Chapter 6 An Old Problem – Chains & Cables

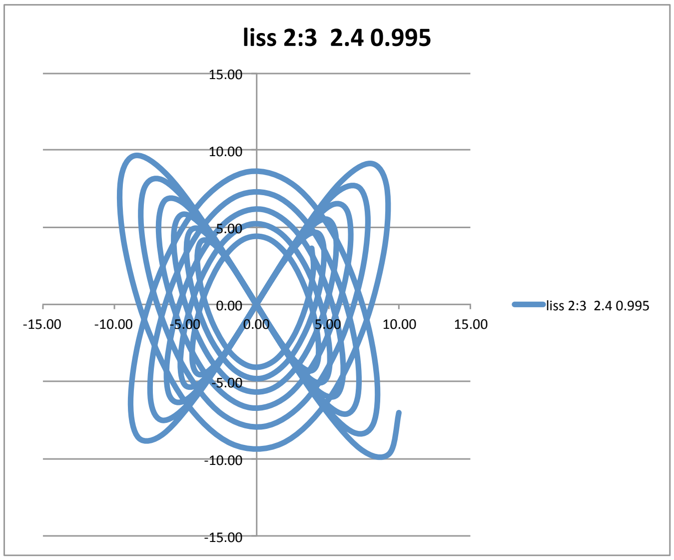


Figure 6-1 Hanging chain

Chains and cables, attached at two endpoints, hang with a characteristic droop. At a glance, the curve seems to be a parabola – or is it?

Galileo observed this curve, and noted that it is approximately a parabola. He was aware in 1638 that it was not exactly a parabola, but the precise mathematical description of the curve had to wait until 1691, when the correct equation was derived by Gottfried Leibniz, Christiaan Huygens, and Johann Bernoulli, all in response to a challenge by Jakob Bernoulli, the older brother of Johann. The particular curve is now known as a catenary, derived from the Latin "catena", meaning chain (cadena in Spanish).

A quick aside on this period in the development of mathematics – there was a time, certainly including 1691, when mathematicians were competitive, much like chess players. They were torn between the desire to get credit for finding a solution to a difficult problem, and the desire to keep secret their methods, for fear of giving a boost to competing mathematicians. Sometimes they published their findings in the form of anagrams – a gigantic jumble of letters. When someone else eventually solved the same problem, they would then unscramble their anagrams, and be able to claim that they had solved the problem some years earlier. This matter of the curve of a chain was in fact the subject of an anagram, published by Robert Hooke in 1675 as an appendix to a book on an entirely unrelated subject, helioscopes. The anagram was not unscrambled publicly until 1705, after Hooke's death. The original was in Latin; its meaning was "As hangs a flexible cable so, inverted, stand the touching pieces of an arch." Hooke had discovered that the curve of the hanging cable, turned upside-down, is the ideal form for building a self-supporting arch.

Finding and proving the exact equation for the curve is a problem for calculus. We can however, use Excel to calculate, link by link, how such a chain will hang. To make the calculation simple, we assume a series of 2 ounce weights, attached together with fishing line of negligible weight. (Whether the weights are 2 ounces or 2 kilograms makes no real difference – but all the weights must be the same.)

We measure the fishing line carefully, so that the distance from one weight to the next, measured center-to-center, is exactly 10 centimeters. (The exact distance is unimportant, but the center-to-center distances must all be the same.) Now the big question – how to get started. If we start with a "chain" of 20 or 30 weights, then we have to figure out how each weight will respond to the forces acting on it, which in turn will depend on the positions of the other weights, which gets horribly complicated quickly.

Simplify!

We can avoid this headache by starting with a "chain" of two weights! Arbitrarily we stretch the two end lines apart until they are 20° from horizontal.

Figure 6-2 Two weight problem

This is much simpler. What's more, it is symmetrical, so if we can analyze one of the weights, we will have taken care of both.

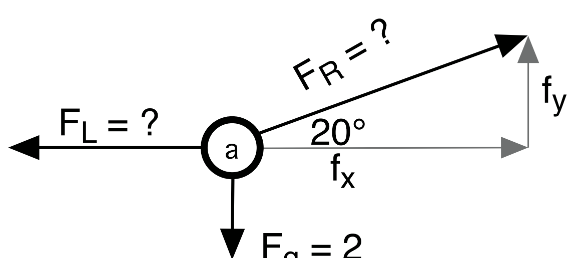

Figure 6-3 Forces on right weight "a"

Looking at weight "a" on the right, there are three forces acting on it: the line on the left is pulling the weight to the left with some unknown force F_L; gravity is pulling the weight down with a force of 2 ounces; and the line on the right is pulling to the right, and up, with some unknown force F_R. (Maybe it is obvious, maybe not: just in case it isn't – the direction of the line is identical to the direction of the force, and vice versa.) If we sketch just this weight, and break the force exerted by the right line into its x and y components, as in Figure 6-3, then mental bells and whistles start going off; we can see how to find the unknown forces!

Remember that the requirement for a stationary object is that all forces must balance. If the forces are given in x, y components, then the x components of the forces must balance, and the y components of the forces must balance. Therefore fy must be 2 ounces in the up direction, to balance the force of gravity F_g, which is 2 ounces directed down.

The only x direction forces are F_L and fx, so they must be equal but opposite. We can use trig to find out exactly what fx is.

tan(20°) = fy/fx = 2/fx

fx = 2 / tan(20°) = 2 / 0.364 = 5.49 ounces

Does it seem odd to you that fx is greater than 2 ounces, the actual weight? If we had picked 45°, then fx and fy would have been equal, both 2 ounces. For an angle smaller than 45°, we have to pull even harder to bring fy up to 2 ounces.

