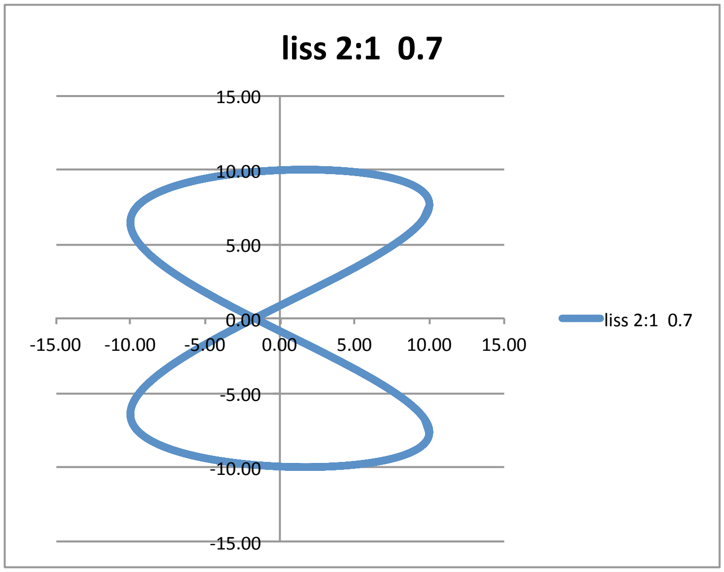
Chapter 3 - The Practical Matter of the Atmosphere



Congratulations! You've made it through a couple of somewhat dry introductory chapters, and in the process become familiar with Excel basic operation. Now let's apply that know-how to a practical matter – how thick is the earth's atmosphere?

The question is poorly stated. We can measure the depth of the ocean at a particular place pretty closely. With the atmosphere, the air gets less and less dense as we go up. There is no well-defined upper surface. We experience the drop in pressure ourselves in a very direct fashion.

- Airlines keep their cabin pressure roughly equal to the pressure at 8000 feet. This is generally sufficient for healthy passengers. (Women in late stages of pregnancy are discouraged from flying.)
- At 12,500 feet, hang glider pilots are advised to use supplemental oxygen.
- 16,700 feet is the elevation of the highest permanently occupied village in the world – La Rinconada in Peru.
- Mountain climbers venture higher – briefly. 95% of the climbers of Mt. Everest (29,029 feet) use supplemental oxygen; only 5% reach the summit without oxygen tanks. Note: climbing Everest without supplemental oxygen doubles the risk of dying on the mountain.

- There are a few species of birds that fly at these altitudes – the bar-headed goose (29,000 feet) and the common crane (33,000 feet) migrate over the Himalayas. Rüppell's griffon vulture (37,000 feet) lives in central Africa. It soars, like most vultures, as a means of spotting food. (37,000 feet? Doubtful it could see a dead whale from that height, but that's the explanation always given for vultures soaring to high altitudes.) Its ability to soar to that extreme height was confirmed by an airliner flying at that altitude, which sucked the unfortunate bird into an engine. The remains were identifiable – barely.

Figure 3-1 Rüppell's Griffon Vulture

- 43,000 feet – maximum service ceiling of a Boeing 777.
- 123,500 feet is the record altitude reached by a jet plane (piloted by Alexandr Fedotov, in a MiG-25M).
- 980,000 feet (~300 kilometers) is the minimum altitude for a satellite. Below this altitude, drag from the atmosphere will cause the satellite to slow and crash back to earth.

So the question really should be – how does air pressure vary with altitude? To answer this question, we start off with a few well-known (or easily Googled) bits of information.

1. Air pressure at sea level 14.7 pounds per square inch
2. Density of air at sea level .0765 pounds per cubic foot
3. Volume of a gas sample is proportional to absolute temperature
4. Volume of a gas sample is inversely proportional to its pressure

Air Pressure

Air pressure is measured directly by an instrument called a barometer. There are several variations of this instrument:

Aneroid barometer – has a small evacuated can, which contracts and expands as the atmospheric air pressure rises and falls (most of the expansion/contraction occurs at one end of the can, which is made of a thinner gauge metal). Levers and gears translate this motion into the rotation of an indicator needle over a dial face.

Figure 3-2 Aneroid Barometer

Electronic – a semiconductor pressure sensor. These are small enough to be incorporated in cell phones and GPS receivers. The early versions of these ICs (integrated circuits) needed additional ICs to translate the output to some useful digital format. Today's ICs integrate the interface logic with the sensor, so that a microprocessor can read the pressure over a serial port. Often these are used for determining altitude.

Mercury barometer – used to be a fixture in college and even high school physics labs, until instructors became alarmed at the incidence of mercury poisoning (mainly among instructors, since their exposure extended over many years.). However, its simplicity makes the concept of air pressure intuitively understandable. The atmosphere presses down on the surface of mercury in the open cup, which pushes the mercury up the inverted tube – until the weight of mercury in the column exerts a pressure equal to the atmospheric pressure. The "empty" part of the tube, at the top, really is empty, a vacuum, no air, no gas of any sort. A scale attached to the tube allows reading the height of the mercury column. Pressure readings are still sometimes given in inches of mercury, or millimeters of mercury, although the use of mercury for pressure measurements has largely been phased out.

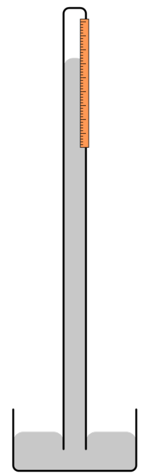


Figure 3-3 Mercury barometer

Because of its long history, barometric air pressure is given in many different units: atm (atmospheres), inches Hg (inches of mercury), mm Hg (millimeters of mercury), torr, bars, millibars, pascals, kPa (kilopascals), hPa (hectopascals), and psi (pounds per square inch). For the sake of familiarity and convenience, we will use psi in our computations, and take the average atmospheric pressure at sea level to be 14.7 psi. This means literally that every square inch of horizontal area at sea level sits under 14.7 pounds of air, which you could imagine as being confined to a column with a cross sectional area of 1 square inch, extending upward for miles and miles.

An Aside on the Confusion of Units for Pressure

Evangelista Torricelli, an Italian mathematician and physicist, invented the mercury barometer in 1643. If the scale on the tube is marked in millimeters, then the average height of the mercury in the tube, at sea level, is 760 mm. To honor Torricelli, one mm of Hg is also referred to as 1 torr; thus average barometric pressure at sea level is 760 torr. To be absolutely accurate, 1 torr is not exactly 1 mm Hg, but the two are so close (within a ten-thousandth of 1%) that they can be used interchangeably for all practical purposes. Note that the meter was not defined until 1795, so Torricelli could not have used millimeters to measure the level of mercury in his instrument. At the time, length was measured in a hodgepodge of different units, and even the same unit might have different definitions in different regions. Traces of that hodgepodge seem to linger in the different ways of measuring barometric pressure:

