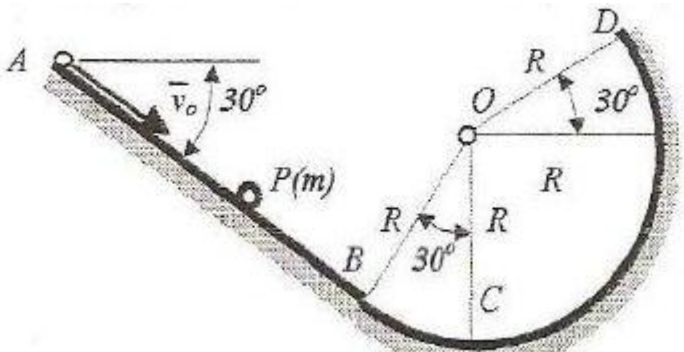


Fig.8.

16.6. Dynamics of the constrained particle with friction.

If the constraints of the particle are with friction the problem of the study of its motion is made in the same way as the problems of particles with ideal constraints but are added to the differential equations resulted from the use of the general theorems the relations corresponding to the friction on the supporting surface:

$$F_{fi} = \mu_i N_i\,; \quad (i = 1,2)$$

Because the particle is in motion the friction force has the maximum value. In fact if are made all the substitutions the problem is solved in the same way as in the previously section (constrained particle with ideal constraints).

Generally in this case we can't use the theorem of the kinetic energy.

16.7. Sample problems

Problem 6. *One particle P by mass m is launched from the point A with initial velocity v_o on an inclined plane with the angle $\alpha = 30^o$ with respect to the horizontal direction. Knowing that the surface is rough and it has the friction coefficient with the particle $\mu = 0,1$ determine the position in which the particle stops on the rough inclined surface.*

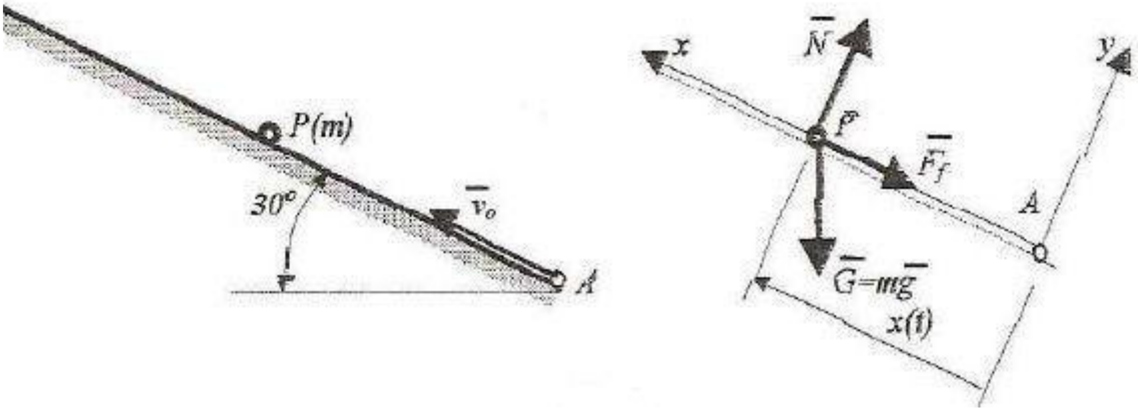


Fig.9.

Solution. *1) We have here a particle with a real constraint namely a constraint with friction and it has one degree of freedom. For to study the motion we shall choose as kinematic parameter the displacement x(t) on the direction of the inclined surface from the initial position to an any position on the surface.*

2) We eliminate the constraint and it is replaced with the normal reaction force $\overline{N}$ *and the friction force* $\overline{F_f}$ *having the opposite sense as the sense of motion.*

3) For to study the motion we shall choose as reference system the Cartesian system with the Ax axis on the direction and the sense of motion of the particle.

4) We shall use the theorem of the linear momentum at which we ad the condition of the friction:

$$m\overline{a} = \Sigma \overline{F_i}$$
$$F_f = \mu N$$

Projecting on the axes of the reference system we have the differential equations:

$$\begin{cases} m\,a_x = -F_f - mg\sin 30^o \\ m\,a_y = N - mg\cos 30^o \\ F_f = \mu N \end{cases}$$

but because the motion is performed only on the direction of the axis Ax and the only one kinematic parameter is x(t) we shall have:

$$\begin{cases} m\ddot{x} = -F_f - mg\sin 30^o \\ 0 = N - mg\cos 30^o \\ F_f = \mu N \end{cases}$$

5) Substituting is obtained the differential equation of the motion:

$$\ddot{x} = -g(\mu\cos 30^o + \sin 30^o)$$

or replacing the values results:

$$\ddot{x} = -0{,}586\,g$$

We shall integrate twice this equation and we obtain:

$$\dot{x} = -0{,}586\,g\,t + C_1 \quad ; \quad x(t) = -0{,}293\,gt^2 + C_1 t + C_2$$

where C_1 and C_2 are two integration constants which will be determined from the initial conditions:

$$t_o = 0; \qquad x(0) = 0\ ;\ \dot{x}(0) = v_o$$

Removing the last two relations are obtained the constants:

$$C_1 = v_o\ ;\ C_2 = 0.$$

Finally we have the law of motion on the inclined surface:

$$x(t) = -0{,}293\ gt^2 + v_o t$$

6) For to determine the position in which the particle stops from its motion we shall consider the condition that its velocity to become zero in that instant:

$$\dot{x}(t_1) = -0{,}586\ gt_1 + v_o = 0 \longrightarrow t_1 = \frac{v_o}{0{,}586g}$$

Removing in the law of motion we shall determine the stop position of the particle in its upward movement:

$$l = x(t_1) = \frac{v_o^2}{1{,}172g}$$

The particle stops in its upward movement but now we have to check if it remains in this position or it will slide down on the inclined surface. For this it is enough to compare the friction force (with opposite sense) with the component, on the direction of the inclined surface, of the weight, component that will pull the particle down.

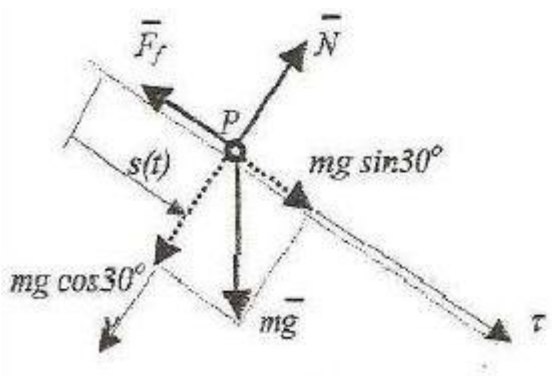


Fig.10.

The friction force has the maximum magnitude:

$$F_{fmax} = \mu N - 0{,}1\ mg \cos 30^o - 0{,}086\ mg;$$

and the component of the weight on the direction of the inclined surface will be:

$$mg\,sin30^{\circ} = 0{,}5\ mg.$$

Comparing the two forces and results that the particle does not stop in this position the particle will start to slide down. The study of this motion will be made in the same way as in the upward motion of the particle, but for to show an example we shall use now the Frenet's system and for the kinematic parameter the space performed by the particle from the stop point on the direction of the inclined surface.

The differential equations resulted from the theorem of the linear momentum and the condition of friction will be:

$$\begin{cases} m\,a_{\tau} = -\,F_f\ -\ mg\,sin30^{\circ} \\ m\,a_{\nu} = N - mg\,cos30^{\circ} \\ F_f = \mu N \end{cases}$$

where considering the corresponding relations in this reference system and the fact that the motion is rectilinear (the radius of curvature is infinity) we will have:

$$\begin{cases} m\ddot{s} = -\,F_f + mg\,0{,}5 \\ 0 = N - mg\,0{,}866 \\ F_f = 0{,}1\,N \end{cases}$$

Making the substitutions we have the differential equation of the motion:

$$\ddot{s} = 0{,}423g$$

that shows us that the motion is uniformly accelerated and consequently the particle does not stop if we don't introduce other conditions (change of the trajectory, introduction of other constraints or forces, etc.).

Problem 7. *We consider the problem 5 in which the inclined surface AB is rough with the friction coefficient μ = 0,15. Considering all the other elements of the motion unchanged determine the same elements of the problem.*

Chapter 17. Dynamics of the rigid body.

17.1. Introduction.

This chapter will be devoted to the study of motion of a rigid body under the action of forces and also considering the mass of it and the distribution of its mass in the body. As for the particle we shall study first the motion of the free rigid body and after the constrained body.

In this chapter we shall study mainly the plane motions of the body.

We have also in the previously chapter three kinds of problems: ***direct problem*** *in which are known the forces acting about the body and is asked to determine its motion,* ***the reverse problem*** *in which we know the motion and is asked to determine the forces which produce that motion, and obviously* ***the mixed problem****.*

17.2. Dynamics of the free rigid body.

One free rigid body has six degrees of freedom in space (and only three degrees of freedom in plane). Consequently the motion of the body will be defined by six scalar independent kinematic parameters (only three in plane. So if are known the forces which act about the body are necessary six scalar independent equations for to define the six parameters of the motion. The six relations can be obtained from the two vector general theorems: the theorem of the linear momentum and the theorem of the angular momentum.

$$\overset{\bullet}{\overline{H}} = \overline{R} \;;\; \overset{\bullet}{\overline{K_O}} = \overline{M_O}$$

Projecting these two theorems on the axes of a convenient reference system we obtain the six scalar equations (or three in plan).

Being a free body it is more advantageous to use the two theorems considering the motion about the mass center of the body, namely to use the theorem of the motion of the mass center and the theorem of the angular momentum about the mass center:

$$M\overline{a}_C = \Sigma \overline{F}_i \;;\; \overset{\bullet}{\overline{K}}'_C = \Sigma \overline{M}_{Oi}$$

Because the angular momentum with respect to the mass center is calculated as the mass center is a fixed point results that this angular momentum will be calculated for a rotation motion about the mass center.

To consider a plane body performing a motion in its plane about its mass center, so performing a rotation motion about that point considered as a fixed point. The angular momentum is a vector perpendicular on the plan containing the velocities and the fixed point (so perpendicular on plane of the motion). From the definition of the angular momentum about a point we have:

$$K'_C = \int_A \left|\overline{r}' \times \overline{v}'\right| dm = \int_A r' v' \, dm = \int_A r'.r' \omega dm = \omega \int_A r'^2 \, dm = \\ = J_C \cdot \omega$$

We remark the fact that if the angular momentum is calculated for a body in rotation motion about a fixed point O then we have obviously:

$$K_O = J_O \, \omega$$

In this way for a free rigid body performing a motion in plane the differential equations will be in Cartesian reference system:

$$\begin{cases} M\ddot{x}_C = \Sigma X_i \\ M\ddot{y}_C = \Sigma Y_i \\ J_C\ddot{\theta} = \Sigma M_{Ci} \end{cases}$$

where θ is the angular parameter of the motion of the body.

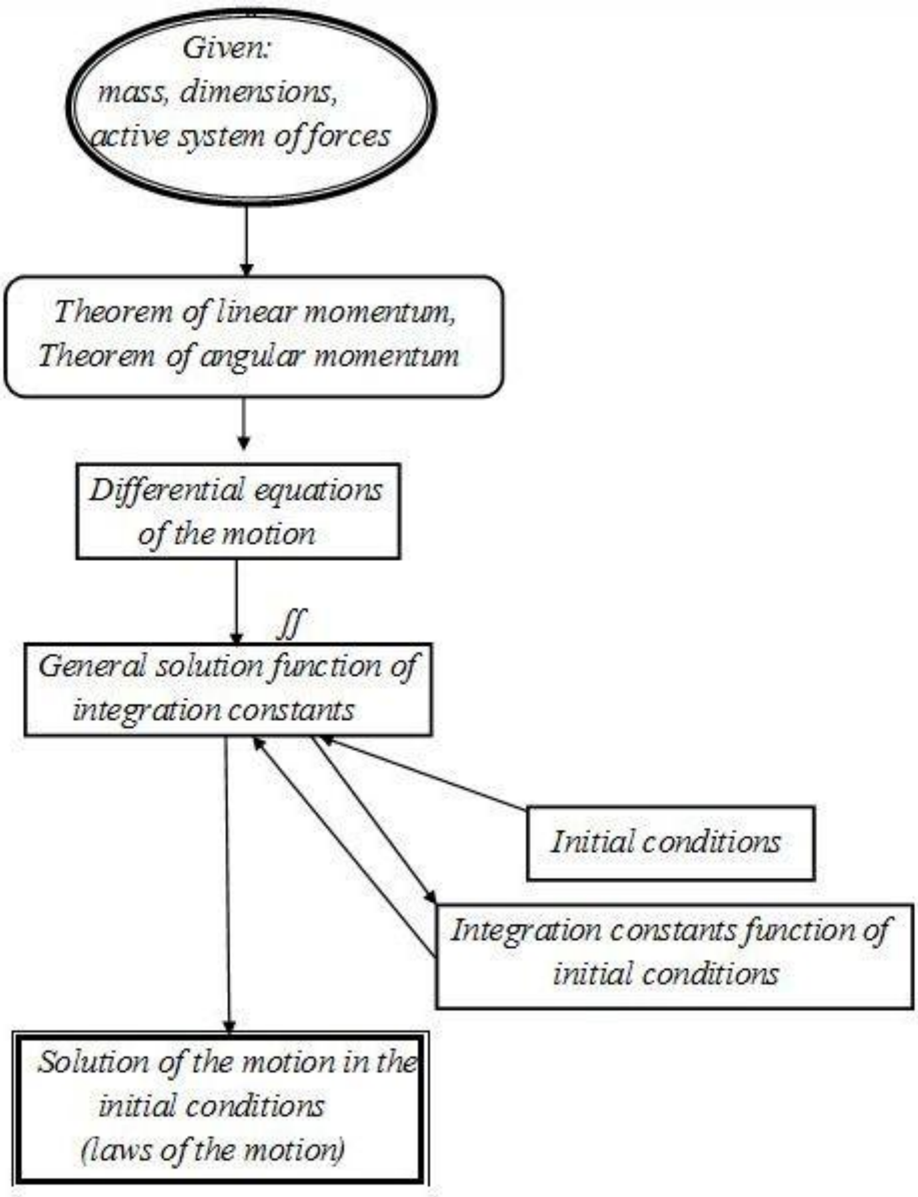


Fig.1.

Integrating these differential equations is obtained the general solution function of six scalar integration constants which will be *calculated from the initial conditions. These constants replaced in the general solution we find the solution in the initial conditions so we have the laws of motion of the free rigid body.*

As we can remark the scheme of solving is identical to the scheme for the free particle and the two vector general theorems solve entirely the dynamics of the free rigid body.

Sometimes one of the scalar differential equations of the motion resulted from these two vector theorems can be removed with the equation resulted from the theorem of the kinetic energy. In plane problems the removed equation is that from the theorem of the angular momentum. The advantage of to use the theorem of the kinetic energy is that we can write this theorem under the finite form so we obtain the differential equation integrated once.

In the theorem of the kinetic energy we use the expression of the kinetic energy in the three particular motions (in plane) so we give here the expressions of them. We start from the theorem of Koenig for kinetic energy:

$$E = \frac{1}{2}Mv^2_C + E'$$

where E' is the kinetic energy of the body as it perform a rotation motion about its mass center considered as a fixed point. Consequently we shall calculate first the expression of the kinetic energy in rotation about the mass center:

$$E' = \frac{1}{2}\int_A v'^2\, dm = \frac{1}{2}\int_A (r'\omega)^2\, dm = \frac{1}{2}\omega^2\int_A r'^2\, dm = \frac{1}{2}J_C\omega^2$$

Now we can give the expressions of the kinetic energy in the three particular motions:

- *In translation motion (ω = 0) we have:*

$$E^{tr} = \frac{1}{2}Mv^2_C\,;$$

- *In rotation motion about a fixed point O the expression is obtained removing the mass center as fixed point with the point O:*

$$E^{rot} = \frac{1}{2} J_O \omega^2$$

- *In plane motion we shall have:*

$$E^p = \frac{1}{2} M v_C^2 + \frac{1}{2} J_C \omega^2$$

17.3. Sample problems

Problem 1. *One disc having the mass M and the radius R is launched in vertical plane with the initial velocity of the mass center v_o on the direction inclined with the angle α with respect to the horizontal direction and is produced an angular velocity ω_o in the clockwise sense. Knowing that the motion is performed in vertical plane in the gravitational field without the resistance of the air study the motion of the disc.*

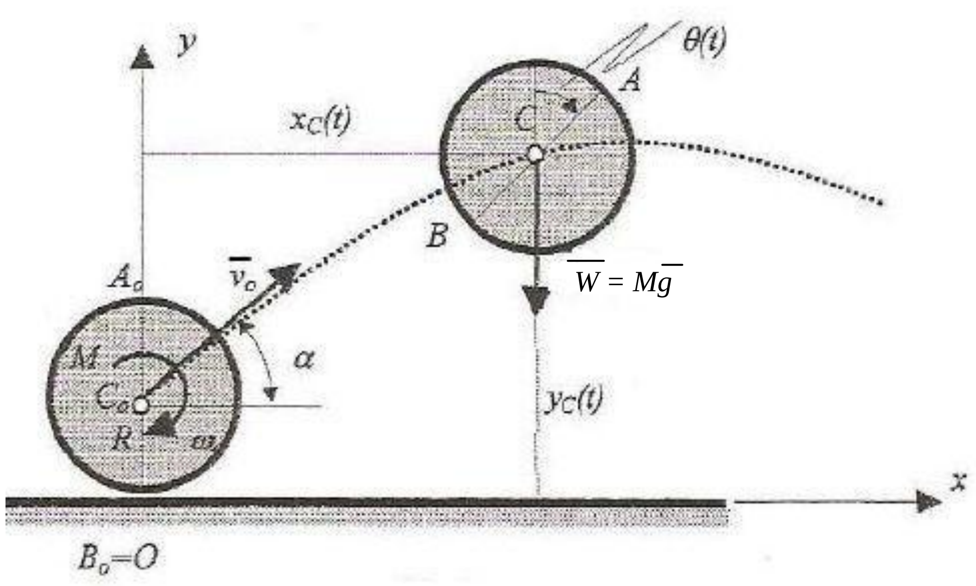


Fig.2.

Solution. *1) In the time of motion the disc does not come in contact with any other bodies so it is a free body. Being a motion in vertical plane the body has three degrees of freedom. We choose for to study the motion a Cartesian reference system with one horizontal axis and one vertically and with the origin in the contact point between the disc and the ground in the initial instant. We shall choose as kinematic parameters the two coordinates of the mass center with respect* *to the chosen reference system and the rotation of the disc, rotation measured between the rotated diameter AB and the vertical direction (its initial position A_oB_o).*

2) About the body acts only one force in the time of motion namely its weight W having as point of application the center of the disc.

3) For to study the motion we shall use the two theorems: theorem of the linear momentum under the form of the theorem of motion of the mass center and the theorem of the angular momentum about the mass center projected on the axes of the chosen reference system:

$$\begin{cases} M\ddot{x}_C = \Sigma X_i \\ M\ddot{y}_C = \Sigma Y_i \\ J_C\ddot{\theta} = \Sigma M_{Ci} \end{cases}$$

or removing with the elements of the problem we have:

$$\begin{cases} M\ddot{x}_C = 0 \\ M\ddot{y}_C = -Mg \\ \dfrac{MR^2}{2}\ddot{\theta} = 0 \end{cases}$$

After the simplification ***the three differential equations of the motion*** *are:*

$$\ddot{x}_C = 0 \ ; \ \ddot{y}_C = -g \ ; \ \ddot{\theta} = 0$$

4) Integrate twice each differential equation and we obtain the general solution depending by six scalar constants:

$$\dot{x}_C = C_1 \ ; \ \dot{y}_C = -gt + C_2 \ ; \ \dot{\theta} = C_3$$

$$x_C(t) = C_1t + C_4 \ ; \ y_C(t) = -\frac{gt^2}{2} + C_2t + C_5 \ ; \ \theta(t) = C_3t + C_6$$

5) The six integration constants will be determined from the initial conditions corresponding to the initial position and initial velocities:

$$t_o = 0 \ ; \longrightarrow \begin{cases} x_C(0) = 0 \ ; y_C(0) = R \ ; \theta(0) = 0; \\ \dot{x}_C(0) = v_{ox} = v_o\cos\alpha \ ; \dot{y}_C(0) = v_{oy} = v_o\sin\alpha \ ; \dot{\theta}(0) = \omega_o \end{cases}$$

Removing in the previously relations we have the six constants:

$$C_1 = v_o\cos\alpha \ ; C_2 = v_o\sin\alpha \ ; C_3 = \omega_o \ ; C_4 = 0 \ ; C_5 = R \ ; C_6 = 0$$

7) We remove these six constants in the general solution and we have the laws of motion in the initial conditions:

$$x_C(t) = v_o\, t \cos\alpha \;\; ; \;\; y_C(t) = -\frac{gt^2}{2} + v_o\, t\, \sin\alpha + R \;;\; \theta(t) = \omega_o\, t$$

8) From the first two equations we remark that the mass center moves on a parabola by the equation:

$$y = -\frac{gx^2}{2v_o^2 \cos^2\alpha} + x\, tg\alpha + R$$

In the same time the disc rotates, in clockwise sense, with the constant angular velocity ω_o.

Problem 2. *O rod AB by length 2l that in the initial instant was horizontally is thrown in vertical plane giving to its ends the initial velocities v_{Ao} = 14 m/s and v_{Bo} = 10 m/s on the directions represented in the figure 3. Knowing that the motion is made in the gravitational field without the resistance of the air and also that l = 0,5 m determine the velocity of the end B at 2 s from the beginning of the motion.*

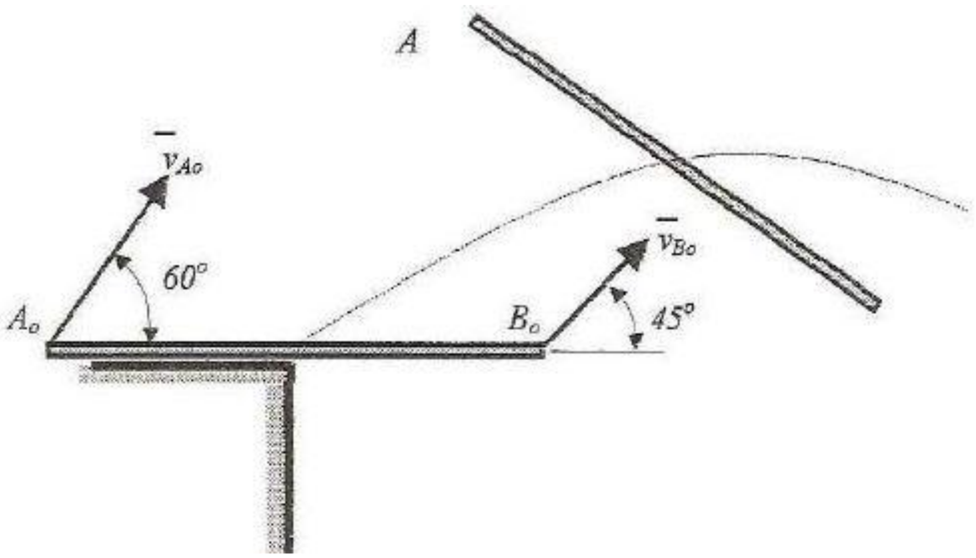


Fig.3.

17.4. Dynamics of the constrained rigid body.

If the rigid body has constraints (which suppress degrees of freedom) the number of the degrees of freedom and also the number of the scalar independent kinematic parameters will be:

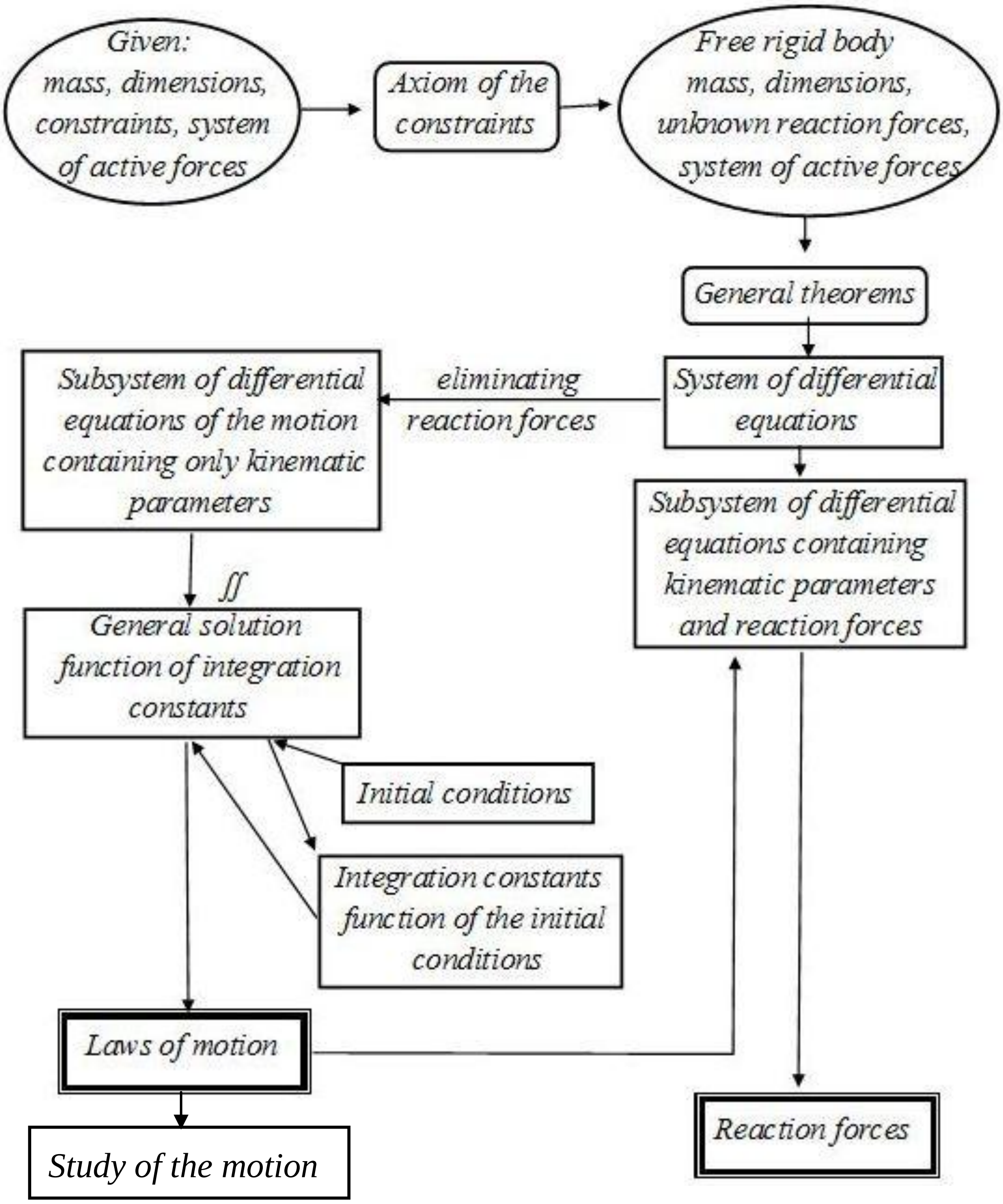


Fig.4.

$$N_{df} = 6 - N_{sc}$$

in space and:

$$N_{df} = 3 - N_{sc}$$

in plane and where we have marked N_{sc} the number of the simple constraints equivalent to the constraints of the body.

Because the constraints can be removed, using the axiom of constraints, with reaction forces which introduce in computation the same number of the scalar unknown as the number of the degrees of freedom eliminated by them, the total number of the unknown remains the same as for a free rigid body. This means that after which we used the axiom of the constraints the body can be studied in the same way as a free body namely using the two vector theorems. The scheme of solving is the same as for the constrained particle but now the difference is that we have more equations.

The theorem of the kinetic energy, in plane problems, is used efficiently when the body has one degree of freedom the constraints are frictionless or are with friction but the body performs rolling motion without sliding. In these cases the differential equation resulted from this theorem give us the differential equation of the motion and the advantage is that this can be obtained in differential form or in integrated form.

We make a remark that if from the system of differential equations of the motion we can solve independently one or more equations then their solutions can be considered as constraints and in this way the number of the degrees of freedom will be less the problem should be solved in another way.

17.5. Sample problems.

Problem 3. *One rectilinear bar AB by length l and mass M has a hinged support in the point O. The motion of the bar is due to its weight and its horizontal initial position. Knowing that the motion is performed in vertical plane without the resistance of the air and the motion starts from rest determine the differential equation of the motion and after calculate the velocity of the point A and the reaction forces from the point O in the instant when the bar is vertically.*

Solution. *1) The body has one degree of freedom (one hinged support eliminate two degrees of freedom) consequently for the study of the motion we shall choose as angular parameter $\theta(t)$ measured between the direction of the bar at one any instant and its initial position.*

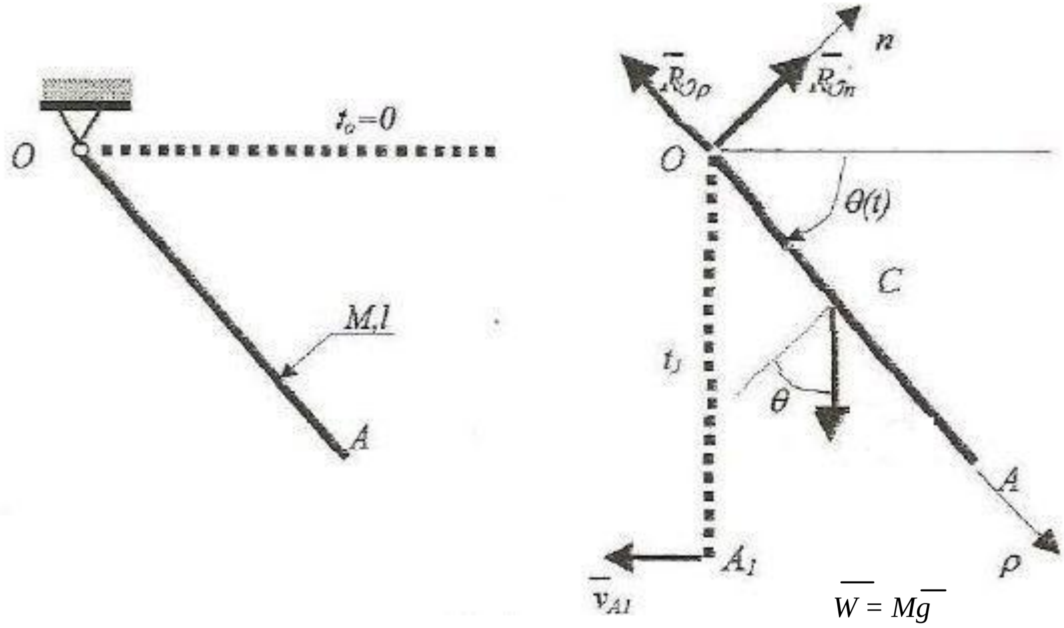


Fig.5.