We can easily calculate the position of the end of the line; this will be the position of the next weight. For this first segment, we pulled the line until the angle reached 20°. Relative to the first weight, the line end will be 10cos(20°) = 9.39 cm in the x direction, and 10sin(20°) = 3.42 cm in the y direction.

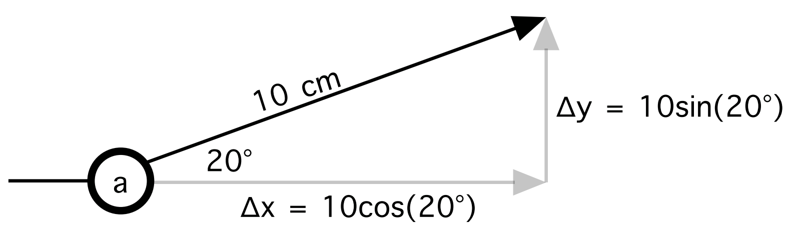

Figure 6-4 Finding the end of a line segment

Now that we have the bottom two weights analyzed, we add the next two weights, as shown. The weights just go at the end of the line segments. The question is where to locate the ends of the next line segments. Again, let's focus our attention on the rightmost weight "b".

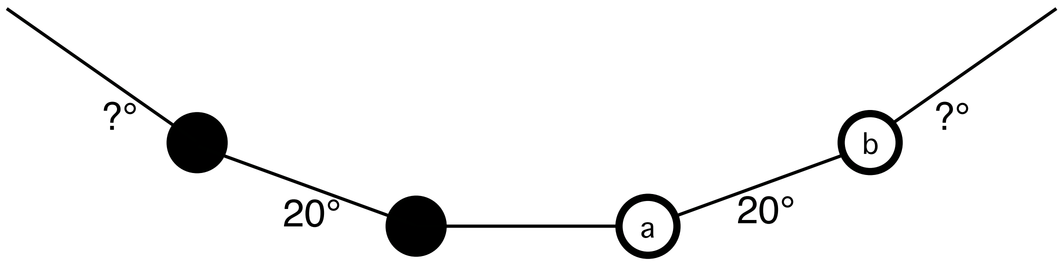

Figure 6-5 Four weight problem...

But before we take on the next weight, let's cover something that may or may not be obvious. Certainly it wasn't always obvious; Newton put it forward as his third law of motion; for every action (force) there is an equal and opposite action (force). In our system, if ball "a" and ball "b" are connected by a line, and ball "a" is pulling on the line with a force F, then ball "b" must be pulling on the line with a force –F. The minus sign signifies that the counterforce is the same magnitude, but opposite direction; this is what Newton means by equal and opposite. If we express the force F as its x and y components, then those force components at ball "a" are equal in magnitude to the x and y components at ball "b", but exactly opposite in direction.

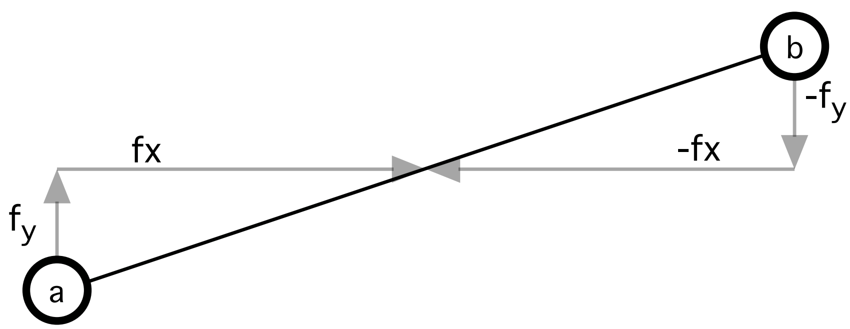

Figure 6-6 Equal and opposite forces

In figure 6-6, the two balls are connected by a line so that the balls are pulling against each other. The line pulls ball "a" up with force f_y, and it pulls ball "b" down with the equal but opposite force $-f_y$. Similarly, the balls are pulled towards each other along the x-axis with equal but opposite forces f_x and $-f_x$. (I haven't been too careful about putting in the minus signs on the diagrams – relying instead on the arrows to indicate the direction of the forces.)

Returning to our catenary chain of weights, we want now to analyze the forces on the next weight "b". The equal and opposite nature of the forces tells us immediately that the force from the line that goes to ball "a" pulls ball "b" to the left with a force of 5.49 ounces, and pulls downward with a force of 2 ounces.

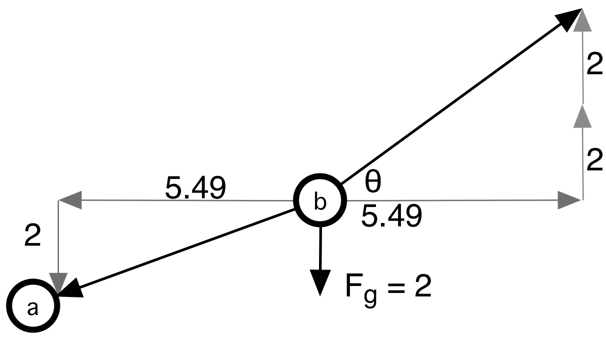

Figure 6-7 Forces on ball "b"

In addition, gravity acts on weight “b” itself, pulling downward with an additional force of 2 ounces.

This is all we need to know to find the forces exerted by the line to the right of weight “b”. The forces have to add to zero, which means:

- the horizontal forces must add to zero, so the new horizontal force is 5.49 ounces to the right. Indeed, no new horizontal force will come into the calculations, so this 5.49 ounce force will just keep getting passed along from one weight to another as long as we care to continue the chain.
- The vertical forces must also add to zero – two ounces passed from the line on the left, plus two ounces from the weight “b” itself – so the line to the right must pull upwards with 2+2 = 4 ounces of force.