1. One atmosphere (abbreviated 1 atm) is the average barometric pressure at sea level at 49° latitude (i.e. Paris, France – and coincidentally the northern border of the state of Washington).
2. Millimeters of mercury – 760 mm Hg is equal to 1 atmosphere. One mm Hg was originally defined to be 1 torr, and is still 1 torr for all practical purposes. Torr is commonly used by people working with vacuum pumps. Mechanical vacuum pumps can achieve vacuums in the range of 0.01 to 0.1 torr (10 to 100 millitorr). Millimeters of mercury, abbreviated mm Hg, is still the standard unit for measuring blood pressure. 120 / 80 gives the systolic / diastolic pressures in mm Hg. If you ask the nurse if 80 torr is normal for diastolic pressure, he or she quite possibly won't know what you're talking about, even though 80 torr is equivalent to 80 mm Hg.
3. Inches of mercury – 760 mm = 29.921 inches
4. Pounds per square inch (abbreviated psi). In the United States, we have a good feel for this particular measure; car tires are filled to 30 or 35 psi, bicycle and aircraft tires to higher pressure – 100+ psi for some bicycles, 200 psi for tires on large aircraft. Footballs – 13 psi for NFL play. Basketballs – 8 psi.
5. The Pascal is the metric unit of pressure, defined as 1 Newton per square meter. The Newton is a unit of force, equal to about 0.225 pounds. One square meter is about 1550 square inches, so a Pascal is only 0.225/1550 = 0.000145 pounds per square inch. One atm comes to 101,325 Pascals. Weather reports sometimes give the barometric pressure in hectopascals (abbreviated hPa); one hectopascal is 100 Pascals. Thus one atm is 1013.25 hPa.

6. The bar is a closely related measure. One bar is 10^5 Pascals. One atm is slightly more than 1 bar (1.01325 x 10^5 Pascals), thus one atm is 1.01325 bars, or 1013.25 millibars. Note that a millibar and a hectopascal are equivalent. Hectopascals are preferred by metric system purists, but both units - millibars and hectopascals – are still in common use.

Table 3-1 Pressure Equivalents

	Atmosphere	Torr or mm Hg	Inches Hg	pounds/ sq inch	hecto-pascal	milli-bar
	(atm)	(Torr)	(in Hg)	psi	hPa	mbar
1 atm	1	760	29.9213	14.696	1013.25	1013.25
1 Torr	0.0013158	1	0.039370	0.019337	1.3332	1.3332
1 in Hg	0.033421	25.4	1	0.49116	33.864	33.864
1 psi	0.068046	51.715	2.0360	1	68.947	68.947
1 hPa	0.00098692	0.75006	0.02953	0.014504	1	1
1mbar	0.00098692	0.75006	0.02953	0.014504	1	1

Use the above table by reading from the left, i.e. 1 psi is the same as 0.068046 atm, or 2.0360 in Hg, or 68.947 hPa. Thus, to convert a pressure of 8.62 psi, to Torr, multiply 8.62 x 51.715 = 445.783 Torr.

There is often a need to distinguish gauge pressure from absolute pressure. When tire pressure is 30 psi, this represents the pressure inside the tire, relative to the pressure outside. If the pressure outside is 14.7 psi – normal atmospheric pressure, then the total pressure inside the tire is actually 44.7 psi. Use gauge pressure if you simply want to inflate your tires correctly, i.e. 30 psi. Use absolute pressure if you are trying to calculate how much air is inside the tire – perhaps for a chemistry or physics problem.

SCUBA divers are concerned with absolute pressure. The weight of a 1 square inch column of sea water 33 feet high is 14.7 pounds; thus each 33 feet of descent into the ocean adds another 1 atmosphere of pressure. A SCUBA regulator (the thingy on the tank with all the hoses) insures that air is delivered to the diver at the ambient pressure. Thus a diver at a depth of 33 feet is breathing air compressed to 2 atmospheres. At 99 feet, the diver gets air compressed to 4 atmospheres. (A diver at 99 feet goes through a tank of air much faster than a diver at 50 feet.) The recommended maximum depth for sport divers varies by country and certifying agency; it is generally around 100 feet, for an absolute pressure of 4 atmospheres. The main problem with depth is that more nitrogen becomes dissolved in the bloodstream and body tissues because of the increased pressure. This nitrogen has to diffuse back out upon return to the surface. Too long a stay at depth, or too rapid an ascent can cause the excess nitrogen to form bubbles in the bloodstream, and in joints – a painful affliction called the bends. A large part of SCUBA training focuses on this gas diffusion problem, and its consequences. The worst case is diving in high mountain lakes – when a return to the lake surface brings the diver to a pressure that is even less than 1 atmosphere.

Density of Air

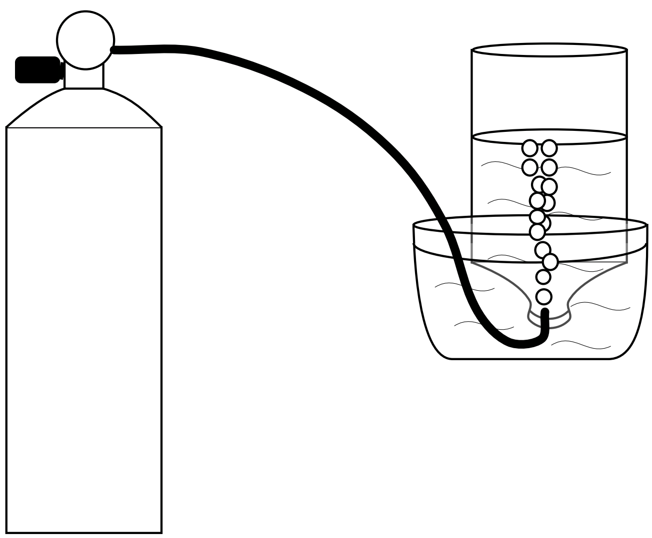


Figure 3-4 Weight of air

By density of air, we really mean simply - what does a cubic foot of air weigh? This is somewhat difficult to measure in practice, because we are surrounded by air. It is like trying to weigh water; it's easy to weigh a cup of water while standing in the kitchen, but more difficult while underwater, for instance while submerged in the deep end of the local swimming pool. Just as buoyancy makes water "weightless" in the pool, buoyancy makes air seem weightless when weighed at the bottom of the sea of air known as the atmosphere.

Here's one way you might approach the problem experimentally. Take a full SCUBA tank. Weigh it. Then let the air out slowly, collecting it by displacement in something handy like a five-gallon jug. SCUBA tanks come in many different sizes; most are in the 50 to 80 cubic foot range.

(The sizes refer to the volume of air at 1 atmosphere that can be compressed and shoved into the tank; the tanks themselves are comparatively small. The air is compressed to 2000 – 3000 psi in the tank, somewhere in the range of 200 atmospheres.)

One cubic foot is about 7.48 gallons, so expect to fill and refill a five-gallon jug many times. If you filled the jug one hundred times, that would come to 500 gallons of air, which would be 66.8 cubic feet of air. Now weigh the SCUBA tank again – how much weight has it lost? Ideally, the tank should weigh about 5 pounds less. The density then is 5/66.8 = 0.0748 pounds per cubic foot.

More careful measurements report the density of air at sea level (i.e. at a pressure of 14.7 psi), with a temperature of 15° Celsius, to be 0.0765 pounds per cubic foot. Since we are working with pressure per square inch, we really need to calculate the density in pounds per cubic inch. Let's use Excel for this step; if an error creeps in somewhere, having it in a spreadsheet makes it easy to find and correct later.

