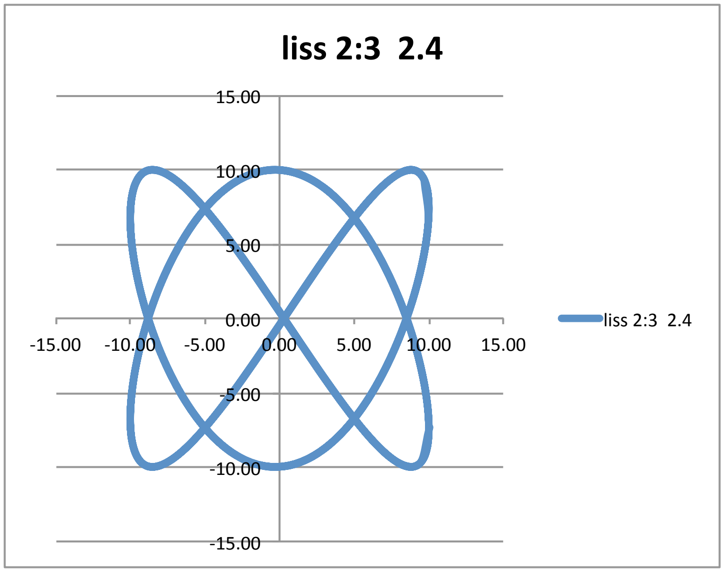
Chapter 5 – Sines, Cosines, and Tangents



Basic Trigonometric Ratios

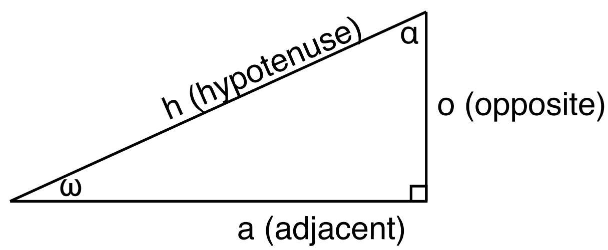


Figure 5-1 Right triangle for trigonometry

A right triangle has one right angle; a right angle is 90°, what a carpenter might call "square". The right angle is often shown as a small square at the angle, as in the figure. I've designated the other angles as α and ω, alpha and omega. The sides are designated as a, o, and h, for adjacent, opposite, and hypotenuse. The terms adjacent and opposite refer to angle ω; the adjacent side is adjacent to the angle ω, and the opposite side is opposite. If we were going to focus on angle α, then we would have to reverse the adjacent/opposite designations. The hypotenuse is always the same; it is the long side of the right triangle. It is convenient (though at first somewhat annoying) to call the angles by Greek letters. When you see a Greek letter, you'll know we're talking about an angle. Normal English letters a, o, h, refer to a side, or to the length of a side.

With three sides, there are six possible ways to make ratios of the lengths of sides, and mathematicians have names for each of these, as tabulated below. Even stodgy mathematicians can get carried away in the heat of the moment. 99% of the time, we can ignore secants and cosecants. Let's focus mainly on the first three: sine, cosine, and tangent. Millions of students remember their definitions by means of a very bad story, whose pun ending is SOH-CAH-TOA, which embodies in one phrase – Sine = Opposite/Hypoteneuse, Cosine=Adjacent/Hypoteneuse, Tangent = Opposite/Adjacent.

Table 5-1 Trigonometric Ratios

ratio	o / h	a / h	o / a	a / o	h / a	h / o
trig	sine	cosine	tangent	cotangent	secant	cosecant
abbrev.	sin(ω)	cos(ω)	tan(ω)	cot(ω)	sec(ω)	csc(ω)

The origin of the word "sine" is buried in Indian history. It was originally a Sanskrit term, which was translated sort of phonetically into Arabic, and then mis-translated into the Latin sinus, from which we get our English words sine and cosine. Tangent refers to a line tangent to a circle – in the same plane as the circle and just touching. It will take another diagram to show exactly how a tangent line relates to the trigonometric ratio o/a.

We read an expression like sin(ω) as "sine of ω", and cos(ω) as "cosine of ω". Tan(ω) may be read as "tangent of ω". Sometimes the last two get shortened to just "cos ω" and "tan ω". Sin(ω) is still usually pronounced "sine ω."

The triangle shown in Figure 5-1 may have given you the impression that trig ratios for angles of 90° and greater are undefined, because we can't have a triangle with two right angles, or a right angle and an obtuse (>90°) angle. True, but we want to define the trig functions for any angle – for 90°, 237°, 873°, whatever real number we can think of. We'll need a bigger protractor.

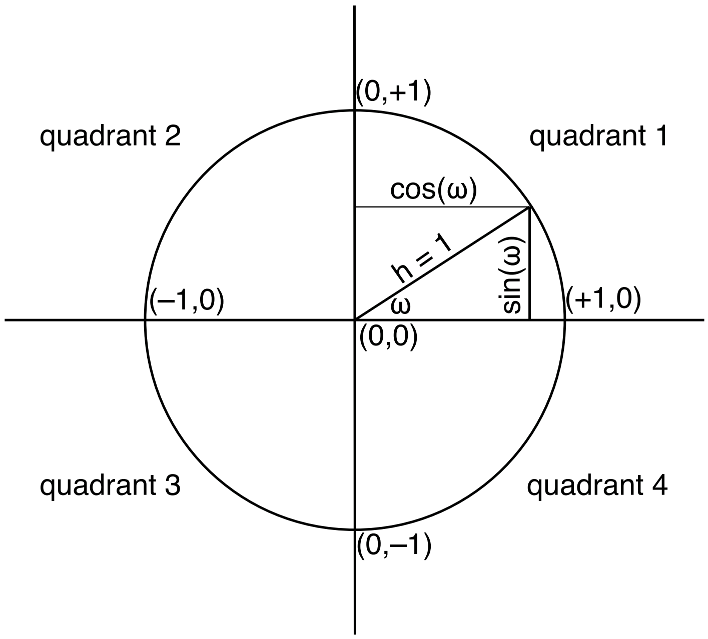


Figure 5-2 Sine and cosine ratios

Consider this confusing figure. We start with the familiar x and y axes, and then draw a circle with a radius = 1, centered on the origin. Then we draw in a radial line from the origin to the circle, at an angle ω to the x axis. With this in place, we achieve three things:

1. The angle ω can be anything. If between 0 and 90°, then the radial line will go into the first quadrant, as shown. If between 90° and 180°, it would be in the second quadrant, etc. If ω is greater than 360°, then just think of the line rotating around more than a full revolution; ω = 390° will put the line in the same position as ω = 30°.
2. If you consider the x,y coordinates of the intersection of the radial line with the circle, you will realize that the coordinates are (cos(ω), sin(ω)). The circle has a radius of 1, so the hypotenuse is always 1, forcing the x and y coordinates to equal the trig ratios.
3. Finally, you can see at a glance that sin(ω) is positive in quadrants 1 and 2 (from 0° to 180°) and negative from 180° to 360°. Similarly, cos(ω) is positive in quadrants 1 and 4, and negative in quadrants 2 and 3.