Knowing the components of the right line force, fx = 5.49, fy = 4, we can find the angle of the right line.

$\theta = \arctan(4/5.49) = \arctan(.729)$

$\theta = 36.1°$

Now you can see how this works; the next weight will add another 2 ounces, and the next angle will be arctan(6/5.49); the one after that will be arctan(8/5.49), etc. So at this point we know enough to calculate the angles of the line from each successive weight.

And we know the distance from one weight to the next; we set that to be 10 centimeters. So the distance from one weight to the next, where the line goes up at an angle θ relative to horizontal, is 10 cos(θ) in the x direction, and 10 sin(θ) in the y direction. (I’ll refer in the spreadsheet to these x and y distances for each line segment as Δx and Δy – which reads as delta x and delta y.)

So we can calculate everything about the positions of the weights by a two-step process:

1. Find the angles:
   $\theta_a = 20°$ ← arbitrary angle of first line segment
   $\theta_b = \arctan(4/5.49) = 36.05°$
   $\theta_c = \arctan(6/5.49) = 47.52°$

θ_d = arctan(8/5.49) = 55.52°

…

2. Find the positions of the line end:

   $x_a = 5.0$ $y_a = 0.0$ ← starting position

   $x_b = x_a + 10\cos(20°)$ $y_b = y_a + 10\sin(20°)$

   $x_b = 5.0 + 9.39 = 14.39$ $y_b = 0 + 3.42 = 3.42$

   $x_c = x_b + 10\cos(36.05°)$ $y_c = y_b + 10\sin(36.05°)$

   $x_c = 14.39 + 8.09 = 22.48$ $y_c = 3.42 + 5.88 = 9.30$

   …

   At this point, we have done the real work of finding a way to calculate the positions of all the weights in the chain. It is just a matter of setting up the spreadsheet to take care of the tedious details. We will go far enough to do a few lines of calculations. First, we create a small block to hold some constants.

	A	B
1	wt	2.00
2	θ	20.00
3	link	10.00
4	Fx	5.49

Figure 6-8 Constants block

The weights are 2 ounces each. Angle θ is the angle with respect to horizontal made by the first line segment – see Figure 6-2. We can set this angle to anything from say, 1° to 89°. The link is set to 10 units (cm?). And Fx is the calculated force in the x direction: Fx = wt/tan(θ) = 2/0.364 = 5.49 ounces. The formula in the cell is:

= $B$1/TAN(RADIANS($B$2))

We can be a little vague about the weight units and length units, so long as all weights are the same and all line segments are the same lengths. Mathematically, it makes no difference whether the weights are 1 gram or 5 tons, we'll end up with the same angles and overall curve. Admittedly there are some practical issues with using 5 ton weights, but the size of the weight doesn't change the calculated angles. The same goes for the link lengths – 10 cm or 3 kilometers. We can use measures that make it impossible to build a model, and Excel won't stop you.

	A	B	C	D	E	F	G	H
10	Weight #	x	y	Fx	Fy	θ	Δx	Δy
11	1	5.00	0.00	5.49	2.00	20.00	9.40	3.42
12	2	14.40	3.42	5.49	4.00	36.05	8.08	5.89
13	3	22.48	9.31	5.49	6.00	47.52	6.75	7.37

Figure 6-9 Catenary spreadsheet

Cells B11 and C11 give the x and y coordinates of the first weight. This sets the origin of the graph at the midway point of the line joining the bottom two weights. Therefore the other bottom weight, 10 cm to the left, will be at (-5, 0). (This isn't shown anywhere on the spreadsheet, but we'll want to keep in mind that for every weight position (x, y) we calculate, there is a mirror image weight located at (-x, y).

Moving on to forces, we calculated that Fx for the first weight must be 5.49 ounces, and indeed that force must get passed along from weight to weight. Arguably we don't really need column D, since all entries are 5.49, but it makes clear how we're carrying out the calculations. The formula in cells D11, D12, D13 … is simply : = $B$4.

The Fy values start at 2 for the first weight, and increase by 2 ounces for each successive weight. Cell E11 contains the formula: = $B$1 (first weight); cell E12 contains the formula = E11 + $B$1. This formula is dragged down through the rest of the cells used in column E, so that each cell just adds 2 to the value in the cell above it.

The angle theta θ is the arctangent of Fy/Fx. In Excel form, to get the angle in degrees, the formula in cell F11 is: =DEGREES(ATAN(E11/D11). Selecting the cell and dragging the selection box downwards repeats the formula for each line, using successive values for Fy/Fx, i.e. E12/D12, E13/D13 etc.

The Δx and Δy displacements for the line segments are in cells G11 and H11 respectively: =10*COS(RADIANS(G2)) and =10*SIN(RADIANS(G2)) . You can see that the formulas would be simpler if we humored Excel and left the angle measurements in radians. I for one find it easy to think in degrees for angle measurements, not so easy in radians. May Euclid forgive me if I am passing my prejudices on to you. Feel free to stick with radians, if you are comfortable with that measure. Replicate the formulas downward in the same fashion as the formula for θ.

Finally, the x and y coordinates of weight #2 are just the x and y coordinates of weight #1 plus the Δx and Δy displacements calculated for the line segment. The formula for cell B12 is: = B11 + G11. The formula for C12 is = C11 + H11. These two formulas are also replicated downward through their columns.

x	y
-22.48	9.31
-14.40	3.42
-5.00	0.00
5.00	0.00
14.40	3.42
22.48	9.31

Figure 6-10 Data Table

We'd like to graph the x and y values; it would be more intuitive to see both sides of the curve. For each weight at (x, y), there is also a weight at (-x, y). Here's the full table of x, y values, and the resulting graph.