	A	B	C	D
1	air density lbs/cubic ft	wikipedia	0.0765	lbs / cubic ft
2	cubic in / cubic foot	12^3	1728	cu in / cu ft
3	air density lbs/cubic in	0.765/1728	=C1/C2	lbs / cubic in

Figure 3-5 Useful constants

Here cell C1 contains the constant value 0.0765, and C2 contains the formula = *12^3*, which evaluates to 1728. C3 contains the formula as shown, which causes the value in C1 to be divided by the value in C2. As soon as the formula is entered (by pressing enter/return) the value is calculated to be 4.4271E-05, or 0.000044271.

We can now calculate the volume of one pound of air, at sea level, at 15° C; the volume is just the inverse of the density. (Density is pounds/inch^3; volume is inch^3/pound.) In excel terms, the formula is: = 1/C3. Let's add this to our Excel table, along with the atmospheric pressure at sea level, and the temperature at sea level.

	A	B	C	D
1	air density lbs/cubic ft	wikipedia	0.0765	lbs / cubic ft
2	cubic in / cubic foot	12^3	1728	cu in / cu ft
3	air density lbs/cubic in	0.765/1728	4.4271E-05	lbs / cubic in
4	volume of 1 pound air			
5	at 14.7 psi, 15°C	1/density	22,588.2	cubic inches/lb
6	pressure at sea level	wikipedia	14.7	psi
7	temp at sea level	wikipedia	288	°K

Figure 3-6 Full set of atmospheric constants

This little block contains all the constants needed for our model of the atmosphere, as well as some indication of where the information came from in column B. In particular, we will use the constants in:

- C5 : 22,588.2 - the volume in cubic inches of 1 pound of air at sea level, 15° C
- C6 : 14.7 - air pressure at sea level
- C7 : 288 - temperature at sea level, in degrees Kelvin – more about the Kelvin temperature scale below…

Gas Laws – Robert Boyle, Jacques Charles

Pressure and volume are inversely related; if the pressure of a gas sample increases, the volume decreases (assuming the temperature remains unchanged). For an "ideal" gas sample kept at a constant temperature, the product of pressure and volume remains constant.

Suppose there is a gas sample whose original PV values are designated P_1 and V_1. If the gas sample is expanded into a larger container, its volume increases to V_2 and its pressure goes down to P_2. If the temperature is kept constant, then $P_1V_1 = P_2V_2$. For example, if the original pressure and volume is 10 psi, and 2 cubic feet, then when the pressure is decreased to 5 psi, what is the new volume?

$P_1V_1 = P_2V_2$ ← Boyle's Law

$10 \text{ psi} \bullet 2 \text{ ft}^3 = 5 \text{ psi} \bullet V_2$

$V_2 = 2 \text{ ft}^3 \bullet 10\text{psi} / 5\text{psi}$

$V_2 = 4 \text{ ft}^3$

This relationship between volume and pressure was first noticed by others, but Robert Boyle was the first to publish the results of experiments confirming the "law", in 1662. The physical explanation of air pressure was not understood for another 200 years.

Is air an "ideal" gas? It is pretty close to ideal, so long as we don't pressurize it or cool it so much that it gets close to becoming a liquid. Most gases behave similarly, though there are exceptions. Acetylene gas for example, explodes if compressed too much, which is very very non-ideal. (SCUBA tanks are filled to 2000 – 3000 psi. Acetylene welding tanks are only filled to around 250 psi, because of the danger of explosion.)

Volume and temperature are directly related; increase the temperature, and the volume will increase as well, provided the pressure is kept constant. For an ideal gas, if the pressure is constant –

$V_1 / T_1 = V_2 / T_2$ ← Charles law

There is one complication here – we have to use absolute temperature. Temperature is a measure of molecular kinetic energy. With a gas held in a container, the molecules are bouncing around, colliding with each other and with the walls of the container. The constant barrage of tiny collisions is how the gas exerts a measurable pressure. The average speed of the molecules determines the temperature. At 0° Celsius, the average speed of the molecules making up air is roughly 1000 miles per hour. As the air is chilled more and more, the molecules travel slower and slower. (This is where we need our "ideal" gas. Air will turn to a liquid if chilled to -196° C. Our hypothetical ideal gas remains a gas, no matter what.) The temperature at which these ideal molecules finally come to rest is absolute zero, defined to be 0° Kelvin, which is -

273.15° C. (A physicist would qualify this statement – even at absolute zero, the molecules will still have a little quantum jitter, so they don't like to say that the molecules actually come to rest. Instead, they would say that all the molecules have come to the lowest possible energy state, and no more heat energy can be extracted.) So temperatures will be expressed in degrees Kelvin, obtained by adding 273.15 to the Celsius temperature. (There is also a Rankine scale, which does for Fahrenheit what Kelvin does for Celsius. Rankine = Fahrenheit + 459.67°. You could use Rankine for these calculations, though you might well be the only person in the world using the Rankine scale for anything…)

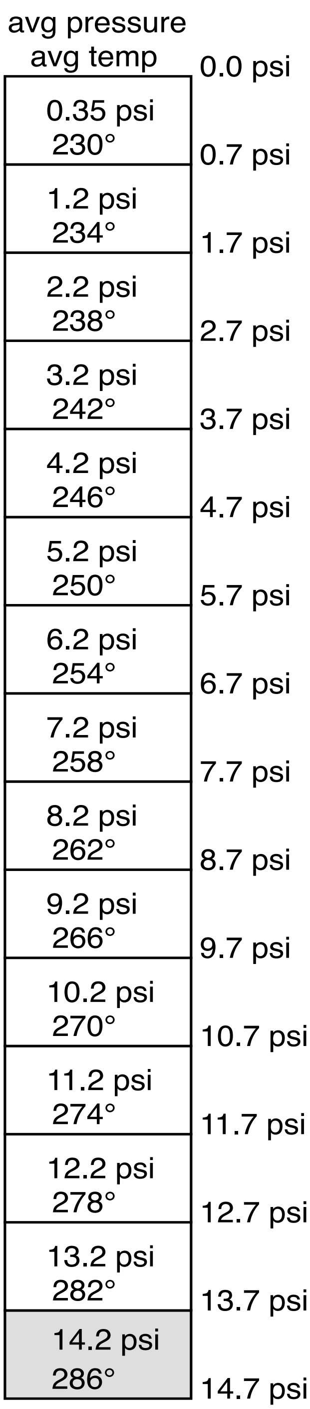


Figure 3-7 Segmenting the atmosphere

Suppose a gas sample has an original volume and temperature of 4 cubic feet at 300° K. If the pressure stays constant and the temperature drops to 250° K, what is the new volume?

$V_1 / T_1 = V_2 / T_2$

4 cu ft / 300° = V_2 / 250°

V_2 = 4 cu ft • 250°/300° = 3.333 cu ft

This relationship was first formulated by Jacques Charles in 1780, who did not publish his work. In 1801, John Dalton published the details of his experiments which pointed to the same law; his work was confirmed a few months later by Joseph Gay-Lussac, who credited the original discovery to the unpublished work of Jacques Charles twenty years earlier. Apparently the general relationship between volume and temperature had been known for a century, but the mathematically precise relationship had to wait until temperature could be measured more accurately.