2) We remove the hinged support from the point O with two components of reaction forces on the directions of the axes of a convenient reference system. The choice of the reference system is made function of the characteristics of the motion. Here the best choice is to take a polar reference system with the origin in the fixed point O, the radial axis Oρ on the direction of the bar and passing through the mass center and the second axis perpendicular on the bar. (If we choose a Cartesian fixed reference system with horizontal and vertical axes then the components of the reactions in point O will be taken on these two directions. Also we can consider a Frenet's system with the origin in point C – the mass center – because we know the trajectory of this point).

3) For to study the motion and to determine the dynamic reaction forces we shall use the two theorems: theorem of linear momentum under the form of the theorem of motion of the mass center and the theorem of the angular momentum about the fixed point O. We prefer the theorem of the angular momentum about the fixed point O because the two unknown reaction forces are passing through this point and in this way they have not moment about this point resulting therefore the directly the differential equation of the motion.

$$M\overline{a}_C = \Sigma \overline{F}_i \; ; \; \dot{\overline{K}}_O = \Sigma \overline{M}_{Oi}$$

4) Projecting on the two axes we have:

$$\begin{cases} M a_{C\rho} = \Sigma F_{i\rho} ; \\ M a_{Cn} = \Sigma F_{in} ; \\ J_O \,\varepsilon = \Sigma M_{Oi} \end{cases}$$

or in polar coordinates and knowing that moment of inertia of the bar with respect to the point O is equal to:

$$J_O = J_C + M \cdot OC^2 = \frac{Ml^2}{12} + M \cdot \frac{l^2}{4} = \frac{Ml^2}{3}$$

we obtain the equations:

$$\begin{cases} M(\ddot{\rho}_C - \rho_C \dot{\theta}^2) = \Sigma F_{ip}\,; \\ -M(2\dot{\rho}_C \dot{\theta} + \rho_C \ddot{\theta}) = \Sigma F_{in}\,; \\ -\dfrac{Ml^2}{3} \ddot{\theta} = \Sigma M_{Oi} \end{cases}$$

The sign (-) in front of the second and the third equation is because the parameter θ is considered with negative variation (in an right hand system the clockwise rotation sense is considered negative). Removing the position of the mass center (that is constant) and performing the projections on the two axes we have the differential equations:

$$\begin{cases} -M\dfrac{l}{2}\dot{\theta}^2 = Mg\sin\theta - R_{Op} \\ -M\dfrac{l}{2}\ddot{\theta} = R_{On} - Mg\cos\theta \\ -\dfrac{Ml^2}{3}\ddot{\theta} = -Mg\dfrac{l}{2}\cos\theta \end{cases}$$

5) The last equation is the differential equation of the motion of the body:

$$\ddot{\theta} = \frac{3g}{2l}\cos\theta$$

Integrating this equation results:

$$\frac{\dot{\theta}^2}{2} = \frac{3g}{2l}\sin\theta + C$$

The integration constant results from the initial conditions namely:

$$t_o = 0\,; \longrightarrow \theta(0) = 0\,;\ \dot{\theta}(0) = \omega_o = 0 \longrightarrow C = 0$$

We can't integrate analytical this differential equation but as we can see it is enough to know the angular acceleration and velocity function of the position of the bar. With these two quantities we can determine all the elements of the motion for different positions of the bar.

6) For to calculate the velocity of the point A in the vertical position of the bar we can write:

$$v_{A1} = l.\omega_1 = l.\dot{\theta}(\theta_1) = l.\,\dot{\theta}\left(\frac{\pi}{2}\right) = l.\,\sqrt{\frac{3g}{l}} = \sqrt{3gl}$$

7) The two reaction forces are found from the first two differential equations:

$$R_{Op} = Mg\sin\theta + M\frac{l}{2}\dot{\theta}^2$$

$$R_{On} = Mg\cos\theta - M\frac{l}{2}\ddot{\theta}$$

The using of the theorem of kinetic energy in this case is optimal in place of the theorem of angular momentum because the body has one degree of freedom and the hinge is frictionless. If is used the theorem under differential form we obtain the differential equation of the motion by second order and under finite form we obtain the integral of this equation.

Theorem of the kinetic energy under differential form is:

dE = dL

where the kinetic energy is corresponding to the body in rotation:

$$E = \frac{J_O\omega^2}{2} = \frac{\frac{Ml}{3}\dot{\theta}^2}{2} = \frac{Ml^2\dot{\theta}^2}{6}$$

The elementary work can be calculated from one of the ways: for example projecting the force on the direction of the elementary displacement and knowing that only the active forces will have work (the reaction forces from ideal constraints does not produce work because these reaction forces are perpendicular on the elementary displacements or they are acting in fixed points as in a hinged support):

$$dL = F_{ar}.\,dr_C = Mg\cos\theta.\,\frac{l}{2}d\theta$$

Differentiating the kinetic energy and replacing in the expression of the theorem we shall obtain the same differential equation of the motion of the bar.

For to calculate the velocity of the point AA in vertical position of the bar we shall use the same theorem in finite form:

$E_1 - E_o = L_{o1}$

where the kinetic energy in the initial instant is zero because the body starts its motion from rest and the kinetic energy in the vertical position of the bar can be expressed function of the velocity of the point A in that position:

$$\omega_1 = \frac{v_{A1}}{l}; \quad E_1 = \frac{J_O \omega_1^2}{2} = \frac{\frac{Ml^2 v_{A1}^2}{3l^2}}{2} = \frac{Mv_{A1}^2}{6}$$

The work of the weight between the two positions of the body is calculated knowing that the weight is a conservative force and the corresponding force function is:

$$\Delta U_W = \pm mg\Delta y$$

relation in which the sign (+) corresponds to the displacement in the sense of the weight of the point of application of the weight and Δy is the difference of the quota between the two positions. Consequently we have:

$$L_{o1} = +Mg \cdot \frac{l}{2}$$

Removing in the expression of the theorem we obtain the value of the velocity of the point A.

Problem 4. *The bar AB by length $l_{AB} = 2l$ and mass M is in the initial instant in vertical position in rest (non stable equilibrium position) with the end A on a horizontal frictionless surface. At an instant (the initial instant of the motion) the bar becomes unbalanced and it begins to move in vertical plane under the action of its weight. Determine the velocity of the point A when the bar makes 45^o with the vertical direction and also the reaction force in the same point in the same instant.*

Solution. *1) The bar has two degrees of freedom because it has only a simple support in point A that removes one degree of freedom. We shall choose two scalar kinematic parameters for to study the motion. Considering the bar in one any position, the bar has obviously o rotation and because the end A slides on the horizontal surface we shall consider also a displacement in one side of the mass center. In this way the two kinematic parameters will be: the angel θ with which the bar rotates with respect to its initial position and the coordinate x_C of the mass center with respect to a fixed Cartesian reference system with its Oy axis collinear with the initial position of the bar. We remark that the coordinate y_C of the mass center is not an independent parameter, because it can be expressed function of the angular parameter:*

$y_C = l \cdot \cos\theta$

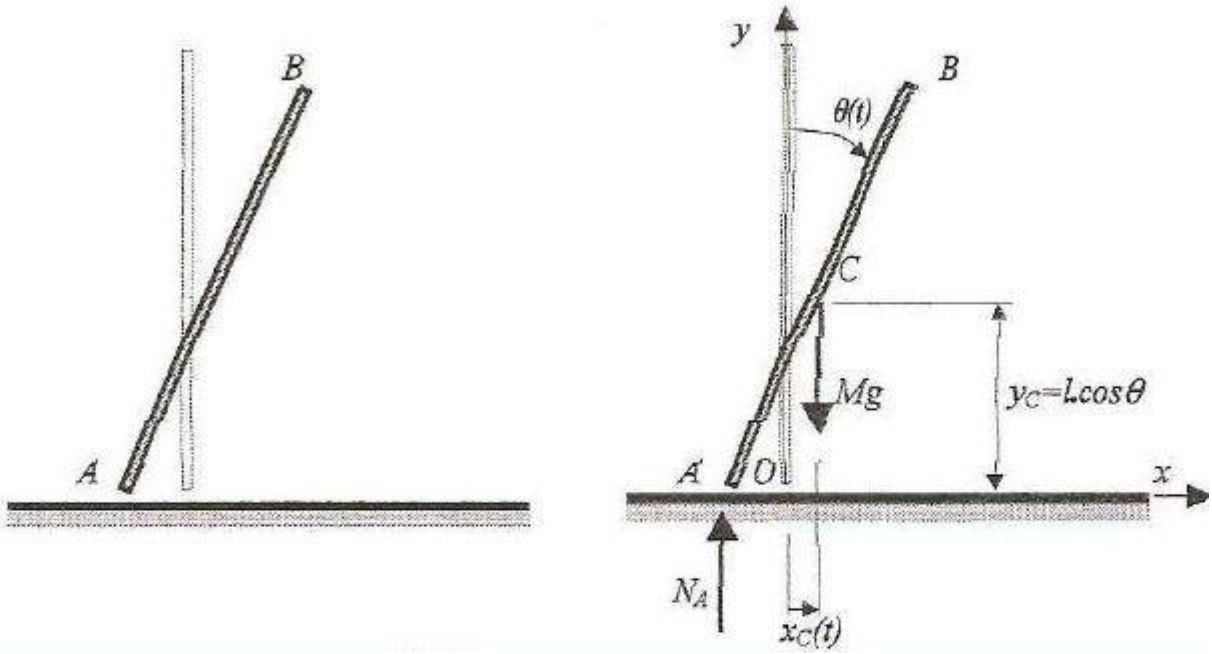


Fig.6.

2) We remove the support from the point A and we replaced with a normal reaction force N_A and representing the weight of the bar is obtained the scheme of the free body for an any position of the bar.

3) We shall use for to study of motion the theorem of linear momentum (under the form of the theorem of motion of the mass center) and the theorem of the angular momentum (about the mass center):

$$M\bar{a}_C = \Sigma \bar{F}_i \, ; \quad \dot{K}_C = \Sigma M_{Ci}$$

or projecting on the two axes:

$$\begin{cases} M\ddot{x}_C = \Sigma X_i \, ; \\ M\ddot{y}_C = \Sigma Y_i \, ; \\ J_C\,\varepsilon = \Sigma M_{Ci} \end{cases}$$

Replacing now function of the kinematic parameters and projecting the forces we have:

$$\begin{cases} M\ddot{x}_C = 0 \, ; \\ -Ml(\theta \sin\theta + \dot{\theta}^2 \cos\theta) = N_A - Mg \\ -\dfrac{M(2l)^2}{12}\ddot{\theta} = -N_A \cdot l \sin\theta \end{cases}$$

4) Substituting the reaction force we separate the two equations as the subsystem of the differential equations of the motion (corresponding to the two degrees of freedom):

$$\begin{cases} \ddot{x}_C = 0 : \\ -l(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) = \dfrac{l}{3\sin\theta}\ddot{\theta} - g \end{cases}$$

5) We remark that these two differential equations are independent and the first equation can be integrated resulting the general solution of this equation:

$$x_C(t) = C_1\, t + C_2$$

The integration constants will be determined from the initial conditions which are:

$$x_C(0) = 0\ ; \dot{x}_C(0) = v_{Co} = 0\ ;$$

because the motion starts from vertical position and rest (there are not initial impulse), the bar being in an unstable equilibrium position any unbalancing state will goes to the motion of the bar.

Results finally the solution of the first differential equation in the initial conditions:

$$x_C(t) = 0$$

This solution shows us that the bar moves so that its mass center of mass falls down on the vertical line. At the other hand this fact can be remarked considering the preservation property of the linear momentum. Remaking that the bar is acted by only two vertical forces, their projections on the horizontal direction are zero so the linear momentum on this direction preserves. But the linear momentum of the body is the linear momentum of the mass center that at initial instant is equal to zero results that the horizontal velocity of the mass center preserves. The initial velocity of the mass center is zero so the mass center will have all the time of motion the same horizontal velocity. The mass center has vertical velocity and the velocity is tangent to the trajectory of the point results that the mass center moves on vertically.

The second differential equation is difficult to integrate because we can't remark the way in which we can integrate. But we may use the following artifice: because from the two differential equations one is integrated and we have its solution that is the law of motion of the mass center this result can be used as a constraint in the motion of the body. Using the fact that the mass center moves on vertical direction (the real state of the body) the bar can be considered in motion with one degree of freedom and the constraints are frictionless, so we can use the theorem of the kinetic energy in finite form resulting the integral of the second differential equation.

The expression of the theorem is:

$$E(\theta) - E_o = L_{ot}$$

The kinetic energy of the bar at the initial instant is equal to zero (the bar starts from rest) and at an instant of the motion it has the expression:

$$E(\theta) = \frac{Mv_C^2}{2} + \frac{J_C\omega^2}{2}$$

because the bar performs a plane motion. In this relation the elements are:

$$v_C = \dot{y}_C = l^2 \dot{\theta}^2 sin^2\theta$$

$$J_C = \frac{M(2l)^2}{12} = \frac{Ml^2}{3}$$

$$\omega = \dot{\theta}$$

The work calculated for the weight of the bar (the normal reaction force being perpendicular on the direction of the displacement in the point of the support has not work) is:

$$L_{o\theta} = Mg\,\Delta y_C = Mg\,l(1 - cos\theta)$$

Now if we replace in the expression of the theorem and if we perform the necessary calculations then we obtain:

$$\dot{\theta}^2 = \frac{6g(1-cos\,\theta)}{l(1+3sin^2\,\theta)}$$

6) Having the expression of the angular velocity of the bar we can determine the velocity of the point A:

$$v_A = \dot{x}_A = \frac{d}{dt}(-l\,sin\theta) = -l\,\dot{\theta}\,cos\theta$$

or removing for the given position we shall obtain:

$$v_A(\frac{\pi}{4}) = 0{,}85\sqrt{gl}$$

7) The reaction force from the point A is found from the second differential equation:

$$N_A = Mg - Ml\,(\ddot{\theta}\,sin\theta + \dot{\theta}^2\,cos\theta)$$

and after we replace the angular velocity corresponding to the given position we have:

$$N_A\left(\frac{\pi}{4}\right) = 0{,}201\,Mg$$

Problem 5. *The bar AB moves so that its ends describe frictionless two straight lines (fig.7.): one horizontal and the other vertical. Knowing that the bar has the length $l_{AB} = 2l$, its mass $M_{AB} = 3M$ and also that the motion starts from vertical position and from rest determine the position in which the bar loses its vertical support.*

Problem 6. *The bar AB by length 2l and mass 2M is joined with two ideal cables by lengths l by two fixed points (Fig.8.). If the bar falls from the horizontal position from rest determine the velocity of the point A when the cables are vertically and the tensions from the cables in the same instant.*

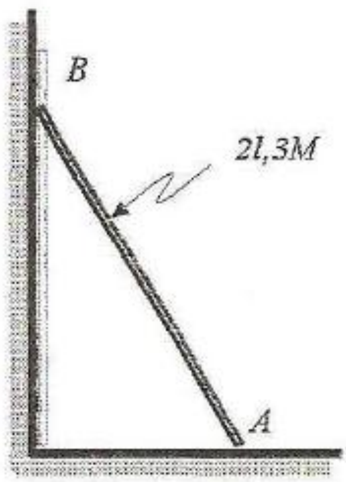


Fig.7.

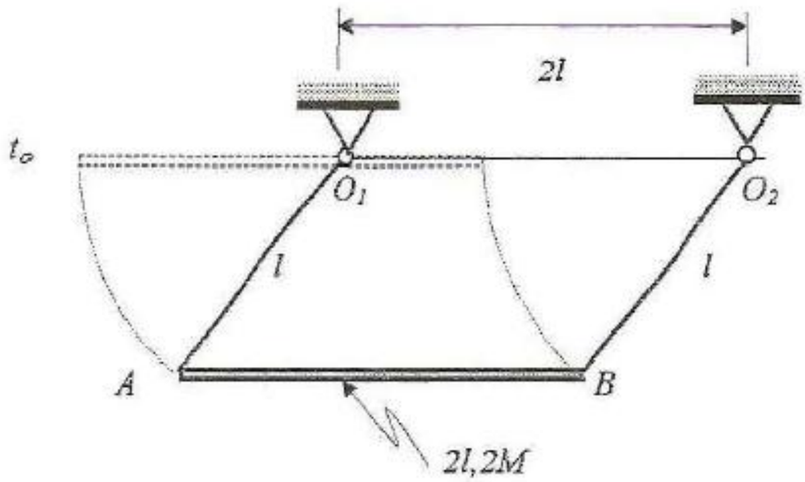


Fig.8.

Chapter 18. Dynamics of the systems

18.1. Introduction.

In this chapter we propose to study the motion of the mechanical systems (of particles or of bodies). As we know a system is an ensemble of particles or rigid bodies between which there are interactions. Namely the motion of a particle or of a body from the system is influenced by the motions of the other particles or bodies from the system.

This propriety of the systems, namely to have always internal connections, makes as the number of the degrees of freedom to be less than the number of the degrees of freedom of the particles or bodies from the system taken separate.

We know that the number of the degrees of freedom is equal to the number of the scalar independent kinematic parameters which define the motion of the system. For a plane system the number of the degrees of freedom will be:

$$N_{df} = 3N_b + N_p - (3N_{fs} + N_{sh} + N_{ss} + N_r)$$

where N_b is the number of bodies, N_p is the number of particles, N_{fs} is the number of fixed supports, N_{sh} is the number of the simple hinges (internal and external), N_{ss} is the number of the simple supports and the internal simple connections and N_r is the number of the restrictions.

We know also that the internal forces from the internal connections are pares, equals and with opposite senses (the principle of the action and the reaction):

$$\overline{F}_{ij} = -\overline{F}_{ji}$$

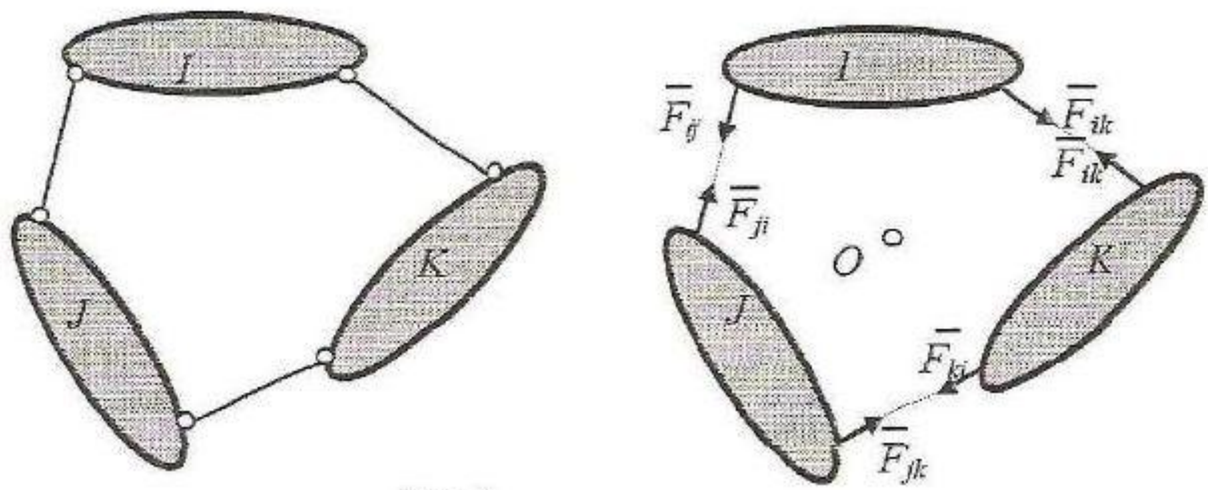


Fig.1.

This propriety of the internal forces makes that the sum of all these forces to be equal to zero and also the resultant moment of these forces about an any point is equal to zero (the equivalent force - couple system of all internal forces is equal to zero):

$$\Sigma\Sigma \overline{F}_{ij} = 0\ ;\ \Sigma\Sigma \overline{M}_O(\overline{F}_{ij}) = 0\ ;\ (i \neq j)$$

Because of this propriety of the internal forces the two vector general theorems (theorem of the linear momentum and the theorem of the angular momentum) if they are used for the entire system then will have the expressions:

$$M\overline{a}_C = \Sigma \overline{F}_{i\,ext}\ ;\ \dot{\overline{K}}_O = \Sigma \overline{M}_{Oi\,ext}$$

namely are considered only the external forces (given and reactions).

Using the propriety of the linear momentum to be a sum of the linear momentums of the components of the system and the theorem of Koenig for the angular momentum we can write these two theorems under the following form:

$$\Sigma M_k \overline{a}_{Ck} = \Sigma \overline{F}_{iext}$$

$$\Sigma \overline{r}_{Ck} \times M_k \overline{a}_{Ck} + \Sigma \overset{\circ}{\overline{K}}_{Ck} = \Sigma \overline{M}_{0i\,ext}$$

where we have marked C_k the mass centers of the body K from the system.

In motions in two dimensions (in plane) the derivative with respect to time of the angular momentum calculated about the mass center of the body is (as we have seen in the previous section):

$$\overset{\circ}{\overline{K}}_{Ck} = J_{Ck}\, \varepsilon_k\, \overline{k}$$

At the other hand the theorem of the kinetic energy has the expression:

$$dE = dL_{ext} + dL_{int}$$

where dL_{int} is the elementary work of the internal forces:

$$dL_{int} = \Sigma\Sigma \overline{F}_{ij} \cdot d\overline{r}_i$$

and dr_i represent the elementary displacements of the points of application of the internal forces.

But if the internal connections of the system are "rigid" connections (namely the relative displacements of the bodies or the particles do not produce), or the connections are ideals (frictionless), or are with friction but the bodies can perform only rolling motions without sliding (in this case the instantaneous relative displacement in the connection point is zero) then the work of the internal forces is zero:

$$dL_{int} = 0$$

18.2. Dynamics of the systems

For the study of the motion of a system of bodies of particles, generally, the use of the two vector theorems considering the entire system (for *to eliminate from the computation of the internal forces) is not enough. This fact results easy from a simple example. Considering a plane system with two bodies with one simple internal connection and without external constraints results that the system will have five degrees of freedom (3 x 2 bodies – 1) but the two theorems in two dimensions for the entire system give us only three scalar independent equations.*

This fact makes that: for to study the motion of a system, generally, we have to divide the system in the component bodies or particles. In this way the solving of the problem of a system will passes the following steps:

- *1) We determine the number of the degrees of freedom and corresponding we choose the independent scalar kinematic parameters. Here we remark that: we shall consider only the active degrees of freedom. The number of the degrees of freedom can be determined or using the previous relation, or considering the simple motions (which are produced with one degree of freedom) and blocking these motions until the entire system is blocked we number of the introduced blockages. Finally the number of the blockages introduced for to stop the motion of the entire system is equal to the number of the degrees of freedom.*

Now we shall choose the reference system that will highlights the main proprieties of the system or of the forces acting about it.

- *2) We make a kinematic study of the system expressing function of the kinematic parameters the accelerations of the particles , the accelerations of the mass centers of the bodies and the angular accelerations of them.*
- *3) The system is divided in the component particles and bodies and all the constraints and internal connections are removed with the equivalent reaction forces and internal forces. Also we evaluate and represent the active (given) forces. Are obtained the free body diagrams (as in statics).*
- *4) For each body or particle we write the differential equations resulted from the two general theorems projected on the axes of the reference system. We remark that for each body or each particle we can choose another convenient reference system. Results a system of scalar differential equations.*
- *5) Using substitutions we divide the system of the differential equations in two subsystems: the first subsystem having the same number of equations as the number of the degrees of freedom and containing*

only the kinematic parameters as unknowns (and obviously their derivatives) and the second subsystem having the same number of equations as the number of the scalar unknown reaction or internal forces containing besides the kinematic parameters also the scalar unknown reaction and internal forces. The first subsystem is the ***subsystem of the differential equations of the motion****.*

- *6) We solve the subsystem of the differential equations of the motion integrating these equations, results the general solution, determining the integration constants from the initial conditions and finally resulting the laws of motion of the system.*
- *7) We study the motion of the system determining the elements of the motion (velocities, accelerations, displacements, trajectories).*
- *8) We determine the dynamic reaction forces and internal forces corresponding to the external constraints and internal connections.*

For the systems which have one degree of freedom and the constraints and connections are frictionless (ideal connections) the motion can be studied with the theorem of the kinetic energy.

As we can see the scheme of solving these kinds of problems is the same as for a constrained rigid body or particle.

18.3. Sample problems.

Problem 1. *Two weights P and Q by the masses 2m and MM are joined between them with an ideal wire (without mass) that passes over a pulley with the center in the fixed point O and mass 6M and radius R. Knowing that at the initial instant the two weights are in rest hanging at the ends of the vertical wires and also the wire does not slides on the pulley determine the acceleration, velocity and displacement of the weight P after two seconds from the beginning of the motion of the system. We shall consider that in the time of the motion the wires remain vertically. Determine also the tensions from the wires and the reaction forces from the hinge O.*

Solution. *1) The system has one degree of freedom because it is made from one body (the pulley) and two particles (the two weights P and Q), has one hinged support in O, two ideal wires (two simple internal connections) and two restrictions (the two wires remain vertically in the time of motion so the two particles P and Q have not horizontal displacements):*

$$N_{df} = 3 \times 1 + 2 \times 2 - (2 \times 1 + 2 + 2) = 1$$

The number of the degrees of freedom can be determined also in the following way: we know that the rotation motion of a body in plane is made with one degree of freedom consequently if we block this motion then is blocked only one degree of freedom. So if we have *blocked the rotation motion of the pulley and knowing that the two weights can move only on vertical direction then in fact we have blocked the entire motion of the system, therefore the system has only one degree of freedom.*

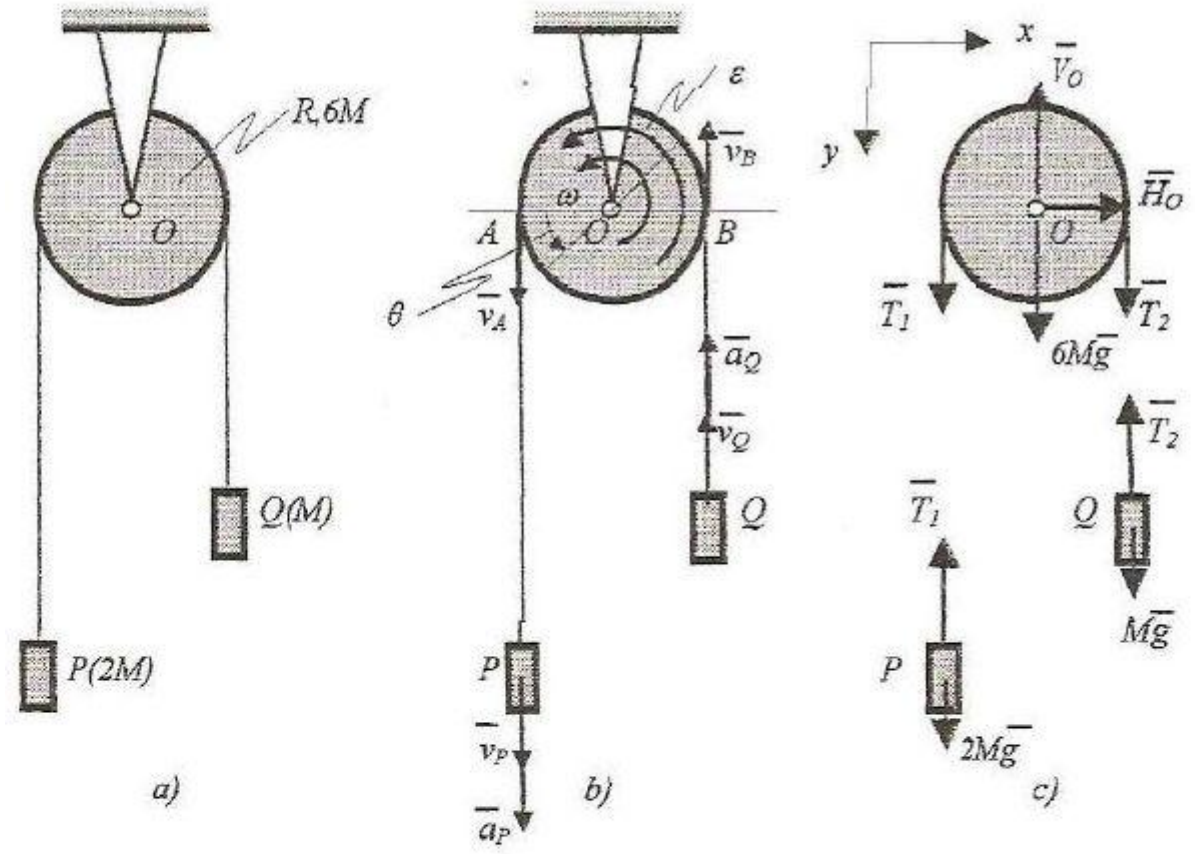


Fig.2

We choose as kinematic parameter of the system's motion the angle θ(t) with which rotates the pulley (the angle between one diameter of the disc and one fixed direction for example the horizontal).

Because we have not any important propriety requiring to choose a certain reference system we shall consider the Cartesian system with the Ox axis horizontal and Oy vertical.

2) We shall make the kinematic analysis using the velocities because we know very well haw to obtain the distribution of velocities in a system with one degree of freedom. We start from the body with known motion namely from the pulley that has rotation motion about its center at which we have defined the kinematic parameter.

The disc performing a rotation motion with the kinematic parameter θ(t) will have the following angular velocity:

$$\omega = \dot{\theta}$$

having the same rotation sense as the sense of variation of the considered parameter. The first derivative with respect to time of the angular velocity is the angular acceleration:

$$\varepsilon = \dot{\omega} = \ddot{\theta}$$

with the same sense of rotation as the angular velocity (because the derivative does not changed the sign)

The fixed point O being the mass center of the pulley has obviously zero acceleration:

$$a_O = 0$$

The motion is transmitted to the two particles through the connections (here the two vertical wires). The ideal wire has the propriety that transmits the velocity with the same intensity direction and sense from one end to the other end. This means that:

$$\bar{v}_A = \bar{v}_P \text{ şi } \bar{v}_B = \bar{v}_Q$$

But these velocities can be calculated from the rotation motion of the disc:

$$v_A = v_B = \omega . R = R\dot{\theta}$$

perpendicular to the OA, or OB and with the same rotation sense about the center O as the angular velocity of the disc.