Off on a tangent

Where does the tangent fit in, and why is it called the tangent? This chapter focuses mainly on the sine and cosine ratios, and their usefulness. However, I don’t remember knowing why the trigonometric tangent is so-named until recently, and I wanted to make sure readers have this little nugget of information early.

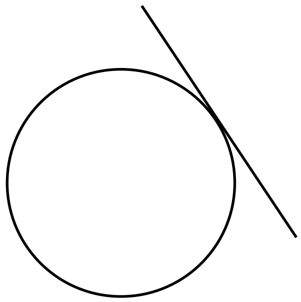


Figure 5-3 A tangent line

The figure on the left illustrates a tangent line. The official definition is that a tangent line is a straight line that intersects a curve at exactly one point, without crossing the curve. The curve can be any curve, not just a circle. And officially, the definition can be extended to planes and curved surfaces; a line could be tangent to a sphere, and a plane could be tangent to an ellipsoid. For the trigonometric tangent ratio, we are specifically concerned with lines tangent to a circle. (The English figure of speech "off on a tangent" refers to a line of thought or speech which starts off on one subject, but veers off into matters only remotely related to the original topic – like when you ask your math teacher a simple question about the homework, and before you know it, he's telling you about the Australian rabbit problem.)

On the right is a partial re-drawing of the figure we used to define the cosine and sine functions. Two lines have been added; horizontal line ED tangent to the circle, which intersects the circle at (0,1). The other is vertical line BC, also tangent to the circle, which intersects the circle at (1,0). The radial line that intersects the circle has been extended far enough to intersect both the horizontal and vertical tangents.

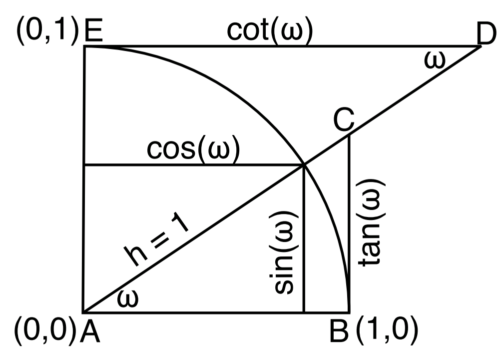


Figure 5-4 Tangent and cotangent ratios

Looking first at triangle ABC, we see that BC/AB is by definition tan(ω). Since AB = 1, BC must be tan(ω).

Similarly, looking at triangle ADE, we first note that the angle at point D must be the same size as ω. Thus by definition, cot(ω) must be ED/EA. Again, since EA=1, ED has to be cot(ω).

Caution – About the Inverse Trig Functions

Excel and most calculators costing $15 or more have built in trig functions. In the olden days, there were tables in which the sines, cosines etc. could be looked up, as well as logarithms of the trig functions. Count yourselves lucky that you're learning trig now, with inexpensive calculators readily available.

A calculator will generally have the so-called inverse trig functions also. These may be labeled arcsin, arccos, arctan, or possibly simply $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$. The second form, the $\sin^{-1}$ style, was chosen mostly to save space, but the notation is unfortunate; anywhere else if you see n^{-1}, it has the meaning 1/n. But $\sin^{-1}$ does not mean 1/sin, it really means arcsin. If sin(30°) = 0.50, then arcsin(0.50) = 30°; arcsin(0.50) means *the arc whose sine is 0.50.*

Excel uses ASIN(x) for the arcsin function. ACOS(x) and ATAN(x) are for arccos and arctan. These functions return the angles in radians; the function DEGREES(n) will convert radians to degrees. Thus = DEGREES(ASIN(0.5) will return the angle whose sine is 0.5, in degrees, i.e. 30°.

Even if you have never had trig or geometry in school, you can see that these definitions, plus the tabulation of trig ratios (or better yet, incorporation into calculators) will allow solving all sorts of surveying problems.

> *Given a right triangle with one of the acute angles measured to be 27°, and the adjacent side measured to be 170 yards long, how long is the hypotenuse?*

And there's the possibility of coming up with an unending stream of trig puzzles.

> *Show that sin(ω)/cos(ω) = tan(ω).*

If that were the extent of the usefulness of trig, I wouldn't have bothered with this chapter.

Measure of Angles

Angles are measured with a variety of units – a wide variety. We are probably most familiar with degrees; an equilateral triangle has three 60° angles.

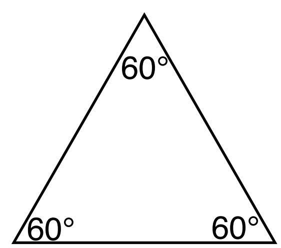


Figure 5-5 Equilateral triangle

Degrees °

We divide a circle into 360 equal divisions; each division is one degree. A right angle is 90°; the angles of an equilateral triangle are each 60°. One degree must be pretty small …

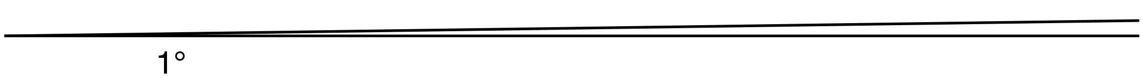


Figure 5-6 1° angle

Looking at it drawn like this, it seems extremely small. But go outside some night and take a look at the full moon. We say the moon "subtends" a certain angle in the sky. If you imagine a giant circle projected on the sky directly "above" the earth's equator, then that circle can be divided into 360 degrees. If the moon happened to lie on our imaginary circle, how many degrees would it span?

It varies a bit, because the moon doesn't have a perfectly circular orbit; sometimes the moon is closer to us and appears larger; it subtends a larger angle. But generally it subtends about ½ degree. The sun happens to subtend roughly ½ degree also, which makes it possible to have solar eclipses, where the moon just exactly blocks the sun for a brief interval. If the moon happens to be close to the earth at the time of eclipse, then the eclipse is total. If the moon is farther out in its orbit, then the eclipse may be annular, with a thin ring of the sun still visible behind the moon.

Usually degrees are further divided into minutes, sometimes called arcminutes or minutes of arc, to avoid confusion with minutes of time. One degree is 60 minutes; 1° = 60′. Minutes are further divided into seconds (arcseconds, or seconds of arc). One minute is 60 seconds; 1′ = 60″. Alternately, an angle may be expressed as a decimal fraction, 47.27° would be the same as 45° 16′ 12″. Many GPS receivers can be set to accept latitudes and longitudes in either format.

Incidentally, whenever you come across something measured in divisions of 60 (or 360), you can generally thank the Babylonians, who lived in and around what is now southern Iraq. They used a base 60 number system. Sixty may seem big and unwieldy, but consider that 60 can be divided by 2, 3, 4, 5, 6, 10, 12, etc. Ten can be divided only by 2 and 5.