We can combine the two equations – the one for volume and pressure, and the one for volume and temperature – into one equation.

$$\frac{V1P1}{T1} = \frac{V2P2}{T2}$$ <- Boyle's & Charles' Laws combined

$$V_2 = \frac{P1}{P2} \bullet \frac{T2}{T1} \bullet V_1$$ Rearranging to solve for V_2

P_1, V_1, T_1 are the original pressure, volume, and temperature.

P_2, V_2, T_2 are the new pressure, volume, and temperature.

Note that the pressure ratio is P_1/P_2, while the temperature ratio is T_2/T_1. Pressure and volume are inversely proportional, while temperature and volume are directly proportional; this leads to the two ratios being opposite…

So we can express the new volume V_2 as the original volume V_1 times a pressure ratio times a temperature ratio, where the ratios are as shown above. This is the formula we will use in Excel to do the real work of modeling the atmosphere.

The Atmospheric Model

Start with a mental image of a one inch square column of air, extending from sea level upwards. To model the atmosphere, we will divide this air column into one pound segments. The figure "Segmenting the atmosphere" just shows how we are segmenting the calculations; it is **not** a scale drawing of the successive segments. Each successively higher segment will be at a lower pressure, and therefore have a larger volume. Exactly how much larger depends on both pressure and temperature – we'll let Excel handle the hard work.

Starting at the bottom of the column, the pressure at the bottom of the first segment is 14.7 psi. At the top of the segment, the pressure is only 13.7 psi; there is one pound of air in the first segment, and therefore only 13.7 pounds of air above. To calculate the volume of the first segment, we'll use the average pressure, i.e. the average of 14.7 and 13.7, which is 14.2 psi. Similarly, the average pressures of the other segments are 13.2, 12.2, 11.2 … down to 1.2 psi. The last segment only contains 0.7 pounds of air; its average pressure is 0.35 psi. We'll hold off on calculations for the last 0.7 pound segment; some additional thought will be needed to model this last bit of the atmosphere.

For temperature, the average sea level temperature is taken to be 15° C, or (273 + 15) = 288° K. We know that temperature declines with altitude. There's no simple way to calculate just exactly what the temperature is for each segment, so we'll take a wild guess, and claim that each successive segment is 4° K colder than the segment below it. This sounds pretty sloppy; the good news is that the pressure drop is much more important to the calculation than the temperature drop. To be consistent, we will use the average temperature of the segment, i.e. for the bottom segment, the average of the temperature at sea level (288°), and the temperature at the top of the segment (284°), is 286°.

Our general strategy is this:

1. Tabulate the average temperature and pressure for the segments
2. Calculate the pressure ratios and temperature ratios for the segments

3. Use the ratios to calculate the volume of each segment
4. Convert the volumes to feet by simply dividing by 12 (this works because we are mentally working on a column of air with a one square inch cross section)
5. Accumulate the thicknesses of the segments (from step 3) into total altitude.
6. Finally add columns giving the temperatures and pressures at each of the calculated altitudes.

We'll execute this step-by-step, so it will hopefully be clear how we get to the end result. One could collapse several of these steps into one step, using a big long formula, but that makes it harder to follow what is going on, and also makes it harder to determine if a mistake has crept into the calculations.

Start by entering 14.2 and 13.2 in column E as shown. Similarly, enter 286 and 282 in column F.

	E	F	G	H
1	P2 Pressure	T2 Temp	P ratio	T ratio
2	psi	°K	P1 / P2	T2 / T1
3				
4	14.2	286		
5	13.2	282		

Figure 3-8 Set-up for temps and pressures

Then select the four-cell block as shown, grab the selection box at the bottom right, and drag it down, extending the average pressure values to 1.2

	E	F	G	H
1	P2 Pressure	T2 Temp	P ratio	T ratio
2	psi	°K	P1 / P2	T2 / T1
3				
4	14.2	286		
5	13.2	282		
6	12.2	278		
7	11.2	274		
8	10.2	270		
9	9.2	266		
10	8.2	262		
11	7.2	258		
12	6.2	254		
13	5.2	250		
14	4.2	246		
15	3.2	242		
16	2.2	238		
17	1.2	234		

Figure 3-9 Drag to fill in pressures and temperatures

→TIP – you can see that we're leaving a blank row, which will come in handy later. Generally, when you realize "too late" that you should have left a blank row or column, you can reposition a block of your spreadsheet quickly and easily by selecting the block, using ⌘x to delete the block, and then ⌘v to paste the block back where you want it to be. You don't have to select a full area for the paste, just select the cell where you want the top left corner of the block to go. Double check the results, just in case a reference doesn't keep up. Alternately you can select an entire column or row (select in the lettered index cells at the very top to select a column, or the numbered index cells at the very left of the spreadsheet to select a row), and then insert a row or column of blank cells. In either case, the existing cells will get bumped over or down, and all the cell references will be updated as needed.

Next we calculate the Pressure ratios and the Temperature ratios. The pressure ratio is P1/P2, where P1 is 14.7 psi (which is entered in cell $C$6), and the P2 values are in column E. The formula we enter in cell G4 is: = $C$6 / E4.

Similarly, the temperature ratio is T2/T1, where T1 is 288 degrees Kelvin, which is in cell $C$7. The formula we enter in cell H4 is: = F4 / $C$7.

	E	F	G	H
1	P2 Pressure	T2 Temp	P ratio	T ratio
2	psi	°K	P1 / P2	T2 / T1
3				
4	14.2	286	=$C$6/E4	=F4/$C$7
5	13.2	282		

Figure 3-10 Formulas for P and T ratios

When cells G4 and H4 are selected, formatted as numbers with 3 decimal places, and pulled down to populate cell in columns G and H down to row 17, we get this:

	E	F	G	H
1	P2 Pressure	T2 Temp	P ratio	T ratio
2	psi	°K	P1 / P2	T2 / T1
3				
4	14.2	286	1.035	0.993
5	13.2	282	1.114	0.979
6	12.2	278	1.205	0.965
7	11.2	274	1.313	0.951
8	10.2	270	1.441	0.938
9	9.2	266	1.598	0.924
10	8.2	262	1.793	0.910
11	7.2	258	2.042	0.896
12	6.2	254	2.371	0.882
13	5.2	250	2.827	0.868
14	4.2	246	3.500	0.854
15	3.2	242	4.594	0.840
16	2.2	238	6.682	0.826
17	1.2	234	12.250	0.813

Figure 3-11 Pressure and temperature ratios

A bit of checking at this point is worthwhile. The pressure ratios start at slightly more than 1 for the first segment, and increase as we go to lower pressures. This is what we expect – lower pressures will cause the volume to increase. The temperature ratio starts at slightly less than 1, and decrease to 0.813 as we get higher in the atmosphere. Also as expected – lower temperatures cause the gas to contract. But notice the difference in magnitude. The pressure ratio is 12.25 for the last segment in the table; the temperature ratio is 0.81. The expansion due to lower pressure is over 1000%, whereas the contraction due to lower temperature is around 20%; the pressure ratio is the dominant factor by far.

Now we can go ahead and calculate the volumes of each segment; the volume is just V1 • Pratio • Tratio, where V1 is the volume 22,588.2 cubic inches, which is stored in

cell $C$5, and the P and T ratios are in columns G and H. So the formula in cell I4 is:
= $C$5 * G4 * H4.