We take these velocities and we represent them in the two particles P and Q. Because these particles perform rectilinear motions their accelerations will be:

$$a_P = \dot{v}_P = R\ddot{\theta} \; ; \; a_Q = \dot{v}_Q = R\ddot{\theta}$$

keeping the directions and the senses of the velocities from which they are coming.

3) We shall consider the system divided in the component bodies. Each body will be loaded with the two kinds of forces: given forces (here the weights, vertically and directed down), the reaction forces from the external constraints (here the hinged support from O that is replaced with the two components) and the internal forces from the internal connections (here the wires AP and BQ which are replaced with pares of tensions equals and with opposite senses). The scheme of the free body diagrams is the same as in statics and it is represented in the figure 2.c.

4) The solving will be made using the two theorems: for all particles we shall use only the theorem of the linear momentum and for all bodies we shall use the theorem of the motion of mass center and the theorem of angular momentum about the mass center. We shall project these theorems on the directions of the two axes and we find:

-for the particle P:

$$\begin{cases} m_P\, a_{P_x} = \Sigma X_i \\ m_P\, a_{P_z} = \Sigma Y_i \end{cases}$$

-for the particle Q:

$$\begin{cases} m_Q\, a_{Qx} = \Sigma X_i \\ m_Q\, a_{Qz} = \Sigma Y_i \end{cases}$$

-for the pulley:

$$\begin{cases} M_s\, a_{Ox} = \Sigma X_i \\ M_s\, a_{Oz} = \Sigma Y_i \\ J_O\, \varepsilon = \Sigma M_{Oi} \end{cases}$$

or removing the accelerations with the expressions function of the kinematic parameter and calculating the projections of the forces and the moments of the forces we shall obtain:

$$\begin{cases} 0 = 0; \\ 2MR\ddot{\theta} = 2Mg - T_1; \\ 0 = 0; \\ -MR\ddot{\theta} = Mg - T_2\,; \\ 0 = H_O; \\ 0 = 6Mg + T_1 + T_2 - V_O\,; \\ \dfrac{6MR^2}{2}\ddot{\theta} = T_1 . R - T_2 . R \end{cases}$$

that represents one system of five differential equations.

5) From these five equations we shall separate one equation representing the ***differential equation of the motion*** *containing only the kinematic parameter and its derivatives. Also we have other four equations used for to determine the reaction forces and internal forces (V_O, H_O, T_1, T_2).*

From the first equation of the system we express the tension T_1:

$$T_1 = 2Mg - 2MR\ddot{\theta}$$

from the second equation the tension T_2:

$$T_2 = Mg + MR\ddot{\theta}$$

and we shall introduce in the last equation (where first we have simplified with R):

$$3MR\ddot{\theta} = 2Mg - 2MR\ddot{\theta} - Mg - MR\ddot{\theta}$$

or after which we reduce and simplify the terms:

$$\ddot{\theta} = \frac{g}{6R}$$

that is the differential equation of the motion of the system.

6) Integrating this equation we obtain:

$$\dot{\theta} = \frac{g}{6R}t + C_1 \ ; \ \theta = \frac{g}{12R}t^2 + C_1 t + C_2$$

The two integration constants will be obtained from the initial conditions:

$$t_o = 0 \ ; \ \theta_o = \theta(0) = 0 \ ; \ \omega_o = \dot{\theta}(0) = 0$$

and results:

$C_1 = C_2 = 0$

meaning that the law of motion of the system will be:

$$\theta = \frac{g}{12R}t^2$$

7) Knowing the law of motion now it is easy to determine the instantaneous velocity of the particle P at the given instant:

$$v_{P1} = R\,\dot{\theta}(t_1 = 2) = \frac{g}{3}\ (m/s) \approx 3{,}3 m/s$$

The displacement of the particle in two seconds from the beginning of the motion will be:

$$y_{P1} = R\,\theta(t_1 = 2) = \frac{g}{3}\ (m) \approx 3{,}3 m$$

8) The internal forces (the tensions from wires) and the reaction forces will be resulted from the remained four equations:

$$T_1 = 2Mg - 2MR\frac{g}{6R} = 1{,}67\,Mg \ ; \ T_2 = Mg + MR\frac{g}{6R} = 1{,}17Mg \ ;$$
$$H_O = 0 \ ; \ V_O = 6Mg + T_1 + T_2 = 8{,}83\,Mg.$$

Remark 1. *As we have seen before in any situations we can use the two theorems for the entire system. Here because the particularities of the system this is possible (Obvious without to determine the tensions from the wires, which will be determined after what we remove them with the tensions). First we use the theorem of the linear momentum:*

$$\Sigma m_i \overline{a}_{Ci} = \Sigma \overline{F}_i$$

and after the theorem of the angular momentum about the fixed point O:

$$\Sigma \overset{\circ}{K}_{Oi} = \Sigma M_{Oi}$$

Projecting on the two axes and calculating the moments we have:

$$\begin{cases} 0 = H_O; \\ 2M\, a_P - M\, a_Q = 2Mg - V_O + 6\, Mg + Mg \\ 2M\, a_P \,.\, R + J_O\, \varepsilon + M\, a_Q \,.\, R = 2Mg \,.\, R - Mg \,.\, R \end{cases}$$

If we remove the accelerations function of the kinematic parameters the last equation is ***the differential equation of the motion*** *and from the first two equations we can determine the reaction forces from the hinge O.*

Remark 2. *Because the system has only one degree of freedom and the connections and constraints are frictionless the differential equation of the motion can be obtained using the theorem of the kinetic energy. In this case the kinematic analysis uses only the velocities, not the accelerations.*

If we want the differential equation by second order we use the theorem under the differential form:

$$dE = dL$$

where:

$$E = \Sigma E_i = E_P + E_t + E_Q = \frac{m_P v_P^2}{2} + \frac{J_O \omega^2}{2} + \frac{m_Q v_Q^2}{2}$$

For the elementary work of the active forces (given forces) we have:

$$dL = 2Mg \,.\, dy_P - Mg \,.\, dy_Q$$

in which the calculation of the displacements is made knowing that:

$$dy_P = v_P \,.dt \;;\; dy_Q = v_Q \,.dt \;;$$

Replacing in the theorem function of the kinematic parameter we obtain the same differential equation of the motion.

Problem 2. *Calculate the velocity of the piston B when it arrives at the half of the distance between the point O and its initial position corresponding to instant when the rod and the crank are horizontal. Are known:* $l_{OA} = l_{AB} = 2l$, $M_{OA} = M_{AB} = 3M$, $M_B = 2M$, $P = 4Mg$.

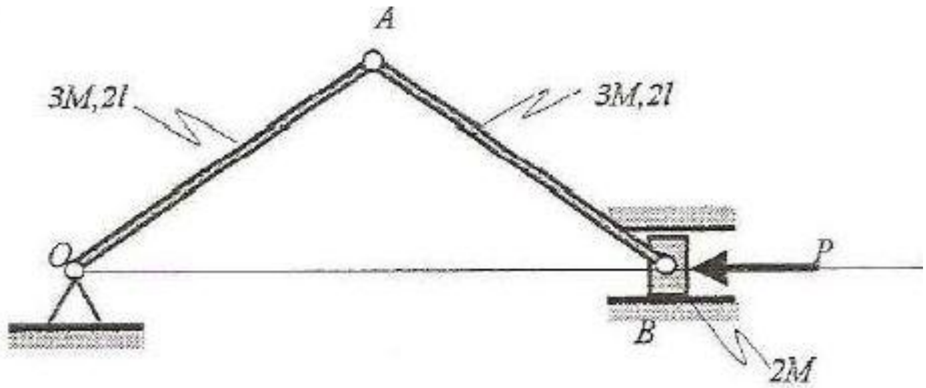


Fig.3.

Problem 3. *Calculate the acceleration of the particle P and the tensions from the wires for the system from the figure 4. In the time of motion the wires remain vertically.*

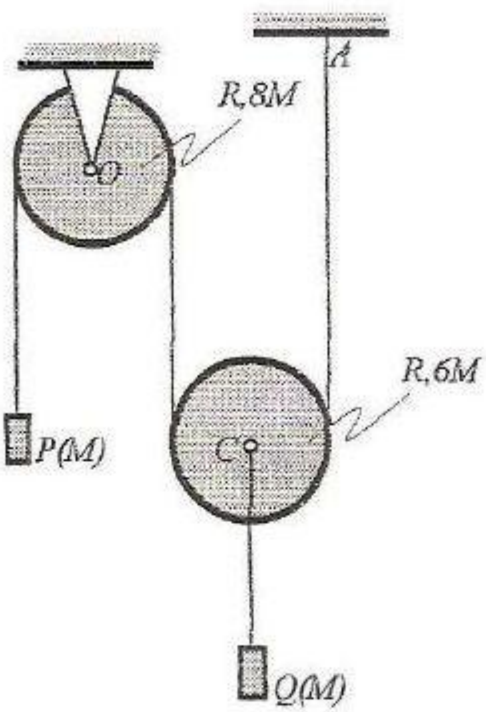


Fig.4.

Problem 4. *Determine the differential equations of the motion for the system represented in the figure 5. Are known: the mass of the collar* $m_P = 2M$, *the mass of the bar* $M_{PA} = 6M$, *the length of the bar* $l_{PA} = 2l$, *the elastic constant of the spring k and the initial length of the* *spring l_o. The collar moves on the horizontal surface without friction and the bar oscillates in vertical plane.*

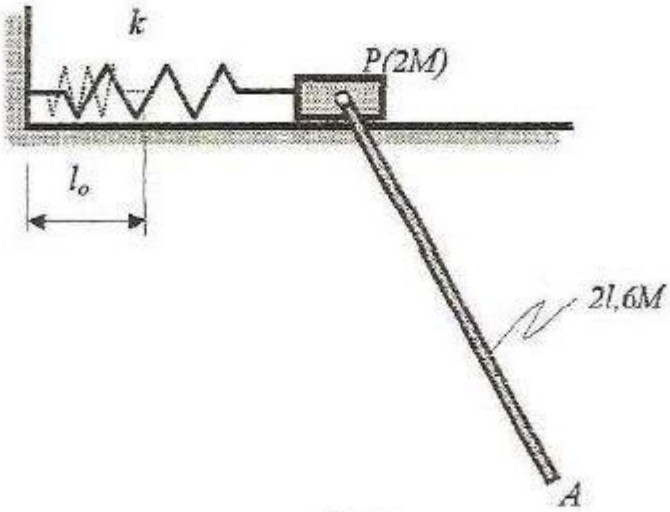


Fig.5.

Problem 5. *Calculate the velocity of the point A when the bar O_1A ia vertically knowing that the motion of the system is performed in vertical plane. The motion starts from the horizontal position of the bar O_1A and from rest.*

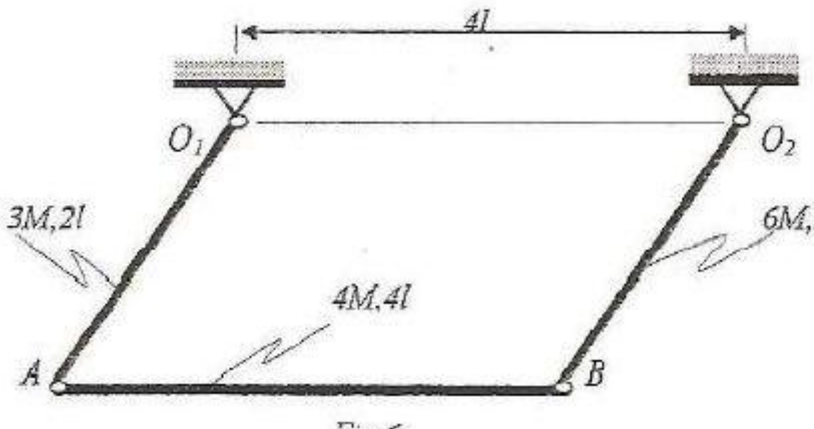


Fig.6

ANALYTICAL MECHANICS

Chapter 19. Generalities.

19.1. Introduction.

The theoretical mechanics studies the motion of a mechanical system (in particular the equilibrium) starting from the fundamental principles of Newton. Based on these principles in theoretical mechanics we develop the general theorems (in particular the conditions of equilibrium) which are used for to study the motions of the mechanical systems.

The major deficiency of the using of these theorems is that the reaction forces or the internal forces corresponding to the external constraints and the internal connections are unknowns (sometimes the internal forces can be eliminated by the general theorems) and make difficult to solve the problems of the motion from the differential equations.

Another deficiency of the using of the general theorems is that the differential equations are very different one to other (we can use three different theorems the equations are projections of forces, or moments about different points, etc.).

These two deficiencies will be eliminated in this chapter in the analytical mechanics. Therefore this part of the mechanics will have as purpose ***to eliminate the reaction forces and internal forces*** *from the equations of motion and to obtain the* ***same kinds of equations*** *(the equations to have the same form with the same number of terms and the term to obtain in the same way).*

The analytical mechanics for to achieves these purposes will develop its methods starting from the other principles as from the theoretical mechanics (but using these principles for to define the state of the mechanical system). Because these principles are mathematical principles this mechanics is called ***analytical mechanics****. In fact in this part we will change the order of the study. In theoretical mechanics first we have defined the state of the system using the fundamental principles and in the second step we have used the mathematical tools for to obtain the motion of the mechanical system. In the analytical mechanics first we will define analytical principles and in the second step we shall use the fundamental principles for to find the motion of the mechanical system.*

The analytical principles used in this part are divided in two categories: ***differential principles*** *which study the mechanical motion at an instant of the motion in an infinitesimal interval of time, and* ***integral principles*** *which study the motion in a finite interval of time.*

The differential principles are: ***D'Alembert's principle, the principle of virtual work*** *and* ***principle of Gauss.***

The integral principles are: ***Hamilton's principle*** *and* ***Maupertui's principle****.*

From these five principles we shall study in this part of the mechanics only two principles: D'Alambert's principle and principle of the virtual work.

Because these two principles are differential principles and because in this part we shall use a few new notions we have to make some specifications about the constraints, about the positions and the state of the mechanical system and about the displacements in the conditions of the differential principles.

19.2. Constraints.

In the vector mechanics (the theoretical mechanics) the constraints were classified about more criteria. We remind that the constraints in theoretical mechanics are: simple constraints (removing one degree of freedom) or multiple (removing more degrees of freedom), unilateral (eliminating the displacement in one sense of a direction) or bilateral (eliminating the possibilities of displacements in the both senses of a direction) *and also they can be ideal constraints (punctual and frictionless) or real (with friction).*

In the analytical mechanics all the constraints will be considered ***simple, bilateral*** *and* ***ideal constraints****. We can see that the generality of the problem is not affected because: the multiple constraints can be considered as combinations of the simple constraints, then one unilateral constraint how long is working can be considered as a bilateral constraint so it or is working as a bilateral constraint or does not exist, and finally if the constraint is real (has friction force) the friction force can be considered as a given force and in this way the constraint can be considered as an ideal constraint.*

Results that in the analytical mechanics we have to make another classification of the constraints. This classification will be made function of the analytical behavior of them.

Thus one simple, bilateral and ideal constraint can be expressed under finite form (in Cartesian reference system) as a relation between the coordinates of the connection point (as in statics) and represents the condition that the point of connection to be located on a surface (the equation of the constraint):

$$f(x,y,z) = 0$$

Under differential form the constraint will have the following expression:

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0$$

representing the differential of the previous relation and the condition as the infinitesimal displacement on the direction of the normal direction to the surface that is the simple constraint.

Without going into details we shall give a new classification of the constraints in analytical mechanics giving examples with the finite form of the constraint.

Besides the coordinates of the connection point in the expression of a constraint may interfere the projections of the velocities and *also the parameter time. This is because the constraint can be deformable and in motion. We remark that in statics we have used fixed and "rigid" constraints.*

Considering all these the constraint can be: ***holonomy constraint*** *if the expression does not depend directly by time:*

$$f(x,y,z) = 0, \quad or \quad f(x,y,z,\dot{x},\dot{y},\dot{z}) = 0$$

and ***non-holonomy constraint*** *if the expression depends directly to the time:*

$$f(x,y,z,t)= 0, \quad or \quad f(x,y,z,\dot{x},\dot{y},\dot{z},t)= 0$$

The constraints also can be classified in: ***scleronomy constraint*** *if the expression of it does not contain the components of the velocity:*

$$f(x,y,z) = 0, \quad or \quad f(x,y,z,t) = 0$$

and ***reonomy constraint*** *if the expression depends by the components of the velocity:*

$$f(x,y,z,\dot{x},\dot{y},\dot{z})=0, \quad or \quad f(x,y,z,\dot{x},\dot{y},\dot{z},t)=0$$

In this part of the mechanics we shall use only holonomy and scleronomy constraints namely the "rigid" constraints, therefore the same constraints as in statics.

19.3. Generalized coordinates, generalized velocities.

For to explain these new notions we shall consider a system of ***n*** *particles P_i (I = 1,…,n). If the system is made from free particles then the position of the system, with respect to a reference system, for example the Cartesian system, will be defined by the coordinates:*

$$x_1,y_1,z_1,x_2,\ldots,z_{n-1},x_n,y_n,z_n$$

*Consequently the position of the system in any instant of the motion is defined by **3n** independent coordinates, generally by **3n** scalar independent parameters (because we can use other system of reference or other position parameters). In the same time the number of the degrees of freedom is also **3n**.*

*Suppose now that the system has **l** simple ideal constraints or connections. As we know each simple connection or constraint eliminates one degree of freedom consequently eliminates one independent scalar position parameter from the definition of the position of the system (through the relation that we write between the coordinates). This means that the number of the scalar independent position parameters which define the position of the system (as the number of the degrees of freedom) will be:*

$$s = 3n - l$$

*The **s** independent position parameters will be renamed and renumbered and we can write:*

$$q_1, q_2, q_3, \ldots, q_{s-1}, q_s$$

*These **s** scalar independent position parameters are called **generalized coordinates**. These coordinates can have different natures (length or angle).*

***Remark.** If we imagine a fictional space with **s** dimensions, then the **s** generalized coordinates can be considered as defining the position of one fictional free particle in this space. We remark that if we can define the proprieties of this free particle then we have eliminated from the study the reaction forces from the constraints and the internal forces from the connections. This space is called **configurations space**.*

With this marks the position of any particle from the mechanical system will be defined in the following way:

$$\bar{r}_i = \bar{r}_i(q_1, q_2, \ldots, q_s)$$

*By analogy with the knowledge from kinematics the first derivatives with respect to time of the generalized coordinates are called **generalized velocities**:*

$$\dot{q}_1, \dot{q}_2, \dot{q}_3, \ldots, \dot{q}_{s-1}, \dot{q}_s$$

Using these velocities the velocity of a particle from the system will be:

$$\overline{v}_i = \dot{\overline{r}}_i = \frac{\partial \overline{r}_i}{\partial q_1}\dot{q}_1 + \frac{\partial \overline{r}_i}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial \overline{r}_i}{\partial q_s}\dot{q}_s + \frac{\partial \overline{r}_i}{\partial t}$$

19.4. Displacements

Here we shall show a few elements joined to the displacements considered in analytical mechanics. We make the remark that due to we shall study only the first two differential principles in this part of the mechanics we shall refer only about the infinitesimal displacements. Also we shall consider only holonomy and scleronomy constraints.

The displacements will be explained for a particle but after will be extended for the rigid body and the systems of particles and bodies.

In the analytical mechanics we shall use three kinds of displacements: possible displacements, real displacements and virtual displacements.

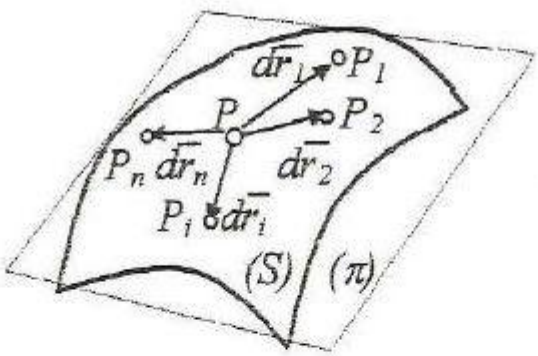


Fig.1.

The possible displacement *is a change of the position of the particle that might produce in any conditions. For example if about a particle should act one any force in a short interval of time then the particle modifies its position. This change depends by the force, or the system of forces, and the* *interval of time. It is obviously that we have an infinity possibilities to produce the displacement of the particle. The common characteristics of all possible displacements are that they are infinitesimal and compatibles with the constraints of the particle. This last propriety means that all the possible displacements are situated in the tangent plan to the surface representing the simple constraint.*

The real displacement *is the displacement (here infinitesimal) performed by the particle in an interval of time* **dt** *under the action of a given force (or system of forces).*

This displacement is one of the possible displacements and it has the following proprieties:

- *it is infinitesimal;*
- *it is compatible with the constraints;*
- *it is produced by forces;*
- *it is produced in time.*

The real displacement is marked $\mathbf{d\overline{r}}$ *and it is a variation of the position vector of the particle in the interval of time* **dt.** *From mathematical point of view this displacement is the differential of the position vector.*
In Cartesian reference system this displacement has the expression:

$$d\overline{r_i} = dx_i\,\overline{i} + dy_i\,\overline{j} + dz_i\,\overline{k}$$

If the position vector is expressed function of the generalized coordinates (for holonomy and scleronomy constraints) then the expression of the real displacement of any particle from a mechanical system will be:

$$d\overline{r_i} = \sum_{k=1}^{s} \frac{\partial \overline{r_i}}{\partial q_k} dq_k$$

The virtual displacement *is an* ***imaginary*** *displacement. This displacement is considered with the following proprieties:*

- *it is infinitesimal;*
- *it is compatible with the constraints and connections;*
- *it is instantaneous that is not produced in time;*
- *it is not produced by forces, it is imaginary;*

- *it has arbitrary magnitude, direction and sense.*

From these proprieties results that the virtual displacement is one of the possible displacements, and because it is imaginary when it is convenient we can take it equal to the real displacement.

The virtual displacement is marked ***δr*** *where δ is a differential operator that consider the parameter time as a constant.*

In Cartesian reference system the virtual displacement of an any point of the mechanical system will be:

$$\delta \overline{r_i} = \delta x_i \overline{i} + \delta y_i \overline{j} + \delta z_i \overline{k}$$

where δx_i, δy_i, δz_i *are the virtual variations of the coordinates of that point.*

If we use the generalized coordinates then we have:

$$\delta \overline{r_i} = \sum_{k=1}^{s} \frac{\partial \overline{r_i}}{\partial q_k} \delta q_k$$

Chapter 20. D'Alembert's principle.

20.1. The inertia force.

For the beginning suppose a free particle P by mass ***m*** *acted by the force* $\overline{\mathbf{F}}$ *and performing a motion with the instantaneous acceleration* $\overline{\mathbf{a}}$*. Based on the second principle of mechanics (or the theorem of the linear momentum) we may write:*

$$\overline{F} = m\overline{a}$$

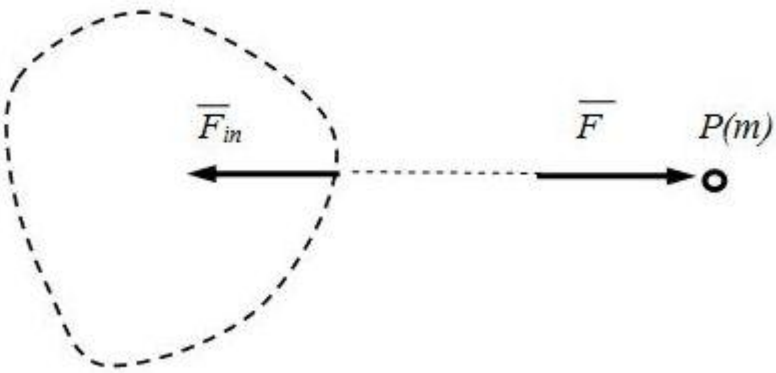


Fig.1.

In accordance with the third principle of mechanics (principle of action and the reaction) the particle, about which is acting the force F, will act about the system that has produced this force with a force equal in magnitude but with opposite sense. This force marked:

$$\overline{F}_{in} = -m\overline{a}$$

*is called **inertia force** and represents the answer of the particle to the action of the force $\overline{\boldsymbol{F}}$ about the particle P by mass **m**.*

Remark: as we can see this inertia force does not act about the particle P, this acts about the mechanical system that have produced the force $\overline{F}$.

20.2. D'Alembert's principle

*Suppose a particle P by mass **m** acted by a system of active forces) given forces) having the resultant force $\overline{\boldsymbol{F}}$, having a set of simple ideal constraints and performing a motion with the instantaneous acceleration $\overline{\boldsymbol{a}}$. According to the axiom of constraints the particle can be considered as a free particle if the constraints are replaced with reaction forces. We mark the resultant of the reaction forces with $\overline{R}$.*

The Newton's second principle will give us the relation:

$$\overline{F} + \overline{R} = m\overline{a}$$

or binging all the terms in the left part:

$$\overline{F} + \overline{R} + (-m\overline{a}) = 0$$

and marking the inertia force we obtain finally:

$$\overline{F} + \overline{R} + \overline{F}_{in} = 0$$

*This last relation expresses the **D'Alembert's principle: The active forces, the reaction forces and the inertia forces form a system in equilibrium.***

Here we have to make an important remark: the inertia force is not a real force for the particle P because as we have seen in the previous section this force acts about the system (or the body) that have produced the active forces and not about the particle. In conclusion the inertia force for the particle is a fictional force for the particle. This means that the equilibrium *expressed by the previous relation (D'Alembert's principle) is also a* ***fictional equilibrium*** *called* ***dynamic equilibrium****.*

If we suppose now a system of particles P_j , for each particle we may write the same kind of relation expressing the dynamic equilibrium:

$$\bar{F}_j + \bar{R}_j + \bar{F}_{inj} = 0$$

Summing all these relations and computing the resultant moment about one any fixed point we shall find the relations:

$$\Sigma\bar{F}_j + \Sigma\bar{R}_j + \Sigma\bar{F}_{inj} = 0$$
$$\Sigma\bar{M}_O(\bar{F}_j) + \Sigma\bar{M}_O(\bar{R}_j) + \Sigma\bar{M}_O(\bar{F}_{inj}) = 0$$

that expresses the dynamic equilibrium of the three kinds of forces acting about the system of particles.

Any rigid body can be considered as a continuous and rigid (non-deformable) system of particles by elementary masses ***dm*** *so the dynamic equilibrium will be expressed by the same kinds of relations but the inertia forces are acting in each points of the body and consequently we have to determine the force - couple system of the inertia forces acting about a rigid body.*

20.3. The force - couple system of the inertia forces.

The dynamic equilibrium in the case of a rigid body will have the expression:

$$\Sigma\bar{F}_j + \Sigma\bar{R}_j + \int_D(-\bar{a}dm) = 0$$
$$\Sigma\bar{M}_O(\bar{F}_j) + \Sigma\bar{M}_O(\bar{R}_j) + \int_D \bar{r}\times(-\bar{a}dm) = 0$$

where the last terms form the force – couple system of the inertia forces in the fixed point O.

If we express the motion of the body using the two vector theorems (theorem of the linear momentum and the theorem of the angular momentum) we shall obtain the following two vector equations:

$$\Sigma \overline{F}_j + \Sigma \overline{R}_j = \dot{\overline{H}}$$
$$\Sigma \overline{M}_O(\overline{F}_j) + \Sigma \overline{M}_O(\overline{R}_j) = \dot{\overline{K}}_O$$

Comparing the two pairs of vector equations (those resulted from D'Alembert's principle and those from the two general theorems) we obtain the two vectors of the force – couple system of the inertia forces:

$$\tau_O(\overline{F}_{in}) = \begin{cases} \overline{R}_{in} = \int_D (-\overline{a}dm) = -\dot{\overline{H}} \\ \overline{M}_{Oin} = \int_D \overline{r} \times (-\overline{a}dm) = -\dot{\overline{K}}_O \end{cases}$$

Expressing function of the motion of the mass center of the body we may write:

$$\tau_O(\overline{F}_{in}) = \begin{cases} \overline{R}_{in} = -M\overline{a}_C \\ \overline{M}_{Oin} = -\overline{r}_C \times M\overline{a}_C - \dot{\overline{K}}_C \end{cases}$$

The second term of the resultant moment of the inertia forces is called ***inertia couple in the mass center****:*

$$\overline{C}_{inC} = -\dot{\overline{K}}_C$$

In the plane problems this couple has the expression:

$$C_{inC} = -J_C\,\varepsilon$$

Now we cam remark that the D'Alembert's principle represents in fact the use of the two theorems bringing the right terms in the left part, renaming them and in this way transforming the differential equations in equilibrium equations.

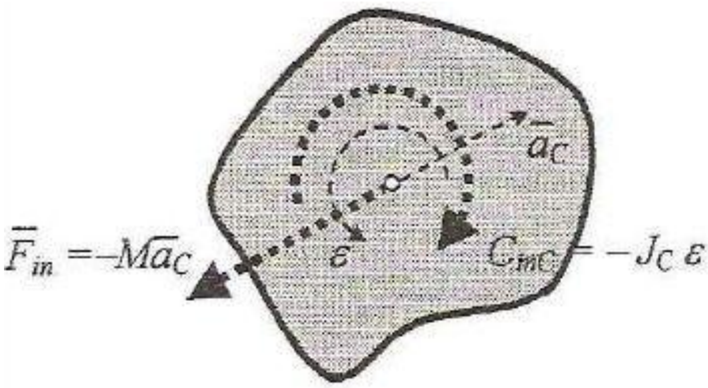


Fig. 2.

20.4. Kineto – static method.

Using the principle of D'Alembert we can obtain a method of study the motion of the mechanical systems through equilibrium equations. We shall make in the following way (as we can see the steps to solve does not essentially differ by the steps in the case of using the general theorems):

-1) We determine, first, the number of the degrees of freedom and we choose the kinematic parameters corresponding to those.

-2) We make the kinematic study of the system expressing the accelerations of the particles, the accelerations of the mass centers of the bodies and the angular accelerations of the bodies function of the chosen kinematic parameters We shall represent these accelerations.

-3) We decompose the system in the component bodies and each particle and body will be loaded with the three kinds of forces: active forces (given forces), reaction and connection forces and the force couple system of the inertia forces having the senses opposite as the senses of the accelerations.

-4) We choose a reference system for each particle and each body and we write the equilibrium equations (for each particle two projection equations on the two axes of the reference system and for each body three equations: two projections and a moment equation about a fixed point or about

the mass center of the body). we remark that the moment equations are ***only about the fixed points or about the mass center*** *of the corresponding body.*

-5) The resulted differential equations are divided in two subsystems: one subsystem is of the ***differential equations of the motion*** *(containing the same number of equations as the number of the degrees of freedom and having as unknowns only the kinematic parameters) and one subsystem of differential equations used for to determine the reaction forces corresponding to the external constraints and the internal connections.*

-6) We integrate the differential equations of the motion obtaining finally the laws of motion of the system (function of the initial conditions).