Firearms manufacturers measure the accuracy of their rifles in minutes of arc, or MOA. A rifle with a 1 MOA rating can group its shots inside a one arcminute circle, which works out to be 1.047 inches at 100 yards. Telescopic sights for rifles usually have adjustments that click in increments of ½ or ¼ MOA.

Circumference of a circle of radius 100 yards =

2π • 100 yards • 36 inches/yard = 22619.5 inches

1 degree arc of that circle = 22619.5/360 = 62.832 inches

1 minute arc of that circle = 62.832/60 = 1.047 inches

Incidentally, there are two "frames of reference" in common usage for degrees. In the world of math and physics, 0° implies a line starting at the origin, pointing to the right along the x-axis. 90° is a line pointing straight up along the y-axis. As the angle increases, the line rotates counterclockwise.

To a navigator however, 0° is due North, and 90° is due East. As the compass heading increases, the direction rotates clockwise. These two systems couldn't be much more different! Be sure you're using the appropriate system, especially if you're steering a sailboat at night.

Gradians^g

This measure is often found on calculators, and is used by surveyors and French artillery officers. It is similar to degrees, except that a circle is divided into 400 gradians. A right angle is 100^g. Inconveniently, 30° and 60° angles are 33.333^g and 66.666^g. The unit was introduced along with the other units of the metric system, but didn't gain wide acceptance. Confusingly, the French word centigrade meant 1/100 of a gradian; to avoid confusion, the temperature scale once known as Centigrade is now called Celsius. Today the grad, or gradian, also goes by the name gon. All this should seem a bit confusing, but unless you become a surveyor, or sign up for the French Foreign Legion, you will probably encounter gradians/grads/gons infrequently.

Radians

The other angle measure in common use (favored by mathematicians and physicists) is the radian. Imagine the apex of the angle at the center of a circle. The angle then defines an arc of the circle. The length of the arc is measured along the curve. This would be difficult in practice, but it is not really necessary; we're really just defining how to assign a value to the size of the angle ω. The arc length divided by the radius is the measure of the angle, in radians; ω = arc_length/radius . Note that the arc length is some length, let's say in centimeters, and the radius is some length, also in centimeters. The ratio of the two lengths is a pure number, really without units of measure. It's a ratio. We say it is some number of radians, but really radians are sort of a non-unit.

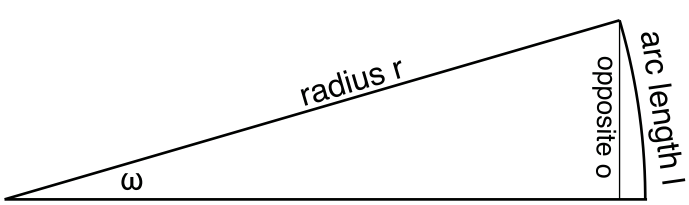


Figure 5-7 Measuring angles in radians

The circumference of any circle is $2\pi r$, so a 360° angle (i.e. one whose arc is the entire circle) is $2\pi r/r = 2\pi$ radians. **2π radians is 360°.**

Thus one radian = 360°/2π = 57.29578°. Furthermore:

Table 5-2 Degrees and radians

degrees	360°	180°	90°	60°	45°	30°	15°	10°
π radians	2π	π	π/2	π/3	π/4	π/6	π/12	π/18
radians	6.2832	3.1416	1.5708	1.0472	0.7854	0.5236	0.2618	0.1745

This does take some getting used to. I was about to write that I had never seen a protractor calibrated in radians; however, a little browsing reveals that one can be purchased cheaply enough on Amazon.

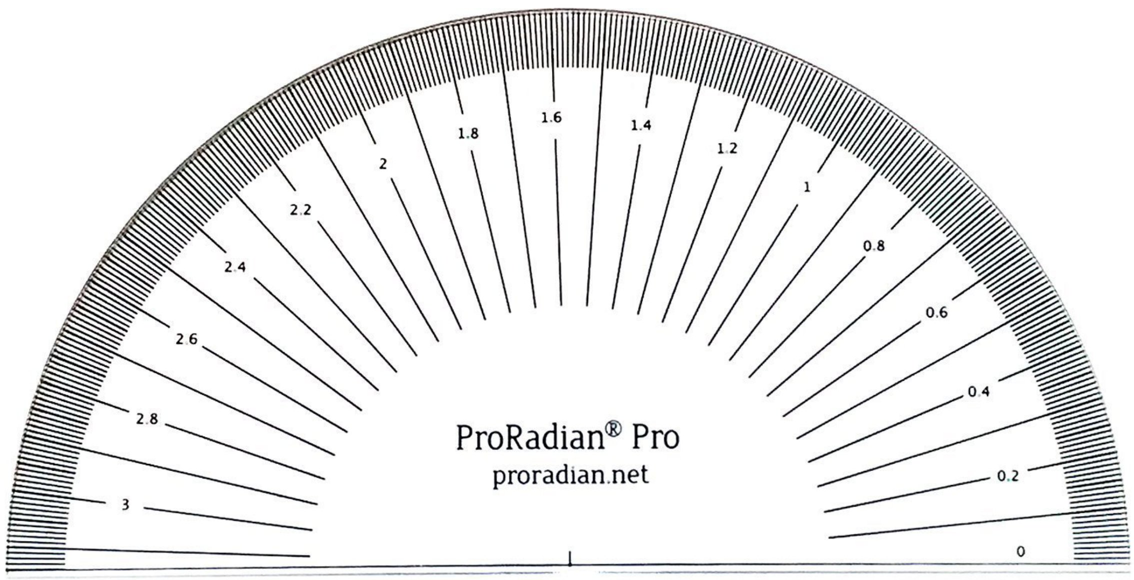


Figure 5-8 Radian protractors do exist!

Referring back to Figure 5-7, you can see that the angle in radians = arc_length/r is defined in a similar way to the sine of an angle, $\sin(\omega) = o/r$. And you can see from the illustration that as the angle ω gets smaller and smaller, the arc_length and the opposite side get closer and closer; therefore ω and $\sin(\omega)$ get closer and closer . For sufficiently small angles sin(x) is very close to the angle x measured in radians. The same is true for tan(x) and the angle x in radians. This is sometimes useful in simplifying formulas in physics, in situations where one can guarantee that the angles are never going to be large.

The following table compares the angle in radians to the sine and tangent of the angle. You can see numerically how close the three are for angles less than a few degrees. That closeness is useful to engineers and scientists, but there is an additional huge reason why radians are favored, which we'll come to in a few chapters.

