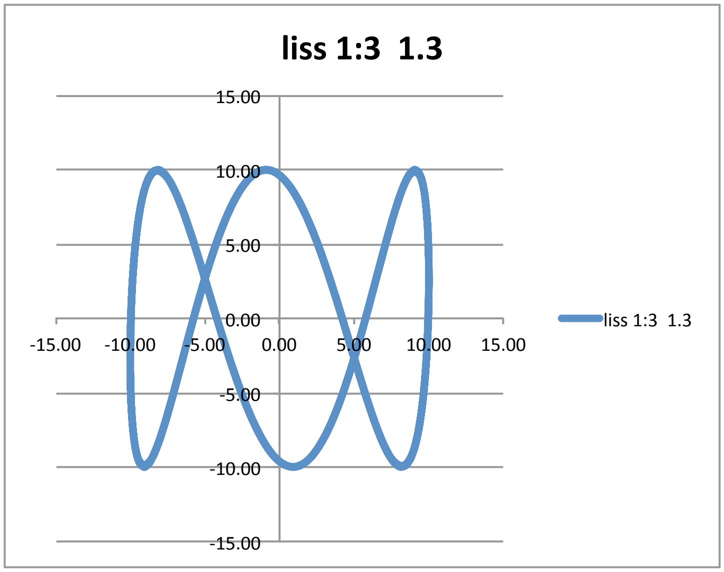
# Chapter 4 – A Number Most Curious




Often when math teachers wish to start at the beginning, they put up the familiar number line.

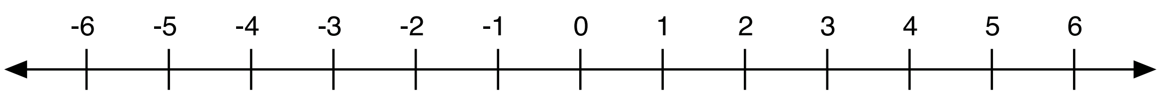



Figure 4-1 The number line

To an engineer, this is a homogenous line, with about as much internal structure as a potato. Any point on the line can be measured and given a value, 4.7238 for instance. A rather small number of decimal places are sufficient to give the measurement, to within the practical limits of machining. Looked at the other way, we normally only measure mechanical objects to the nearest thousandth or perhaps ten thousandth of an inch. Optical surfaces are more precise – their curved surfaces differ from the ideal by only a few millionths of an inch, but still that amounts to only a couple more decimal places.

But if we leave behind the practical matter of brake rotors, pistons, and telescope mirrors, then even a middle school math student knows that not all the numbers on the number line can be expressed as decimal fractions. Something as simple as 1/3 is stubbornly difficult to write as a decimal fraction, without the dodge of writing 0.333… , with those three dots signifying that the 3s continue indefinitely. But if we can write a number as a ratio of two integers, then it is a rational (from ratio) number. That takes care of 1/3, 3/7, etc. That should cover it, right? No – something as simple as $\sqrt{2}$ proves to be not rational; we can't find an integer ratio x/y which gives 2 when squared. What's worse, we find that the number line has far more of these irrational numbers than it does rational numbers. Still, with rational numbers and irrational numbers, we do have the number line covered. Nothing else is going to pop up and surprise us. (Well, there are imaginary numbers, but they don't exactly sit on the number line.)

And that should be that. Of course it isn't though. To a mathematician, the number line is as densely populated with bizarre denizens as the Amazon rainforest. Let's go have a close look at one of these creatures.

## A Calculator Trick

Using a calculator that has a reciprocal function (1/x), do the following:

Pick a number, 0 or higher.

1. Add 1
2. Take the reciprocal 1/x

Repeat steps 1 and 2 for many iterations, maybe 25 or 30. You should see the calculator converge slowly on a number 1.x, whose reciprocal is 0.x.