-7) Having the laws of motion we study the motion of the system.

-8) From the second subsystem we determine the reaction forces.

20.5. Sample problems.

Problem 1. *One disc by mass M and radius R has wrapped on its periphery one ideal wire and its and is fixed in the point A. Knowing that at the initial instant the wire is vertically and the disc is in rest calculate the acceleration of the mass center of the disc, the tension in the wire at an any instant of the motion. We know also that in the time of motion the wire remains vertically.*

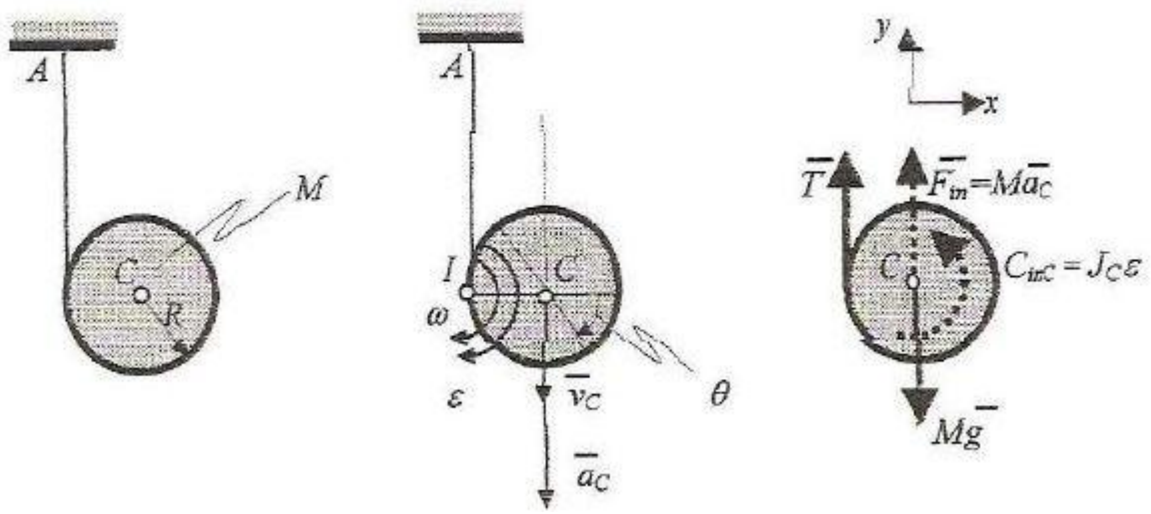


Fig.3.

Solution. *1) We have one single body that has only one degree of freedom because the wire removes one degree of freedom and the restriction as the body does not oscillate (the wire remains vertically) eliminates another degree of freedom.*

For the study of the motion we shall choose as kinematic parameter the rotation of one any diameter of the disc with respect to its initial position (which will be considered horizontally).

2) We shall determine now the acceleration of the mass center of the disc and its angular acceleration function of the kinematic parameter. Because the vertical wire is fixed in the fixed point A this means that the other end of this wire (point I the tangent point between the wire and the disc) will has the same velocity as the point A:

$$v_I = v_A = 0$$

consequently the point I is the instantaneous center of rotation of the disc.

The angular velocity of the disc will result (from the definition of the angular velocity):

$$\omega = \dot{\theta}$$

and will have the same sense of rotation as the sense of rotation of the disc, namely the clockwise sense.

Deriving again the angular velocity is obtained the angular acceleration of the disc:

$$\varepsilon = \dot{\omega} = \ddot{\theta}$$

that will keep the sense of the angular velocity (because through derivation the sense does not changed.

The velocity of the mass center will be:

$$v_C = IC \cdot \omega = R\dot{\theta}$$

having vertical direction and with the sense directed down (this the sense of rotation of the angular velocity about the instantaneous center of rotation). Because the center C performs a rectilinear motion the acceleration of this point will have the magnitude:

$$a_C = \dot{v}_C = R\ddot{\theta}$$

with the same direction and sense as its velocity.

3) We remove the constraint (here the wire) and load the disc with: the weight of the disc (vertical and acting in the mass center C) the tension from the wire (vertical and directed up in the tangent point between the wire and the disc) and the force-couple system of the inertia forces made from the inertia force with the magnitude:

$$F_{in} = M\,a_C = M\,R\,\ddot{\theta}$$

having the direction of the acceleration of the mass center but with opposite sense and the inertia couple in the mass center having the magnitude:

$$C_{inC} = J_C\,\varepsilon = \frac{MR^2}{2}\ddot{\theta}$$

with opposite sense as the angular acceleration of the disc.

4) The dynamic equilibrium equations will be:

$$\Sigma X_i = 0\ ;\ 0 = 0\ ;$$
$$\Sigma Y_i = 0\ ;\ T + F_{in} - Mg = 0;$$
$$\Sigma M_{Ci} = 0\ ;\ C_{inC} - T\,.\,R = 0$$

5) Removing the magnitudes of the inertia force and inertia couple and eliminating the tension from the two equations results the differential equation of the motion (also simplifying the resulted equation):

$$\ddot{\theta} = \frac{2g}{3R}$$

6) We remark that for to determine the acceleration of the mass center is not necessary to integrate the differential equation of the motion but only to remove the angular acceleration:

$$a_C = R\,\ddot{\theta} = \frac{2g}{3} \approx 6\ m/s^2$$

7) The tension from the wire will result the angular acceleration in the second equation:

$$T = \frac{MR}{2}\ddot{\theta} = \frac{Mg}{3}$$

Problem 2. *The system from the figure 4. is made from three rods hinged among them and the environment in the two hinged supports O and C. The system performs motions in vertical plane under the action of the weights of the bodies. Knowing that the motion begins from the horizontal position of the rod OA and from rest calculate the velocity of the point A when the bar Oa becomes vertically and also the reaction forces from the two external hinges in the same instant. Are known* $M_{OA} = M_{BC} = 3M$, $M_{AB} = 4M$, $l_{OA} = l_{BC} = 2l$, $l_{AB} = 3l = AC$.

Solution. *1) The system of bodies has one degree of freedom because:*

$$N_{df} = 3 \cdot 3 - (2 \cdot 4) = 1$$

Also we remark that the bars OA and BC perform rotation motions (the rotation motion is performed always in plane with one degree of freedom). If we block the motion of one bar4 , from two, then the entire motion of the system is blocked consequently the system has one degree of freedom.

We shall choose as kinematic parameter of the motion the angle between the horizontal direction (the initial direction of the bar OA) and the direction of the bar OA at an any instant.

2) The system being made from three bodies we shall determine function of the chosen parameter the angular accelerations of the bodies and the accelerations of the centers of them. In this way the angular accelerations will be:

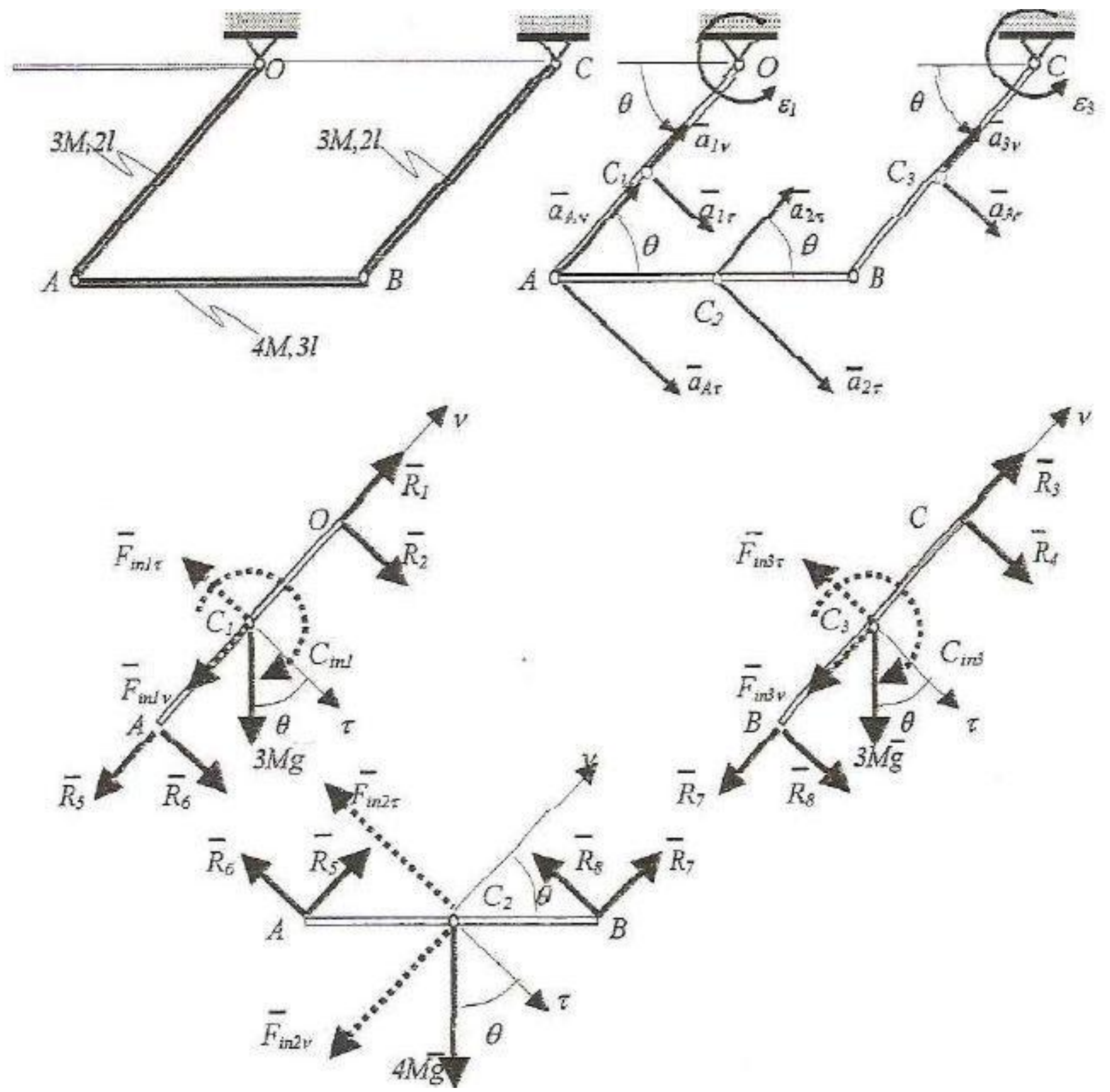


Fig.4.

$$\varepsilon_1 = \varepsilon_{OA} = \ddot{\theta};\ \varepsilon_2 = \varepsilon_{AB} = 0\ ;\ \varepsilon_3 = \varepsilon_{BC} = \ddot{\theta}$$

because for the bars OA and BC (parallel in the time of motion), by definition, the angular acceleration is the second derivative with respect to time of the angle between an any line from the body and a fixed straight line from the plane, and the bar AB performs a translation motion. The senses of the two angular accelerations are the same with the supposed sense of the motion of the bars (because we have not changed the sign through the derivation).

The mass centers C_1 and C_3 of the bars OA and BC perform circular motions and consequently their accelerations will have two components, one tangent component with the magnitude:

$$a_{1\tau} = a_{3\tau} = OC_1 \,.\, \varepsilon_1 = CC_3 \,.\, \varepsilon_3 = l.\ddot{\theta}$$

having the directions perpendicular on OC_1 and CC_3 and with the same sense of rotation about the points O and C as the angular accelerations.

The normal components of the accelerations of the mass centers will have the magnitudes:

$$a_{1\nu} = a_{3\nu} = OC_1 \,.\, \omega_1^2 = CC_3 \,.\, \omega_3^2 = l.\dot{\theta}^2$$

having the directions of the two bars and with the senses directed toward the rotation centers O and C.

The bar AB performs a translation motion (it remains parallel to itself in the time of motion) and consequently the acceleration of the center C_2 is equal to the acceleration of any point from the bar. So we shall calculate for example the acceleration of the point A and after will transfer this acceleration in the point C_2. In this way the acceleration of the point A (performing a circular motion) has two components:

$$a_{A\tau} = a_{2\tau} = OA \,.\, \varepsilon_1 = 2l \,.\, \ddot{\theta}$$
$$a_{A\nu} = a_{2\nu} = OA \,.\, \omega_1^2 = 2l \,.\, \dot{\theta}^2$$

having the same directions and senses as the components of the accelerations of the mass centers of the two bars OA and BC.

3) We undo the system in three bodies and we shall load each body with the three kinds of loads: given forces (active forces, here the weights acting in the mass centers of the bars having vertical directions and directed down), reaction forces corresponding to the removed constraints and connections (in the two hinged supports O and C two components an two convenient directions, here the directions of the bars and on perpendicular directions on the bars, and in internal hinges two pairs of two unknown internal force having the same directions but opposite senses and an the same directions as in the hinges O and C) and the force – couple systems of the inertia forces. For the two bars in rotation motion the force – couple system of the inertia forces is made from the inertia force with two components with the magnitudes:

$$F_{in1\tau} = F_{in3\tau} = 3M \cdot a_{1\tau} = 3M \cdot a_{3\tau} = 3M \cdot l \cdot \ddot{\theta}$$
$$F_{in1\nu} = F_{in3\nu} = 3M \cdot a_{1\nu} = 3M \cdot a_{3\nu} = 3M \cdot l \cdot \dot{\theta}^2$$

with the same directions as the corresponding components of the accelerations but with opposite senses , and the inertia couples in the mass centers with the magnitudes:

$$C_{in1} = C_{in3} = J_{C1} \cdot \varepsilon_1 = J_{C3} \cdot \varepsilon_3 = \frac{3M(2l)^2}{12}\ddot{\theta} = Ml^2\ddot{\theta}$$

and having opposite senses as the corresponding angular accelerations.

For the bar AB the force – couple system of the inertia forces contains only the inertia force (the angular acceleration is equal to zero) having two components with the magnitudes:

$$F_{in2\tau} = 4M \cdot a_{2\tau} = 4M \cdot 2l \cdot \ddot{\theta} = 8M \cdot l \cdot \ddot{\theta}$$
$$F_{in2\nu} = 4M \cdot a_{2\nu} = 4M \cdot 2l \cdot \dot{\theta}^2 = 8M \cdot l \cdot \dot{\theta}^2$$

with the directions of the corresponding accelerations and with opposite senses.

4) For each body we shall write three equilibrium equations: two projections on the axes of a convenient reference system, here the Frenet's system with the origin in the mass center of the body and a moment equation about the mass center for the body AB and about the fixed points O and C for the other two bodies. We choose the fixed points for the bodies OA and BC because about these points we have moment for a single reaction force and the removal operations will be simplified.

We have for the body OA the following dynamic equilibrium equations:

$$\Sigma F_{i\tau} = 0\ ;\ R_2 + R_6 - 3Ml\ddot{\theta} + 3Mg\cos\theta = 0;$$
$$\Sigma F_{i\nu} = 0\ ;\ R_1 - R_5 - 3Ml\dot{\theta}^2 - 3Mg\sin\theta = 0;$$
$$\Sigma M_{Oi} = 0\ ;\ R_6 . 2l - 3Ml\ddot{\theta} . l - Ml^2\ddot{\theta} + 3Mg . l\cos\theta = 0;$$

For the body AB the equations will be:

$$\Sigma F_{i\tau} = 0\ ;\ -R_6 - R_8 - 8Ml\ddot{\theta} + 4Mg\cos\theta = 0;$$
$$\Sigma F_{i\tau} = 0\ ;\ R_5 + R_7 - 8Ml\dot{\theta}^2 - 4Mg\sin\theta = 0;$$
$$\Sigma M_{C2i} = 0\ ;\ -R_5 . 2l\sin\theta + R_7 . 2l\sin\theta - R_6 . 2l\cos\theta + R_8 . 2l\cos\theta = 0;$$

And finally for the body BC the equations are:

$$\Sigma F_{i\tau} = 0 \; ; \; R_4 + R_8 - 3Ml\ddot{\theta} + 3Mg\cos\theta = 0;$$
$$\Sigma F_{i\nu} = 0 \; ; \; R_3 - R_7 - 3Ml\dot{\theta}^2 - 3Mg\sin\theta = 0;$$
$$\Sigma M_{O_1} = 0 \; ; \; R_8 . 2l - 3Ml\ddot{\theta} . l - Ml^2\ddot{\theta} + 3Mg . l\cos\theta = 0.$$

5) In this way we have obtained a system of nine differential equations which contain the eight reaction forces from the constraints and connections and the kinematic parameter. The differential equation of the motion of the system of bodies will be obtained through substitutions. From the third equation is removed R_6, from the last equation the reaction force R_8 function of the kinematic parameter and they are introduced in the fourth equation from that results the differential equation of the motion:

$$\ddot{\theta} = \frac{7g}{12l}\cos\theta$$

6) Integrating once this equation we have:

$$\frac{\dot{\theta}^2}{2} = \frac{7g}{12l}\sin\theta + C$$

where the integration constant is find from the initial conditions:

$$\dot{\theta}(0) = 0 \; ; \; \theta(0) = 0 \; ; \longrightarrow C = 0.$$

7) The velocity of point A at the given instant will be:

$$v_A(\theta_1 = \frac{\pi}{2}) = 2l . \dot{\theta}(\frac{\pi}{2}) = \sqrt{\frac{14gl}{3}}$$

8) The reaction forces in the same instant from the hinged supports are R_1, R_2 R_3 and R_4. These reaction forces will be determined in the following way: after which we remove from the third and the last equations the two reaction forces R_6 and R_8, these are introduced in the first and the seventh equations from which we take out the reactions R_2 and R_4. After this the two reactions R_6 and R_8 are introduced in the sixth and fifth equations from which result the reaction forces R_5 and R_7. The reaction force R_5 introduced in the second equation will give us the reaction R_1 and R_7 introduced in the eighth equation results the reaction R_3. The solving becomes easier if we remove first the value of the parameter for the given instant. Finally we have:

$$R_1(\frac{\pi}{2}) = R_3(\frac{\pi}{2}) = 13{,}17Mg \; ; \; R_2 = R_4 = 0.$$

Problem 3. *The system from the figure 5. moves so that the wires remain vertically. Knowing the masses of the bodies and the radii of the pulleys calculate the distance covered* *by the particle P in an interval by the two seconds from the beginning of the motion if the motion starts from rest.*

Solution. *1) The system of pulleys is made from one fixed pulley with the center in the fixed point O, a mowing pulley and a particle and it has two degrees of freedom because has a hinged support, two wires and two restrictions (the wires remain vertically in the time of motion and so the horizontal displacements of the particle and the moving pulley are removed):*

$$N_{df} = 3 \cdot 2 + 2 \cdot 1 - (2 \cdot 1 + 2 + 2) = 2$$

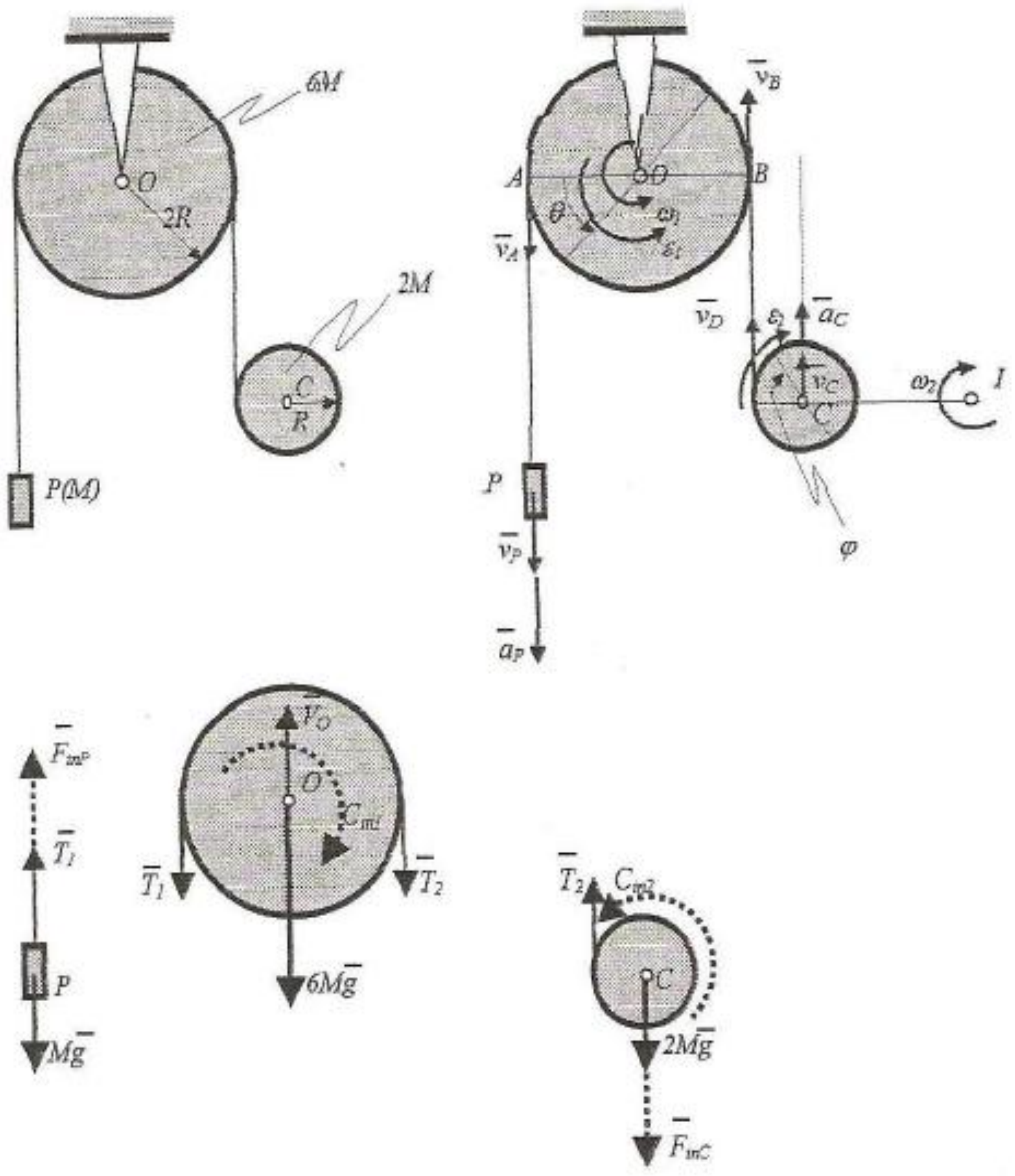


Fig.5.

In other way we remark that the fixed pulley performs a rotation motion that if it is blocked is stopped the vertical motion of the particle and also the motion of the wire that join the moving pulley. But now the moving pulley has one other possibility of motion: the rolling motion without sliding on vertical direction, motion that has also one degree of freedom. If we block this rolling motion then the entire system is stopped, so the system has two degrees of freedom. This last way for to determine the number of the degrees of freedom allows us to determine easier the independent kinematic parameters of the motion because the two blocked motions are independents. Consequently we shall choose the following two kinematic parameters: the first parameter will be the angle θ between one diameter of the fixed pulley and the horizontal direction describing the rotation motion of this disc and the second parameter will be the angle φ between one diameter of the moving pulley and the horizontal direction describing the rolling motion of this pulley. The senses of rotation of the two pulleys are arbitrary however if we see the real senses then is better to take those senses. Here we have taken the trigonometric sense for θ and the clockwise sense for φ.

2) The kinematic study of the system have to obtain the acceleration of the particle P, the accelerations of the centers of the two pulleys and the angular accelerations of these pulleys function of the kinematic parameters.

The calculation starts from the fixed pulley (has fixed center in O) that has a rotation motion that is defined by the parameter θ. The angular velocity and acceleration of this disc will be:

$$\omega_1 = \dot{\theta}; \ \varepsilon_1 = \dot{\omega}_1 = \ddot{\theta}$$

with the same sense of rotation as the parameter from which are coming because they do not changed the signs after the derivation.

For transmission of the motion to the particle P with the wire AP we shall calculate first the velocity of the point A that results:

$$v_A = OA \,.\, \omega_1 = 2R\,\dot{\theta}$$

having the sense directed down (that is in the sense of rotation of the angular velocity about the rotation center O). This velocity is transmitted to the point P with the entire intensity, direction and sense:

$$v_P = v_A = 2R\,\dot{\theta}$$

Because the particle P performs a rectilinear motion we have the magnitude of its acceleration:

$$a_P = \dot{v}_P = 2R\,\ddot{\theta}$$

keeping the direction and sense of the velocity because through the derivation we have not changed the sign.

Now let to transmit the motion to the moving pulley. We shall calculate first the velocity of the point B:

$$v_B = OB \cdot \omega_1 = 2R\dot{\theta}$$

With the sense directed up. This velocity is transmitted with the entire intensity, direction and sense in the point D:

$$v_D = v_B = 2R\dot{\theta}$$

The moving pulley is a body that performs a plane motion. Because we have taken the angular parameter that defines the motion results the angular velocity and acceleration of this disc:

$$\omega_2 = \dot{\varphi};\ \varepsilon_2 = \dot{\omega}_2 = \ddot{\varphi}$$

with the same sense of rotation as the sense of rotation of the angular parameter (the clockwise sense).

We shall use the propriety of the instantaneous center of rotation for to determine the velocity of the point C (that performs a vertical rectilinear motion). But we have to determine the position of this center. Knowing one velocity of a point of the body (here the point D) and the angular velocity of the body we can write:

$$v_D = ID \cdot \omega_2$$

relation from which is obtained the position of the instantaneous center of rotation:

$$ID = \frac{v_D}{\omega_2} = \frac{2R\dot{\theta}}{\dot{\varphi}}$$

The instantaneous center of rotation is located in the right part of the point D because only from this position the sense of rotation of the angular velocity and of the point D are the same about this point. Having the position of the instantaneous center of rotation we shall calculate the velocity of the point C that has the magnitude:

$$v_C = IC \cdot \omega_2 = (ID - R) \cdot \omega_2 = 2R\dot{\theta} - R\dot{\varphi}$$

Vertical direction and with the sense directed up (the sense of rotation about the instantaneous center of rotation is the same as the sense of rotation of the angular velocity).

Through derivation we have the acceleration of the point C.

$$a_C = \dot{v}_C = 2R\ddot{\theta} - R\ddot{\varphi}$$

with the same sense and direction as the velocity of the point.

3) Having all the accelerations of the centers and the angular accelerations of the bodies defined function of the kinematic parameters we shall calculate the force - couple system of the inertia forces for all bodies from the system. For the particle P we have only the inertia force:

$$F_{inP} = M\,a_P = M\,2R\,\ddot{\theta}$$

with vertical direction and with the sense directed up (opposite as the sense of the acceleration of the particle).

The fixed pulley will be loaded with the inertia couple only:

$$C_{in1} = J_O \cdot \varepsilon_1 = \frac{6M(2R)^2}{2}\ddot{\theta} = 12MR^2\,\ddot{\theta}$$

having opposite sense as the angular acceleration ε_1 because the center of the disc is fixed and the acceleration in this point is zero so we have not inertia force.

The moving pulley will be loaded with inertia force:

$$F_{inC} = 2M\,a_C = 2MR(2\ddot{\theta} - \ddot{\varphi})$$

with the sense directed down (opposite as the sense of the acceleration of its mass center C) and acting in the mass center and also with the inertia couple:

$$C_{in2} = J_C \cdot \varepsilon_2 = \frac{2MR^2}{2}\ddot{\varphi} = MR^2\,\ddot{\varphi}$$

with its sense opposite as the sense of the angular acceleration of this pulley.

4) We divide the system of bodies in the tree independent bodies: two pulleys and the particle P removing the constraint from O with the corresponding two components of reaction (here because the body is not acted by horizontal forces the horizontal component of the reaction is also equal to zero) and the connections (here the two wires) with the internal forces (pare tensions with the same intensities and opposite senses). Also we shall load each body with their weights and the calculated inertia force – couples systems.

5) We shall write the following dynamic equilibrium equations:

-for the particle P:

$$\Sigma X_i = 0\ ;\ 0 = 0\ ;$$
$$\Sigma Y_i = 0\ ;\ F_{inP} + T_1 - Mg = 0;$$

-For the fixed pulley:

$$\Sigma X_i = 0 \; ; \; 0 = 0 \; ;$$
$$\Sigma Y_i = 0 \; ; \; V_O - T_1 - 6\,Mg - T_2 = 0;$$
$$\Sigma M_{O1} = 0 \; ; \; T_1 \,.\, 2R - C_{in1} - T_2 \,.\, 2R = 0 \; ;$$

-For the moving pulley:

$$\Sigma X_i = 0 \; ; \; 0 = 0 \; ;$$
$$\Sigma Y_i = 0 \; ; \; T_2 - 2\,Mg - F_{inC} = 0;$$
$$\Sigma M_{O1} = 0 \; ; \; - T_2 \,.\, R + C_{in2} = 0 \; ;$$

In this way we have obtained a system of five differential equations in which two equations are the differential equations of the motion corresponding to the two kinematic parameters and three are used for to determine the three unknown reaction forces.

*6) If we take out from the first equation the tension T_1 and from the last equation the tension T_2 and if we remove in the third and fourth equations we have the **subsystem of the differential equations of the motion**:*

$$\begin{cases} 8\ddot{\theta} + \ddot{\varphi} = \dfrac{g}{R} \\ -4\ddot{\theta} + 3\ddot{\varphi} = \dfrac{2g}{R} \end{cases}$$

7) As we can see first we solve the system as an algebraic system of equations and we have the angular accelerations of the two pulleys:

$$\ddot{\theta} = \frac{g}{28R} \; ; \; \ddot{\varphi} = \frac{5g}{7R}$$

The acceleration of the particle P will be:

$$a_P = 2R\,\ddot{\theta} = \frac{g}{14} = 0{,}7\ m/s^2$$

Knowing that the particle performs a rectilinear motion and its acceleration is constant we obtain through integration:

$$y_P = a_P \frac{t^2}{2} + C_1\,t + C_2$$

where the integration constants are determined from the initial conditions:

$C_1 = C_2 = 0$

The distance passed by the particle in the two seconds from the start of the motion will be:

$$D = y_P(2) = 1{,}4\ m$$

Problem 4. *The system from the figure 6 is made from a disc with its center in the fixed point O having the radius R and its mass 4M, a bar AB hinged by the disc in point A located on the periphery of the disc having the length R and mass 3M and a slider B by its mass M hinged at the end of the bar AB and performing a motion on a horizontal fixed straight line passing through the point O. Knowing that the motion is made in vertical plane under the action of the weights and is start from rest when the bar AB is vertically determine the differential equation of the motion of this system the velocity of the slider B and the reactions from the external constraints when the bar AB becomes horizontal.*

Problem 5. *The system of pulleys from the figure 7. moves so that in the time of motion the wires remain vertically. Determine the acceleration of the particle P the tensions from the wires at one any instant of the motion. The motion is performed in vertical plane under the actions of the weights.*

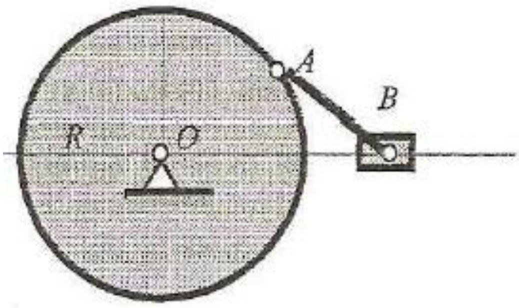


Fig.6.