That gives the volume in cubic inches, but numerically it is also the thickness of the segment of the air column. We would rather have this measurement in feet, so we'll use column J to convert to feet. The formula in J4 is:
= I4/12.

Enter these formulas, and drag down through all rows of the table. That should produce the following spreadsheet. (Remember – if your spreadsheet entries misbehave, more often than not it is a cell reference problem.)

	E	F	G	H	I	J
1	P2 Pressure	T2 Temp	P ratio	T ratio	V2	thickness
2	psi	°K	P1 / P2	T2 / T1	cubic inches	feet
3						
4	14.2	286	1.035	0.993	23,221	1,935
5	13.2	282	1.114	0.979	24,631	2,053
6	12.2	278	1.205	0.965	26,272	2,189
7	11.2	274	1.313	0.951	28,206	2,350
8	10.2	270	1.441	0.938	30,519	2,543
9	9.2	266	1.598	0.924	33,335	2,778
10	8.2	262	1.793	0.910	36,838	3,070
11	7.2	258	2.042	0.896	41,314	3,443
12	6.2	254	2.371	0.882	47,233	3,936
13	5.2	250	2.827	0.868	55,430	4,619
14	4.2	246	3.500	0.854	67,529	5,627
15	3.2	242	4.594	0.840	87,191	7,266
16	2.2	238	6.682	0.826	124,727	10,394
17	1.2	234	12.250	0.813	224,824	18,735

Figure 3-12 Find the segment volumes and thicknesses

An aside on constants

There are three referenced cells used in the calculations:

1. $C$5 is 22,588.2 – the volume of one pound of air in cubic inches
2. $C$6 is 14.7 - the average sea level air pressure in pounds/square inch
3. $C$7 is 288 - the average sea level temperature in degrees Kelvin (15° C)

You could just enter the constants in the cell formulas. But by putting the constants in cells somewhere, you can change the constant later by just changing that one cell entry, instead of having to update the formulas in a range of cells (and possibly missing some cells that need to be updated).

We now have the thickness of each atmospheric segment – each one pound segment of the 14.7 pounds of air above every square inch of the earth's surface. It would be more convenient to have the cumulative altitude – the height of the current segment plus all the lower segments. That's easy:

- Enter a 0 in cell K3
- In cell K4 enter the formula = K3 + J4. This adds the current segment height (J4) to the previous total height.
- Extend the formula to the bottom of the spreadsheet.

	E	F	G	H	I	J	K
1	P2 Pressure	T2 Temp	P ratio	T ratio	V2	thickness	altitude
2	psi	°K	P1 / P2	T2 / T1	cubic inches	feet	feet
3							0
4	14.2	286	1.035	0.993	23,221	1,935	1,935
5	13.2	282	1.114	0.979	24,631	2,053	3,988
6	12.2	278	1.205	0.965	26,272	2,189	6,177
7	11.2	274	1.313	0.951	28,206	2,350	8,528
8	10.2	270	1.441	0.938	30,519	2,543	11,071
9	9.2	266	1.598	0.924	33,335	2,778	13,849
10	8.2	262	1.793	0.910	36,838	3,070	16,918
11	7.2	258	2.042	0.896	41,314	3,443	20,361
12	6.2	254	2.371	0.882	47,233	3,936	24,297
13	5.2	250	2.827	0.868	55,430	4,619	28,917
14	4.2	246	3.500	0.854	67,529	5,627	34,544
15	3.2	242	4.594	0.840	87,191	7,266	41,810
16	2.2	238	6.682	0.826	124,727	10,394	52,204
17	1.2	234	12.250	0.813	224,824	18,735	70,939

Figure 3-13 Almost done ...

Let's assess what we have so far. The altitude column tells us that the first segment of the atmosphere (the first pound of air) starts at 0 feet above sea level, and extends to 1935 feet. The next segment starts at 1935, and ends at 3988 feet. We can continue in this way to the last segment, which starts at 52,204 feet and extends up to 70,939 feet. The altitudes we have calculated represent the boundaries between the segments.

Finally, for completeness we'd like to have the pressures and temperatures at these boundary altitudes. This doesn't require any new calculation; we stated at the outset that the pressures at the boundaries would be 14.7 psi, 13.7 psi, … 0.7 psi. We can add a column for pressure at the boundaries just by entering 14.7 in cell L3 (corresponding to 0 feet altitude), and 13.7 in cell L4 (for altitude of 1935 feet). Then select the two cells L3 and L4, and drag down to the bottom of the table to extend the series down to 0.7 .

Similarly, the temperatures are 288°K, 284°K, etc. Celsius is more common for atmospheric measurements. In order to more easily compare our calculations to real-world data, let's go back to Celsius. Instead of 288, 284 … we'll use 15°C, 11°C … Just as we did for pressure, enter 15 in cell M3, and 11 in cell M4. Select the two cells, and drag down to extend the series. The result should look like this:

	E	F	G	H	I	J	K	L	M
1	P2 Pressure	T2 Temp	P ratio	T ratio	V2	thickness	altitude	Pressure	Temperature
2	psi	°K	P1 / P2	T2 / T1	cubic inches	feet	feet	psi	°C
3							0	14.7	15
4	14.2	286	1.035	0.993	23,221	1,935	1,935	13.7	11
5	13.2	282	1.114	0.979	24,631	2,053	3,988	12.7	7
6	12.2	278	1.205	0.965	26,272	2,189	6,177	11.7	3
7	11.2	274	1.313	0.951	28,206	2,350	8,528	10.7	-1
8	10.2	270	1.441	0.938	30,519	2,543	11,071	9.7	-5
9	9.2	266	1.598	0.924	33,335	2,778	13,849	8.7	-9
10	8.2	262	1.793	0.910	36,838	3,070	16,918	7.7	-13
11	7.2	258	2.042	0.896	41,314	3,443	20,361	6.7	-17
12	6.2	254	2.371	0.882	47,233	3,936	24,297	5.7	-21
13	5.2	250	2.827	0.868	55,430	4,619	28,917	4.7	-25
14	4.2	246	3.500	0.854	67,529	5,627	34,544	3.7	-29
15	3.2	242	4.594	0.840	87,191	7,266	41,810	2.7	-33
16	2.2	238	6.682	0.826	124,727	10,394	52,204	1.7	-37
17	1.2	234	12.250	0.813	224,824	18,735	70,939	0.7	-41

Figure 3-14 The atmosphere - pressure and temperature vs. altitude

Note that our spreadsheet effectively stops when we reach 70,939 feet, at a pressure of 0.7 psi. If our results are close to measured values, maybe we can find a way to extend the model to cover the last 0.7 psi.

Real-world Measurements

The American Vacuum Society (AVS) has published online a table of measured pressures and temperatures at altitudes ranging from 0 to 2 million feet. To compare the model against their measured values, I've combined both sets of data into the table below.