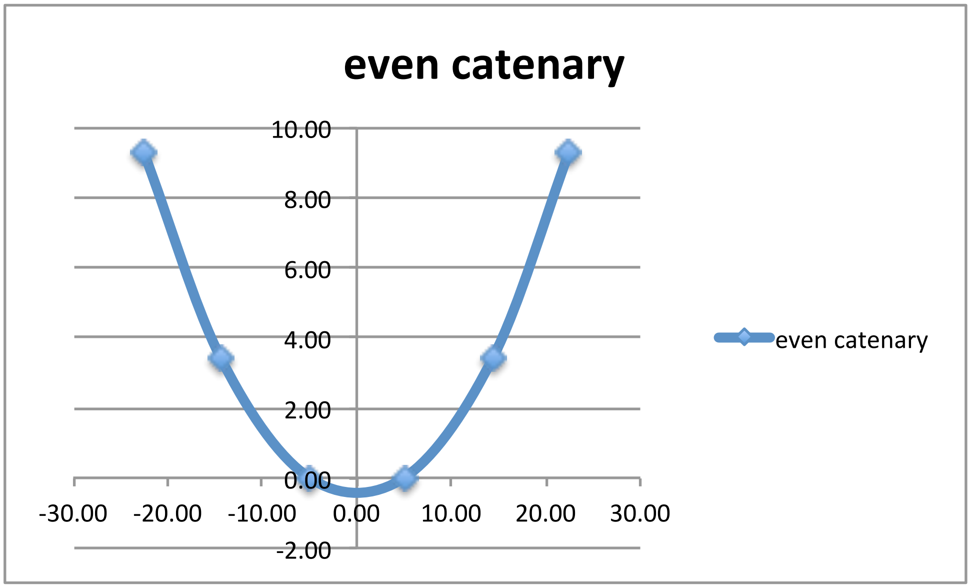

Figure 6-11 Catenary with 6 weights

Catenary with Center Weight

It would be nice to have one additional weight, at the center bottom of the curve. I didn't put it in the original spreadsheet, because it is a little bit of a special case, and I didn't want to muddy the waters of the overall approach with one more detail. But now that we've got the basics under control, it's easy to go back and add this in.

Why bother? We will assign this center weight the x,y coordinates of (0, 0). Then we can look at successive weights, and see easily whether or not they lie on a parabola. If the curve is a parabola, then the equation of the curve will be in the form $y = Kx^2$. We should be able to take the coordinates of each point – except (0,0) – and compute y/x^2, which gives K. If we get the same K for all points, then the curve is a parabola. This simple approach only works if the curve's lowest point is (0,0).

With no center weight, we aren't quite sure where the bottom of the curve really is. The graph above shows the line between the bottom two weights as a curved line that (maybe) dips below 0 on the y-axis, but only because we told Excel to use curved line segments; we don't really know exactly how much the curve dips between the two weights. Even if we build a model, with fishing line and weights, we won't find out; the line between the two bottom weights will just go straight across. Adding the center weight gives us a true bottom point, which we can assign to (0,0).

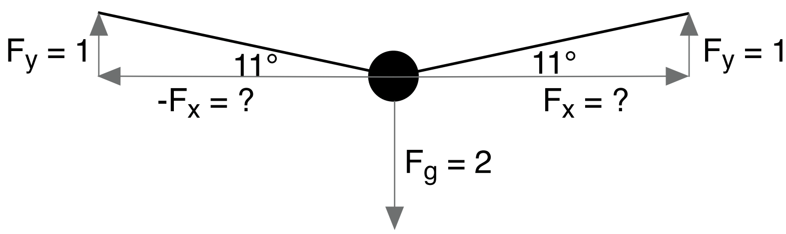

Figure 6-12 Adding a center weight

This figure shows just the center weight, which is pulled down by gravity with a force of two ounces. The counter force to gravity comes from the two support lines, each of which must counteract one half of the gravitational force, one ounce each. That is really the only thing that is different about this center weight; the force countering gravity is distributed between two lines. I've picked the angle of the lines to be 11°, which will keep the horizontal forces at roughly 5 ounces.

We can find Fx much as we did before:

tan(11°) = Fy/Fx = 1/Fx

Fx = 1/tan(11°) = 1/0.1944 = 5.14 ounces

This 5.14 ounces is the sole horizontal force, which will be passed along the lines linking subsequent weights. And each subsequent weight will add another 2 ounces of vertical force. We repeat what we did before, adding weights a pair at a time. The next weight on the right side will be positioned at the end of the line, at:

x = 10cos(11°), y = 10 sin(11°). And the angle of the next line segment will be:

arctan(3/5.14), followed by arctan(5/5.14), arctan(7/5.14), etc.

I'll show the spreadsheet through 11 weights. If you have entered the first spreadsheet, the one with no center weight, you can copy it and just change the first line to get this center weight version.

	A	B
1	wt	2.00
2	θ	11.00
3	link	10.00
4	Fx	5.14

Figure 6-13 Constants block

The only change in the constants block is the calculation for Fx. The formula is now : = $B$1/(2*TAN(RADIANS($B$2))) . The sole difference is the extra 2 in the denominator, because each supporting line segment is only supporting ½ the weight.