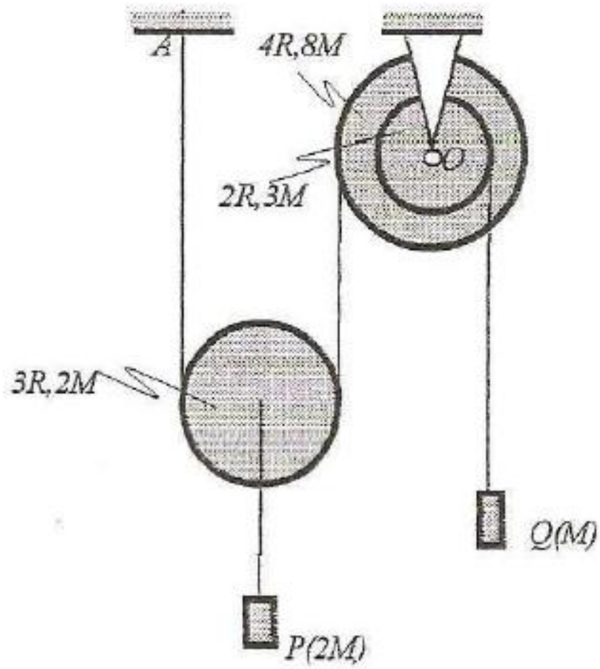


Fig.7.

Problem 6. *The system from the figure 8. is made from the slider P by mass M that slide without friction on a vertical rod and the bar PA by the length 2l and mass 6M hinged by the slider P. Knowing that the motion of the system is performed under the action of the weights in vertical plane write the differential equations of the motion of the system.*

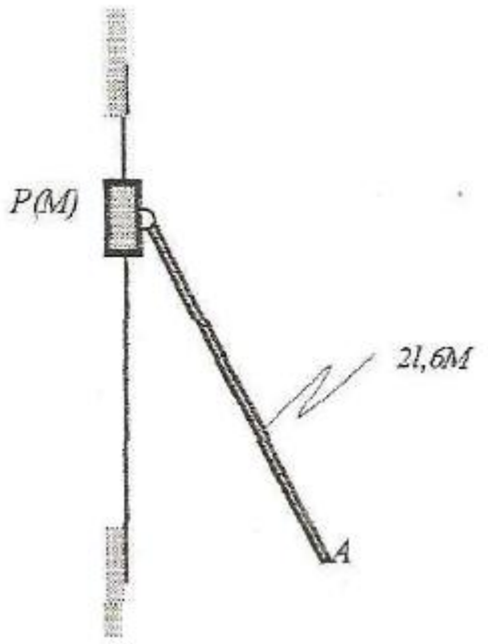


Fig.8.

Chapter 21. Principle of virtual work

21.1. Introduction.

The principle of virtual work is the most used principle in the analytical mechanics for to study the systems of bodies. This principle is used in the solving of all problems involving conditions of equilibrium. This principle can be used in two ways: ***the principle of virtual work with virtual displacements*** *(and real forces) and* ***the principle of virtual work with virtual forces*** *(and real displacements).*

In this chapter we shall study only the principle of virtual work with virtual displacements called in the following the principle of virtual work. We make the remark that in this chapter we shall study only the equilibrium in the state of rest of the systems. The state of motion will be studied in the following chapter using the same principle.

We shall call ***virtual work*** *and will be marked* ***δL*** *the scalar quantity equal to the scalar product between the force and the virtual displacement of the point of application of the force:*

$$\delta L = \overline{F} \cdot \delta \overline{r}$$

21.2. Principle of the virtual work for the state of rest.

Consider a particle P, in equilibrium, acted by a system of forces with the resultant F and having a simple ideal and bilateral constraint.

Using the axiom of the constraints the simple constraint of the particle can be removed with one normal reaction force N with which we can express the equilibrium of the particle:

$$\overline{F} + \overline{N} = 0$$

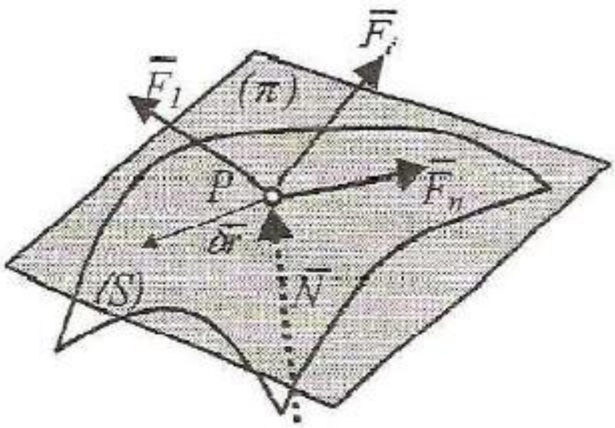


Fig.1.

where:

$$\overline{F} = \Sigma \overline{F}_i$$

Let be one virtual displacement of the particle P. This displacement being compatible with the constraint will be performed in the tangent plane to the surface representing the simple constraint therefore it will be perpendicular on the normal reaction force corresponding to the constraint. In this way we have:

$$\delta L = \overline{N} \cdot \delta\overline{r} = 0$$

*meaning that: **the virtual work of the reaction forces of the ideal constraints is always equal to zero**. This statement and relation represents the first form of the principle of virtual work.*

To consider now that we multiply of the particle with the virtual displacement the vector equation of equilibrium:

$$\overline{F} \cdot \delta\overline{r} + \overline{N} \cdot \delta\overline{r} = 0$$

from which we obtain, if we make zero the term containing the reaction force, the second form of ***the principle of the virtual work****:*

$$\delta L = \overline{F} \cdot \delta\overline{r} = 0$$

namely: ***the necessary and sufficient condition as a particle having ideal constraints to be in equilibrium is that the virtual work of the active forces (given forces) to be equal to zero.***

We remark that from the condition of equilibrium we have eliminated the reaction forces, the condition containing only the active forces. This is an important step in our study: to eliminate from the equations the reaction forces.

If we have a system of particles in equilibrium then for each particle (after which we use the axiom of the constraints) we can express the equilibrium condition with this kind of equation. Summing for all particles of the system is obtained the condition of equilibrium of the entire system under the form:

$$\delta L = \Sigma \overline{F_i} \cdot \delta\overline{r_i} = 0$$

where $\overline{F_i}$ *are the active forces which act about the system and* δr_i *represents the virtual displacements of the points of application of the active forces.*

For rigid body, where the forces can be reduced in a few points, through elementary transformations is obtained the equilibrium conditions under the form:

$$\delta L = \Sigma \overline{F_i} \cdot \delta\overline{r_i} + \Sigma \overline{M_j} \cdot \delta\overline{\theta} = 0$$

Relation in which $\overline{M_j}$ *represents the concentrated couples and* $\delta\overline{\theta}$ *is the virtual rotation of the body.*

Finally, for one any mechanic system (system of particles or rigid bodies) the equilibrium condition will have the form:

$$\delta L = \Sigma \overline{F_i} . \delta \overline{r_i} + \Sigma \overline{M_j} . \delta \overline{\theta_j} = 0$$

And we can sets out the final form of the principle of virtual work: ***the necessary and sufficient condition as a mechanical system having ideal constraints to be in equilibrium is that as the virtual work of the active forces (and couples) to be equal to zero.***

In the following section we shall shout the way in which the principle of virtual work can be used for to determine the reaction forces from the constraints of a statically determined and stable mechanical system.

21.3. Calculation of the reaction forces from the constraints of a Gerber beam.

A Gerber beam is a system of rigid bodies made from straight bars having their axes on the same straight line. Because the Gerber beam is a particular system of bodies it has some particularities which simplify the determination of the reactions. We shall show how we can determine the reactions using the principle of the virtual work.

At the first sight looks like the determination of the reaction forces cannot be calculated using this principle because the Gerber beam is a statically determined and stable system namely it is fixed and consequently the possible displacements are all equal to zero. So we have not virtual displacements because they are compatibles with the constraints of the system.

For to calculate one reaction force we can make in the following way: the system is transformed in a mechanism with one degree of freedom removing one simple constraint with the corresponding reaction force (or moment). In this way the system that initial was a statically determined and stable system becomes a mechanism. The unknown reaction force corresponding to the removed constraint will be considered unknown active force and because the system is in equilibrium we can write the condition of equilibrium using the principle of the virtual work. From the resulted equation results the reaction force.

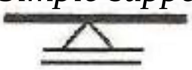
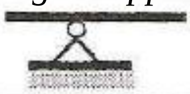
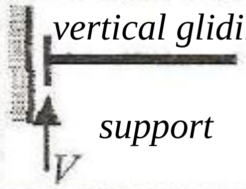
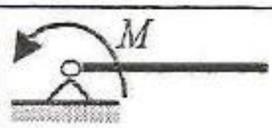
The main characteristic of the system called Gerber beam is that the vertical displacements do not produce horizontal displacements and reverse the horizontal displacements do not produce vertical displacements. This is the propriety that separates this system to the other systems (frames and trusses).

In the following we shall present the way to use the principle of virtual work for to determine the reaction forces for a Gerber beam loaded with vertical forces only.

1)First we check if the system is or not a statically determined and stable system.

2)We transform the beam in a mechanism with one degree of freedom removing the simple constraint corresponding to the searched reaction. This is made in the way shown in the table T1, table obtained for the situation in which the beam is loaded with vertical forces only.

Table T1

Constraint	*Searched reaction*	*Transformation*
Simple support 	*Vertical reaction force*	
Hinged support 	*Vertical reaction force*	
Fixed support 	*Vertical reaction force*	 *vertical gliding support*
	The reaction moment	

We can remark that because we have not horizontal action the simple support and the hinged support will have the same behavior.

3)We eliminate from the mechanism the fixed bodies. The cases when a body is fixed are: body with one fixed support, body with one hinged support (or fixed hinge) and a vertical simple support, body with two vertical simple supports and body with two fixed hinges.

We remind that the fixed hinges have the same behavior as the vertical simple supports and opposite the simple supports will have the same behavior as the fixed hinges and also that an internal hinge becomes a fixed hinge if it is in contact with a fixed body.

4)The remained bodies can perform only two kinds of motions: or a vertical translation motion if the body has one vertical gliding support or a rotation motion when the body has one simple support or a fixed hinge. We make the remark that the internal mobile hinges do not impose the way of motion of the body they make the contact between the bodies transmitting the motion from a body to another.

Now we draw the diagrams of the virtual displacements of the mechanism. His is made in the following way: we start from one arbitrary body for which we consider the corresponding motion (translation motion or rotation motion) in one any sense. Knowing the kinds of motion of all bodies and that the bodies are connected with internal hinges we draw the diagram of the virtual displacements for the entire mechanism.

5)We mark the rotations of the bodies and the vertical displacements corresponding to the vertical concentrated forces (the distributed forces are concentrated on each body separately).

We choose one kinematic parameter (for example the first rotation) and we express all the rotations of the bodies and all displacements function of this parameter. The rotations and displacements will be considered with senses. For to express the rotations and the displacements it is considered that the virtual displacements are infinitesimal and consequently the rotation of the body is the slope of the diagram of the virtual displacement of it. In this way for to calculate the rotation of a body we shall use the relation:

$$d_{left} \cdot \delta\theta_{left} = d_{right} \cdot \delta\theta_{right}$$

These kinds of relations are written in the internal hinges where d_{left} and d_{right} are the distances from the rotation centers of the bodies (the simple supports or the fixed hinges) to the internal hinges (common to the two *neighbor bodies) and the* $\delta\theta_{left}$ *and* $\delta\theta_{right}$ *are the rotations of the two bodies joined by that internal hinge.*

The relation expresses the equality of the displacements on the two bodies in the internal hinge.

The virtual displacement of the point of application of a force will be determined with the relation:

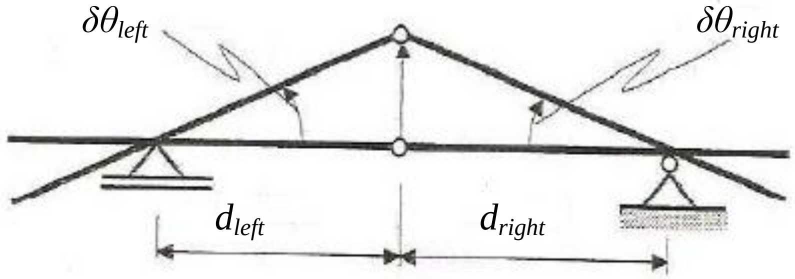


Fig.1.

$$\delta y_i = d_i \cdot \delta\theta_{body}$$

where d_i *is the distance from the rotation center of the body to the point of application of the force and* $\delta\theta_{body}$ *is the virtual rotation of the body on that is acting the force.*

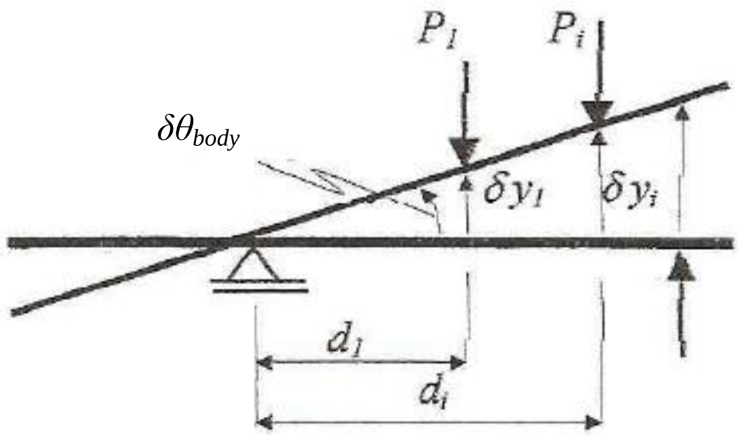


Fig.2.

6)It is calculated the virtual work with the relation:

$$\delta L = \Sigma Y_i \cdot \delta y_i + \Sigma M_j \cdot \delta \theta_j$$

The sign of the virtual work of a force is determined comparing the sense of the force with the sense of the displacement and if they have the same senses then the virtual work is positive and if the senses are opposite then the virtual work is negative. For the virtual work of a concentrated moment is compared the sense of rotation of the virtual rotation of the body with the sense of rotation of the concentrated moment and if they are in the same sense then the sign of the work is positive and in opposite senses the work is negative.

Positive work *Negative work* *Positive work* *Negative work*

Fig.3.

7)We shall consider the condition of equilibrium:

$\delta L = 0$

and because the virtual displacement is arbitrary as magnitude, therefore we can consider it different to zero we can simplify the equation with the kinematic parameter and solving the equation results the searched reaction force.

Remark: *because the kinematic parameter is simplified in the finally equation, namely it is divided by itself, we can choose for the beginning this parameter equal to the unit.*

Also we make the remark that for a system of bodies each reaction will be determined using another mechanism, therefore the calculation of the reactions are independent and each calculation represents another problem.

21.4. Sample problems

Problem 1. *Calculate the reactions from the constraints of the Gerber beam from the figure 4 using the principle of the virtual work.*

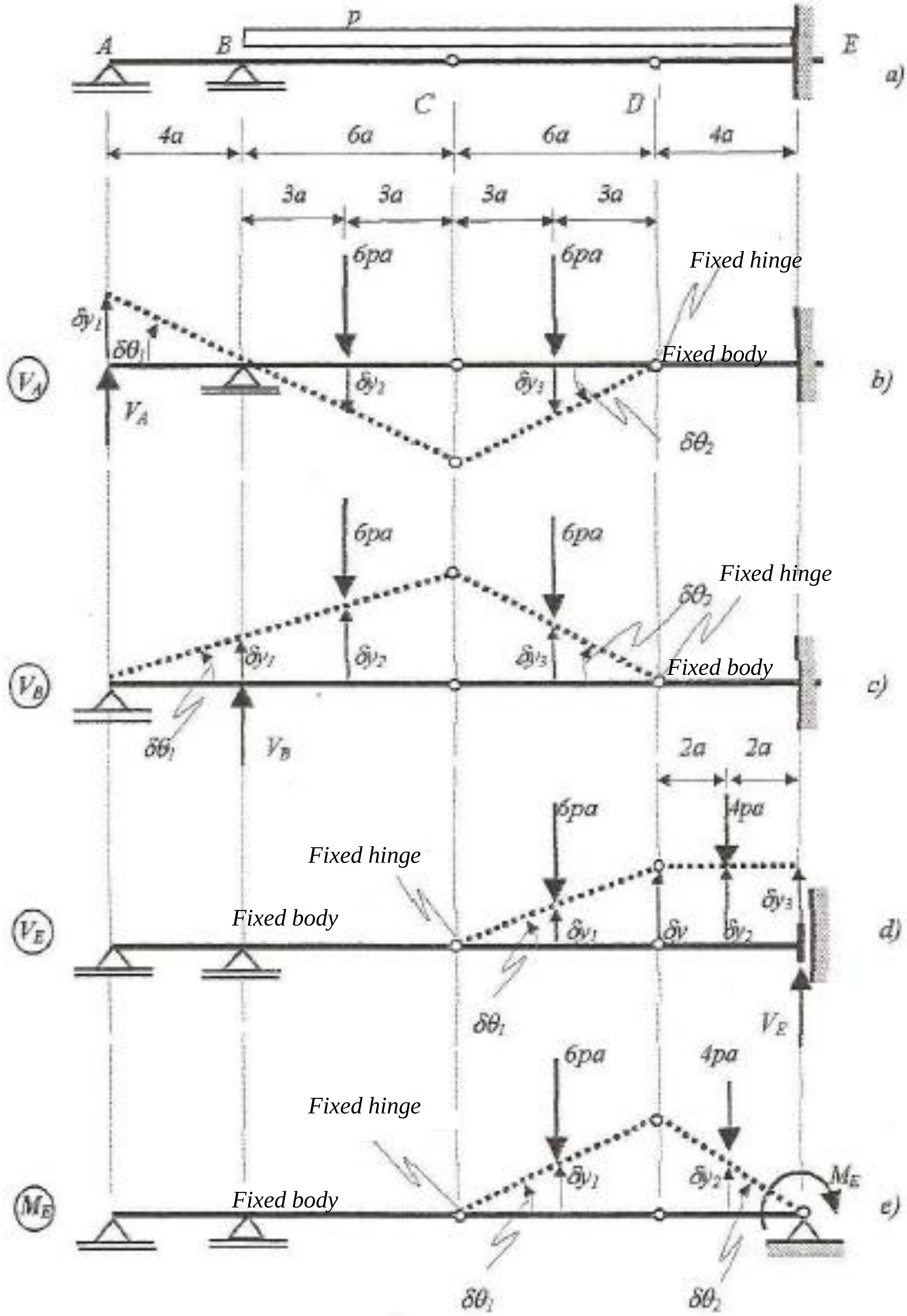


Fig.4.

Solution. *As we have presented each reaction will be determined independently by the other reactions, for each reaction will correspond another mechanism.*

Reaction $\boldsymbol{V_A}$*: 1)As we can see the Gerber beam is statically determined and stable because it is made from three bodies, one fixed support, two hinges and two simple supports. At the other hand it is fixed by its constraints.*

2)The reaction force V_A *is the mechanical equivalent of the simple support from the point A so we shall remove this support with the vertical unknown reaction force resulting the mechanism with one degree of freedom from the figure 4.b. All the other constraints remain unchanged.*

3)The bodies from the mechanism will be in the following states:

-the third body (DE) having a fixed support is a fixed body. The internal hinge from D becomes a fixed hinge;

-the second body (CD) having only a fixed hinge (in D) will perform a rotation motion about this fixed hinge;

-the first body (ABC) having only a simple support in point B performs a rotation motion about this support.

4)The diagram of virtual displacements for the resulted mechanism is started from one arbitrary body (but first we have eliminated the fixed body, here the body DE). We shall start with the first body and we shall imagine a virtual rotation of this body about the simple support from the point B in an arbitrary sense here in clockwise sense. We mark the rotation of the body with $\delta\theta_1$*.The second body (CD) has a common point with the first in the internal hinge C, hinge that moves on vertical direction when the first body rotates. Knowing that the second body rotates about the fixed hinge D and has a common point with the first in C is obtained the diagram of the second body. We shall mark the rotation of the second body (in counterclockwise sense) with* $\delta\theta_2$*.*

5)The uniformly distributed force is concentrated on each body separately (here the first and the second body) and we mark the vertical displacements of the forces.

We shall choose as kinematic parameter, function that we shall express the entire motion (the system has one degree of freedom) the rotation of the first body. As we have presented this parameter can be taken equal to one:

$$\delta\theta_1 = 1$$

The rotation of the second body will be determined from the condition as the vertical displacement in the internal hinge C to be the same indifferent from which body is calculated:

$$\delta y_C^{AC} = \delta y_C^{CD}$$

or:

$$6a \,.\, \delta\theta_1 = 6a \,.\, \delta\theta_2$$

Replacing the kinematic parameter results the magnitude of the rotation of the second body:

$$\delta\theta_2 = 1$$

Now we shall calculate, function of the same parameter, the virtual displacements of the points of application of the forces:

$$\delta y_1 = 4a\ .\ \delta\theta_1 = 4a\ ;\ \delta y_2 = 3a\ .\ \delta\theta_1 = 3a\ ;\ \ \delta y_3 = 3a\ .\ \delta\theta_2 = 3a$$

6)We shall calculate now the virtual work:

$$\delta L = \Sigma Y_i\ .\ \delta y_i = V_A\ .\ \delta y_1 + 6pa\ .\ \delta y_2 + 6pa\ .\ \delta y_2 = V_A\ .\ 4a + 6pa\ .\ 3a + 6pa\ .\ 3a$$

We shall solve the equation:

$\delta L = 0$

that represents the equilibrium condition of the Gerber beam. Results the searched reaction force:

$$V_A = -9pa.$$

Reaction force V_B. *We shall start the calculation from the initial system, statically determined and stable, where we shall remove the simple support from B with the vertical reaction force V_B. In this way we have the mechanism with one degree of freedom from the figure 4.c. where we may remark that the third body is a fixed one. The internal hinge from D becomes a fixed hinge. The first two bodies are in rotation motion, the first about its simple support from the point A and the second about the fixed hinge from D.*

We start with the first body for which we consider a counterclockwise rotation $\delta\theta_1$. Results the rotation $\delta\theta_2$ for the second body and in this way we have the diagram of the virtual displacements represented in the figure 4.c.

After what we concentrate the distributed force on the two bodies we mark the vertical displacements of the forces and we choose as kinematic parameter the rotation of the first body:

$\delta\theta_1 = 1$

The rotation of the second body results from the equality of the displacements in point C:

$$10a\ .\ \delta\theta_1 = 6a\ .\ \delta\theta_2; \longrightarrow \quad \delta\theta_2 = 1{,}67$$

The displacements of the points of application of the concentrated forces are:

$$\delta y_1 = 4a\ .\ \delta\theta_1 = 4a\ ;\ \delta y_2 = 7a\ .\ \delta\theta_1 = 7a\ ;\ \ \delta y_3 = 3a\ .\ \delta\theta_2 = 5a$$

We shall calculate the virtual work in the same way as for the previous reaction force and we shall consider the condition of equilibrium:

$$\delta L = \Sigma Y_i\ .\ \delta y_i = V_B\ .\ \delta y_1 - 6pa\ .\ \delta y_2 - 6pa\ .\ \delta y_2 = V_B\ .\ 4a - 72pa^2 = 0$$

Results:

$V_B = 18\ pa$

Reaction force $\boldsymbol{V_E}$. *For this reaction force we shall remove the fixed support from E with one gliding support and the vertical reaction force* V_E *(Fig.4.d.).*

Because the first body has two vertical simple supports (that at the Gerber beams loaded with vertically forces behavior as hinged supports) it is a fixed body and consequently the internal hinge from C becomes one fixed hinge. The second body (CD) having one fixed hinge will perform a rotation motion and the third body (DE) having one gliding support will perform a vertical translation motion. Considering the virtual rotation of the body (CD) as $\delta\theta_1$ *is obtained the diagram of the displacements for the resulted mechanism.*

The rotation of the body (CD) will be considered as the kinematic parameter of the motion:

$$\delta\theta_1 = 1$$

The virtual displacements of the points of application of the concentrated forces will be:

$$\delta y_1 = 3a\ .\ \delta\theta_1 = 3a\ ;\ \delta y_2 = \delta y_3 = \delta y = 6a\ .\ \delta\theta_1 = 6a\ ;$$

with which the virtual work will be:

$$\delta L = \Sigma Y_i . \delta y_i = -\ 6pa\ .\ \delta y_1 - 4pa\ .\ \delta y_2 + V_E\ .\ \delta y_3 = V_E\ .\ 6a - 42pa^2 = 0$$

from which results finally:

$V_E = 7\ pa.$

The reaction $\boldsymbol{M_E}$ ***.*** *In this case we shall remove the fixed support from E with one fixed hinge and a unknown concentrated moment (Fig.4.e.). The first body is a fixed one and the internal hinge from C is fixed. The next two bodies having each of them one fixed hinge will perform rotation motions and considering for one from them one any rotation will results for the other the corresponding rotation ad finally the diagram of the virtual displacements.*

The kinematic parameter will be:

$$\delta\theta_1 = 1$$

and for the following body writing the equality of the displacements in point D we have:

$$6a\ .\ \delta\theta_1 = 4a\ .\ \delta\theta_2;\ \longrightarrow\ \delta\theta_2 = 1{,}5$$

The virtual displacements of the forces will be:

$$\delta y_1 = 3a\,.\,\delta\theta_1 = 3a\ ;\ \delta y_2 = 2a\,.\,\delta\theta_1 = 3a\ ;$$

displacements with which we have the virtual work:

$$\delta L = \Sigma Y_i . \delta y_i + \Sigma M_j . \delta\theta_j = -\,6pa\,.\,\delta y_1 - 4pa\,.\,\delta y_2 + M_E\,.\,\delta\theta_2 = 0$$

From this equation results:

$M_E = 20\ pa^2$.

Problem 2. *Using the principle of virtual work determine the vertical reaction forces for the Gerber beam from the figure 5.*

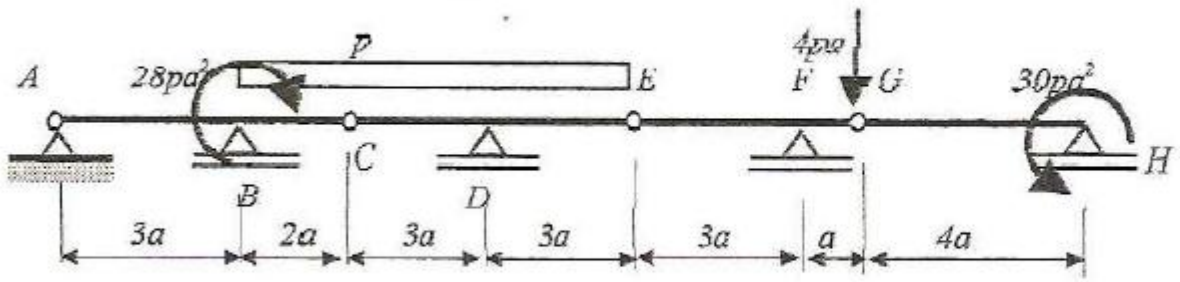


Fig.5.

21.5. Calculation of the reaction forces in the constraints of the frames.

The calculation of the reaction forces using the principle of virtual work for the frames is based, as for the Gerber beam, on the transformation of the statically determined and stable system of bodies in a mechanism with one degree of freedom. The transformations for to obtain a mechanism with one degree of freedom are represented in the table T2.

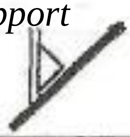
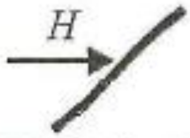
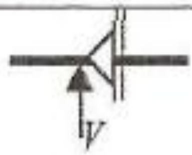
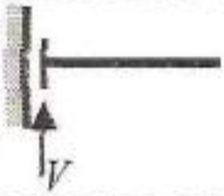
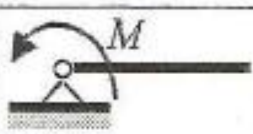
The determination of the virtual displacements for an arbitrary system of rigid bodies is based on the fact that: the laws of variations of the elementary displacements are the same with the laws of the variations of the distributions of velocities:

$$d\bar{r} = \bar{v} \, . \, dt$$

or for the virtual displacements if we shall consider the interval of time ***δt*** *as a constant quantity we have:*

$$\delta \bar{r} = \bar{v} \, . \, \delta t$$

Table T2

Constraint	*The searched reaction*	*Transformation*
Vertical simple support 	*Vertical reaction force*	
Horizontal simple support 	*Horizontal reaction force*	
 *Hinged support*	*Vertical reaction force*	
	Horizontal reaction force	
 *Fixed support*	*Vertical reaction force*	
	Horizontal reaction force	
	Reaction moment	

We make the remark that the gliding support allows to perform only translation motion of the body consequently the absolute rotation center of the body having this kind of constraint is located at infinity distance on the normal direction on the possible displacement. Because the displacement is on the direction of the searched reaction force this absolute rotation center will be on the normal direction on the reaction force.

Fig.6.

We give the steps for solving the problem of calculation of a reaction of a frame using the principle of virtual work:

1)Is cheeked if the system of bodies is or not statically determined and stable;

2)The system is transformed in a mechanism with one degree of freedom removing the simple constraint corresponding to the searched reaction;

3)Are eliminated the fixed bodies. We remind that: there are four simple cases in which the bodies are fixed. These cases are: one body with one fixed support, one body with one hinged support (fixed hinge) and one simple support, one body with three simple supports (not parallel all the three, not concurrent in the same point all the three) and finally two bodies each of them with one fixed hinge and one internal hinge between them (the three hinges are not collinear).

4)For the remained bodies in the mechanism we determine the rotation centers. The knowledge of all absolute rotation centers is necessary but the given relative rotation centers generally are enough for to find the virtual displacements.

