Packing Problem (Simplified)

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Introduction:

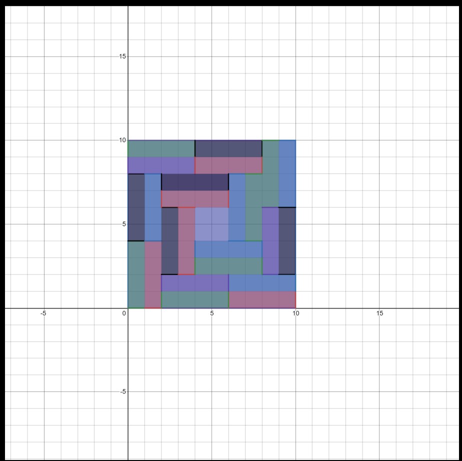
As an undergraduate Maths student, I love seeing problems that are easy to understand. This in no way implies that these problems are easy or uninteresting to solve (an extreme example would be Collatz Conjecture). This article presents the simplified version of a problem in Engel, A. (1998). Problem-solving strategies . Springer..

The Problem:

Can we place 25 rectangles of size 1×4 on a square of size 10×10 without cutting any rectangle?

Trial:

Let's begin by trying to fit the rectangles in the square. Here is one of my tries (powered by Desmos).



It seems very difficult to come up with a way to place 25 rectangles without overlapping. This is very surprising. Time to analyse.

Analysis:

Let's begin by making a few observations.

Observations:

(1) 25 rectangles of size 1×4 have a total area of 100, and a square of size 10×10 has an area of 100. So, if 2 rectangles overlap, then the area covered by the rectangles will be less than 100. **This means the square cannot be completely covered by our rectangles if the rectangles overlap.**

(2) Since we cannot cover the square if our rectangles overlap, and we cannot cut our rectangles, if we lay our rectangles on the square in a diagonal manner, we will be left with an uncovered part of the square. So, **the rectangles must be either vertical or horizontal.**

Now that we have some rules we need to follow if we want to cover the square. Let's make use of them. We start by numbering each square in the 10×10 square.

$$\text{Square} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 \\ 31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 & 39 & 40 \\ 41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 & 49 & 50 \\ 51 & 52 & 53 & 54 & 55 & 56 & 57 & 58 & 59 & 60 \\ 61 & 62 & 63 & 64 & 65 & 66 & 67 & 68 & 69 & 70 \\ 71 & 72 & 73 & 74 & 75 & 76 & 77 & 78 & 79 & 80 \\ 81 & 82 & 83 & 84 & 85 & 86 & 87 & 88 & 89 & 90 \\ 91 & 92 & 93 & 94 & 95 & 96 & 97 & 98 & 99 & 100 \end{bmatrix}$$

Now, from our observations, it is clear that if we place a rectangle in the square, it would either take up 4 slots numbered consecutively between 1 and 10 or 11 and 20 or 21 and 30, and so on, or it would take up 4 slots numbered in the form $n, 10+n, 20+n, 30+n$ for some $1 \leq n \leq 70$ (i.e. 4 consecutive slots vertically down).

Now, we want to find a way to visualize how rectangles can simultaneously exist in the square. Let's start by visualizing vertically placed rectangles.

If we label the squares in our 10×10 square in the following way:

$$\text{Square} = \begin{bmatrix} R & R & R & R & R & R & R & R & R & R \\ G & G & G & G & G & G & G & G & G & G \\ B & B & B & B & B & B & B & B & B & B \\ Y & Y & Y & Y & Y & Y & Y & Y & Y & Y \\ R & R & R & R & R & R & R & R & R & R \\ G & G & G & G & G & G & G & G & G & G \\ B & B & B & B & B & B & B & B & B & B \\ Y & Y & Y & Y & Y & Y & Y & Y & Y & Y \\ R & R & R & R & R & R & R & R & R & R \\ G & G & G & G & G & G & G & G & G & G \end{bmatrix}$$

We see that any vertical rectangle we place covers the letters 'R', 'G', 'B' and 'Y'. Note that the order in which these letters are covered might change from rectangle to rectangle, but all vertical rectangles, definitely cover the letters 'R', 'G', 'B' and 'Y'.

Now, there are 30 'R's, 30 'G's, 20 'B's and 20 'Y's in the above representation of the 10×10 square. But, since all vertical rectangles, definitely cover the letters 'R', 'G', 'B' and 'Y', we know that we definitely cannot place more than 20 rectangles vertically in a 10×10 square. (This is because if we could place 21 rectangles vertically, that would mean we have 21 of each 'R', 'G', 'B' and 'Y' in our 10×10 square.)

This is encouraging. Can we modify the above labelling such that it helps visualize horizontally placed rectangles too? Let's try the following labelling of the 10×10 square:

$$\text{Square} = \begin{bmatrix} R & G & B & Y & R & G & B & Y & R & G \\ G & B & Y & R & G & B & Y & R & G & B \\ B & Y & R & G & B & Y & R & G & B & Y \\ Y & R & G & B & Y & R & G & B & Y & R \\ R & G & B & Y & R & G & B & Y & R & G \\ G & B & Y & R & G & B & Y & R & G & B \\ B & Y & R & G & B & Y & R & G & B & Y \\ Y & R & G & B & Y & R & G & B & Y & R \\ R & G & B & Y & R & G & B & Y & R & G \\ G & B & Y & R & G & B & Y & R & G & B \end{bmatrix}$$

We see that, like before, any vertical rectangle we place covers the letters 'R', 'G', 'B' and 'Y'. But, now, any horizontal rectangle we place also covers the letters 'R', 'G', 'B' and 'Y'.

From our observations, we know if we can cover the 10×10 square with 25 1×4 rectangles, then all rectangles that cover the 10×10 square are either vertical or horizontal. So, we know that if we can cover the 10×10 square with 25 1×4 rectangles, then we must have 25 of each 'R', 'G', 'B' and 'Y'.

But, our new labelling only has 24 'Y's! This means we cannot place 25 rectangles of size

1×4 on a square of size 10×10 without cutting any rectangle.

I would like to give you, the maths enthusiasts reading this article, an opportunity to solve some extended versions of this problem. I hope you find them interesting.

1. Can we pack 250 bricks of size $1 \times 1 \times 4$ into a box of size $10 \times 10 \times 10$ without cutting or crushing any brick? [This is the packing problem that I simplified for this article. It is Problem 7 in Chapter 2. Coloring proofs of Engel, A. (1998). Problem-solving strategies . Springer.]

2. Suppose n is an integer such that 100 is a multiple of n. What are the values of n for which we can place $\frac{100}{n}$ rectangles of size $1 \times n$ into a square of size 10×10?

3. And, finally, suppose m and n are integers such that m^2 is a multiple of n. What can we say about the values of n for which we can place $\frac{m^2}{n}$ rectangles of size $1 \times n$ into a square of size $m \times m$?

1 Reference:

Engel, A. (1998). Problem-solving strategies . Springer.