	A	B	C	D	E	F	G	H	I
1	Weight #	x	y	Fx	Fy	θ	Δx	Δy	y/x^2
2	1	0.00	0.00	5.14	1.00	11.00	9.82	1.91	…
3	2	9.82	1.91	5.14	3.00	30.25	8.64	5.04	0.020
4	3	18.45	6.95	5.14	5.00	44.18	7.17	6.97	0.020
5	4	25.63	13.92	5.14	7.00	53.69	5.92	8.06	0.021
6	5	31.55	21.97	5.14	9.00	60.25	4.96	8.68	0.022
7	6	36.51	30.65	5.14	11.00	64.94	4.24	9.06	0.023
8	7	40.75	39.71	5.14	13.00	68.41	3.68	9.30	0.024
9	8	44.43	49.01	5.14	15.00	71.07	3.24	9.46	0.025
10	9	47.67	58.47	5.14	17.00	73.16	2.90	9.57	0.026
11	10	50.57	68.04	5.14	19.00	74.85	2.61	9.65	0.027
12	11	53.18	77.69	5.14	21.00	76.23	2.38	9.71	0.027

Figure 6-14 Catenary spreadsheet with center weight

I've added a final column with the formula $= y/x^2$, where the y and x values start with weight #2. This will yield the K from $y = Kx^2$. If K is truly constant – the same value for each weight, then the curve is in fact a parabola. As you can see, the constant K isn't constant; it slowly grows with each successive weight.

To make the case visually, I've added a parabola to the graph of the center-weight catenary. The parabola is adjusted so that it agrees with the catenary at the middle weight at (0,0) and the end weights of the chain. The two curves are certainly close, but not quite coincident. Here's the data table and graph.

Aside on creating the data table

	A	B	C
1	x	y	y parabola
2	-53.18	77.69	77.69
3	-50.57	68.04	70.25
4	-47.67	58.47	62.43
5	-44.43	49.01	54.22
6	-40.75	39.71	45.61
7	-36.51	30.65	36.62
8	-31.55	21.97	27.34
9	-25.63	13.92	18.04
10	-18.45	6.95	9.36
11	-9.82	1.91	2.65
12	0.00	0.00	0.00
13	9.82	1.91	2.65
14	18.45	6.95	9.36
15	25.63	13.92	18.04
16	31.55	21.97	27.34
17	36.51	30.65	36.62
18	40.75	39.71	45.61
19	44.43	49.01	54.22
20	47.67	58.47	62.43
21	50.57	68.04	70.25
22	53.18	77.69	77.69

Figure 6-15 Data table - catenary vs parabola

If you try to copy & paste the x and y columns of the spreadsheet to another location on the spreadsheet, to prepare the full table for graphing, you'll run into problems immediately – the cells contain formulas with relative references, that will now point to bad locations. (It generally works if you copy an entire block containing all the data cells, but copying just a column or two usually causes problems.) To get around this, copy as usual, and "Paste Special" from the Edit tab. Paste_special brings up a menu that allows you to copy the values only, not the formulas.To create the half of the data table that has the negative x values, copy and paste_special the x,y columns again, from x=9.82 to x=53.18. You'll have to manually put in the – signs on the x values. After that, the only remaining problem is that the data points are in the wrong order – you want the final table to go from x = -53.18 to x = +53.18. Select the block of x and y values you wish to reorder, then click Sort… on the Data tab on the menu bar. This will bring up a menu that allows sorting by the values of either the x column, or the y column, in either ascending or descending order. If you select column_y, sorting with descending values, then you'll get the x and y columns sorted, from y = 77.69 to y = 1.91, as shown in figure 6-15.

This is actually a flexible way to reverse any column of numbers. If you had a column with disorderly values like: 17, 23, 12, 55 – and you wanted to reverse the column to 55, 12, 23, 17, then a simple sort will disappoint; you'll get 12, 17, 23, 55 or its inverse. You can get around this by adding a column beside the column you want to reverse. Populate the new column with 1, 2, 3… , select the new column and the original data column together, and then sort on the 1-2-3 column. This allows you to reverse your original column of data quickly.

Alternately, you can go through the cells in the data table, entering formulas like = -B12, = -B11, etc. This is laborious for the top half of the table, but if you change a number in the constants block, then your data table will track the change, which is handy.

Finally, to create the y parabola column, I used the formula: $y = Kx^2$ - and for K, I used the value computed in the spreadsheet in cell I12, which is $77.69/53.18^2$. This assures that the uppermost points of the catenary and the parabola will coincide. The Excel formula in cell C2 is simply: = $I$12* A2^2 . This formula is then dragged down through the rest of the table to give all the parabolic points. This allows a good representation of how a chain would hang (the catenary curve) if it were suspended from the two endpoints of the graphed parabola, with just enough chain to reach y = 0.