5)Are drawn the diagrams of the virtual displacements on two convenient directions: on horizontal and on vertical directions. The diagrams are started from an arbitrary body and the virtual displacement of this body is also arbitrary. The diagrams are obtained in the same way as the diagrams of velocities;

6)We concentrate the distributed forces on each body separately and the inclined forces are removed with two components on the directions of the displacements (horizontal and vertical);

7)We mark the rotations of all bodies from the mechanism and represent and mark all the displacements of the forces on the corresponding directions;

8)We choose a kinematic parameter and we express all the rotations and all the displacements function of this parameter. As in the previous section we can take this parameter equal to one.

9)Is calculated the virtual work of all forces and couples and we express the condition of equilibrium;

10)Solving the resulted equation we have the searched reaction.

21.6. Sample problems.

Problem 3. *Using the principle of virtual work calculate the reactions from the constraints for the system of rigid bodies from the figure 7.*

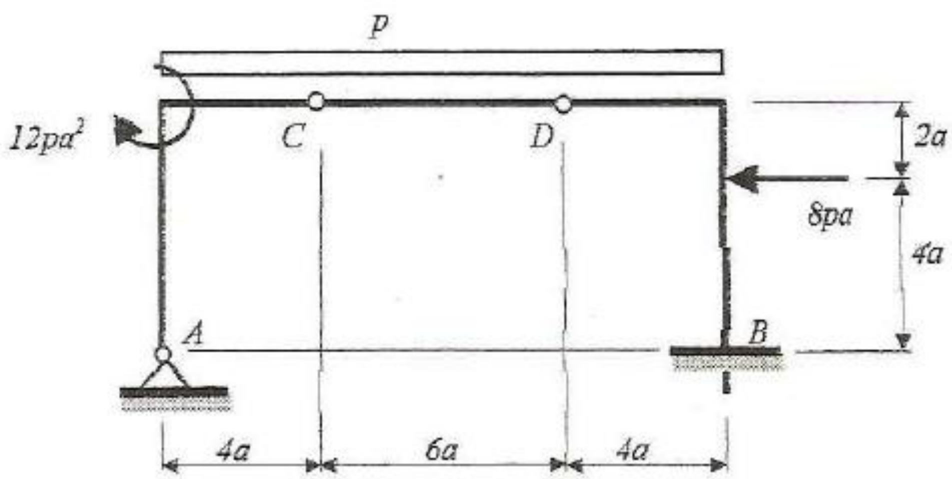


Fig.7.

Solution. *As we have seen each reaction will be solved from an independent calculation. Therefore we have to solve in fact five independent problems.*

The reaction force V_A. *1) The system is statically determined and stable being made from three bodies and having a fixed support, one hinged support and two internal hinges:*

$$3 \cdot 3 = 3 \cdot 1 + 3 \cdot 2$$

and also the bodies are fixed (the third body has one fixed support, and the first two bodies have three non- collinear hinges from which two are fixed hinges.

2)For to determine the vertical reaction force from the hinged support from A we shall remove this hinged support with the vertical reaction force and horizontal simple support (Fig.8.). In this way we obtain a mechanism with one degree of freedom.

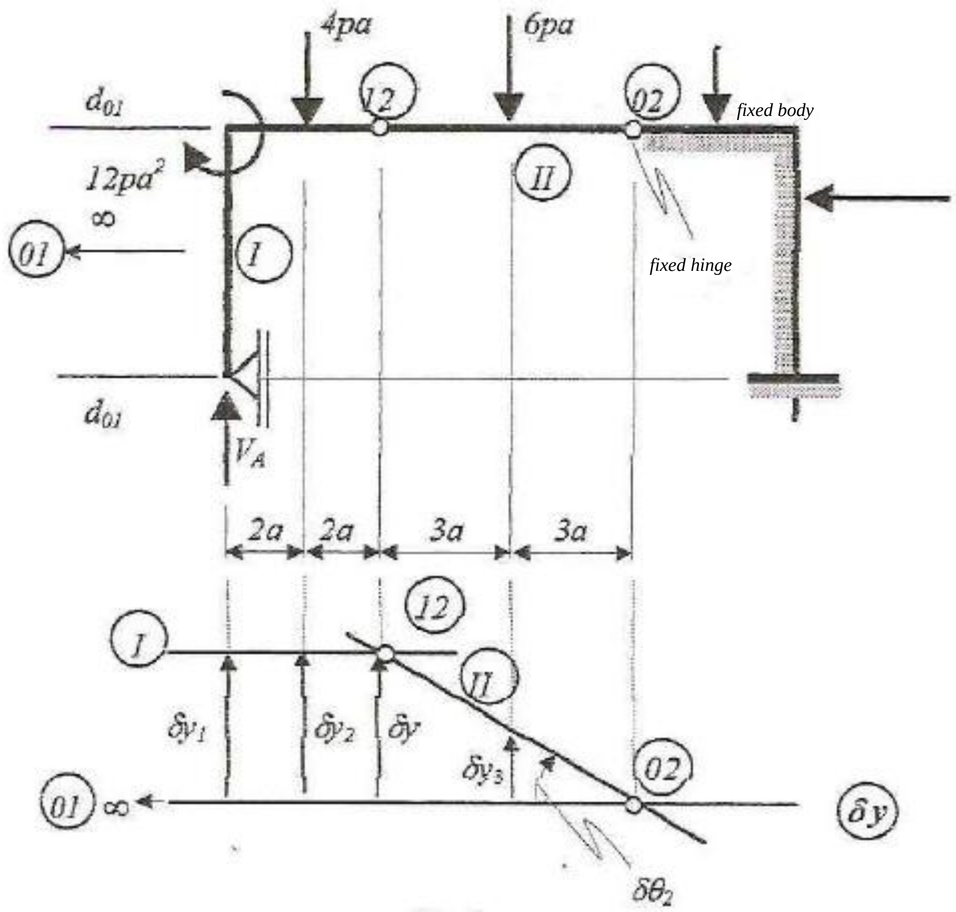


Fig.8.

3)We remark that the third body is fixed having one fixed support and in this way the internal hinge from D becomes a fixed hinge. The mechanism will have only two bodies.

4)For the mechanism having two bodies we shall determine the rotation centers (as in kinematics). First we number the bodies and we see that: the simple support from A shows us that on the horizontal straight line passing through the point A is located the absolute rotation center of the first body, the internal hinge from C is the relative rotation center (12) and the fixed hinge from D is the absolute rotation center of the body (II).

For to determine the absolute rotation center (01) we shall use the theorem off co linearity of the rotation centers (or the rule of the indexes) and we see that on the line passing through the centers (12) and (02) is located the absolute rotation center (01). It results the absolute rotation center (01) at infinity distance on the horizontal direction.

5)Having all the absolute rotation centers we can draw the diagrams of the displacements on the two directions (horizontal and vertical). But remarking that the forces acting about the mechanism are only vertically we shall draw only the diagram of the vertical displacements (δy).

First we project on the horizontal reference line the two absolute rotation centers and we start from the body (II) for which we consider a virtual rotation (here in clockwise sense) $\delta\theta_2$. *Projecting the relative rotation center (12) on the diagram of the body (II) results the diagram of the body (I) as o horizontal straight line (parallel with the reference line because the absolute rotation center (01) is located at infinity distance on horizontal direction).*

6)For the three vertical forces which act about the mechanism we mark the displacements:δy_1, δy_2, δy_3.

We shall choose as kinematic parameter the rotation of the body (II):

$$\delta\theta_2 = 1$$

with that we calculate the virtual displacements of the points of application of the forces:

$$\delta y_1 = \delta y_2 = \delta y = 6a\delta\theta_2 = 6\alpha; \ \delta y_1 = 3a \ \delta\theta_2 = 3a.$$

7)The virtual work is calculated with the relation:

$$\delta L = \Sigma X_i \, \delta x_i + \Sigma Y_j \, \delta y_j + \Sigma M_k \, \delta\theta_k$$

where in this case we have:

$$\delta L = V_A \, . \, \delta y_1 - 4pa \, . \, \delta y_2 - 6pa \, . \, \delta y_3 = 0$$

Removing we find the searched reaction force:

$$V_A = 7 \ pa.$$

The reaction force H_A. *We shall remove this time the hinged support from A with one vertical simple support and the horizontal reaction force (Fig.9).*

The body DB is fixed because it has a fixed support and in this way the internal hinge from D becomes a fixed hinge.

Making as for the previous reaction force we obtain the rotation centers (the absolute rotation center (02) is in point D, the relative rotation center (12) is located in point C and the absolute rotation center (01) is find at the intersection of the vertical line passing through the point A (the direction of the simple support) and the horizontal line passing through the centers (02) and (12)).

The two diagrams of the virtual displacements are obtained considering first a virtual rotation for one arbitrary body (here the counterclockwise rotation $\delta\theta_1$ of the body (I)0.

We choose as kinematic parameter:

$\delta\theta_1 = 1$

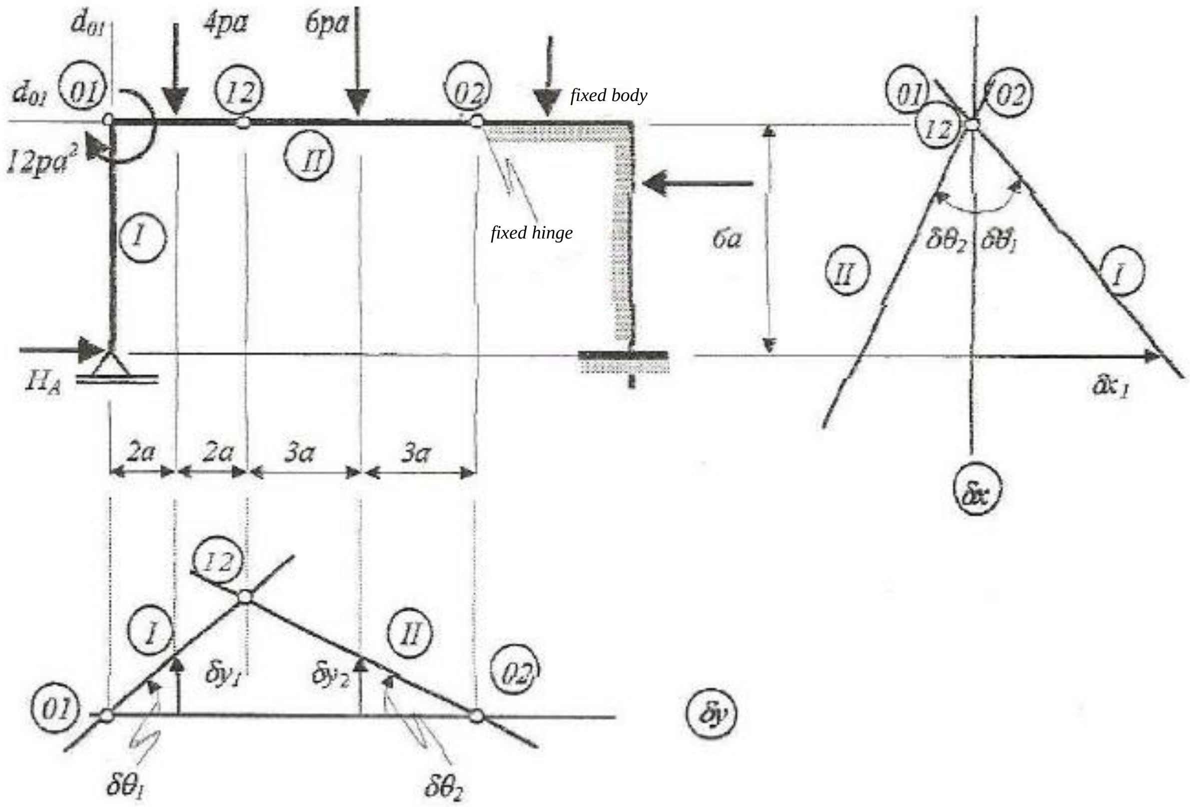


Fig.9.

The rotation of the second body will be calculated from the equality in the relative rotation center (12):

$$4a\,.\,\delta\theta_1 = 6a\,.\,\delta\theta_2$$

from which results:

$$\delta\theta_2 = 0{,}667$$

The virtual displacements of the forces will be:

$$\delta x_1 = 6a\,.\,\delta\theta_1 = 6a;\ \delta y_1 = 2a\,.\,\delta\theta_1 = 2a;\ \delta y_2 = 3a\,.\,\delta\theta_2 = 2a;$$

The virtual work results:

$$\delta L = H_A \cdot \delta x_1 - 4pa \cdot \delta y_1 - 6pa \cdot \delta y_2 - 12pa^2 \cdot \delta\theta_1 = 0$$

from which we have the reaction force:

$H_A = 5{,}33\ pa.$

The reaction force V_B. *In the point B in place of the fixed support we shall consider one vertical gliding support and the reaction force V_B.*

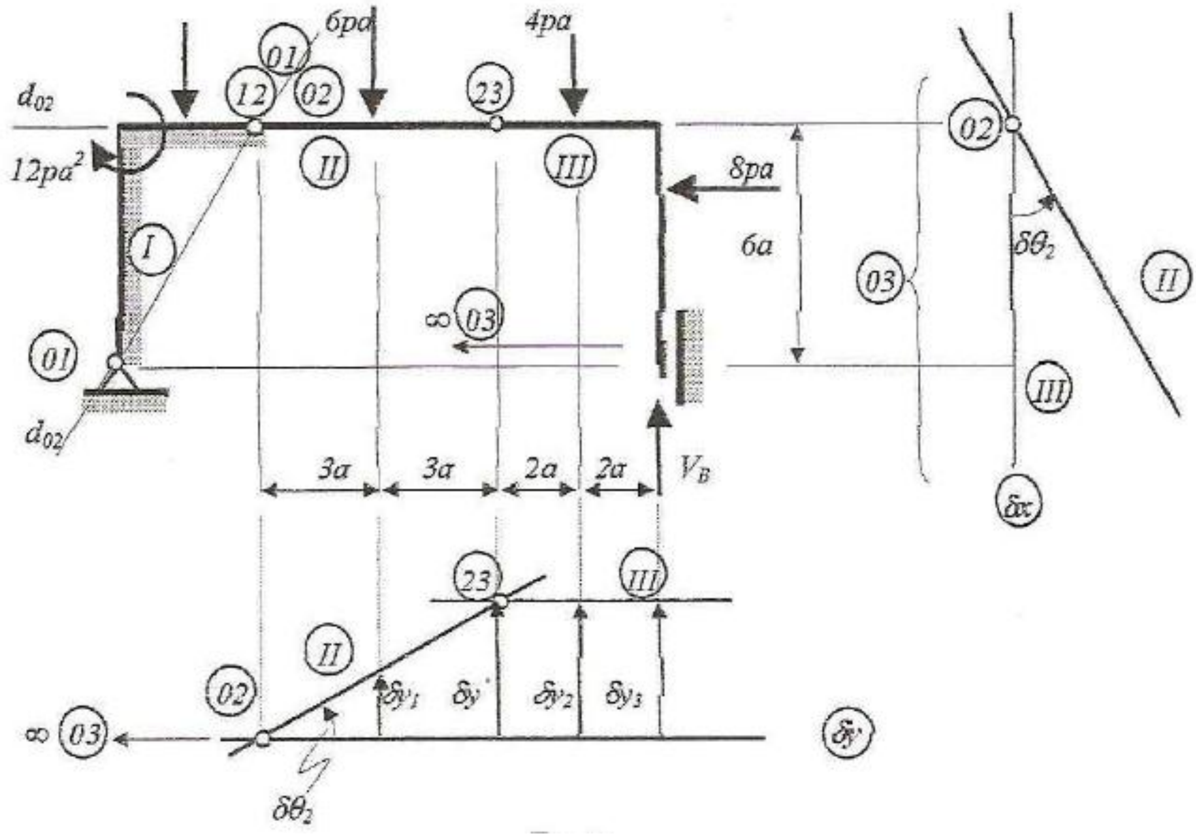


Fig.10.

First we remark that in this case we have not fixed bodies consequently we have a mechanism with three bodies.

We shall have the following rotation centers: the absolute rotation center (01) in the hinged support from the point A, the relative rotation center (12) in the internal hinge C, the relative rotation center (23) is located in the internal hinge from D, and the body DB will have the absolute rotation center (03) at infinity distance on horizontal direction (because this body has one vertical gliding support it will perform only a vertical translation motion so the instantaneous center of rotation is located at infinity distance on horizontal direction).

We have to find the absolute rotation center (02). For this we shall use the theorems of co linearity (or the rule of indexes) finding first one straight line d_{02} that passes through the centers (01) and (12) and after another straight line passing through the centers (03) and (23) *namely a horizontal line passing through the center (23). The intersection of these two lines is located in the point representing the relative center (12) so in this point is located the absolute rotation center (02) also. This position shows us the following situation: we know that the relative rotation center (12) is the point in which the velocities (or the displacements) from the two bodies are the same but the body (II) has here zero velocity (or displacement) because here is located its absolute rotation center. This means that the body (I) has here the same velocity namely zero velocity so it has the absolute rotation center (01) also. We can state the following propriety:* ***if one absolute rotation center is located in one relative rotation center with one common index then in the same point is located the absolute rotation center corresponding to the non common index.***

But the first body (I) has its absolute rotation center in another position and a body can have only one absolute rotation center. It results also another propriety: ***if a body has two or more absolute rotation centers then the body immobilized.*** *So the body (I) can be considered fixed and eliminated from the mechanism.*

In conclusion the mechanism has only two bodies: (II) and (III).

We draw the diagrams of the virtual displacements and we choose as kinematic parameter the rotation:

$$\delta\theta_2 = 1$$

The virtual displacements of the points of application of the forces will be:

$$\delta y_1 = 3a \cdot \delta\theta_2 = 3a\ ;\ \delta y_2 = \delta y_5 = \delta y = 6a \cdot \delta\theta_2 = 6a$$

with which we have the virtual work:

$$\delta L = -6pa \cdot \delta y_1 - 4pa \cdot \delta y_2 + V_B \cdot \delta y_5 = 0$$

The searched reaction force will be:

$$V_B = 7\ pa.$$

The reaction force H_B. *For this reaction force we remove the fixed support from B with the horizontal gliding support and the horizontal reaction force H_B and we obtain a mechanism with three bodies (Fig. 11).*

The position of the absolute rotation center (02) results writing one likeness relation between the triangles (02)(12)(23) and (12)(01)E:

$$\frac{6a}{y_2} = \frac{4a}{6a} \longrightarrow y_2 = 9a$$

After which we draw the diagrams of displacements we choose as kinematic parameter the rotation of the body (I):

$$\delta\theta_1 = 1$$

and then we write the equality in the relative rotation center (12):

$$4a \,.\, \delta\theta_1 = 6a \,.\, \delta\theta_2$$

from which we have:

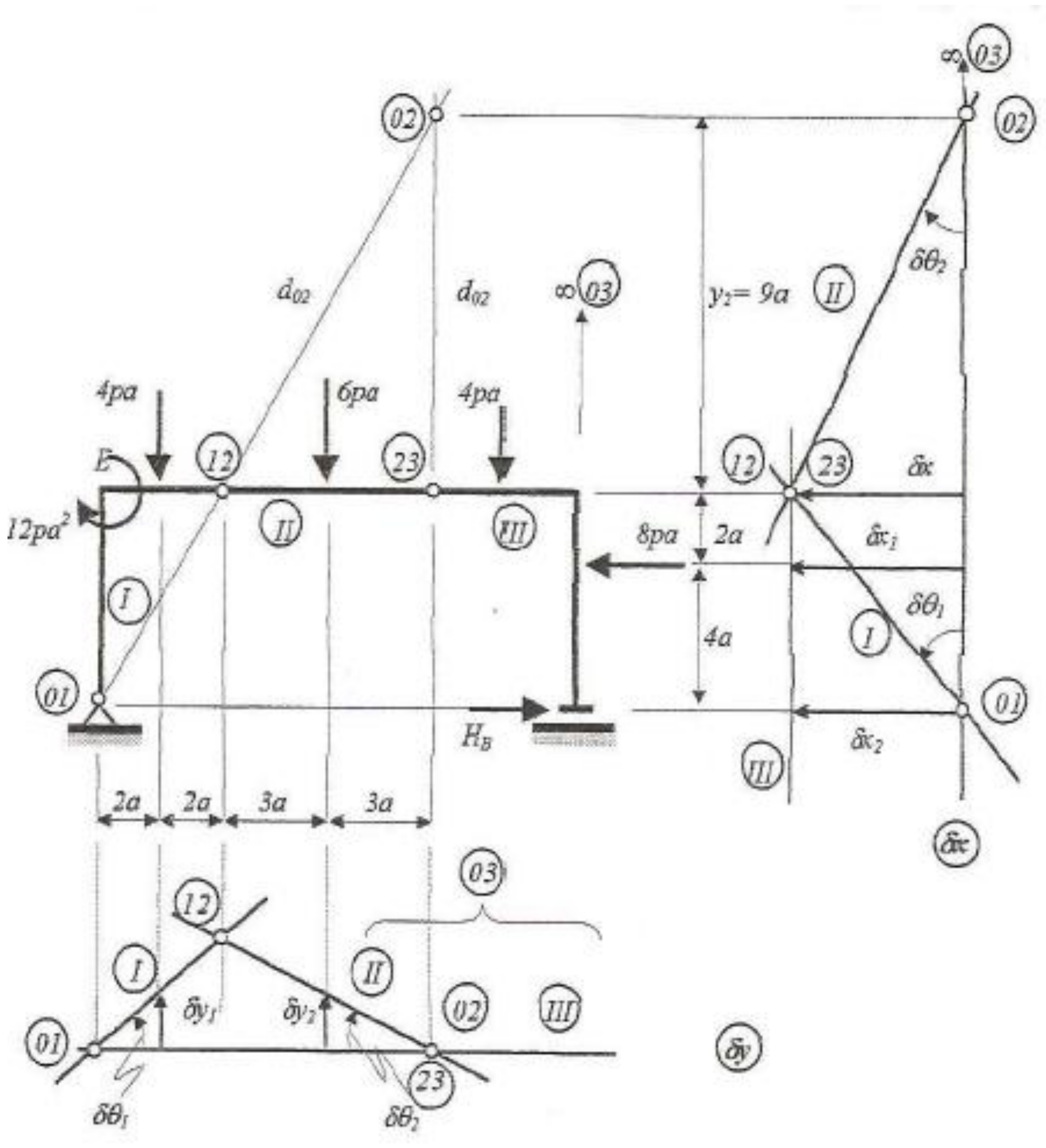


Fig.11.

$$\delta\theta_2 = 0{,}667$$

With these values we have the virtual displacements of the forces:

$$\delta x_1 = \delta x_2 = \delta x = 9a \cdot \delta\theta_2 = 6a;\ \delta y_1 = 2a \cdot \delta\theta_1 = 2a;\ \delta y_2 = 3a \cdot \delta\theta_1 = 2a$$

The virtual work will be:

$$\delta L = 8pa \cdot \delta x_1 - H_B \cdot \delta x_2 - 4pa \cdot \delta y_1 - 6pa \cdot \delta y_2 - 12pa^2 = 0$$

equation from which results the reaction force:

$H_B = -2{,}67\ pa.$

The reaction moment M_B. *We remove the fixed support from B with a hinged support and the unknown concentrated moment M_B. The mechanism has one degree of freedom and three bodies (Fig.12.).*

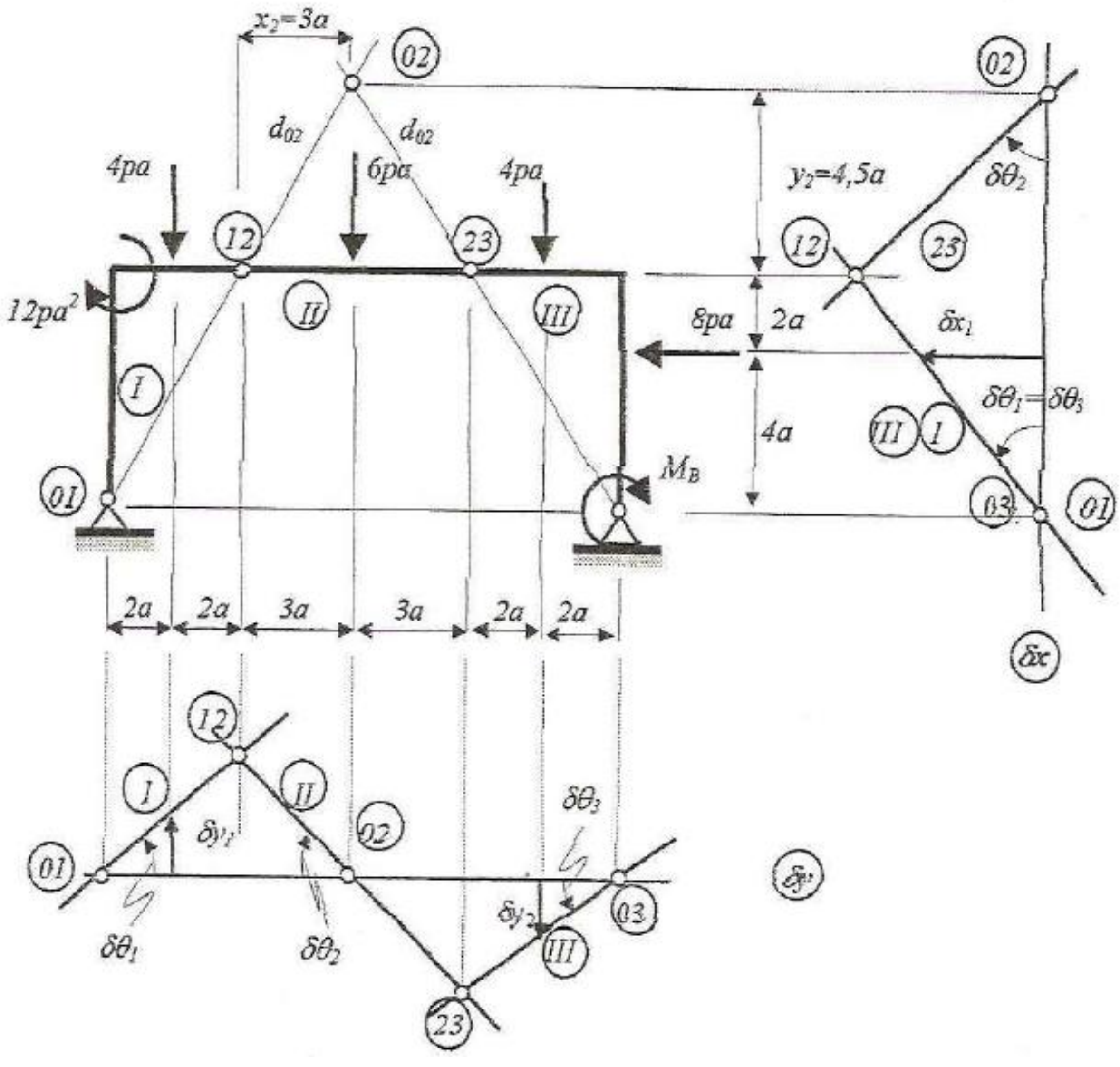


Fig.12.

The position of the absolute rotation center (02) that is find at the intersection of two straight lines d_{02} is determined writing two likeness relations (each of them for one inclined line). From the system of two equations we have:

$$x_2 = 3a;\ y_2 = 4,5a$$

We draw the diagrams of the virtual displacements and we choose as kinematic parameter the rotation of the body (I) (equal to the rotation of the body (III)):

$$\delta\theta_1 = \delta\theta_3 = 1$$

For the body (III) we write the equality in the relative rotation center (12) (or in (23)):

$$4a \cdot \delta\theta_1 = 3a \cdot \delta\theta_2 \longrightarrow \delta\theta_2 = 1,33$$

With these rotations we shall calculate the virtual displacements in the points of application of the forces:

$$\delta x_1 = 4a \cdot \delta\theta_3 = 4a;\ \delta y_1 = 2a \cdot \delta\theta_1 = 2a;\ \delta y_2 = 2a \cdot \delta\theta_3 = 2a$$

The condition of equilibrium will be:

$$\delta L = 8pa \cdot \delta x_1 - 4pa \cdot \delta y_1 + 4pa \cdot \delta y_2 - 12pa^2 \cdot \delta\theta_1 - M_B \cdot \delta\theta_3 = 0$$

from which we have:

$M_B = 20\ pa^2$

Problem 4. *Using the principle of the virtual work calculate the reaction forces from the constraints of the system from the figure 13.*

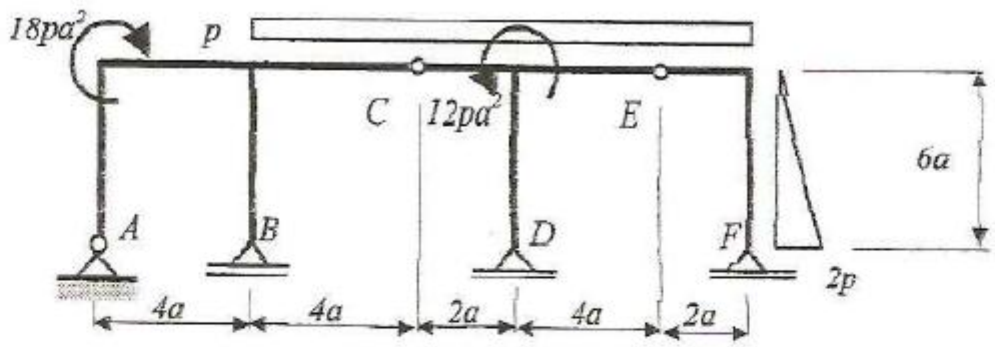


Fig.13.

21.7. Calculation of the internal forces from the members of a truss.

The calculation of the internal forces in the members of a truss using the principle of the virtual work is made in the same way as for the reaction forces for the systems of rigid bodies. The differences in the calculations are coming from the fact that now we calculate a internal forces and also that the members can be considered or bodies or simple internal connections.

When we want to determine the internal force from a member for to transform the truss in a mechanism with one degree of freedom we shall remove the corresponding member with a pair of unknown internal forces on the direction of the member having opposite senses acting about the two joints corresponding to that member. As we know the internal forces will be considered as tensions because in this way the sign of the result corresponds to the sign of the convention for the axial internal forces.