I've run this loop in Excel, for the purpose of demonstrating the iteration. Here are the first few lines.

| | A | B | C |
|---|---|---|---|
| 1 | 1 + 1/r | 1/r | |
| 2 | | 3.00000000 | <--initial guess |
| 3 | 4.00000000 | 0.25000000 | |
| 4 | 1.25000000 | 0.80000000 | |
| 5 | 1.80000000 | 0.55555556 | |
| 6 | 1.55555556 | 0.64285714 | |
| 7 | 1.64285714 | 0.60869565 | |
| 8 | 1.60869565 | 0.62162162 | |
| 9 | 1.62162162 | 0.61666667 | |
| 10 | 1.61666667 | 0.61855670 | |

Figure 4-2 Calculator trick

You can see that the iteration is heading towards the pair 1.61-something, 0.61- something. Carried out for another 20 passes, the eventual numbers come out to be 1.61803399 and 0.61803399. These two numbers are reciprocals of each other. Once you have either of these numbers entered in your calculator, you can just keep pressing the reciprocal 1/x key, and the display just toggles between the two numbers. It appears that the only change is that the leading digit toggles between 0 and 1, the fractional part of the number remains constant.

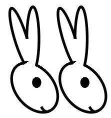
## Leonardo Pisano, aka Fibonacci

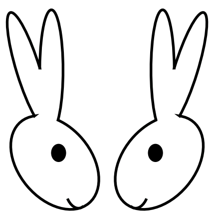
Leonardo Pisano, now universally known as Fibonacci, was born in Pisa, Italy in the 12$^{th}$ century. He spent his childhood in North Africa, where he was educated by the Moors.

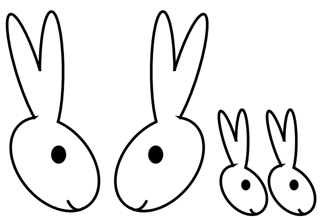
He travelled widely on business through Algeria, Syria, Greece, Sicily, and Egypt. Around 1200 he returned to Pisa, where he published a book which introduced the decimal number system (0, 1, 2, 3, 4, … instead of I, II, III, IV…) to the Latin-speaking world. By all rights, that is what he should be remembered for. But no, we remember him for a side topic in his book, on the mathematics of breeding rabbits.

## Fibonacci's rabbits

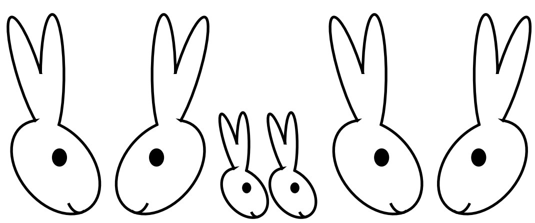
Fibonacci modeled the rabbit population explosion with this line of reasoning.

Start with a pair of young rabbits:  total – 1 pair

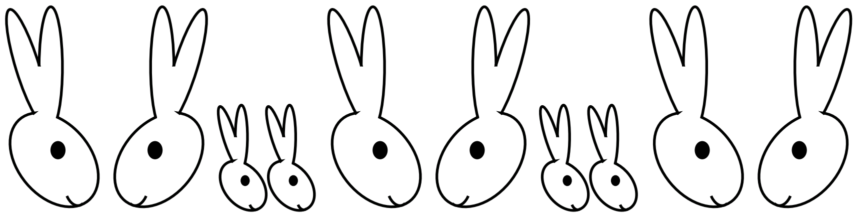
1 month later – they've grown  total – 1 pair

1 month later – proud parents  total – 2 pair

1 month later – parents again, and first children have grown…

 total – 3 pair

1 month later – 2 sets of parents, 1 pair children have grown



total – 5 pair

So the rabbits multiply, obeying 2 simple rules:

1. Baby rabbits take 1 month to become mature rabbits.
2. Mature rabbits produce a new pair of babies every month.

If you continue this process, month by month, you find that the pairs of rabbits multiply according to the Fibonacci series – 1, 1, 2, 3, 5, 8, 13, 21, … - where each new element is the sum of the previous two entries. The previous entry is last month's total number of rabbit pairs. And the entry before that is the number of rabbit pairs which will give birth in the current month.