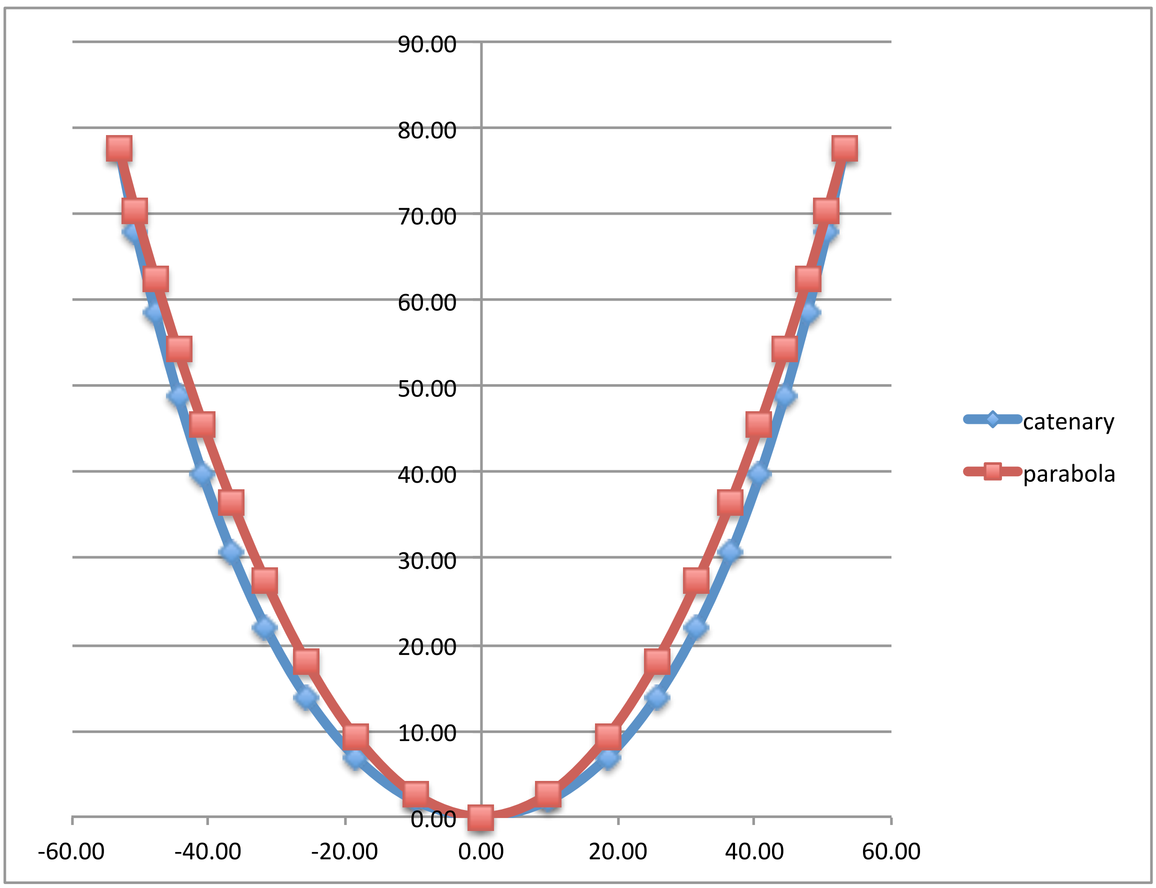

Figure 6-16 Graph of catenary vs parabola

Catenary Lab

The work so far suggests some form of hands-on activity, with a general goal of showing that a chain does not quite hang in a parabolic curve. I'm not going to list step-by-step instructions this time, but instead offer some general thoughts about how to effectively make the point.

My first thought was to replicate with fishing line and weights the general exercise we just did with Excel. However, just writing it up became tedious; I'm sure actually trying to measure the x and y coordinates of hanging weights would be mind-numbing and frustrating. Plus, as the differences between a parabola and a catenary are subtle, there is no guarantee that students would conclude that there is a difference.

A cleaner faster approach would be to start with an accurate graph of a parabola. Then hang a chain in front of the graph, adjust until the middle and ends coincide, and compare. (Tiffany's has some nice platinum chains. In a pinch, you could find something less expensive in a hardware store. The little bead chains used for sink drain plugs and "dog tags" would work, and are available from Amazon in quantity.)

You might be able to stage the demonstration to a whole classroom by projecting a parabola onto the wall or a screen using an overhead projector, and then hang a larger chain in front of the image – just make sure that the projected image is still a parabola (sometimes the optics and alignment of projectors distort the image; maybe include a few extra dots or lines on your parabola slide, so that you can be sure that dots separated by equal distances on your slide result in equidistant dots on the projection. I do like the drama of using a larger chain, with students holding the ends.

From Chains to Arches

Suppose we turn the whole problem upside down. Instead of a chain, now we want to build an arch, high in the middle, anchored to the ground at the ends. We'll model this arch as consisting of weights connected by rigid sticks, and add the condition that the arch will be strongest if all the force is directed along the sticks (so that the force isn't trying to bend the sticks). As soon as we start considering the forces involved …

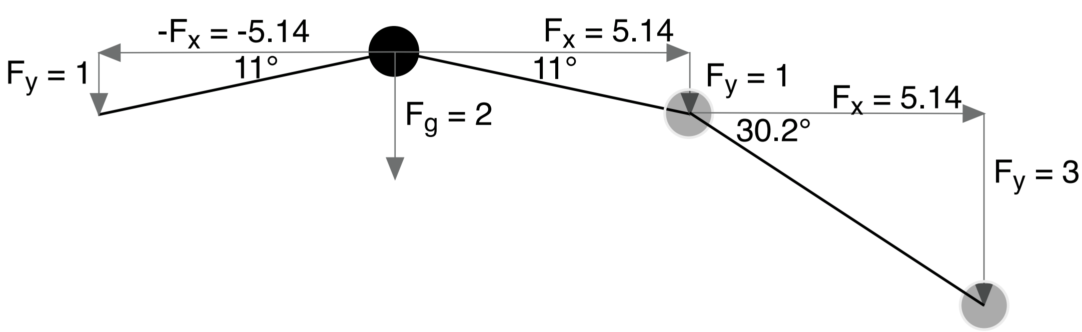

Figure 6-17 Analysis of an arch

We realize that we've seen this scheme before. Everything is going to mirror the analysis we already did for the hanging chain. This brings us back exactly to Robert Hooke's cryptic anagram – "As hangs a flexible cable so, inverted, stand the touching pieces of an arch." If we can calculate the curve for a chain, we can use the same curve for building an arch. For that matter, a craftsman might do away with the need for calculation entirely. Just hang a chain and mark its path on a suitable surface, plywood perhaps. Then build a form with that shape, and construct the arch over the form (with bricks or stones and mortar). Remove the form, and voila – the arch remains standing.