Table 5-3 Angle in radians vs. sine ratio

angle ω		sine	tangent
degrees	radians	sin(ω)	tan(ω)
90	1.57079633	1.00000000	
60	1.04719755	0.86602540	1.73205080
45	0.78539816	0.70710678	1.00000000
30	0.52359878	0.50000000	0.57735027
15	0.26179939	0.25881905	0.26794919
10	0.17453293	0.17364818	0.17632698
5	0.08726646	0.08715574	0.08748866
3	0.05235988	0.05233596	0.05240778
2	0.03490659	0.03489950	0.03492077
1	0.01745329	0.01745241	0.01745506
0.5	0.00872665	0.00872654	0.00872687
0.2	0.00349066	0.00349065	0.00349067

Excel has functions to find the trigonometric values for angles, SIN(ω), COS(ω), TAN(ω), and it expects these angles to be expressed in radians. This would be pretty annoying, since those radian protractors are still an oddity, however there is another Excel function RADIANS(α) which converts α in degrees to radians. So the radians column in the table above just has the formula = RADIANS(A2), and the sine column has the formula =SINE(B2). If you are working with angles in degrees, and don't really care about radians, you can save some space and just nest the formulas =SINE(RADIANS(A3)) – this will return the sine of A3 degrees directly.

Sine & Cosine functions

Before going farther, it is worth tabulating the sine and cosine functions, and graphing them. We will encounter these "waveforms" in many situations. Notice that the sine and cosine values for 360° and 0° are the same. The same is true for 375° and 15°, 390° and 30°, etc. The sine and cosine functions are continuous, and repetitive; sine(360° + 30°) = sine(30°). Excel will handle anything you feed it. The formula = SIN(RADIANS(N)) will handle N = 0, -43, 5278.3 – any real number you care to evaluate.

	A	B	C	D
1	degrees	radians	sine	cosine
2	0	0.000	0.000	1.000
3	15	0.262	0.259	0.966
4	30	0.524	0.500	0.866
5	45	0.785	0.707	0.707
6	60	1.047	0.866	0.500
7	75	1.309	0.966	0.259
8	90	1.571	1.000	0.000
9	105	1.833	0.966	-0.259
10	120	2.094	0.866	-0.500
11	135	2.356	0.707	-0.707
12	150	2.618	0.500	-0.866
13	165	2.880	0.259	-0.966
14	180	3.142	0.000	-1.000
15	195	3.403	-0.259	-0.966
16	210	3.665	-0.500	-0.866
17	225	3.927	-0.707	-0.707
18	240	4.189	-0.866	-0.500
19	255	4.451	-0.966	-0.259
20	270	4.712	-1.000	0.000
21	285	4.974	-0.966	0.259
22	300	5.236	-0.866	0.500
23	315	5.498	-0.707	0.707
24	330	5.760	-0.500	0.866
25	345	6.021	-0.259	0.966
26	360	6.283	0.000	1.000
27	375	6.545	0.259	0.966
28	390	6.807	0.500	0.866

Figure 5-9 Sine and cosine functions

The graph shows that the sine and cosine functions have similar shapes, but they are "out of phase." If we evaluated sin(x+90°), it would exactly overlay the cosine graph; you can convince yourself of this just by looking at the table of values. Whenever some quantity shows this sort of variation, it is almost always referred to as a sine wave, or sinusoid. Rarely does anyone call something a cosine wave. We'll get back to sine waves in more depth in a couple of chapters. For now, let's focus on the direct utility of the sine and cosine functions for vector arithmetic.

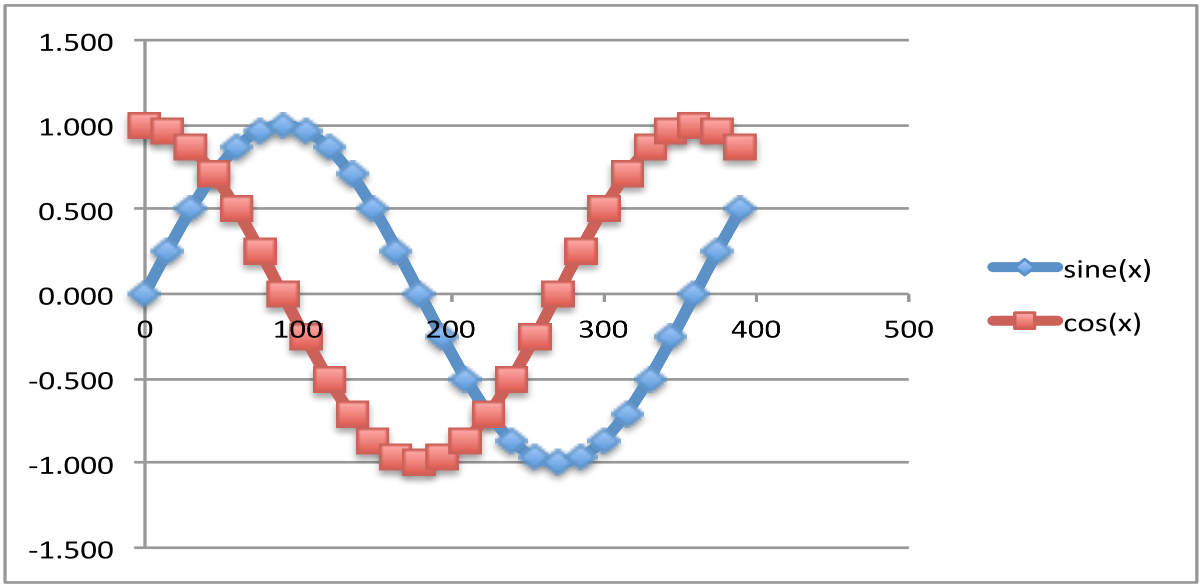


Figure 5-10 Graph of sine and cosine functions

Vectors

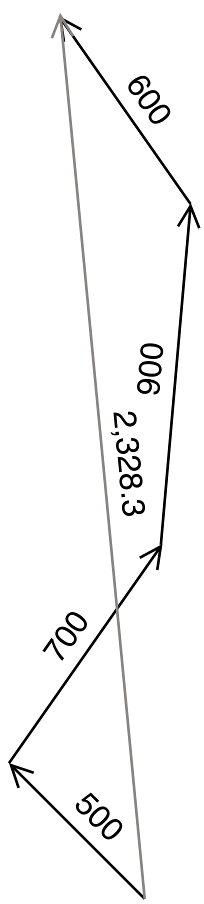


Figure 5-11 Graphic addition of vectors

What is this all good for? Well, a great many things; manipulating vectors is high on the list. A vector is a mathematical entity, consisting of a magnitude coupled with a direction. It can be as simple as say: 500 meters, Northwest. Let's stick with this example for a moment. Instead of Northwest, let's give the directions in the math framework – East is 0°, North is 90°, West is 180°, South is 270°. So suppose a boat travels:

- 500 meters @ 135° (Northwest)
- 700 meters @ 55°
- 900 meters @ 85°
- 600 meters @ 125°

Where is the boat after the last leg, relative to its starting point? The way a navigator might solve this problem is to add the vectors on the chart, by drawing them one after the other, "head to tail", using a protractor to set the direction, and a ruler to mark the length of the vector.