Obviously this is a simplified model, which ignores lots of important biology. The rabbits never die, and they give birth to exactly two offspring, every month, like clockwork. But even with these simplistic assumptions, some truths about rabbits do emerge. Australians probably wish their forebears had paid more attention to Fibonacci before they imported a few dozen rabbits in 1857 – 1859. Ten years later, the population had exploded. By that time, they were shooting and trapping more than 2 million rabbits every year, with no noticeable decrease in the rabbit population. In 1950 myxoma virus was introduced to control the population, which was then estimated to be around 600 million. The virus worked well at first, dropping the population to around 100 million. However, some of the rabbits were genetically resistant; by 1990 the population was back up to an estimated 200-300 million. Control efforts continue.

Australia's rabbit problem aside, what does the Fibonacci series have to do with the number 1.618033989 ? As the numbers of the series get larger and larger, the ratio of the N+1 term to the N term takes on a familiar look. We can resort to Excel to show this easily.

| | F | G |
|---|---|---|
| 1 | fib series | ratio of terms |
| 2 | 1 | |
| 3 | 1 | 1.00000000 |
| 4 | 2 | 2.00000000 |
| 5 | 3 | 1.50000000 |
| 6 | 5 | 1.66666667 |
| 7 | 8 | 1.60000000 |
| 8 | 13 | 1.62500000 |
| 9 | 21 | 1.61538462 |
| 10 | 34 | 1.61904762 |
| 11 | 55 | 1.61764706 |
| 12 | 89 | 1.61818182 |
| 13 | 144 | 1.61797753 |
| 14 | 233 | 1.61805556 |
| 15 | 377 | 1.61802575 |
| 16 | 610 | 1.61803714 |
| 17 | 987 | 1.61803279 |
| 18 | 1597 | 1.61803445 |
| 19 | 2584 | 1.61803381 |
| 20 | 4181 | 1.61803406 |
| 21 | 6765 | 1.61803396 |
| 22 | 10946 | 1.61803400 |
| 23 | 17711 | 1.61803399 |
| 24 | 28657 | 1.61803399 |
| 25 | 46368 | 1.61803399 |

Figure 4-3 Fibonacci series & ratio of terms

A pair of 1s are entered at the top of column F. Then, for the third element, we enter the formula = F2 + F3, which gives 2 when entered (the sum of 1 + 1). When we drag that formula down to repeat it in column F, the relative cell references insure that each new entry is the sum of the previous two, and we get the Fibonacci series, as far as you and Excel care to go.

Cell G3 contains the formula = F3/F2, which is just the ratio of the current Fibonacci element (1) to the previous element (1). That formula is repeated down through column G. So the ratio of 13/8 is 1.625, for example. By the 25$^{th}$ entry, the ratio has reached 1.61803399.

This result isn't really dependent on the numbers in the first two cells – the 1s in cells F2 and F3. Regardless of the starting numbers, the resulting series will be Fibonacci-ish, and the ratio of successive values will approach 1.618… (OK – the two starting numbers can't both be 0; that gives a very boring series.) This result was proved by the astronomer/mathematician Johannes Kepler around the year 1600. If you have replicated this series in Excel, you can easily tinker with it – try changing the initial values to bigger integers, fractions, or even a positive and a negative number.

## Newton's Method

There are faster ways to arrive at this number. One way, which relies on a touch of calculus, is to guess a value n, and then calculate the next n – let's call it n' – as: $n' = (n^2 + 1)/(2n-1)$. Repeat this iteratively, each time using the latest value of n as the starting value for the next calculation. Here's how to get it started:

| | A | B | D | C |
|---|---|---|---|---|
| 1 | $n^2 + 1$ | 2n - 1 | $(n^2+1)/(2n-1)$ | |
| 2 | | | 3.00000000 | <- initial guess |
| 3 | 10.00000000 | 5.00000000 | 2.00000000 | |
| 4 | 5.00000000 | 3.00000000 | 1.66666667 | |

Figure 4-4 Newton's method

The initial guess is 3. The formulas used in the columns are given at the heads of the columns. $n^2 + 1$ formatted for Excel is = D2^2 + 1, 2n – 1 is = 2*D2 – 1, etc. How many iterations do you anticipate needing to get to 1.618033989 ? Our previous Excel treatments have taken over 20 iterations. I've suggested that this will be faster, and Newton seems to have been a pretty smart cookie. I've shown the first 2 iterations. Will it take (a) five iterations or fewer total, (b) 6 to 10 iterations total, or (c) 11 to 15 iterations total? A clever student might be able to place a wager with an unsuspecting math teacher, resulting in treats for the class.