Table 3-2 Atmospheric model vs measured values

MODEL			AVS MEASURED VALUES		
altitude	temp	pressure	altitude	temp	pressure
0	15	14.7	0	15	14.7
1,935	11	13.7	2,000	11	13.66
3,988	7	12.7	4,000	7	12.69
6,177	3	11.7	6,000	3	11.78
8,528	-1	10.7	8,000	-1	10.91
11,071	-5	9.7	10,000	-5	10.1
13,849	-9	8.7	15,000	-14	8.29
16,918	-13	7.7			
20,361	-17	6.7	20,000	-24	6.76
24,297	-21	5.7	25,000	-34	5.46
28,917	-25	4.7	30,000	-44	4.37
34,544	-29	3.7	35,000	-54	3.47
41,810	-33	2.7	40,000	-57	2.73
52,204	-37	1.7	50,000	-57	1.69
70,939	-41	0.7	70,000	-55	0.65
			90,000	-59	0.26
			100,000	-46	0.16
			150,000	-	0.021

Looking at the raw data, a few things are notable:

1. Our temperature guess worked very nicely up to 10,000 feet or so. But by 20,000 feet, the measured temperature has dropped more than the model expected, and by 35,000 feet the measured temperature is 25° C lower than predicted. Also, the measured temperature doesn't drop much after 35,000 feet. This altitude is considered to be the lower boundary between the troposphere (bottom layer of the atmosphere, where "weather" occurs) and the stratosphere (next higher level, which includes the ozone layer).
2. It is a little hard to compare the two pressure data sets. Our model steps through pressure in one pound increments; AVS steps through the data in altitude increments of thousands of feet. Sometimes the AVS data happens to hit pressures close to n.7 psi, sometimes not. Offhand, the model and the measured values seem pretty close.

Excel Graphing

A quick way to compare the data sets is by graphing both on a common graph. To do this, we first copy the AVS measurements into our spreadsheet. They can be copied to columns beside our model data, or anywhere really; we'll tell Excel where the data is for the graph.

Excel will do the graphing for us; however different versions of Excel let you access the graphing options in different ways, so it is hard to give step-by-step instructions that will work for everyone. Generally what you want to do is to add an x-y graph; my version of Excel lets you add this by clicking the "scatter" icon under the Charts tab. This gives several options – a "marked" line indicates where the data points are. "Straight" connects the data points with straight line segments; "smooth" lets the lines curve as needed to produce a smooth looking graph. For the charts below, "smooth, marked scatter" was chosen. (Be sure no active cells of the spreadsheet are selected when you select smooth marked scatter; otherwise Excel will take the selected data cells and graph them, but not necessarily the way you want.) Once you have picked an option, Excel will present you with a empty square where your chart will be. A right-click on the square will allow you to pick Select Data from the menu, which in turn opens a Select Data Source box. With the Select Data Source box open, (with my version of Excel):

1. Click the Add button. This will name a new series of data points as "Series 1".
2. There is a Name field. Type "model" there, to name the data "model" instead of "Series 1".
3. Put the cursor in the X-values box. Then go select the spreadsheet model cells that have the altitude – everything from cell K3 to K17. This should automatically enter the range of cells into the X-values box.
4. Put the cursor in the Y-values box. Select the spreadsheet model cells that have the temperatures – everything from cell M3 to M17.

At this point, the graph should display the points for model temperature vs. altitude. You can now add a second series of points for the measured values. If the Select Data Source box is not still open, select the graph, right click, and Select Data again. With the Select Data Source box open again, repeat steps 1 – 4, this time naming the new series "measured". Select the altitudes and temperatures from the AVS table for the X-values and Y-values respectively; you can stop at 70,000, since we don't have model data above that to compare. This should plot a second line on the graph, this time of the AVS data.

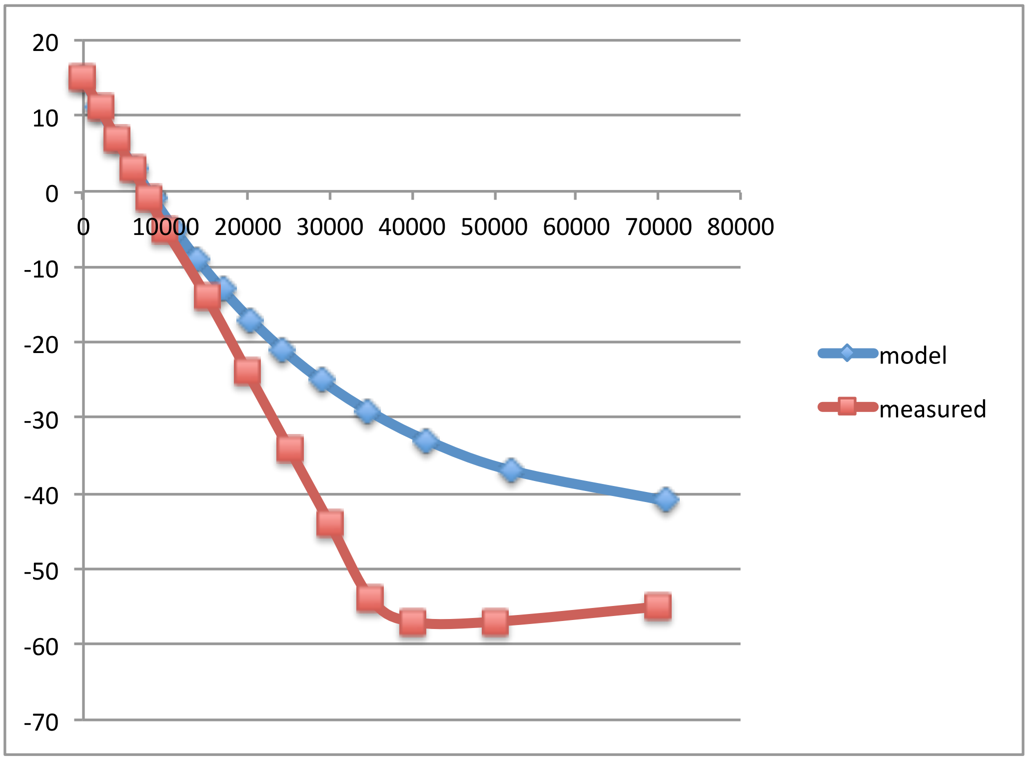


Figure 3-15 Temperature vs altitude, model vs measured values

The graph shows very nicely the difference between the modeled temperature and the measured temperature. We might have made a more intelligent "guess" about the decline in temperature with altitude, but it seems very unlikely that we would have predicted that the temperature would level off and even increase above 40,000 feet. Despite this difference, the next graph showing the model vs measured pressures shows good agreement.

(We get the pressue graph by repeating the steps above, selecting the altitude cells again for the X-values, and this time selecting the pressures for the Y-values.)