Here we've drawn in the vectors scaled to a convenient size, with the angles as specified. When all four are added in this way, the final position, as measured with ruler and protractor, is found to be ~2,328 meters at an angle of ~95°. This method is a little tedious, but it is an acceptable if a small number of vectors are involved.

Note that the order of addition doesn't really matter; the end point will be the same.

A second way to do this is to take the first two vectors – the 500 meter and the 700 meter vector – and treat them as two sides of a triangle. Then use trigonometry to find the third side, let's call it V_{12}, which will itself be a vector from the start of the 500 m vector to the end of the 700 m vector. Then use this new vector, in combination with the 900 m vector to find V_{123}, which is the vector from the start of the 500 m to the end of the 900 m vector. Then use V_{123} in combination with the 600 m vector to find V_{1234}, which is the sum of all 4 vectors. This is a horrible and tedious way to add the vectors. Forget we even mentioned it.

But there's a nicer way to do this. Consider the first vector, 500 meters @ 135°. Instead of using it in that raw form, suppose we find a pair of vectors that add together to give that vector. That would just be more work, unless we think of something clever:

We will treat the vector as though it is the sum of a vector parallel to the x-axis, plus a vector parallel to the y-axis. We can readily calculate these vectors. The x-component is 500 cos(ω); the y-component is 500 sin(ω). We'll do this for all four vectors, then we can add the x and y vectors separately. It may sound like a lot of work, but Excel automates the process.

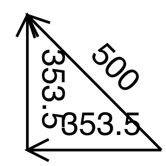


Figure 5-12 X & Y components of a vector

	A	B	C	D
1	heading	distance	x	y
2	135	500	-354	354
3	55	700	402	573
4	85	900	78	897
5	125	600	-344	491
6				
7		SUM	-218	2315

Figure 5-13 Adding X and Y vectors

The numbers in column C are from the formula =B2*COS(RADIANS(A2)). Column D uses the formula = B2*SIN(RADIANS(A2)). Those formulas are dragged down to repeat for all four vectors.

The sum in C7 is = SUM(C2:C5), and similarly the sum in D7 is = SUM(D2:D5). The final result gives the position of the boat in x,y coordinate form, (-218, 2315).

If you want the position in the form heading and distance, Excel has a very useful function ATAN2 that takes in x, y coordinates, and gives back the angle from the x-axis. The formula = ATAN2(G6, H6) will give the heading, in radians. To get the answer in degrees, use:

= DEGREES(ATAN2(G6, H6))

And the Pythagorean theorem will give us the length, it is just $\sqrt{x^2 + y^2}$, which in Excel form is = SQRT(G6^2 + H6^2). Let's add these two formulas to the spreadsheet. The formulas are in cells C9 and C10; they are also entered in text form in column D, just as a reminder of how the conversion works.

	A	B	C	D
1	heading	distance	x	y
2	135	500	-354	354
3	55	700	402	573
4	85	900	78	897
5	125	600	-344	491
6				
7		SUM	-218	2315
8				
9		Heading	95	=DEGREES(ATAN2(C7,D7)
10		Distance	2325	=SQRT(C7^2 + D7^2)

Figure 5-14 Result in x,y form, and distance-heading form

This is not in perfect agreement with the results of our original graphic addition (2325 meters instead of 2328 meters), but it is close. Doing things with protractor and ruler is never going to give a perfect answer. The Excel answer is presumably closer, but in reality, this problem was about the movement of a boat. None of the numbers are very precise, neither the headings nor the distances travelled. The distances could all be off by 10 meters; the headings could be off by a few degrees. Either approach is sufficiently accurate, given the nature of the data.

Let's summarize this section on vectors:

- a vector has a magnitude and a direction, i.e. 300, 60°
- alternately, it can be given in the form (x-component, y-component)
- to go from magnitude, direction to x,y
  x = magnitude*COS(RADIANS(direction°))
  y = magnitude*SIN(RADIANS(direction°))
- to go from x, y to magnitude, direction
  magnitude = SQRT(x^2 + y^2)
  direction° = DEGREES(ATAN2(x,y))

Vector Lab

This is a common lab exercise for physics classes. It is equally suitable for a math class. The objective is to experimentally measure a set of force vectors that are expected to add to 0, and verify that they do so by adding them graphically, and by using x, y vectors.

Materials

- Three or more identical spring balances, max force 2 Kg / 4 pounds or less (Doesn't matter if calibrated in ounces, grams, or newtons, but should not be heavy duty scales like fisherman use that can weigh up to 40 lbs.)
- Small metal ring – available at lock shops for key rings
- Protractor
- Ruler
- Large sheet of paper – butcher paper, kraft paper …

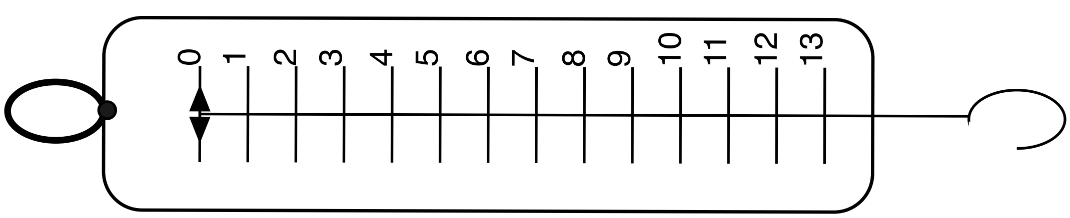


Figure 5-15 Spring balance

Procedure

1. Connect the hooks of 3, 4, or 5 spring balances to the metal ring.
2. Position the balances over a large sheet of paper.
3. Have the students each pull on their balances. Adjust the pull strengths so that all balances are in range, somewhere between 0 and max pull. You could have the balances equally spaced, but it is more interesting to use more varied angles.

4. On the paper, mark:
    a. The center of the metal ring
    b. The direction of each scale
    c. The force registered on each scale