## Fibonacci Backwards

We generate the Fibonacci series by adding the previous two elements to generate the next element, like this:

1

1

1+1 = 2

2+1 = 3

3+2 = 5

5+3 = 8 etc.

You can readily see that we could drop into the series at an arbitrary point, and generate the preceding elements, like so:

8

5

8-5 = 3

5-3 = 2

3-2 = 1

2-1 = 1

1-1 = 0 and there is really nothing that forces us to stop here, so…

1-0 = 1

0-1= -1

1-(-1) = 2

-1 -2 = -3

2-(-3) = 5

-3-5 = -8 etc.

We find we have a strange extension to the Fibonacci series, with the same numbers as before, but alternating signs. Is this just a curiosity? It doesn't shed any new insights on rabbits – that's for sure! But will it pop up again somewhere?

### Euclid and a Host of Others

The number 1.618… was known in antiquity. It has a way of popping up in seemingly unrelated areas; we'll cover a few more in this section. It comes up so often that it has been given a symbol, in the same way that 3.14159… has been given the symbol π. The symbol for 1.618033989 is φ, a lower case letter of the Greek alphabet, usually pronounced fy (rhyming with fly), though pronounced by some as fee. (Purists point out that Greeks pronounce it as fee. However, Greeks also pronounce π as pee, so if you demand linguistic purity, you will be in for a lifetime of confusion and snickering.) The same symbol φ also stands for a variety of other things in other contexts; it isn't so unambiguous as π, which seems always to represent 3.14159.

It was first designated by the Greek letter φ in the early 1900's. Apparently φ was chosen to represent this number, because it was the first letter of the name of the Greek mathematician/sculptor Phidias, who was known to have used it to proportion sculptures for the Parthenon. Plato apparently was familiar with it, and Euclid (365 – 300 BC) included it in his book of geometry, *Elements*. "A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the lesser." Perhaps a diagram will clarify this unfamiliar wording.

Euclid is saying that we have cut the line according to this special ratio when:


Figure 4-5 Euclid's definition

$\frac{length\ of\ whole\ line}{longer\ segment} = \frac{longer\ segment}{shorter\ segment}$ or, in terms of 1 and x shown in the figure:

$\frac{x+1}{x} = \frac{x}{1}$ convert this to standard quadratic form

$x^2 - x - 1 = 0$ now substitute into quadratic equation

$x = \frac{1 \pm \sqrt{1+4}}{2}$

$x = \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2} = 1.61803399..., -0.61803399...$

(And we sort of ignore the negative value, since we are looking for a positive line length.) This may be the quickest way to find the value of $\varphi$; find the square root of 5, add 1, and divide by 2.

This number has come to be known as the golden ratio. Another common way of introducing it is by this construction:

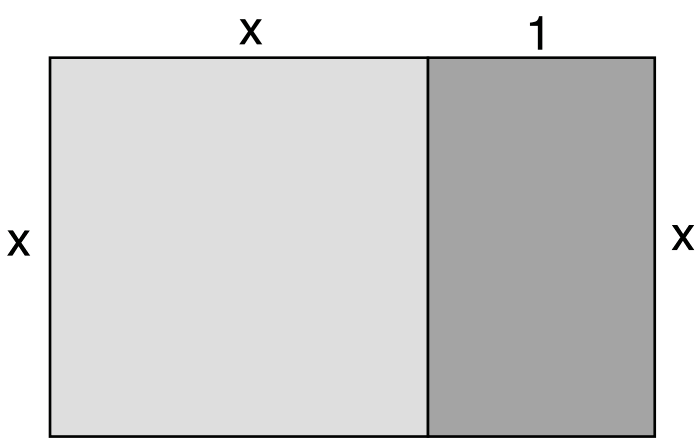



Figure 4-6 Golden rectangle

The constraint is that the length to width ratio of the small rectangle, namely $\frac{x}{1}$, must be the same as the length to width ratio of the overall rectangle, $\frac{x+1}{x}$, that is: $\frac{x+1}{x} = \frac{x}{1}$