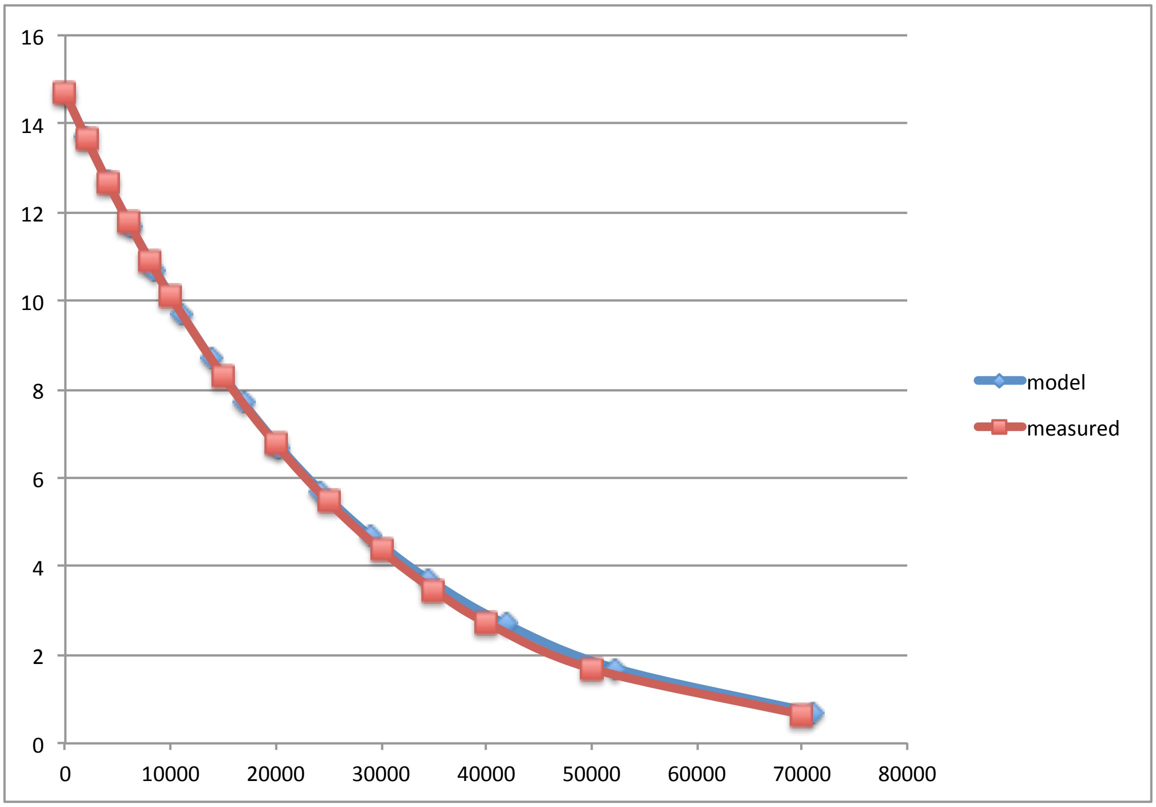


Figure 3-16 Pressure vs altitude, model vs measured values

Let's take a moment to bask in the warm glow of seeing our model of atmospheric pressure agree so well with measurements. We started with just two constants – the air pressure at sea level, and the density of air at sea level. Then using only the gas laws relating volume to pressure and temperature, we were able to calculate the atmospheric pressure from sea level up to 70,000 feet. Some tinkering was required to set up the spreadsheet correctly, and along the way you had to put up with the untimely death of an unfortunate African vulture, but in the end Excel came through and did all the repetitive drudgery for us.

Our model did stop being useful after 70,000 feet. Can you see a way to extend the model, and calculate more accurately the distribution of the last 0.7 pounds of air? How would you go about calculating the altitude at which the pressure is 0.6, 0.5, … 0.1 pounds per square inch? Give it a try on your own. There may be more than one approach, especially considering the "guess" we used for temperature. I've included one approach to extending the model in the table below.

	E	F	G	H	I	J	K	L	M
15	3.2	242	4.594	0.840	87,191	7,266	41,810	2.7	-33
16	2.2	238	6.682	0.826	124,727	10,394	52,204	1.7	-37
17	1.2	234	12.250	0.813	224,824	18,735	70,939	0.7	-41
18	0.65	232	22.615	0.805	41,116	3,426	74,366	0.6	-41
19	0.55	231	26.727	0.803	48,507	4,042	78,408	0.5	-42
20	0.45	231	32.667	0.802	59,184	4,932	83,340	0.4	-42
21	0.35	231	42.000	0.801	75,962	6,330	89,670	0.3	-42
22	0.25	230	58.800	0.799	106,163	8,847	98,517	0.2	-43
23	0.15	230	98.000	0.798	176,631	14,719	113,236	0.1	-43

Figure 3-17 Extending the model to 0.1 psi

Without belaboring all the details, the graph comparing the high altitude points of the model and measurements should come out like this, more or less. (Don't forget that the volume calculations for the new rows with average pressures of 0.65, 0.55 etc. must be multiplied by 1/10, since these rows are for 1/10 pound of air.)

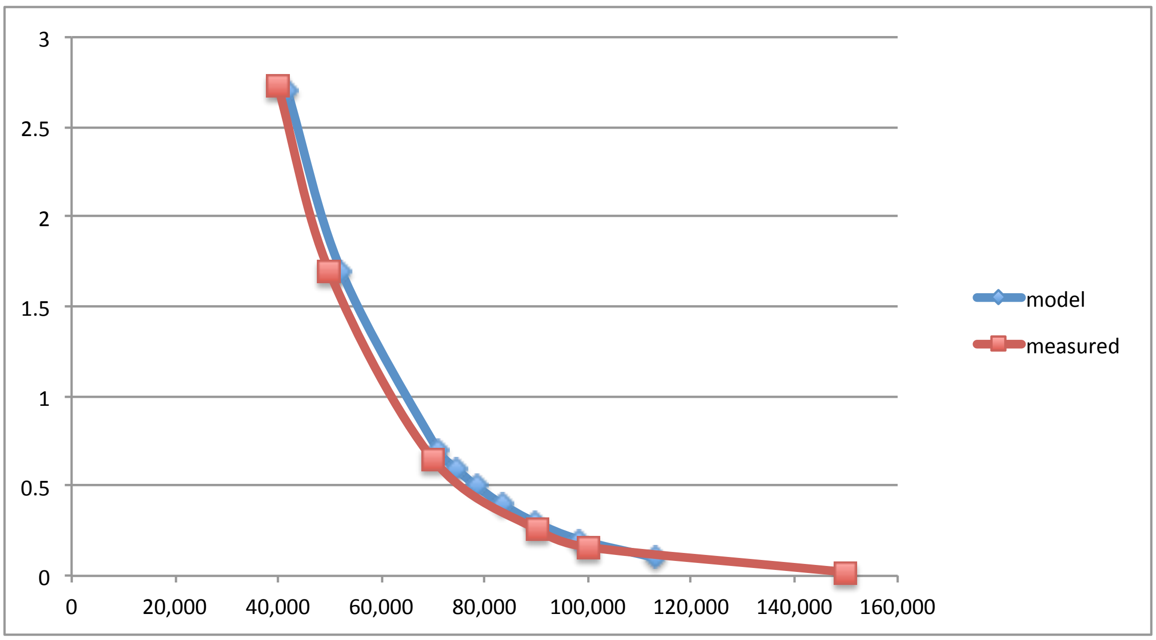


Figure 3-18 Pressure vs altitude - to 0.1 psi - model vs measured

Agreement between model and measurements is still good, all the way up to 113,000 feet (and 0.1 psi). Do you notice a similarity in the shape of the first graph (which covered altitudes from 0 to 70,000 feet), and this graph, which goes from about 40,000 feet up to 120,000 feet. This characteristic shape is typical of an exponential function; we'll see it again later.

We still have the last 0.1 pound of air unaccounted for. We could divide it up into 1/100 pound increments and run another series of calculations. At some point other factors will come into play, and the model will stop agreeing with measurements. I'm inclined to stop here and declare victory – that our model has agreed with measurements closely up to an altitude of 21 miles, and a pressure of 0.1 psi.