Of course, with enough rebar – steel reinforcing rods – you can build an arch with any shape you want. Most of those spectacular cathedrals in Europe use flying buttresses to keep their walls from bowing out and collapsing, because the walls are not catenary arches. But the point is that in a catenary arch, the support forces exactly align with the arch. If this design goal is achieved, then the blocks of a stone archway could be stacked without mortar, and would remain standing, until disturbed by an external force – wind, earthquake, your little brother… The following photograph shows such an arch, built of adobe bricks. You can see the wooden form lying behind the arch, which was used to hold the bricks up until the arch was complete. (The, lower parts of the arch are built more massively than the top part, so one could argue that the full arch may not be a pure catenary curve. But the top 60% of the arch is uniform in cross section, and therefore catenary.) It is impressive to see a free standing arch this high with nothing but gravity holding the bricks in place.

Figure 6-18 Catenary arch with adobe bricks

Larger pottery kilns are sometimes built with catenary arches. The extreme heat in the kiln degrades most forms of mortar, making a self-supporting arched structure very attractive.

Perhaps the best-known arch in the United States is the Gateway Arch near the Mississippi river in St. Louis Missouri. At a height of 630 feet, it is the tallest arch in the world. It differs from the arch we have calculated only because the top sections are smaller in cross section than the sections nearer the ground, and therefore less massive. It would not be hard to modify our spreadsheet to use smaller masses in the top sections. Indeed, that is the only modification needed; the angles and forces will take care of themselves if the masses are altered.

The Gateway Arch commemorates the westward expansion of the United States, symbolizing St. Louis as the gateway to the west (one critic, a Kansas City resident, has described it as more of an exit from the east). It is officially dedicated to “the American people.” One supposes that the native american peoples are less enthusiastic.

Not Every Hanging Cable is a Catenary!

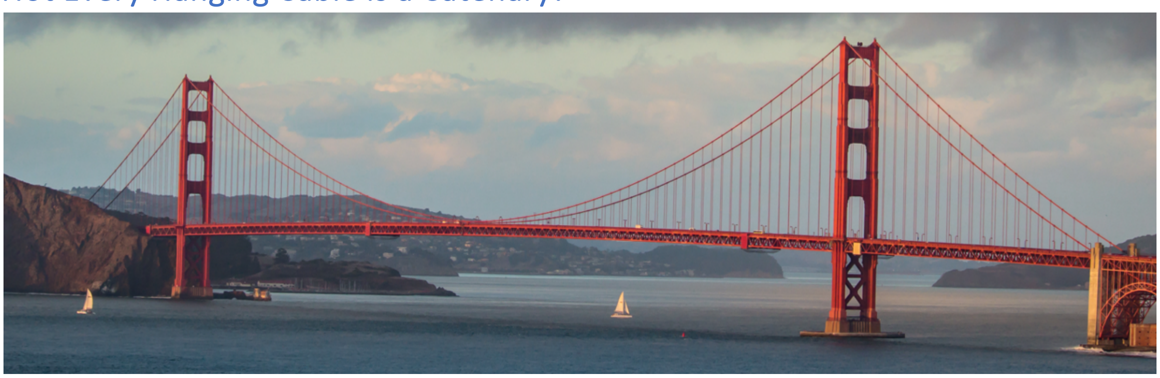

Figure 6-19 Golden Gate Bridge

It is natural to look at a suspension bridge, and think that the big cables supporting the bridge hang in a catenary curve. There’s a cable; it’s hanging from one support to another. However …

The bulk of the load on a suspension bridge cable is not the weight of the cable, but the weight of the roadway structure – tons of steel and concrete. And when you look at the smaller vertical cables that connect the roadway structure to the big suspension cables, you notice that they are evenly spaced along the roadway. On the golden gate bridge, there is a pair of cables every 50 feet, one on each side of the roadway. And that’s the difference; for a catenary curve, the weight is distributed uniformly along the length of the cable. Here, in a suspension bridge, the weight is distributed uniformly along the roadway, effectively along the x-dimension of the cable.

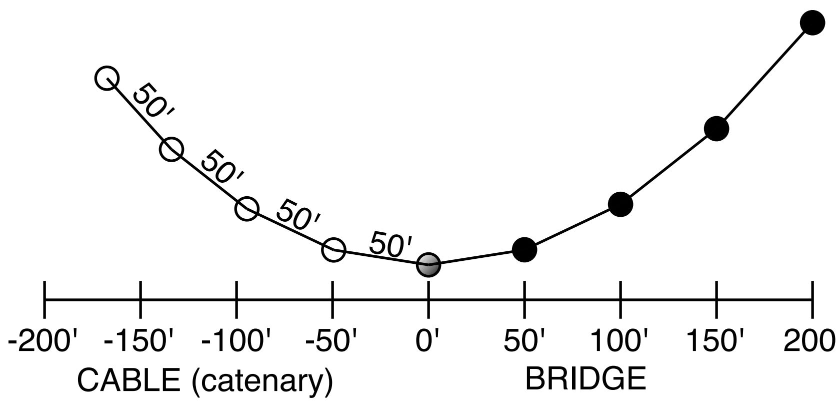

Figure 6-20 Cable load vs. bridge load

The figure above makes the distinction clear. With a cable or chain, every 50 feet along the cable adds another increment of weight – 2 ounces or 200 tons or whatever. With bridge loading, every 50 feet along the bridge adds another increment of weight. As the angle of the cable becomes steeper and steeper, the length of cable between loads (where the vertical cables attach) can be much longer than 50 feet.