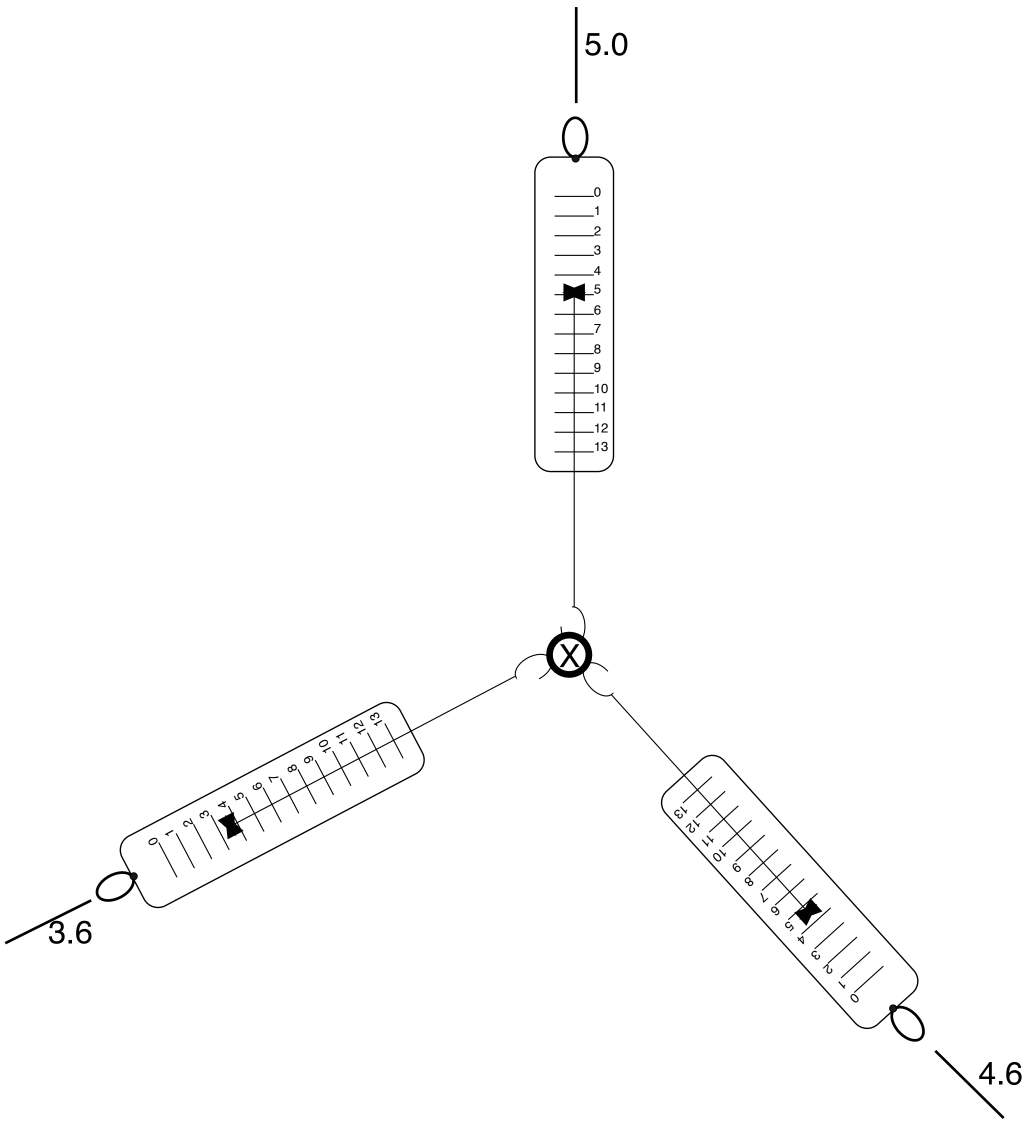


Figure 5-16 Force vectors - setup and marks

5. Remove the balances.
6. Draw lines from the center point to each of the scale direction marks.
7. Use the protractor to measure the angles of the lines drawn in (6). Pick any one of the lines to start, call it 0°, and measure the other lines relative to that first line.

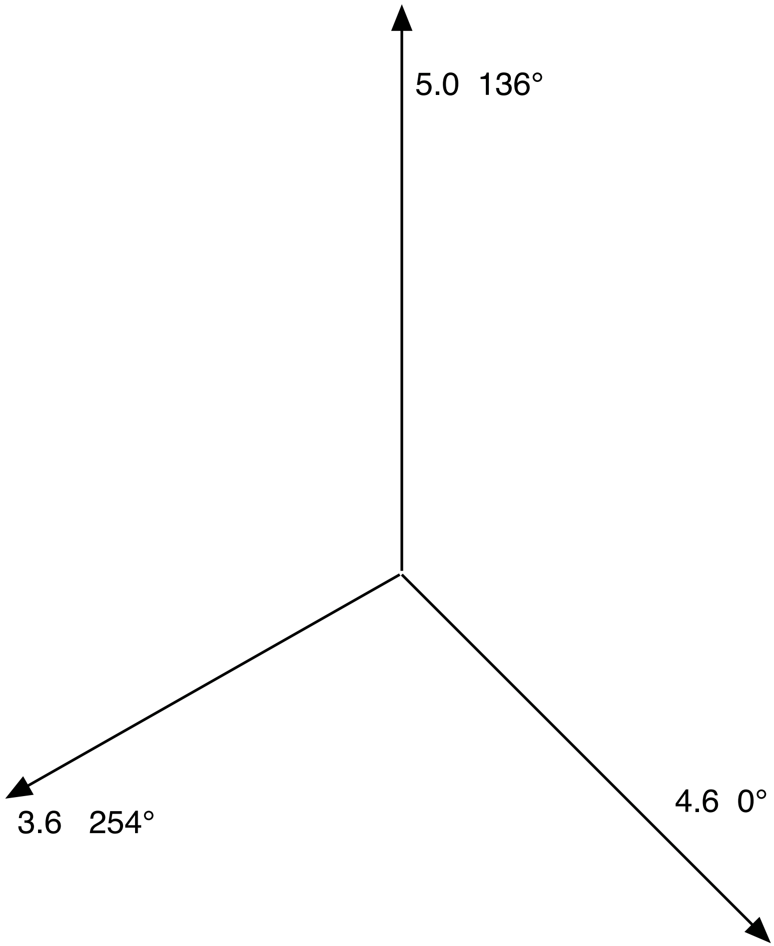


Figure 5-17 Force vectors from direct measurement

8. You now have measured a set of force vectors – one vector for each spring balance. In the example we've been showing, we have put the vectors into table 5-4. Tabulate your vectors in a similar form.

Table 5-4 Force vectors

force	direction
4.6	0°
5.0	136°
3.6	254°

9. Add your set of vectors together graphically, using a compass and ruler. Draw the first vector; then draw the second vector starting at the end of

the first vector; then draw the third vector starting at the end of the second vector; etc. Ideally, the last vector should end at the start of the first vector. (If it is way off, the usual problem is misreading the protractor. Looking at lines drawn through the origin on an x-y graph, lines in the first quadrant are 0° – 90°, second quadrant are 90° - 180°, etc.)

10. And finally, once you have the graphic vector addition straightened out, use Excel to add the vectors together using the x-y components of the vectors.

Geography and GPS – an aside on angle measurement on earth's surface

The earth spins one revolution per day, 15° per hour, 1° every four minutes. The sun, moon, and stars move through the sky at that rate. Thus the moon, only ½° wide, moves across the sky by its angular diameter, ½°, every two minutes. If you can estimate how many degrees separate the moon and the western horizon, you can estimate how long it will take for the moon to set; 3° ➔ 12 minutes, maybe a bit more if the moon is moving toward the horizon on a slanted course. (The moon, sun, and stars don't move at exactly the same rate however. In a year's time, the earth rotates 366.25 full turns relative to the stars. The earth's orbital motion "unwinds" one turn from the sun's apparent motion, so we get only 365.25 solar days. The moon's orbital motion adds another twist; the moon rises about 50 minutes later each night, due to its orbital motion.)