This is the equation we just solved with Euclid's line segments, so of course we again arrive at x = 1.618033989… , and the rectangle is known as the golden rectangle.

Much has been made of this rectangle as being in some sense perfectly proportioned. Certainly artists and sculptors have used it from time to time in proportioning their works. In the context of art and architecture, φ is commonly called the golden ratio, and is $\frac{1+\sqrt{5}}{2}$, which in decimal form is 1.61803399…

Where else does it show up? Pentagons and pentagrams, for example:

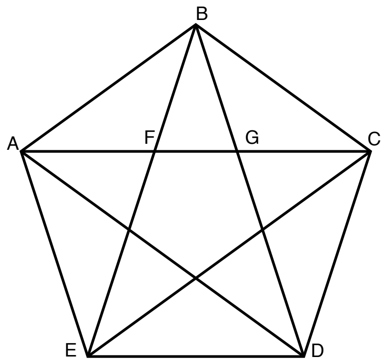


Figure 4-7 Pentagons and the golden ratio

The ratio of the length of diagonal AC to side AB is φ. The ratio of length AB to length BG is φ. The ratio of length BG to length FG is φ. Segment AG has the same length as side AB, so AG has the same relationships to AC and BG that AB has.

Johannes Kepler was certainly a "fan" of φ. As mentioned earlier, he was the first to prove that ratio of successive elements of the Fibonacci series approaches $\varphi$. Furthermore, he wrote:

> *Geometry has two great treasures: one is the theorem of Pythagoras, the other the division of a line into mean and extreme ratio. The first we may compare to a mass of gold, the second we may call a precious jewel.*

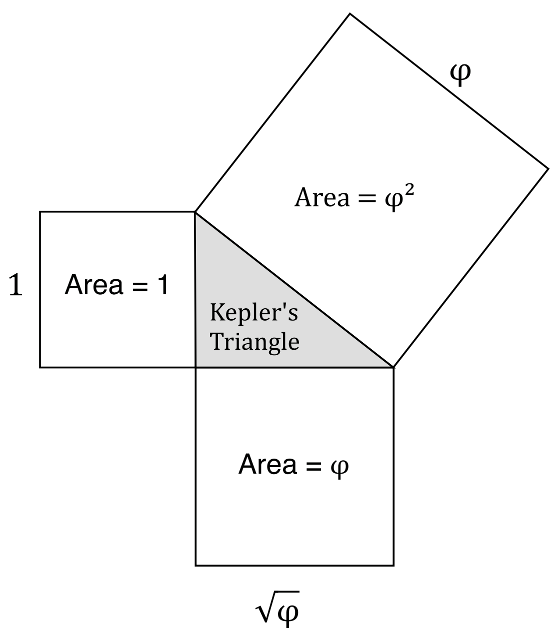


Figure 4-8 Kepler's triangle

Because $\varphi^2 = 1 + \varphi$ (see below), he realized he could construct a right triangle with sides of length 1, $\sqrt{\varphi}$, and φ. This triangle is known as a Kepler triangle, and combines what he considered to be the gold and jewel of mathematics into one entity.