Incidentally, the golden gate bridge has a rich history. It was not built without controversy; various lawsuits tried to halt its construction. Notable was a lawsuit by Southern Pacific Railroad, which saw the bridge as a threat to its profitable ferry business. Only a massive boycott of the ferry service stopped the Southern Pacific legal team. Construction began in 1933, and was completed in 1937 – all after the 1929 stock market collapse, during the depression years.

The main span of the golden gate bridge is 4,200 feet long, which made it the longest suspension span in the world until 1964. The total length, from abutment to abutment, is 8,981 feet. The main suspension cables are slightly over three feet in diameter. Each cable is 7,650 feet long, and weighs over 10,000 tons. As it is not feasible to handle or transport such a cable, it was manufactured in place, with a loom-type machine that shuttled back and forth laying strands of steel wire in place to form the cables, a process that took a little over six months. The individual strands are 0.192 inches in diameter; the finished cable consists of 27,572 strands, which are grouped into 61 bundles. The vertical cables that support the roadway are 2.688 inches in diameter; there are 250 pairs of them.

Specifications and history aside, can we model this suspension cable? Of course! The scheme we've been using to calculate the force and angles remains valid; we'll do that part the same way as before. But the way we calculated the position of the next weight will have to change. With a free chain and 10 centimeter links, we calculated the Δx and Δy distances to be $10\cos(\theta)$ and $10 \sin(\theta)$ respectively, where θ is the angle we calculated from the x and y forces. But with the constraints added by bridge construction, we know that Δx will always be 10. And Δy will be 10tan(θ) – see figure below.

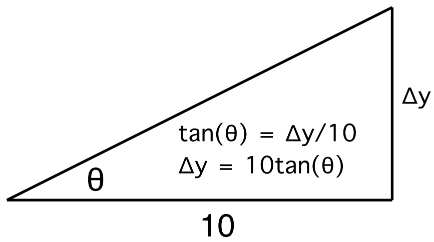

Figure 6-21 Finding Δy for bridge cable

I've copied the previous center weight spreadsheet, and changed only the calculations for Δx (now always 10) and for Δy (which is now 10tan(θ)). I've also added a column for y/x^2, to test whether the weights now lie on a parabola. Here are the first few lines:

	A	B	C	D	E	F	G	H	I
1	Weight #	x	y	Fx	Fy	θ	Δx	Δy	y/x^2
2	1	0.00	0.00	5.14	1.00	11.00	10.00	1.94	...
3	2	10.00	1.94	5.14	3.00	30.25	10.00	5.83	0.019
4	3	20.00	7.78	5.14	5.00	44.18	10.00	9.72	0.019

Figure 6-22 Calculations for bridge suppport cable

(Note: the table displays only hundredths for x, y positions, but excel is keeping track of everything to many decimal places.)

The second two weights suggest that the bridge support cable is a parabola, since y/x^2 = 0.019 for both. Or to put it another way, the equation of the curve seems to be $y = 0.019x^2$. I'll leave it to you to extend the table to include more weights to verify – or not – that it really is a parabola. Expressing y/x^2 as a number with more decimal places – six or so – will help you reach a conclusion. Is the bridge support cable a parabola?

To summarize – we've used just a bit of trigonometry, plus the x & y vector trick, to analyze a complex problem. It is regarded as complex mainly because it requires a lot of tedious calculation. With excel to eliminate the tedium, we can get back to focusing on the underlying principles, which are fairly simple and straightforward. With a spreadsheet program as part of the arsenal, it may be time to rethink which problems are complex.

It is also worth noting that a mathematician might be prone to delay discussion of this topic until enough calculus is on hand to derive the equations of the curves. But the spreadsheet we've used here for chains, arches, and now suspension bridges is more adaptable. If one wanted to build an arch, and have the top sections lighter than the lower sections, it is easy to work that into the spreadsheet.

Suggested Project

Do you have the mathematical tools (and woodworking tools) now that will enable you to build an arch? If you have a two-by-four, and some kind of saw that will allow you to make cuts reasonably accurately at angles, then can you devise a way to cut 9, or 11, or

13 lengths of two-by-four, with the ends angled so that they can stand as a self-supporting arch? You could take this on in several ways:

1. Try to make all the sections the same length.
2. Try to make the sections all contribute the same angular bend. (If the bottoms of the arch are inclined at 10°, then over the span of the arch, there is a cumulative bend of 180 – 2•10 = 160°. So you could partition the arch into 17 sections, with 16 junctions between sections, with each junction cut so that it adds 10° of bend to the arch. This would simplify cutting; all cuts could be made at a 95° setting (with maybe the exception of the two base pieces, which need to be cut at 100°, to give the initial 10° tilt at the base). Sections closer to the base would be longer; sections closer to the top of the arch would be shorter.
3. Do some free-form cutting, maybe a mix of 1 and 2, or a scheme of your own.

What is crucial is that the forces be directed straight through the junctions. If you imagine the arch drawn out, then the cuts at the end of the sections need to be perpendicular to the path of the arch. This is a little tricky from a carpentry standpoint. It is standard procedure to make perpendicular cuts on straight pieces, but the arch is a curved structure. Essentially if you want to make a cut on the curve at some point, what you need is the tangent to the curve, and then make the cut perpendicular to the tangent line. You may be able to do this by eye – or you may want to tinker with the spreadsheet to try to calculate the cutting angles for the sections.

Also note that the force in the x direction will be passed all the way to the base; it will be necessary to anchor the two final sections to the floor, so that they don't just slip apart and allow the arch to collapse.