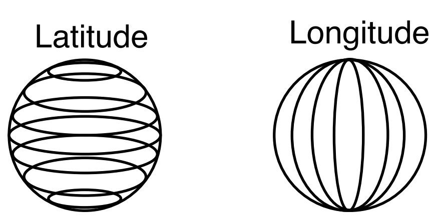


Figure 5-18 Latitude and Longitude

Mapmakers have divided the earth with lines of longitude (the north-south lines) and latitude (the east-west lines, parallel to the equator). One degree of latitude is 69 miles. One degree of longitude varies; at the equator it is 69 miles, but if you travel closer to the poles, the distance decreases, because the lines of longitude converge. One arcminute of latitude is 69/60 = 1.15 miles approximately. This was at one time the definition of a nautical mile. Definitions change – the nautical mile is now set by international agreement to be exactly 1,852 meters, but still very close to the length at the surface of the earth of 1 arcminute.

One arcsecond of latitude is 1.15 miles/60, approximately 101 feet. High quality GPS receivers can pinpoint horizontal position with an accuracy of ~11 feet, so they are accurate to roughly 1/10 arcsecond. I mention this mainly to emphasize that we can and do resolve angles to exceedingly fine levels. An even better example is found in astronomy.

Astronomers, the Masters of Angle Measurement – An Aside on Parallax

Measuring the distance to stars using parallax is nothing more than surveying on a grand (dare I say cosmic) scale. In surveying, if you can get a bearing on a distant object from two points separated by a known distance, then you can use trig to find the unknown sides of the triangle, and thus the distance to the distant object, in this case a telephone pole.

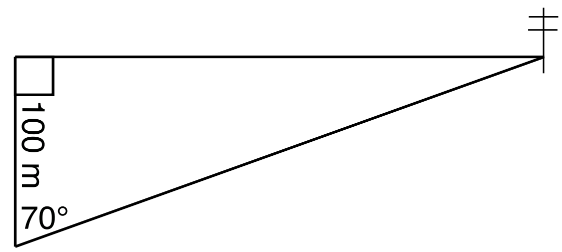


Figure 5-19 Typical surveying problem

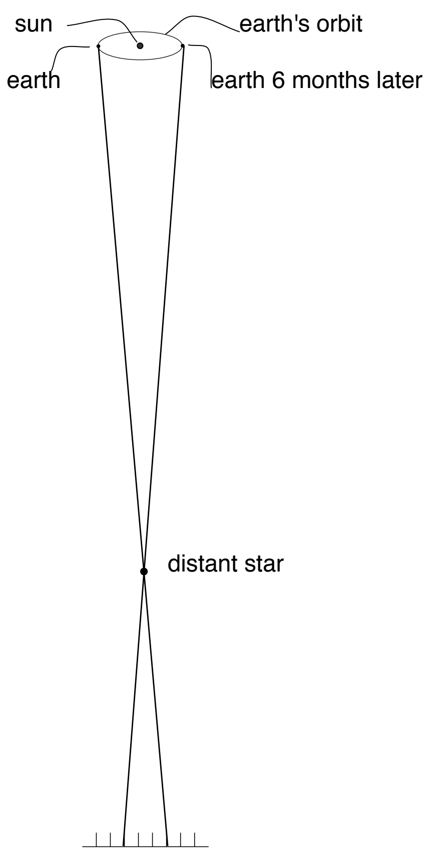


Figure 5-20 Principle of parallax measurement

The complication in astronomy is that even the closest stars are very far away. If you tried to get a bearing on a star from two points on the earth's surface, separated by a few thousand kilometers, you wouldn't detect any difference in the angle as measured from the two points. In order to get a longer baseline, the practice has been to check the angle at six-month intervals. By doing this, the baseline is the diameter of the earth's orbit around the sun, roughly 300 million kilometers. Even with this, the difference in the angle from the two points is less than two seconds of arc for even the closest stars (not counting the sun). This is too small to read from any protractor-like scale fixed to a telescope. (Telescopes used to have setting circles, essentially full 360° protractors, fixed to their mounts. Nowadays this hardware has given way to computers, which display the angles, and control the motion and direction of the telescope. But even these computerized systems are not precise enough to measure arc seconds.) Instead, the change in angle is determined by noting the apparent shift of the star relative to a distant scale, depicted in the figure as a protractor scale. Unfortunately, the ruled scale at the edge of space doesn't exist. But there are lots of stars handy, and most of them are so extremely distant that they show no shift whatsoever. These extremely distant stars are used as the scale.

Given an image with stars of known positions, the scale of the image can be calculated, i.e 1 centimeter on the image = 1 arc minute. The apparent movement of a nearby star relative to a distant star can be measured, and scaled to some number of arc seconds (typically just a fraction of an arc second).

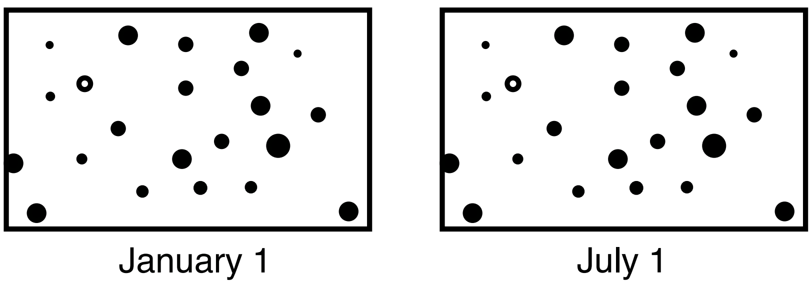


Figure 5-21 Position shift due to parallax

In this figure, all the stars are extremely distant except the one shown as a circle. If you look closely, you will notice that it has shifted slightly in the second picture, while all the others have remained stationary. (With a little practice, you may be able to view these two pictures as a 3D image, if you can persuade your left eye to look at the left image, while the right eye looks at the right image.)

How distant would a star be if it showed one arc second of displacement? The diameter of earth's orbit is 2.992×10^8 Km, so …

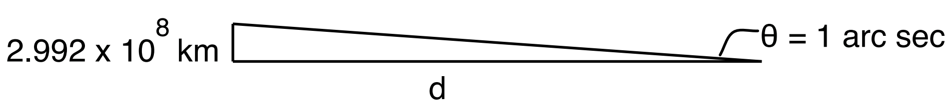


Figure 5-22 Parallax measurement of 1 arc second

$\tan(\theta) = 2.992 \times 10^8 / d$

$d = 2.992 \times 10^8 / \tan(1 \text{ arc second})$ Let's use Excel to do the math…

	A	B
1	diameter of earth's orbit in kilometers	2.992E+08
2	θ in arcseconds	1
3	θ in arcminutes = B2/60	0.016667
4	θ in degrees = B3/60	0.000278
5	θ in radians = RADIANS(B4)	4.84814E-06
6	tan(θ) = θ for small angles, = B5	4.84814E-06
7	tan(θ) = 2.992E+08/d	
8	d = 2.992E+08/tan(θ) = B1/B6	6.171E+13

Figure 5-23 Distance for 1 arcsecond angle

So the distance would be 61.71 x 10^{12} kilometers, or 61.71 x 10^{15} meters. There is a metric prefix for 10^{15}, peta, so we could say 61.7 petameters, although honestly, no one would know what you are talking about, certainly not astronomers.