Proof: Since $\varphi = \frac{1+\sqrt{5}}{2}$,

$$\varphi^2 = (\frac{1+\sqrt{5}}{2})^2 = \frac{1+5+2\sqrt{5}}{4}$$

$$= \frac{4+2+2\sqrt{5}}{4} \quad \text{rearranging terms}$$

$$= 1 + \frac{1+\sqrt{5}}{2}\text{, or simply } 1+\varphi. \text{ Thus } \varphi^2 = 1 + \varphi$$

## Beware the Mumbo Jumbo!

There is a branch of superstition known as numerology, which associates divine mystical relationships between numbers and one's fate. While it is clearly hooey, most of us are not entirely immune. Do you perk up when your birth day-of-the-month (i.e. 12, if you were born on the 12th of the month) happens to come up in some unrelated context – maybe as the jersey number of a football player? Are you superstitious about the number 13? (This is formally called triskaidekaphobia, which is a mash-up of Greek words meaning three-and-ten-fear. If you like that word, you'll really like fear of Friday the 13th, which throws in the Norse goddess Frigg, for whom Friday is named – friggatriskaidekaphobia.) There are so many people who fear the number 13 that buildings with more than 13 floors often have no button marked "13" in the elevators. Trump Tower in Chicago labels the 13th floor as the Mezzanine floor, thus dodging the issue. φ has gotten more than its share of attention from numerologists, due to its unexpected emergence from so many lines of numerical reasoning. In particular, the pentagram (the five pointed star) is itself associated with a number of religions, some of them leaning towards the occult. But quite aside from the mumbo jumbo, it is an interesting number, and it has been studied by mathematicians from Pythagoras (570 B.C.) to Kepler (1571) to Roger Penrose (1931).

## Geometric and/or Fibonacci Series

By now, we've learned a few things about φ. The calculator trick that kicked off this chapter showed that the reciprocal of 1.618… is 0.618… In terms of φ, $1/\varphi = \varphi - 1$. And in the section on Kepler's triangle, we showed that $\varphi^2 = 1 + \varphi$. We've got four elements of some kind of series, but is it Fibonacci or geometric?

$\varphi^{-1} = \varphi - 1$ (remember that $x^{-1}$ is the same thing as $1/x$)

$\varphi^0 = 1$ (because anything to the 0 power is 1, except ironically 0)

$\varphi^1 = \varphi$

$\varphi^2 = \varphi + 1$

Let's start with a generalized geometric series, one that extends in both directions:

… $x^{-3}$, $x^{-2}$, $x^{-1}$, $\underline{1, \ x, \ x^2}$, $x^3$ …

If we want a geometric series which doubles as a Fibonacci-ish series, then we can say that each element must be the sum of the previous two elements, for instance:

$x^2 = x + 1$

$x^2 - x - 1 = 0$ the same equation we keep running into …

$x = \varphi$

So we have a series that we can write in the form of a geometric series, or in the form of a Fibonacci-ish series.

| geometric | Fibonacci | decimal |
|---|---|---|
| $\varphi^{-4}$ | $-3\varphi + 5$ | |
| $\varphi^{-3}$ | $2\varphi - 3$ | |
| $\varphi^{-2}$ | $-1\varphi + 2$ | 0.38196601 |
| $\varphi^{-1}$ | $1\varphi - 1$ | 0.61803399 |
| 1 | $0\varphi + 1$ | 1.00000000 |
| $\varphi$ | $1\varphi + 0$ | 1.61803399 |
| $\varphi^{2}$ | $1\varphi + 1$ | |
| $\varphi^{3}$ | $2\varphi + 1$ | |
| $\varphi^{4}$ | $3\varphi + 2$ | |

Figure 4-9 Geometric/Fibonacci-ish series

If you aren't quite convinced, you can:

1. Use Excel (or a calculator) to evaluate the elements in the geometric and Fibonacci series, and show that they come out to be the same, when expressed as decimal fractions.
2. Or use the value of $\varphi = \frac{1+\sqrt{5}}{2}$, and verify through algebraic manipulations that the terms are equivalent.

Finally, study the entries in the Fibonacci column. The extended Fibonacci series ( …+5, –3, +2, –1, +1, 0, +1, +1, +2, +3…) is showing up again, both as the coefficients of $\varphi$, and the constants. The way the Fibonacci terms alternate signs to give the $\varphi^{-n}$ values is really quite beautiful – a Keplerian gem.