Finally, let's have one last table that combines all the atmospheric pressure and temperature parameters with human (and avian) performance:

Table 3-3 Altitude vs. biology

Altitude feet	Pressure psi	Pressure %atm	Temp °C avg	Description
0	14.70	100	15	sea level
5000	12.23	83%	5	Denver CO, Albuquerque NM Newfound Gap in Smokies Mt Katahdin MA
8000	10.91	74%	-1	Mammoth Lakes CA airline cabin pressure
10,000	10.1	69%	-5	Leadville CO, Cuzco Peru
15,000	8.29	56%	-14	12,500 – O_2 needed for hang gliders 13,420 – Potosi silver mine, Bolivia 15,980 – Wenquan, Tibet 16,700 – La Rinconada, Peru – highest permanent settlement
20,000	6.76	46%	-24	19,341 - Mt Kilamanjaro, Tanzania 19,974 - Huyana Potosi, Bolivia 20,310 - Denali, Alaska
25,000	5.46	37%	-34	All peaks 25,000 and above are in Himalayas – China, India, Nepal, Pakistan
30,000	4.37	30%	-44	29,035 – Mt Everest – 95% of climbers use oxygen tanks Bar-headed geese and common crane fly this high!

Altitude feet	Pressure psi	Pressure %atm	Temp °C avg	Description
40,000	2.73	19%	-57	37,000 – maximum altitude reached by any bird (Rüppel's vulture) 40,000 – upper cruising altitude for commercial aircraft
50,000	1.69	11%	-57	At this altitude, breathing pure O_2 would not be sufficient to keep one from passing out from hypoxia. Lower reaches of stratosphere. Finally safe from vultures.
80,000	0.40	2.8%	-52	60,000 – max altitude Concorde 70,000 – max altitude U2 spy plane 85,000 – max altitude SR-71 Pressure lower than vapor pressure of H_2O at body temp => if exposed one might experience saliva boiling away, just before passing out.
100,000	0.16	1.1%	-46	96,800 – max alt. Helios solar plane 123,520 – record alt. jet plane
200,000	0.003	0.022%	-	264,000 – USAF awards astronaut wings to pilots exceeding this limit (50 miles), incl. eight X-15 pilots

We often hear of scientists modeling parts of the natural world. Meteorologists model the atmosphere in order to predict weather. Climatologists model on a larger scale to make predictions about climate. Astronomers model the motions of stars and galaxies. This simple spreadsheet of atmospheric pressure is a mathematical model also. Though it is far simpler than any model that can predict weather, it gives a very hands-on idea of the nature of a mathematical model.

Project Ideas

- Determine the density of air, using the SCUBA tank approach outlined in this chapter. You may be able to obtain a "pony" tank – a small backup tank that holds only a few cubic feet of air. This would allow use of a smaller (and more accurate) scale for weighing the tank, as well as reducing the tedium of filling a five-gallon jug over and over.

- Determine "absolute zero", the temperature at which an ideal gas has a volume of zero. Start with an air sample at 0°C (in an ice bath). Then measure the volume of the same sample when the temperature is increased, say to 40°C. Graph the results, volume on the y-axis, temperature on the x-axis. Draw a line through the two data points, and see where the line intersects the x-axis. This sounds too simplistic to possibly work – but it actually works pretty well.

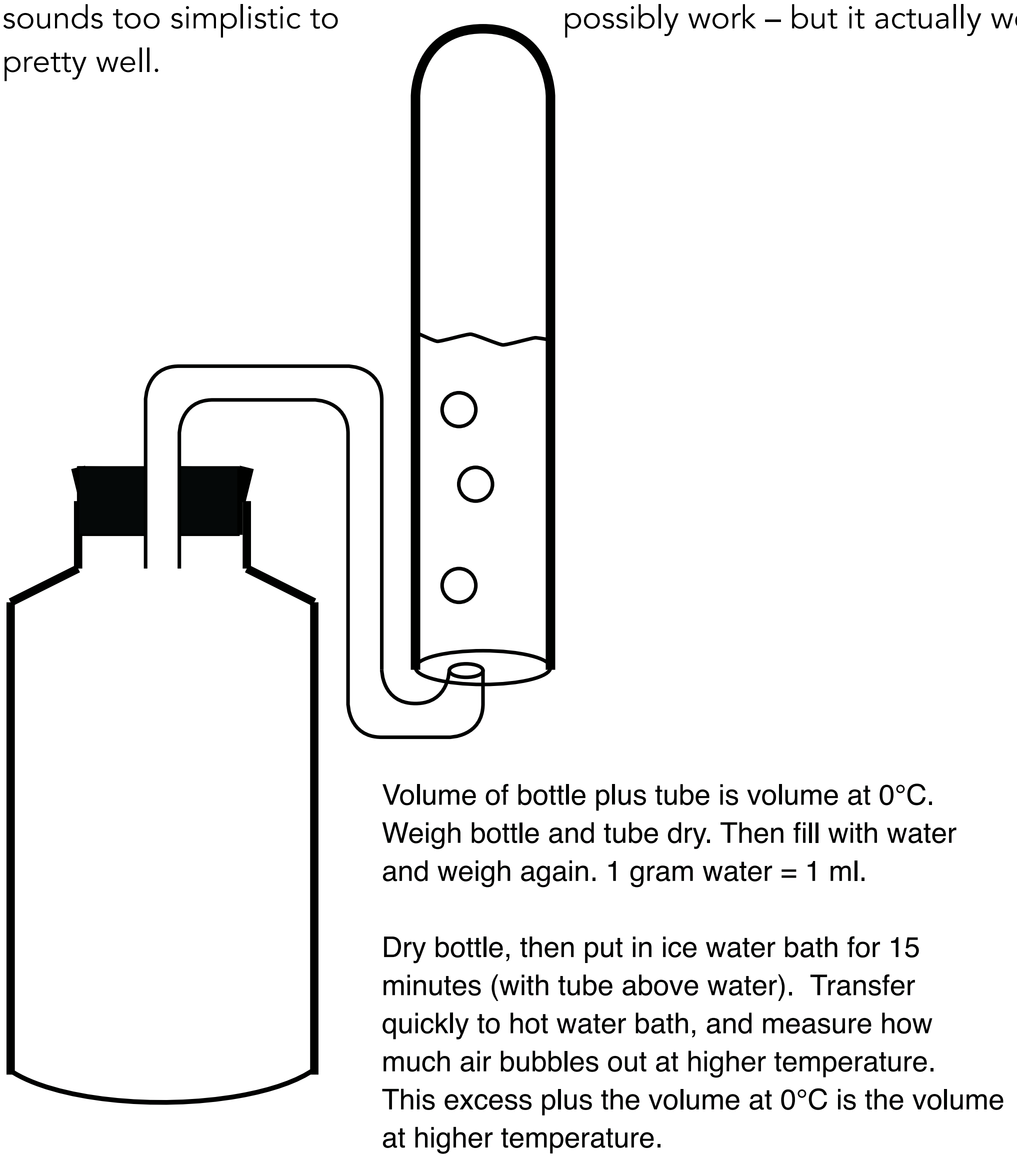


Figure 3-19 Finding "absolute zero"