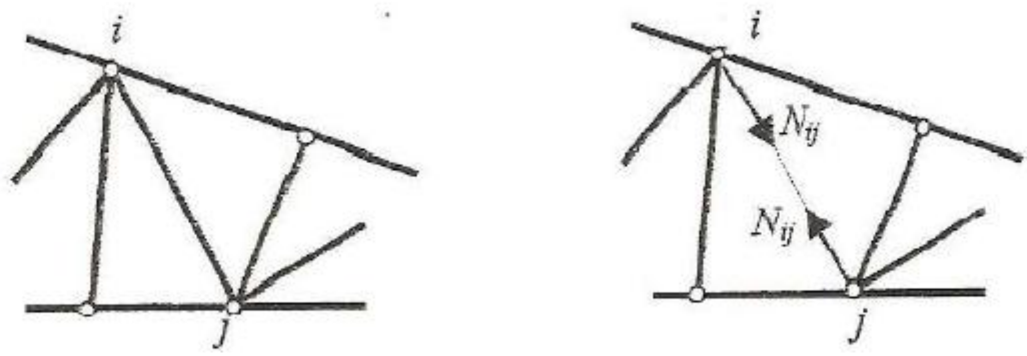


Fig.14.

We shall consider that the independent members which are unloaded will be considered as simple internal connections. Also the members of the truss which make non-deformable geometric configurations will be considered parts of the bodies represented by those non-deformable parts. One *independent member will be considered as body if it cannot be considered unloaded by external forces or internal forces.*

We make also the remark that if the internal force has other direction as the horizontal or vertical directions then it will be decomposed in two directions corresponding to the directions of the virtual displacements, namely on the horizontal and vertical directions.

21.8. Sample problems

Problem 5. *Using the principle of virtual work determine the internal forces from the marked members for the simple truss from the figure 15.*

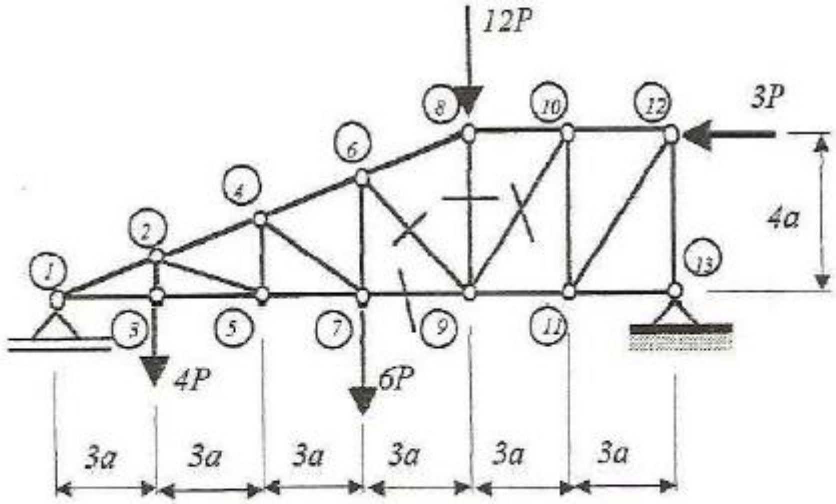


Fig.15.

Solution. *First we shall number in one any order the joints of the truss.*

The internal force N_{79}. *We remove the marked member with a pair of unknown tensions with opposite senses on the direction of that member. it is obtained one mechanism with one degree of freedom in which we can consider the parts 1-6-7 and 6-8-12-13-9 as the two component bodies because these parts being made from triangles side by side they are non-deformable parts.*

In conclusion we have one mechanism made from two bodies. We make the remark that the joints (which are internal hinges) have behaviors of internal hinges if they allow rotations (relative or absolute rotations) namely only the joints between the resulted bodies will be considered as internal hinges (here the joint (6)).

After the determination of the rotation centers we draw the diagrams of the virtual displacements and we shall choose as kinematic parameter the rotation of the first body:

$$\delta\theta_1 = 1$$

We calculate the rotations of the bodies and the displacements of the forces as for any other system of bodies and we have:

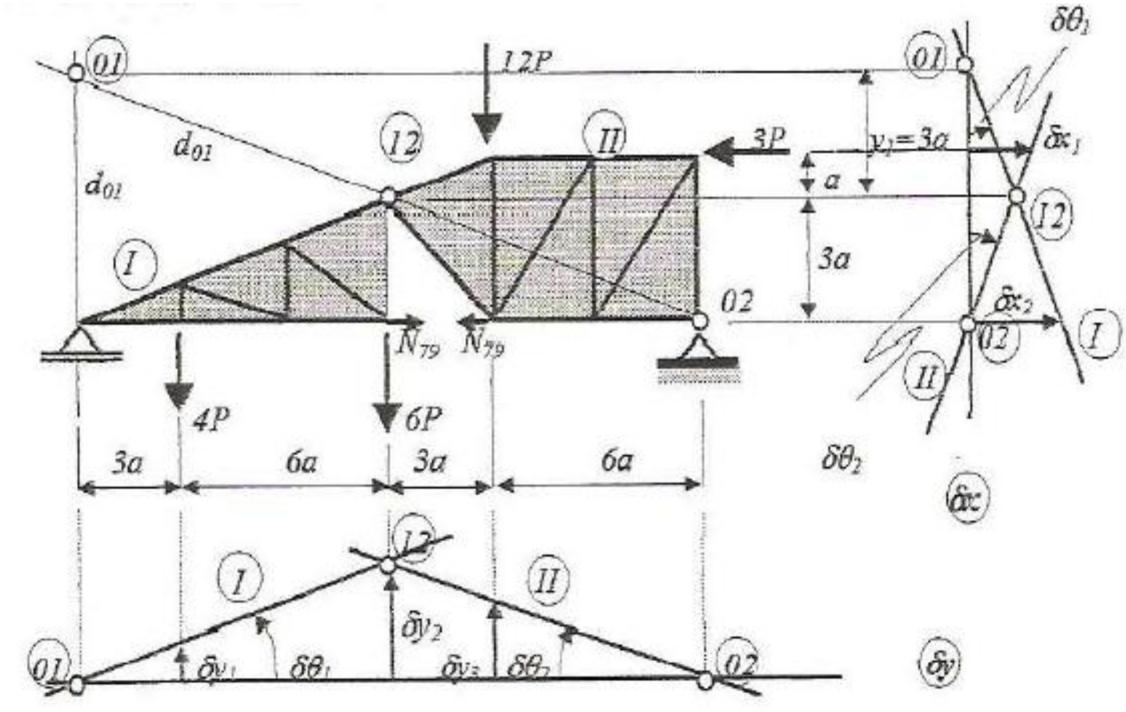


Fig.16.

$$\delta\theta_2 = 1;\ \delta x_1 = 4a\ \delta\theta_2 = 4a;\ \delta x_2 = 6a\ \delta\theta_1 = 6a;\ \delta y_1 - 3a\ \delta\theta_1 - 3a;$$
$$\delta y_2 = 9a\ \delta\theta_1 = 9a;\ \delta y_3 = 6a\ \delta\theta_2 = 6a;$$

with which we calculate the virtual work of the all the forces from the mechanism:

$$\delta L = -3P\ .\ \delta x_1 + N_{79}\ .\ \delta x_2 - 4P\ .\ \delta y_1 - 6P\ .\ \delta y_2 - 12P\ .\ \delta y_3 = 0$$

that is the condition of equilibrium and from which results the internal force:

N_{79} *= 25P*

The internal force N_{69}. *We remove the member with the pair of the unknown internal forces N_{69}.*

The mechanism has two bodies: the non-deformable part 1-6-7 and 8-12-13-9 and two independent members 6-8 and 7-9 which can be considered unloaded (the vertical force 12P and the internal force N_{69} from the joint (9) are considered acting about the body (II) and the internal force N_{69} from the joint (6) as the force 6P are acting about the body (I). We shall consider only the joints (6), (7), (8) and (9) as internal hinges. Consequently the two members will be considered as simple internal connections and on their directions will be find the relative rotation center (12).

We remark that the absolute rotation center (01) is located in the relative rotation center (12) that means that in this point is located also the absolute rotation center (02). But the body (II) has another absolute rotation center, so this body may be considered "fixed body" and it will be eliminated from the mechanism remaining only one body in motion.

For the single body in motion (I) we draw the diagrams of the virtual displacements and we have:

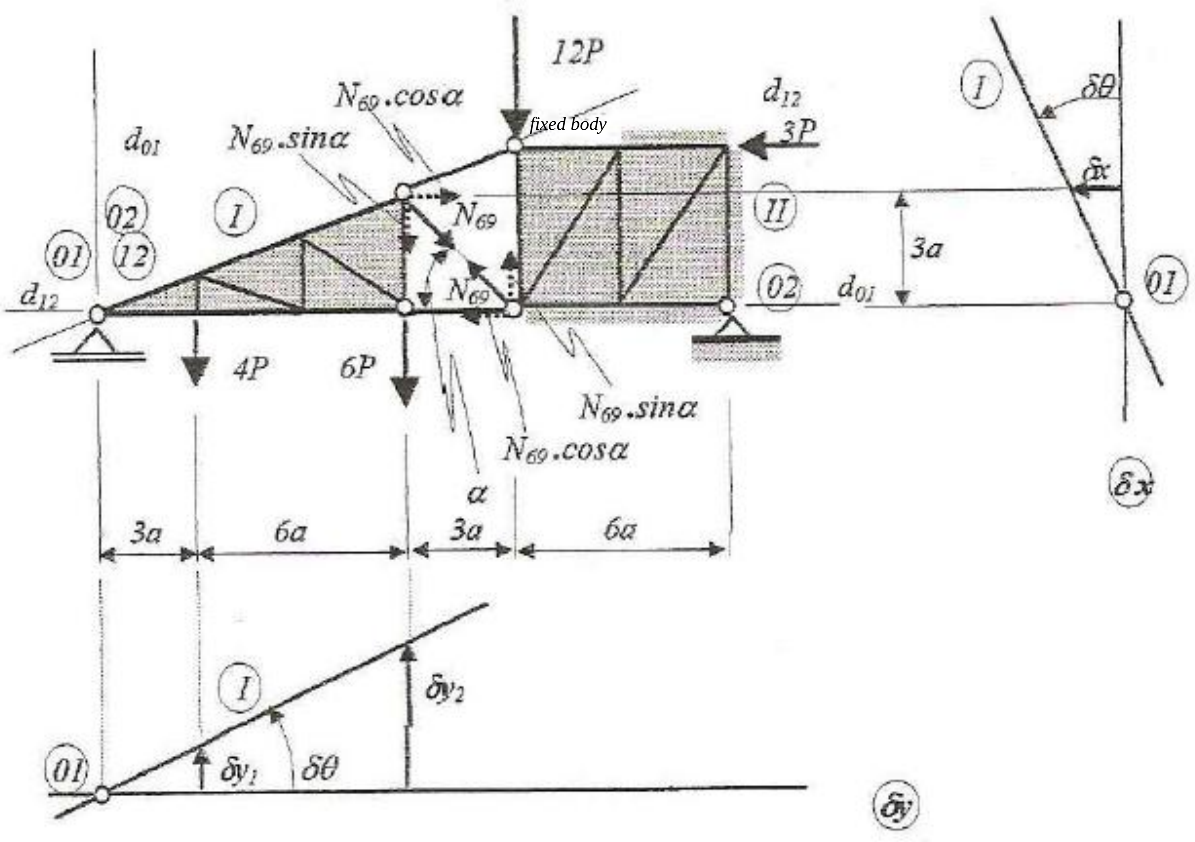


Fig.17.

$$\delta\theta = 1\ ;\ \delta x = 3a\,.\,\delta\theta = 3a\ ;\ \delta y_1 = 3a\,.\,\delta\theta = 3a\ ;\ \delta y_2 = 9a\,.\,\delta\theta = 9a$$

For to calculate the virtual work we shall decompose the internal forces N_{69} in two components with which we shall calculate the condition of equilibrium:

$$\delta L = -\,N_{69}\cos\alpha\,.\,\delta x - 4P\,.\,\delta y_1 - 6P\,.\,\delta y_2 - N_{69}\sin\alpha\,.\,\delta y_2 = 0$$

from which we have:

$N_{69} = -7{,}78\ P$.

The internal force N_{89}. *We remove (we eliminate) the member 8-9 and we replace with the pair of tensions N_{89} in the two joints (Fig.18).*

The mechanism that resulted will contain three bodies: The non-deformable part 1-6-9, the part 10-12-13-9 and one of the two independent members 6-8 and 8-10. We can see the two forces (the forces 12P and N_{89}) which are acting in the joint (8) have to considered acting or on the member 6-8 or on the member 8-10. In this problem we shall consider these forces acting on the body 6-8, consequently the member 8-10 will be considered as a simple internal connection.

After which we determine the rotation centers for the resulted mechanism from three bodies we draw the diagrams of the virtual displacements.

The kinematic parameter will be:

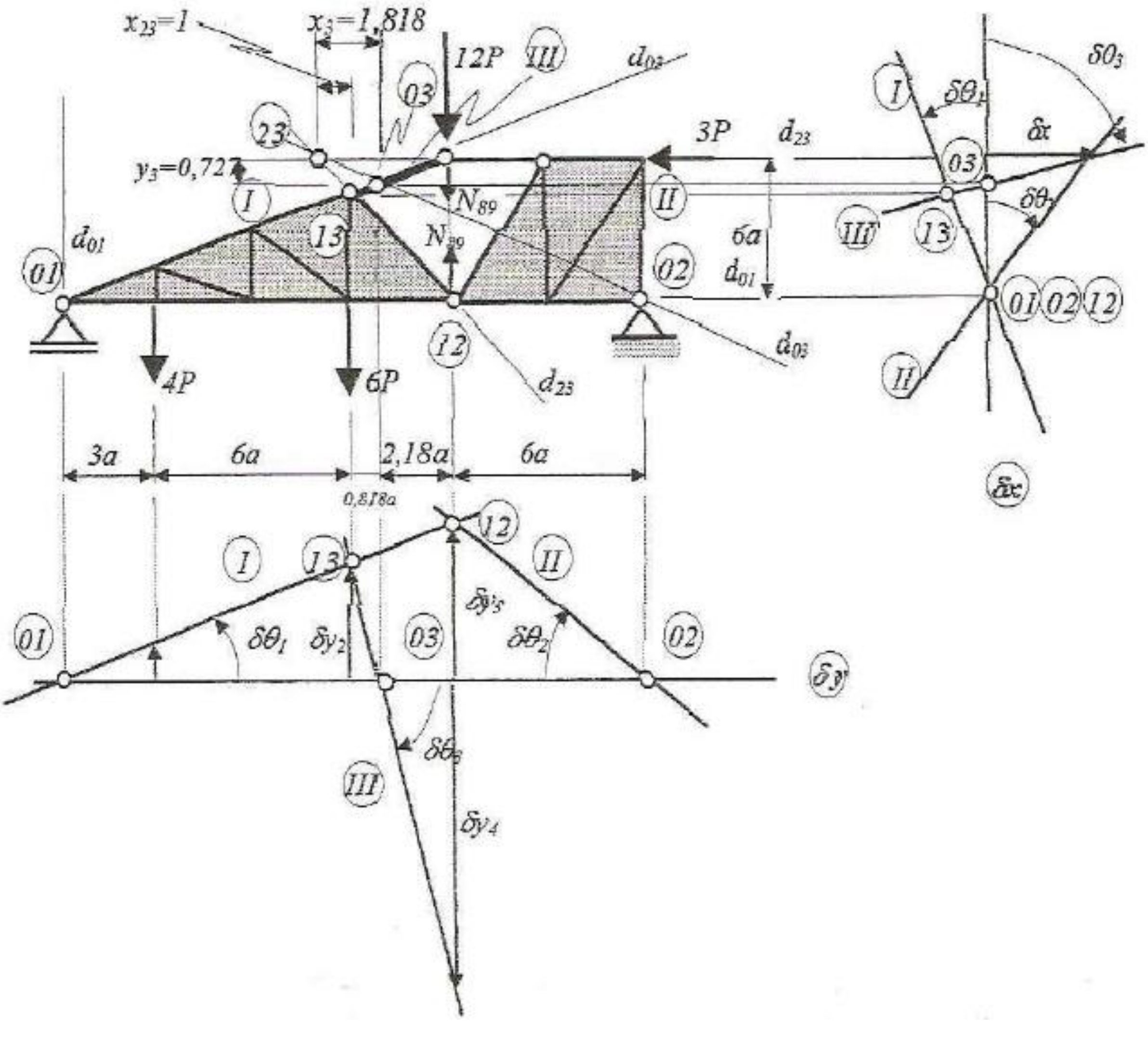


Fig.18.

$$\delta\theta_1 = 1$$

Function of this we shall calculate the other rotations and the virtual displacements of the forces which are acting about the mechanism:

$$12a\,.\,\delta\theta_1 = 6a\,.\,\delta\theta_2 \longrightarrow \delta\theta_1 = 2\,;$$
$$9a\,.\,\delta\theta_1 = 0{,}818a\,.\,\delta\theta_3 \longrightarrow \delta\theta_3 = 11;$$
$$\delta x = 4a\,.\,\delta\theta_2 = 8a\,;\ \delta y_1 = 3a\,.\,\delta\theta_1 = 3a\,;\ \delta y_2 = 9a\,.\,\delta\theta_1 = 9a\,;$$
$$\delta y_3 = 12a\,.\,\delta\theta_1 = 12a\,;\ \delta y_4 = 2{,}18a\,.\,\delta\theta_3 = 24a$$

With these we shall calculate the virtual work and we shall consider the condition of equilibrium:

$$\delta L = -3P.\,\delta x - 4P.\,\delta y_1 - 6P.\,\delta y_2 - 12P.\,\delta y_3 - N_{89}.\,\delta y_3 - N_{89}.\,\delta y_4 = 0$$

From this equation results:

$N_{89} = -6{,}5\,P$.

The internal force N_{910}. *It is eliminated the member 9-10 and it is removed by the pair of the unknown tensions N_{9910} (Fig.19).*

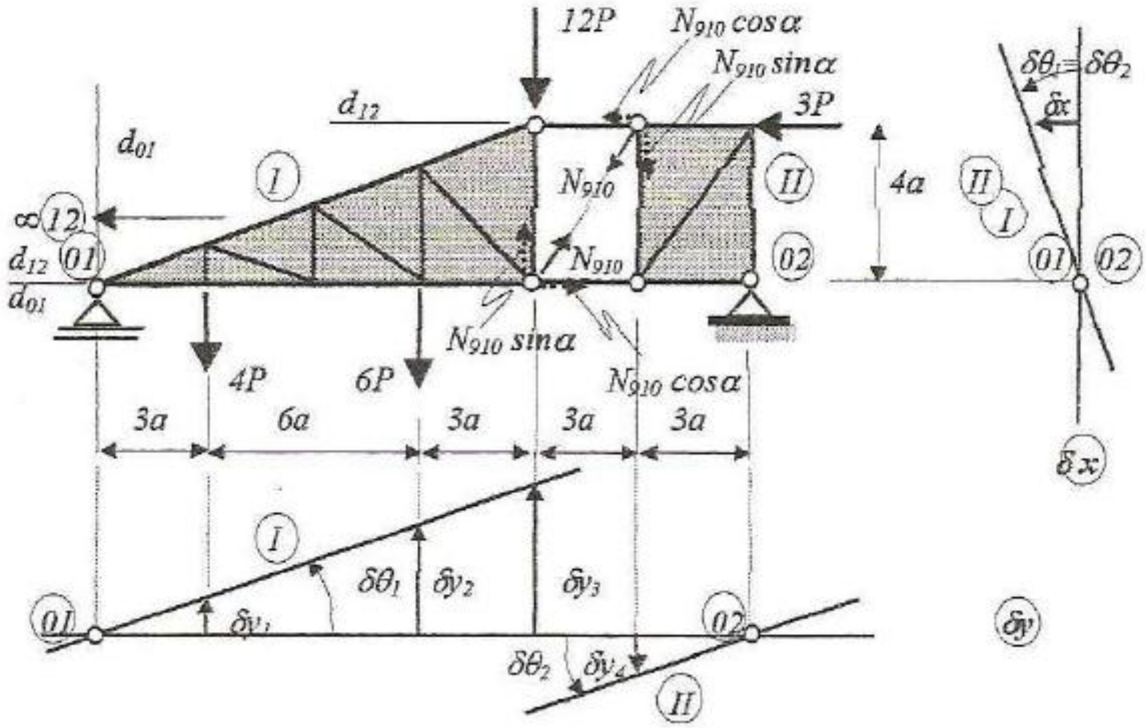


Fig.19.

The resulted mechanism is made from two bodies joined among them with two pendulums (the internal simple connections). As we can see the relative rotation center of the two bodies results at infinity distance (on the horizontal direction). This means that the two bodies will have equal displacements at infinity distance. Consequently the diagrams of the two bodies are parallel (or collinear) the bodies having the same rotations.

After which we have the diagrams of the virtual displacements result the displacements of the forces:

$$\delta\theta_1 = \delta\theta_2 = 1\ ;\ \delta x = 4a\,.\,\delta\theta_2 = 4a\ ;\ \delta y_1 = 3a\,.\,\delta\theta_1 = 3a\ ;$$
$$\delta y_2 = 9a\,.\,\delta\theta_1 = 9a\ ;\ \delta y_3 = 12a\,.\,\delta\theta_1 = 12a\ ;\ \delta y_4 = 3a\,.\,\delta\theta_2 = 3a$$

The equilibrium condition is written under the form:

$$\delta L = 3P.\,\delta x + N_{910}\cos\alpha.\,\delta x - 4P.\,\delta y_1 - 6P.\,\delta y_2 - 12P.\,\delta y_3 +$$
$$+ N_{910}\sin\alpha.\,\delta y_3 + N_{910}\sin\alpha.\,\delta y_4 = 0$$

condition from which results the internal force:

$N_{910} = 13{,}75\ P.$

Problem 6. *Using the principle of virtual work calculate the internal forces from the marked members of the simple truss represented in the figure 20.*

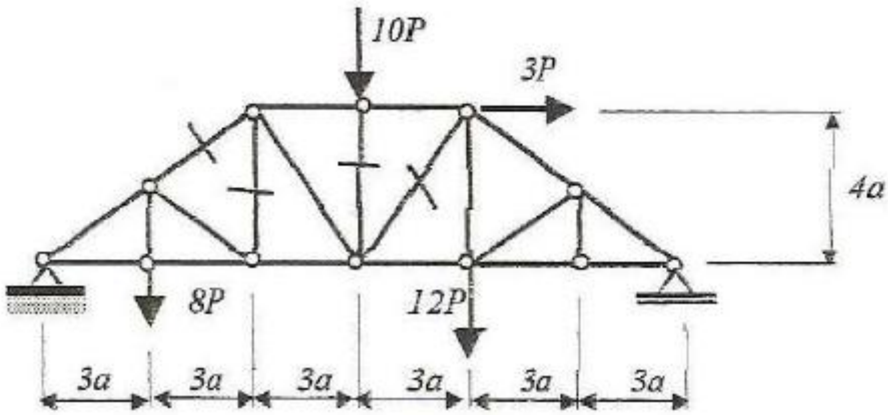


Fig.20.

21.9. The principle of virtual work in generalized coordinates.

The drawback of the principle of virtual work presented in the previous sections is that it offers us one single scalar equation for to express the condition of equilibrium of a mechanical system. This drawback is a big problem in the case of a system with more degrees of freedom. In these cases the number of the scalar independent position parameters is equal to the number of *the degrees of freedom, namely one single equation cannot solve these kinds of problems.*

This drawback can be eliminated if we express the virtual displacement function of the generalized coordinates (in fact function of the independent position parameters).

We know that the position vector of one any point of the mechanical system can be expressed function of the generalized coordinates in the following way:

$$\overline{r_i} = \overline{r_i}\,(q_1, q_2, \ldots, q_s)$$

where s is the number of the degrees of freedom of the mechanical system.

Because the virtual displacement is a virtual variation of the position vector (δ is a differential operator that consider the time as a constant) we can write:

$$\delta \overline{r_i} = \sum_{k=1}^{s} \frac{\partial \overline{r_i}}{\partial q_k} \delta q_k$$

where δq_k is the virtual variation of the generalized coordinate q_k.

Let to remove this expression of the virtual displacement in the virtual work produced by the active forces for a system of particles:

$$\delta L = \sum_{i=1}^{n} \overline{F_i} \cdot \delta \overline{r_i} = \sum_{i=1}^{n} \overline{F_i} \cdot \sum_{k=1}^{s} \frac{\partial \overline{r_i}}{\partial q_k} \delta q_k$$

In this last relation we shall change the order of the summarization:

$$\delta L = \sum_{k=1}^{s} \left(\sum_{i=1}^{n} \overline{F_i} \cdot \frac{\partial \overline{r_i}}{\partial q_k} \right) \delta q_k$$

We shall mark the parenthesis that is a scalar quantity:

$$\sum_{i=1}^{n} \overline{F_i} \cdot \frac{\partial \overline{r_i}}{\partial q_k} = Q_k$$

This expression represents the ***generalized force*** *corresponding to the* ***k*** *degree of freedom (or corresponding to the generalized coordinate* q_k*). We see that the generalized force has the dimension of force if the generalized coordinate has the dimension of length and has the dimension of moment if the generalized coordinate is an angle.*

We can remark that also the generalized force can be calculated in the following way:

$$Q_k = \frac{\delta L}{\delta q_k} = \frac{\partial L}{\partial q_k}$$

where $\partial L = dL$ *is the elementary work of the active forces.*

The equilibrium condition of the mechanical system:

$$\delta L = \sum_{k=1}^{s} Q_k \cdot \delta q_k = 0$$

Because the virtual variations are arbitrary as magnitudes and the generalized coordinates are independent we can give different values for these variations. In this way if we consider the following values for these variations:

$$\delta q_1 \neq 0 \; ; \; \delta q_2 = \ldots = \delta q_s = 0$$

then results that the equation (the condition of equilibrium) is verified for the following value of the generalized force:

$Q_1 = 0$

Making in the same way with all the generalized coordinates (their variations) is obtained the following system of equations:

$$\begin{cases} Q_1 = 0\,; \\ \vdots \\ Q_k = 0\,; \\ \vdots \\ Q_s = 0 \end{cases}$$

This system of equations expresses the equilibrium of a mechanical system with **s** *degrees of freedom. We can state also:* ***the enough and necessary condition as a mechanical system with ideal constraints to be in equilibrium is that all the generalize forces to be equal to zero.***

Chapter 22. Lagrange equations

22.1. Principle of virtual work for the state of motion.

Suppose a system of particles P_j by masses $\mathbf{m_j}$ *in motion. We shall suppose the system having ideal constraints and acted by a system of forces $\overline{F_j}$ ($\overline{F_j}$ is the resultant of the system of concurrent forces acting about the particle P_j).*

Using the D'Alembert's principle, after which the reaction forces have replaced the corresponding constraints, we can express the dynamic equilibrium of the system. For the particle P_j we have:

$$\overline{F_j} + \overline{R_j} + \overline{F_{inj}} = 0$$

where $\overline{R_j}$ represents the resultant force of the reaction forces acting about the particle and $\overline{F_{inj}}$ is the inertia force corresponding to the particle P_j:

$$\overline{F_{inj}} = - m_j \overline{a_j}$$

We calculate the scalar product of the equilibrium equation with one virtual displacement and we add for all particles from the system. We shall find considering also that the virtual work of the reaction forces is equal to zero for the ideal constraint the dynamic equilibrium condition for the system of particles:

$$\delta L = \sum_{j=1}^{n} (\overline{F}_j + \overline{F}_{inj}) \cdot \delta \overline{r_j} = 0$$

This condition can be state: ***for a mechanical system with ideal constraints the virtual work of the active forces and inertia forces is equal to zero.***

22.2. Lagrange equations by first kind

Removing, in the condition of dynamic equilibrium, the inertia force with its expression we obtain:

$$\delta L = \sum_{j=1}^{n} (\overline{F}_j - m_j \overline{a}_j) \cdot \delta \overline{r_j} = 0$$

If the system of particles has more degrees of freedom it is convenient to express the motion and the virtual displacements function of the generalized coordinates. It is obtained the equation:

$$\sum_{j=1}^{n} (\overline{F}_j - m_j \overline{a}_j) \cdot \sum_{k=1}^{s} \frac{\partial \overline{r}_j}{\partial q_k} \delta q_k = 0$$

For to disconnect the dynamic condition of equilibrium in equations corresponding to the degrees of freedom (or corresponding to the generalized coordinates) we shall make as for the condition of equilibrium in the state of rest, namely first we change the order of the summarization:

$$\sum_{k=1}^{s} [\sum_{j=1}^{n} (\overline{F}_j - m_j \overline{a}_j) \cdot \frac{\partial \overline{r}_j}{\partial q_k}] \delta q_k = 0$$

We mark the straight parenthesis:

$$\sum_{j=1}^{n} (\overline{F}_j - m_j \overline{a}_j) \frac{\partial \overline{r}_j}{\partial q_k} = Q_k^*$$

representing ***the generalized force for the state of motion*** *and the previous equation becomes:*

$$\sum_{k=1}^{s} Q_k^* \cdot \delta q_k = 0$$

Considering all the virtual variations equal to zero excepting one (as we have made in the state of rest) is obtained the system of differential equations:

$$\begin{cases} Q_1^* = 0 ; \\ \vdots \\ Q_k^* = 0 ; \\ \vdots \\ Q_s^* = 0 \end{cases}$$

Removing function of the active forces and the accelerations of the particles we have:

$$\begin{cases} \sum_{j=1}^{n} (\overline{F}_j - m_j \overline{a}_j) \dfrac{\partial \overline{r}_j}{\partial q_1} = 0; \\ \vdots \\ \sum_{j=1}^{n} (\overline{F}_j - m_j \overline{a}_j) \dfrac{\partial \overline{r}_j}{\partial q_k} = 0; \\ \vdots \\ \sum_{j=1}^{n} (\overline{F}_j - m_j \overline{a}_j) \dfrac{\partial \overline{r}_j}{\partial q_s} = 0 \end{cases}$$

This system of equations represents the ***system of the differential equations of the motion*** *of the system of particles in which the equations are called* ***Lagrange equations by first kind****.*

If we integrate these equations are obtained the independent kinematic parameters (the laws of motion) of the mechanical system:

$$q_k = q_k(t) \quad , \quad k = 1,\dots,s$$

We can see that we have obtained the differential equations directly without to intervene the reaction forces from the constraints and connections and the equations are the same form.

It is more convenient in problems as in place of the accelerations to use the velocities of the points. This fact can be obtained through a few transformations of these equations.