## Fibonacci_3 Series, and a Quirky Approach to Solving a Cubic Equation

Possibly in your high school math class, the teacher has briefly covered solving cubic equations, equations of the form $ax^3 + bx^2 + cx + d = 0$. Math classes today may be different, but when I was introduced to cubics, there was always something fairly easy that could be factored out, like $(x + 2)$ for instance. After the first round of factoring, what remained was a normal quadratic equation, which could be further factored, or plugged into the quadratic formula if the next round of factoring was difficult.

In the worst case however, there may not be a simple factor to pick out to get things started. Let's try working with one of these cases, a special one closely related to the Fibonacci series.

Suppose we have a Fibonacci_3 series, where each term is the sum of the preceding three terms. If this were a geometric series and a Fibonacci series, we could write $x^3 = x^2 + x + 1$, or in standard form

$x^3 - x^2 - x - 1 = 0$

And this equation turns out to be a booger to solve. There isn't a cubic formula to plug into like there is for quadratics. There is however a process which can be followed to eventually arrive at the answer. It is not an easy process to follow, which is evident from the final solutions for x. There are two imaginary roots, and one real root; let's focus on the real root for now – it is:

$x = \frac{1}{3}(1 + \sqrt[3]{19 - 3\sqrt{33}} + \sqrt[3]{19 + 3\sqrt{33}})$, which works out to 1.839286755…

Can we arrive at this number by tinkering with the Fib_3 series, in the same way we found φ by playing with the Fibonacci series?

The initial values are not too important, but let's start with 1, 1, 2 – so our series would be:

1, 1, 2, 4, 7, 13, …

If we reason that this should behave like the Fibonacci series with larger and larger values, then we hope that the ratio of successive terms will be a solution for $x^3 - x^2 - x - 1 = 0$. Here's the start of a spreadsheet that calculates the Fib_3 series, and tracks the ratios of successive terms. This will be almost identical to the spreadsheet in Figure 4-3 Fibonacci series, except that terms are the sum of the preceding 3 terms.

| | F | G |
|---|---|---|
| 1 | fib_3 series | ratio of terms |
| 2 | 1 | |
| 3 | 1 | 1.00000000 |
| 4 | 2 | 2.00000000 |
| 5 | 4 | 2.00000000 |
| 6 | 7 | 1.75000000 |
| 7 | 13 | 1.85714286 |
| 8 | 24 | 1.84615385 |
| 9 | 44 | 1.83333333 |
| 10 | 81 | 1.84090909 |

Figure 4-10 Fibonacci 3 series

If you continue this series, how far do you have to go to reach a ratio that no longer changes with later elements in the series?

And is that ratio 1.839286755… ? Or is it just a lucky but misleading break that the ratio seems to be heading in that direction at the ninth element, i.e. 81/44 ?

Don't trust anyone! Does that expression for x, with the cube roots and square roots, really evaluate to 1.839286755… ? It's worth double-checking with Excel. Note that there is no cube root function, but you can raise a number to the 1/3 power, i.e. =N^(1/3) which is just another way of specifying cube root. It is in general more powerful; you can ask Excel to find the 17$^{th}$ root in the same way, should you ever need to know, i.e. = N^(1/17) . Note: by putting the 1/3 inside parentheses, you insure that Excel will find the 1/3 power; otherwise it will find N^1, and divide that by 3.

Finally, check to see that the number 1.839286755 does satisfy the conditions that we have specified, namely that if we generate a geometric series based on the number – i.e. $1/1.839286755^2$, 1/1.839286755, 1.0000, 1.839286755, $1.839286755^2$… then it is also true that each term is equal to the sum of the previous three terms.

How far can you push this method? It's a good bet that you could use it to find a root of $x^4 - x^3 - x^2 - x - 1 = 0$. But could you use it to find a root for $x^3 - x^2 - x - 3 = 0$ ? In this case, you're hoping for a Fibonacci-ish series where a term $x^3$ is equal to the sum of the previous two terms ($x^2 + x$) plus three times the term before that (the constant 3 in the equation). If you find an "answer" via Fibonacci, you can test it by substituting into the original equation. This seems to invite some mathematical tinkering…