Astronomers measure nearby objects (within the solar system) in astronomical units. 1 A.U. is the average distance from the sun to the earth, 1.496 x 10^{8} meters or 1.496 x 10^{5} kilometers.

Larger distances – to stars, other galaxies, etc. – are measured in parsecs. One parsec is just exactly one half the distance we calculated in the Excel snippet above – 3.086 x 10^{13} kilometers. Why is it half? Instead of using the full diameter of the earth's orbit in the calculation (2.992 x 10^{8} kilometers), astronomers use one A.U., which is just half of 2.992 x 10^{8} kilometers. Furthermore, the parallax of an object is defined to be ½ of the change in angle measured from the earth at opposite sides of its orbit. In practice, the angles are measured whenever weather permits, and when time on the telescope is available. Whatever the earth's displacement happens to be between two observations, the parallax angle for a given object is scaled to what it would be if the earth's displacement were exactly 1 A.U.

And once the parallax angle is obtained, the distance in parsecs is just the reciprocal of the angle. 1 arcsecond => 1 parsec, ½ arcsecond => 2 parsecs, 1/10 arcsecond => 10 parsecs, 0.22 arcsecond => 1/0.22 = 4.55 parsecs, etc. Note that this works perfectly well for these small angles. But don't expect it to work if you're dealing with big angles in your trigonometry class. Tan(70°) is not 2x tan(35°) (2.75 ≠ 2 x 0.70). But when the angles are measured in arc seconds, the tangent function is very linear, i.e.

tan(2 arc seconds) ≅ 2*tan(1 arc second)

0.000009696 ≅ 2 * 0.000004848)

For angles of an arcsecond or less, the approximation is nearly perfect.

For unknown reasons, the parsec unit of measurement has never really caught on outside the astronomical community. In the popular press, the light year is more commonly used. One light year is simply the distance a light beam can travel through empty space in one year. Light travels roughly 300,000 km per second. Again resorting to excel …

	A	B	C
1	speed of light = 299,800 Km/sec	2.998E+05	
2	1 light second = B1	2.998E+05	kilometers
3	1 light minute = B2 * 60	1.799E+07	kilometers
4	1 light hour = B3 * 60	1.079E+09	kilometers
5	1 light day = B4 * 24	2.590E+10	kilometers
6	1 light year = B5 * 365.25	9.461E+12	kilometers

Figure 5-24 One light year, in kilometers

Although scientists pride themselves on being logical, the variety of length units proliferating in astronomy shows a lot of history creeping in. I've put together a table below that will help you convert from one measure to another. Sometimes understanding the units is half the battle in grasping new material.

	kilometers	astronomical units A.U.	light years	parsecs
1 kilometer	1	6.68×10^{-9}	1.06×10^{-13}	3.24×10^{-14}
1 A.U.	1.496×10^{8}	1	1.58×10^{-5}	4.84×10^{-6}
1 light year	9.46×10^{12}	6.32×10^{4}	1	0.307
1 parsec	3.09×10^{13}	2.06×10^{5}	3.26	1

Figure 5-25 Common distance units for astronomy

Astronomers had sought to detect parallax shift for decades, if not centuries. Indeed, the inability to detect the parallax shift was a strong argument for placing the earth at the center of things with the sun orbiting around it, rather than the having the earth orbiting the sun. Galileo was certainly aware of the principle, but actual detection had to await better instruments. Friedrich Bessel, the German astronomer-mathematician, was the first to find the distance to a star by parallax measurements, in 1838. The star was 61 Cygni; its distance is 11 light-years. Bessel measured its parallax to be 0.3136 arcseconds. He later refined his measurements, and concluded that the true angle was 0.348 arcseconds. Modern measurements reveal that his first measurement was closer;

the value is now taken to be 0.287 arcseconds. Before his measurement, stellar distances were purely a matter of speculation and opinion.

Note that photography was in its infancy in 1838. The first astronomical photograph would not be taken until 1851. Bessel had to measure the displacement at the eyepiece of his telescope, probably using a bifilar micrometer – sort of a crosshair reticle, with an extra moveable crosshair, allowing precise measurement of the angular separation of two stars. The process of getting a reliable parallax measurement is somewhat more involved than I have made it sound; it took him several years to get a measurement that he was comfortable with publishing.

A British astronomer, Thomas Henderson, published a second parallax distance measurement two months later. Having been assigned a post in the southern hemisphere (at the Cape of Good Hope, a place he described as a "dismal swamp" infested with "insidious venomous snakes") he could observe Alpha Centauri, a bright star not visible to European observers in the northern hemisphere. His first estimate of the parallax was slightly more than one arcsecond. Later measurements have trimmed the parallax down to 0.76 arcseconds, making the distance 4.3 light-years. It has turned out to be our closest neighbor. (It is a triple-star, consisting of two large stars Alpha Centauri A and Alpha Centauri B, and a third dim red giant star Proxima Centauri. You may have read that Proxima Centauri is the closest star, but all three are close to each other, and thus all at nearly the same distance.)

The ability to measure these small displacements is limited by earth's atmosphere, which "fuzzes" the images of stars. Hipparcos, a satellite launched by the European Space Agency ESA in 1989, has measured these parallax angles more accurately than is possible by earth-based telescopes. Its resolution in measuring parallax is 0.001 arcseconds. Hipparcos is both an acronym – **Hi**gh **P**recision **Par**allax **Co**llecting **S**atellite – and a reference to the ancient Greek astronomer Hipparchus, who was known for his application of trigonometry to astronomy. One result of the Hipparcos satellite measurements is the Hipparcos Catalog, which gives the parallax measurements of more than 100,000 stars. Note that the precision of the measurements limits it to stars within roughly 1000 parsecs. As our Milky Way galaxy is over 30,000 parsecs in diameter, the stars in the catalog are all within the Milky Way. (Back in the 1960's, when I first learned a little about astronomy, the number of stars close enough for their parallax to be determined was only a few dozen.)

The ESA launched a more sophisticated Gaia spacecraft in late 2013. It has several mission goals, including measuring the positions of 1 billion stars, roughly 1% of the stars in the Milky Way, with an accuracy of 20 microarcseconds for the brighter stars. The observations will be made over a five year period. A first catalog is expected to be released in 2016, but parallaxes will not be included in this first catalog, as further observations are necessary to separate the parallax displacements from displacements due to the stars' motions in their orbits within the galaxy.