22.3. Lagrange equations by second kind.

Because all the equations are identically as form we shall work only with one equation. First we rewrite the equations undoing the parenthesis from the left side and bringing one term in the right side:

$$\begin{cases} \vdots \\ \sum_{j=1}^{n} m_j \overline{a}_j \cdot \frac{\partial \overline{r}_j}{\partial q_k} = \sum_{j=1}^{n} \overline{F}_j \cdot \frac{\partial \overline{r}_j}{\partial q_k} \qquad (k) \\ \vdots \end{cases}$$

The term from the left side can be developed in the following way:

$$\sum_{j=1}^{n} m_j \overline{a}_j \cdot \frac{\partial \overline{r}_j}{\partial q_k} = \sum_{j=1}^{n} m_j \frac{d\overline{v}_j}{dt} \cdot \frac{\partial \overline{r}_j}{\partial q_k} = \frac{d}{dt}\left(\sum_{j=1}^{n} m_j \overline{v}_j \cdot \frac{\partial \overline{r}_j}{\partial q_k}\right) - \sum_{j=1}^{n} m_j \overline{v}_j \cdot \frac{d}{dt}\left(\frac{\partial \overline{r}_j}{\partial q_k}\right)$$

Because the velocity of an arbitrary particle can be expressed function of the generalized coordinates and the generalized velocities:

$$\overline{v}_j = \sum_{k=1}^{s} \frac{\partial \overline{r_j}}{\partial q_k} \dot{q}_k$$

and because the generalized velocities and the generalized coordinates are independent we have the equality:

$$\frac{\partial \overline{v}_j}{\partial \dot{q}_k} = \frac{\partial \overline{r}_j}{\partial q_k}$$

And because the constraints are holonomy and scleronomy we can write:

$$\frac{d}{dt}\left(\frac{\partial \overline{r}_j}{\partial q_k}\right) = \frac{\partial}{\partial q_k}\left(\frac{d\overline{r}_j}{dt}\right) = \frac{\partial \overline{v}_j}{\partial q_k}$$

Removing in the equations we have:

$$\sum_{j=1}^{n} m_j \overline{a}_j \cdot \frac{\partial \overline{r_j}}{\partial q_k} = \frac{d}{dt}\left(\sum_{j=1}^{n} m_j \overline{v}_j \cdot \frac{\partial \overline{v}_j}{\partial \dot{q}_k}\right) - \sum_{j=1}^{n} m_j \overline{v}_j \cdot \frac{\partial \overline{v}_j}{\partial q_k}$$

We can see that in the first term from the right part the parenthesis can be arranged under the form:

$$\sum_{j=1}^{n} m_j \overline{v}_j \cdot \frac{\partial \overline{v}_j}{\partial \dot{q}_k} = \frac{\partial\left(\sum_{j=1}^{n} \frac{m_j v_j^2}{2}\right)}{\partial \dot{q}_k} = \frac{\partial E}{\partial \dot{q}_k}$$

and the second term can be considered under the form:

$$\sum_{j=1}^{n} m_j \overline{v}_j \cdot \frac{\partial \overline{v}_j}{\partial q_k} = \frac{\partial\left(\sum_{j=1}^{n} \frac{m_j v_j^2}{2}\right)}{\partial q_k} = \frac{\partial E}{\partial q_k}$$

If we see that the right member of the differential equation is the generalized force for the state of rest:

$$\sum_{j=1}^{n} \overline{F}_j \cdot \frac{\partial \overline{r_j}}{\partial q_k} = \frac{\partial L}{\partial q_k} = Q_k$$

then removing in the initial equations we shall obtain the following form of the differential equations:

$$\begin{cases} \vdots \\ \frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}_k}\right) - \frac{\partial E}{\partial q_k} = Q_k \quad ; k = 1, \dots , s \\ \vdots \end{cases}$$

These equations are the ***Lagrange equations by second kind*** *and they represent in the same time the system of the differential equations of the motion of the mechanical system.*

Suppose that the mechanical system is acted by a system of conservative forces. This means that the forces are coming from the force functions and the elementary work of these forces are:

$$dL = dU$$

where:

$$U = U(q_1, \dots , q_s) = \Sigma U_j$$

is the resultant force function and depends only by the position of the mechanical system. This means that we have obviously:

$$Q_k = \frac{\partial U}{\partial q_k}; \quad \frac{\partial U}{\partial \dot{q}_k} = 0$$

Considering these proprieties and marking the scalar quantity:

$$\mathfrak{L} = E + U = E - V$$

called ***function of Lagrange*** *or* ***kinetic potential*** *and the Lagrange equations by second kind can be written under the form:*

$$\frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathfrak{L}}{\partial q_k} = 0\,,\ k = 1, \ldots, s$$

22.4. Steps to solve problems using Lagrange equations by second kind.

1)We determine the number of the degrees of freedom of the mechanical system and function of this we choose the independent kinematic parameters which define the motion.

2)After which we define the motions of the bodies we calculate function of the kinematic parameters the velocities of the particles, the velocity of one any point of the body in translation motion, the angular velocities of the bodies in rotation motions and the velocity of the mass center and the angular velocity of the body in plane motion.

3)It is calculated the kinetic energy of the mechanical system.

4)After which we represent the active forces we calculate the elementary work of these forces.

5)We calculate the derivatives of the kinetic energy for each equation and each degree of freedom (three derivatives for each kinematic parameter) and we obtain the terms from the left part of the differential equations (Lagrange equations).

6)We calculate the generalized forces. We make the remark that: it is enough to group the mechanical work after the differentials of the kinematic parameters and the coefficients which multiply these differentials are the corresponding generalized forces.

7)We remove the terms in the Lagrange equations and we obtain the system of the differential equations of the motion of the system.

8)Integrating the system of the differential equations of the motion are obtained the laws of motion of the mechanical system.

22.5. Sample problems.

Problem 1. *The system made from a disc by radius R and mass 2M and a rectilinear bar by length $l_{AB} = 3R$ and mass 4M moves so that the disc performs a rolling motion without sliding on a straight horizontal line and the extremity A of the bar describes a vertical fixed straight line. Determine the differential equation of the motion of the system.*

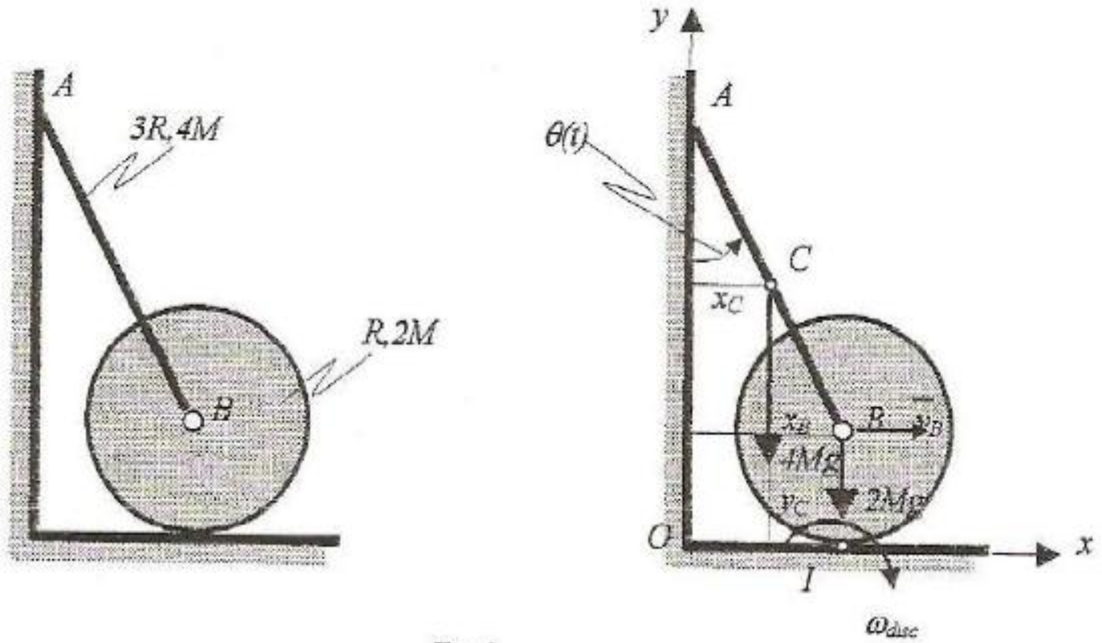


Fig.1.

Solution.*1)The system has one degree of freedom:*

$N_{df} = 3 \cdot 2 - (2 \cdot 1 + 2 + 1) = 1$

because it is made from two bodies it has one internal hinge (in B) and two simple supports (in A and the support of the disc on the horizontal straight line) and one restriction (rolling motion without sliding).

We choose as kinematic parameter the angle θ(t) made between the bar AB and the fixed vertical straight line.

2)The two bodies make plane motions (they have not fixed points and one straight line from the body modify its direction in the time of motion). Consequently we shall calculate the angular velocities of the bodies and the velocities of the mass centers function of the kinematic parameter.

We shall start with the bar AB at which the angular velocity is:

$$\omega_{AB} = \dot{\theta}$$

because by definition the angular velocity is the first derivative with respect to time of the angle made by a line from the body with a fixed line from space (here from plane).

For the calculation of the velocity of the mass center C we shall choose the simplest way namely we shall choose a fixed reference system (here xOy) we express the coordinates of the point with respect this system of reference and deriving the coordinates we obtain the projections of the point velocity. Now squared the coordinates and adding we obtain the square of the velocity of the mass center. We have:

$$\begin{cases} x_C = \frac{3R}{2} \sin\theta; \\ y_C = R + \frac{3R}{2} \cos\theta; \end{cases} \xrightarrow{\text{derivare}} \begin{cases} \dot{x}_C = 1{,}5R\, \dot{\theta} \cos\theta; \\ \dot{y}_C = -\, 1{,}5R\, \dot{\theta} \sin\theta; \end{cases}$$

$$v_C^2 = \dot{x}_C^2 + \dot{y}_C^2 = 2{,}25\, R^2\, \dot{\theta}$$

For the disc we shall make in the following way: first we determine the velocity of the center B using the same way as in the previous calculation:

$$\begin{cases} x_B = 2R \sin\theta; \\ y_B = R \end{cases} \longrightarrow \begin{cases} \dot{x}_B = 3R\, \dot{\theta} \cos\theta; \\ \dot{y}_B = 0 \end{cases}$$

$$v_B^2 = 9\, R^2\, \dot{\theta}^2 \cos^2\theta;$$

For the calculation of the angular velocity we know that in the rolling motion without sliding the instantaneous center of rotation is in the contact point between the disc and the surface on which is made the rolling motion. Having the velocity of the point B we can write:

$$v_B = IB \cdot \omega_{disc}$$

from which results:

$$\omega_{disc} = \frac{v_B}{IB} = 3\dot{\theta} \cos\theta$$

3)The kinetic energy of the system will be calculated with the relation:

$E = E^{bar} + E^{disc}$

where:

$$E^{bar} = \frac{M_{AB} v_C^2}{2} + \frac{J_C \omega_{AB}^2}{2} = \frac{4M.2{,}25R^2\dot{\theta}^2}{2} + \frac{\frac{4M(3R)^2}{12}\dot{\theta}^2}{2} =$$

$$= 6MR^2\dot{\theta}^2;$$

$$E^{disc} = \frac{M_{disc} v_B^2}{2} + \frac{J_B \omega_{disc}^2}{2} = \frac{2M.9R^2\dot{\theta}^2\cos^2\theta}{2} + \frac{\frac{2MR^2}{2}9\dot{\theta}^2\cos^2\theta}{2} =$$

$$= 13{,}5MR^2\dot{\theta}^2\cos^2\theta$$

Finally we have the kinetic energy of the system:

$$E = 6MR^2\dot{\theta}^2 + 13{,}5MR^2\dot{\theta}^2\cos^2\theta$$

4)The active forces that act about the system are the two weights of the two bodies. Being vertical forces they have the projections only on the Oy axis:

$$Y_1 = -4Mg \; ; \; Y_2 = -2Mg$$

The elementary work will be calculated with the relation:

$$dL = \Sigma Y_i \, . \, dy_i$$

where dy_i represents the differential of the coordinate of the point of application of the corresponding force. We have:

$$y_1 = y_C = R + 1{,}5R\cos\theta \; ; \; y_2 = y_B = R$$

whence will result:

$$dy_1 = -\,1{,}5R\sin\theta \, . \, d\theta \; ; \; dy_2 = 0.$$

Removing in the previous formula results the elementary work of the active forces:

$$dL = 6MgR\sin\theta \, . \, d\theta$$

5)The Lagrange equation has the form:

$$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{\theta}}\right) - \frac{\partial E}{\partial \theta} = Q_\theta$$

in which we have three derivatives:

$$\frac{\partial E}{\partial \theta} = 13{,}5\, MR^2 \dot{\theta}^2 . 2\cos\theta . (-\sin\theta);$$

$$\frac{\partial E}{\partial \dot{\theta}} = 6MR^2 . 2\dot{\theta} + 13{,}5MR^2 . 2\dot{\theta} . \cos^2\theta;$$

$$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{\theta}}\right) = 12MR^2 . \ddot{\theta} + 27MR^2 . \ddot{\theta} . \cos^2\theta + \\ + 27MR^2 . \dot{\theta} . 2\cos\theta . (-\sin\theta) . \dot{\theta}$$

6)As we know the work in generalized coordinates has the expression:

$$dL = \Sigma Q_k . dq_k = 6MgR\sin\theta . d\theta$$

so the generalized force will be:

$$Q_\theta = 6MgR\sin\theta$$

7)Removing the find terms in the Lagrange equation results the differential equation of the motion of the given system of bodies:

$$12MR^2 . \ddot{\theta} + 27MR^2 . \ddot{\theta} . \cos^2\theta - 54MR^2 . \dot{\theta} . \cos\theta . \sin\theta - \\ -(-27\, MR^2\dot{\theta}^2 . \cos\theta . \sin\theta) = 6MgR\sin\theta$$

that after simplification becomes:

$$\ddot{\theta}(4 + 9 . \cos^2\theta) - 18 . \dot{\theta}^2 . \cos\theta . \sin\theta = 2\frac{g}{R}\sin\theta$$

Problem 2. *One bar OA having the length 2l and mass 3M rotates in vertical plane about the fixed point O. In the same time on the bar slides a collar P by mass 2M without friction. Knowing that the motion is made under the action of the weights determine the differential equations of the motion of the system.*

Solution*. The system has two degrees of freedom because is made from a body and a particle (the collar) and it has one hinge and one simple support (the bar is the support of the collar):*

$N_{df} = 3 . 1 + 2 . 1 – (2 . 1 + 1) = 2$

Or in the other way: we remark that the bar performs a rotation motion so it has one degree of freedom. If we block this motion then remains only the sliding rectilinear motion of the collar along the bar that has also one degree of freedom. If we block this motion also then the entire system is stopped from the motion consequently the system has two degrees of freedom.

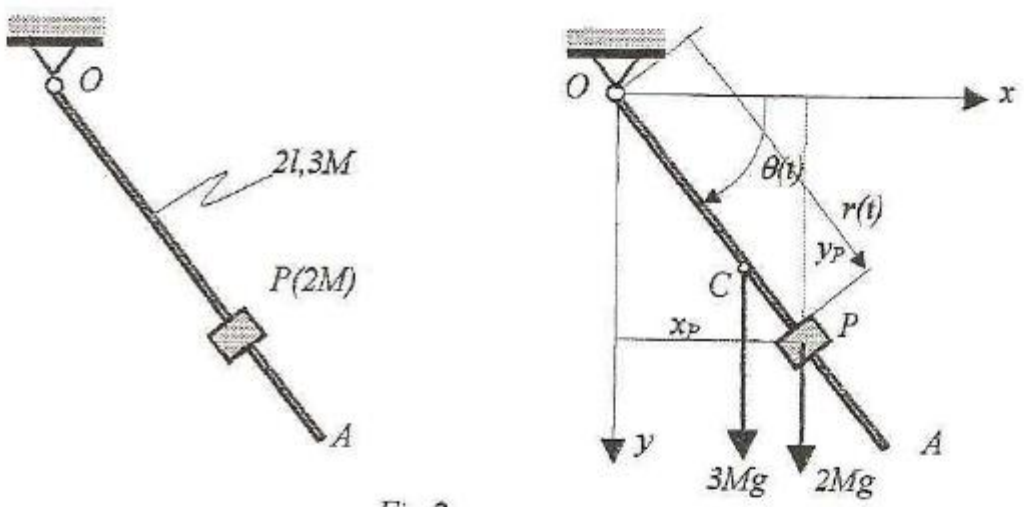


Fig.2.

We choose as kinematic parameters of the system the angle θ(t) made by the bar with the horizontal direction and r(t) the displacement of the collar on the bar.

The kinetic energy of the system will be:

$$E = E_{OA} + E_P$$

where:

$$E_{OA} = \frac{J_O \omega^2}{2};$$

$$E_P = \frac{M_P v_P^2}{2}$$

We shall calculate each element of these energies. First we calculate the moment of inertia of the bar with respect to the end O:

$$J_O = J_C + M_{OA} . OC^2 = \frac{M_{OA} l_{OA}^2}{12} + M_{OA} . OC^2 = 4Ml^2$$

The angular velocity of the bar will be:

$$\omega = \dot{\theta}$$

For the calculation of the velocity of the particle P we shall choose one fixed reference system and we shall express the coordinates of the particle with respect to this system. Deriving the coordinates we obtain finally the velocity of the particle:

$$x_P = r \cdot \cos\theta \ ; \ y_P = r \cdot \sin\theta$$

$$\dot{x}_P = \dot{r} \cdot \cos\theta - r \cdot \dot{\theta} \cdot \sin\theta \ ; \ \dot{y}_P = \dot{r} \cdot \sin\theta + r \cdot \dot{\theta} \cdot \cos\theta$$

$$v_P^2 = \dot{r}^2 + r^2 \cdot \dot{\theta}^2$$

The kinetic energy of the system will be:

$$E = \frac{4Ml^2\dot{\theta}^2}{2} + \frac{2M(\dot{r}^2 + r^2 \cdot \dot{\theta}^2)}{2} = 2Ml^2\dot{\theta}^2 + M\dot{r}^2 + Mr^2\dot{\theta}^2$$

After which we represent the two weights we project them and we have:

$$Y_1 = 3Mg \ ; \ Y_2 = 2Mg$$

and the coordinates of the points of application and their differentials are:

$$y_1 = y_C = l \cdot \sin\theta \ ; \ y_2 = y_P = r \cdot \sin\theta \ ;$$
$$dy_1 = l\cos\theta \cdot d\theta \ ; \ dy_2 = dr \cdot \sin\theta + r \cdot \cos\theta \cdot d\theta$$

With these we calculate the elementary work and putting in order after the differentials of the two kinematic parameters we obtain the generalized forces:

$$dL = \Sigma Y_i dy_i = 3Mg \cdot l\cos\theta \cdot d\theta + 2Mg \cdot (dr \cdot \sin\theta + r \cdot \cos\theta \cdot d\theta) =$$
$$= d\theta \cdot (3Mg \cdot l\cos\theta + 2Mg \cdot r \cdot \cos\theta) + dr \cdot (2Mg \cdot \sin\theta)$$

$$Q_\theta = 3Mg \cdot l\cos\theta + 2Mg \cdot r \cdot \cos\theta \ ; \ Q_r = 2Mg \cdot \sin\theta$$

The two Lagrange equations have the expressions:

$$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{\theta}}\right) - \frac{\partial E}{\partial \theta} = Q_\theta$$
$$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{r}}\right) - \frac{\partial E}{\partial r} = Q_r$$

The six derivatives corresponding to the two equations will be (three derivatives for each equation):

$$\frac{\partial E}{\partial \theta} = 0 : \ \frac{\partial E}{\partial \dot{\theta}} = 4Ml^2 \dot{\theta} + 2Mr^2 \dot{\theta};$$
$$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{\theta}}\right) = 4Ml^2 \ddot{\theta} + 4Mr\dot{r}\dot{\theta} + 2Mr^2 \ddot{\theta};$$
$$\frac{\partial E}{\partial r} = 2Mr\dot{\theta}^2 ; \ \frac{\partial E}{\partial \dot{r}} = 2M\dot{r} : \ \frac{d}{dt}\left(\frac{\partial E}{\partial \dot{r}}\right) = 2M\ddot{r}$$

Removing in the expressions of the two Lagrange equations we obtain the differential equations of the motion (after which we simplify the equations):

$$\begin{cases} 2l^2\ddot{\theta} + 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = g(1{,}5\ l\ +\ r)\cos\theta ; \\ \ddot{r} - r\dot{\theta}^2 = g\sin\theta \end{cases}$$

Problem 3. *Determine the differential equations of the small oscillations of the system made from two straight bars hinged in the fixed points O_1 and O_2 and joined among them with two springs by known elastic constants. Is known also that when the bars are vertically the springs are not loaded (are not tensioned). Are given: $O_1B = 3a$, $AB – O_2D = 2a$, $M_1 = 4M$, $M_2 = 3M$, $k_1 = k$, $k_2 = 2k$.*

Solution. *Because we shall study the small oscillations of the system we shall consider that in the time of motion the two springs remain horizontally.*

The system has two degrees of freedom because the springs are not connections or constraints (they will be removed with active forces).

We shall choose as kinematic parameters the angles made by the two bars with the vertical direction.

The bars perform rotation motions and consequently they will have the angular velocities:

$$\omega_1 = \dot{\theta} ; \ \omega_2 = \dot{\varphi}$$

and the kinetic energy of the system will be:

$$E = E_1 + E_2 = \frac{J_{O_1}\omega_1^2}{2} + \frac{J_{O_2}\omega_2^2}{2}$$

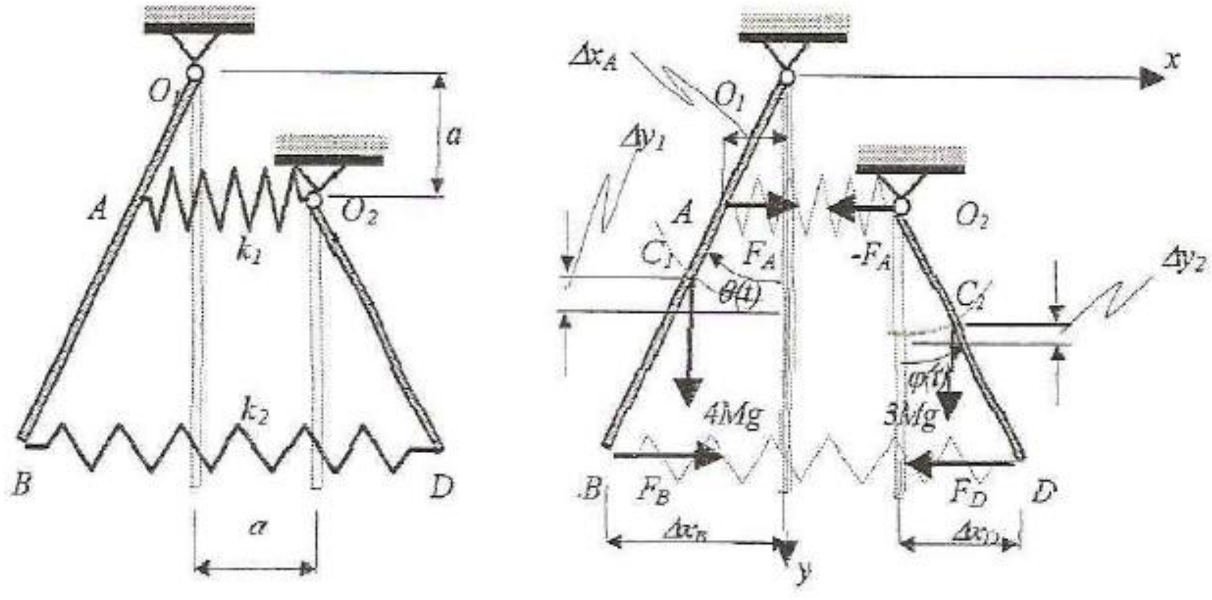


Fig.3.

The moments of inertia of the two bars about their ends will be:

$$J_{O_1} = J_{C_1} + M_1 \cdot O_1C_1^2 = \frac{4M(3a)^2}{12} + 4M(\frac{3a}{2})^2 = 12Ma^2;$$

$$J_{O_2} = J_{C_2} + M_2 \cdot O_2C_2^2 = \frac{3M(2a)^2}{12} + 3Ma^2 = 4Ma^2$$

With these the kinetic energy of the system results:

$$E = 6Ma^2 \dot{\theta}^2 + 2Ma^2 \dot{\varphi}^2$$

Because the active forces which act about the system (the weights and the elastic forces from the springs) are conservative forces we shall calculate the elementary work using the force functions of these forces:

dL = dU

where:

$$U = \Sigma U_i$$

For the weights we have:

$$U_{mg} = \pm\, mg\Delta y$$

namely:

$$U_1 = -\,4Mg\,.\,\Delta y_1 = -\,4Mg\,.\,\frac{3a}{2}(1-\cos\theta);$$
$$U_2 = -\,3Mg\,.\,\Delta y_2 = -\,3Mg\,.\,a(1-\cos\varphi)$$

For the elastic forces we have:

$$U_3 = -\frac{k_1(\Delta x_A)^2}{2} = -\frac{k}{2}(a\sin\theta)^2;$$
$$U_4 = -\frac{k_1(\Delta x_B + \Delta x_D)^2}{2} = -\frac{2k}{2}(3a\sin\theta + 2a\sin\varphi)^2;$$

Finally after which we differential and arrange after the differentials of the two kinematic parameters we have the elementary work of the active forces:

$$dL = \Sigma U_i = (-6Mga\sin\theta - 7ka^2\sin\theta\cos\theta - 4ka^2\sin\varphi\cos\theta)\,d\theta + $$
$$+\,(-3Mga\sin\varphi - 6ka^2\sin\theta\cos\varphi - 4ka^2\sin\varphi\cos\varphi)\,d\varphi$$

For the small oscillations of the system we shall consider the approximations:

$$\sin\theta \approx \theta;\ \cos\theta \approx 1;\ \sin\varphi \approx \varphi;\ \cos\varphi \approx 1$$

with which the generalized forces are:

$$Q_\theta = -(6Mga + 7ka^2)\,\theta - 4ka^2\,\varphi;$$
$$Q_\varphi = -6ka^2\theta - (3Mga + 4ka^2)\,\varphi$$

The Lagrange equations of second kind for the system with two degrees of freedom will be:

$$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{\theta}}\right)-\frac{\partial E}{\partial \theta}=Q_{\theta}$$
$$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{\varphi}}\right)-\frac{\partial E}{\partial \varphi}=Q_{\varphi}$$

The corresponding derivatives for the two equations are:

$$\frac{\partial E}{\partial \theta}=0:\ \frac{\partial E}{\partial \dot{\theta}}=12Ma^2\dot{\theta};\ \frac{d}{dt}\left(\frac{\partial E}{\partial \dot{\theta}}\right)=12Ma^2\ddot{\theta};$$
$$\frac{\partial E}{\partial \varphi}=0:\ \frac{\partial E}{\partial \dot{\varphi}}=4Ma^2\dot{\varphi};\ \frac{d}{dt}\left(\frac{\partial E}{\partial \dot{\varphi}}\right)=4Ma^2\ddot{\varphi}$$

Removing in the Lagrange equations we shall find the differential equations of the small oscillations of the system of bodies (after simplifications):

$$\begin{cases}\ddot{\theta}+\left(\frac{g}{2a}+\frac{7k}{12M}\right)\theta+\frac{k}{3M}\varphi=0;\\ \ddot{\varphi}+\frac{3k}{2M}\theta+\left(\frac{3g}{4a}+\frac{k}{M}\right)\varphi=0\end{cases}$$

As we can see these equations are homogeneous and linear differential equations.

Problem 4. *Determine using the Lagrange equations by second kind the velocity of the particle P knowing that the motion is made in vertical plane under the action of the weights and also that the strings remain vertically in the time of motion (Fig.4.). In the initial position the system will be considered in rest.*

Problem 5. *Determine the differential equations of the motion of the system from the figure 5. Knowing that the disc by mass 2M and radius R performs a rolling motion without sliding on a horizontal fixed straight line and the bar AB by length l_{AB} = 4R and mass 3M is hinged in the center of the disc. The motion is performed in vertical plane under the action of the weights.*

Problem 6. *Determine the differential equations of the motion of the small oscillations of the bar AB that has two vertical springs (Fig.6.). The bar by length 2l and mass 6M in the horizontal initial position is in rest and the horizontal displacements are neglected with respect to the vertical displacements. Are known: k_1 = 2k, k_2 – 3k.*

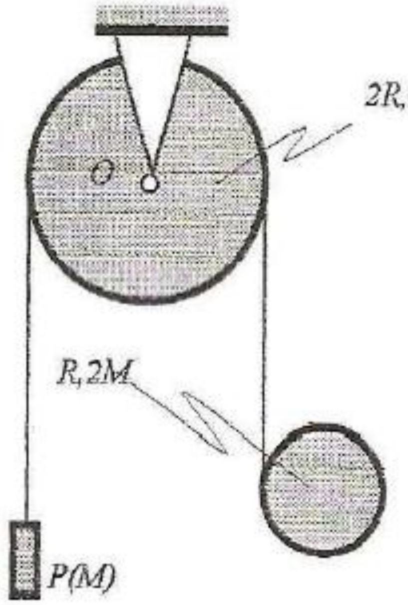


Fig.4.

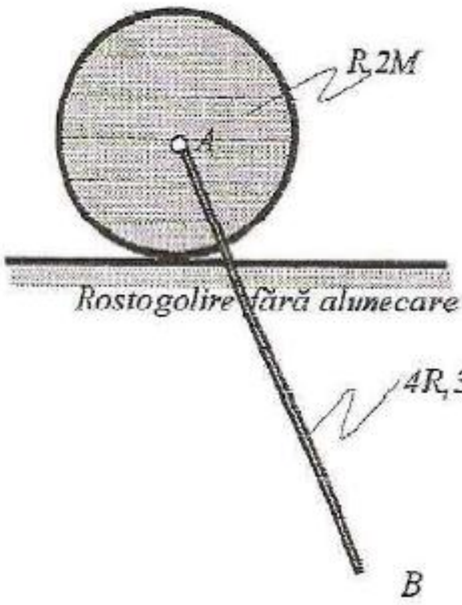


Fig.5.

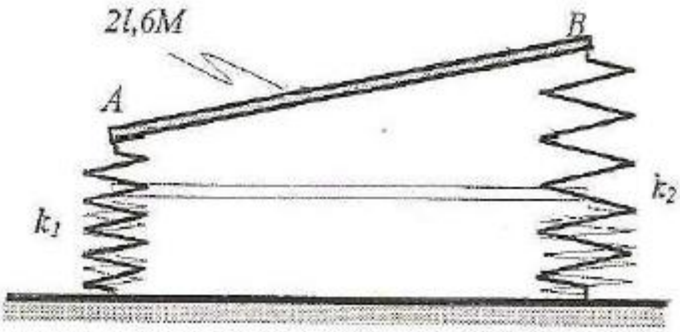


Fig.6.