Approximating integrals with respect to stationary probability measures and applications

Italo Cipriano

Facultad de Matemáticas
Pontificia Universidad Católica de Chile
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Motivation

A paper by Jenkinson and Pollicott.

Abstract- “We show how ideas originating in the theory of dynamical systems inspire a new approach to numerical integration of functions. Any Lebesgue integral can be approximated by a sequence of integrals with respect to equidistributions, i.e. evenly weighted discrete probability measures concentrated on an equidistributed set. We prove that, in the case where the integrand is real analytic, suitable linear combinations of these equidistributions lead to a significant acceleration in the rate of convergence of the approximate integral. In particular, the rate of convergence is faster than that of any Newton-Cotes rule.” [1]

[1] Oliver Jenkinson and Mark Pollicott (2007), A dynamical approach to accelerating numerical integration with equidistributed points. Proceedings of the Steklov Institute of Mathematics 256.1: 275- 289.

Iterated Function System (IFS)

Let $\mathcal{I} := \{1, \dots, N\}$ for some $N \in \{2, 3, \dots\}$.

Definition

We say that $\Phi = \{\phi_i\}_{i\in\mathcal{I}}$ is an IFS consisting of Lipschitz contractions $\phi_i : [0,1] \to [0,1]$ if

$$\sup_{x,y} \frac{|\phi_i(x) - \phi_i(y)|}{|x-y|} < 1.$$

Definition

The unique, non-empty and compact set $\Lambda \subset [0,1]$ such that

$$\Lambda = \bigcup_{i\in\mathcal{I}} \phi_i(\Lambda)$$

is called the attractor of Φ.

Stationary Probability Measure (SPM)

Definition

Given a probability vector $\mathbf{p} = (p_1, p_2, \ldots, p_N)$, (meaning that each $0 < p_i < 1$ and $\sum_{i \in \mathcal{I}} p_i = 1$), the unique probability measure $\mu = \mu^{(\Phi, \mathbf{p})}$ such that

$$\int \varphi \mathrm{d}\mu = \sum_{i \in \mathcal{I}} p_i \int \varphi \circ \phi_i \mathrm{d}\mu \tag{1}$$

for every continuous function $\varphi : [0, 1] \to \mathbb{R}$ is called the SPM associated to Φ and $\mathbf{p}$. Its support is the attractor of Φ.

Main goal today

Given a SPM μ on the unit interval and a continuous function $g : [0,1] \to \mathbb{R}$, construct a "Jenkinson - Pollicott " algorithm to approximate

$$\mu(g) := \int g d\mu.$$

Why?

Under some hypotheses on the IFS (in many cases) dynamics properties of the SPM μ depend only on $\mu(g)$ for a specific g.

Example 1

Theorem (A. H. Fan and K.-S. Lau)

If the IFS $\Phi = \{\phi_i\}_{i\in\mathcal{I}}$ satisfies the open set condition (i.e. there exists a non empty open set $V \subset [0,1]$ such that $\cup_{i\in\mathcal{I}}\phi_i(V) \subset V$ and $\phi_i(V) \cap \phi_j(V) = \emptyset$ for $i \neq j$) and $\mathbf{p} = (p_1, \ldots, p_N)$ is a probability vector. Then the Hausdorff dimension of $\mu = \mu^{(\Phi,\mathbf{p})}$ satisfies

$$\dim_H \mu := \inf\{\dim_H(E) : \mu(E) = 1\} = \frac{-\sum_{i\in\mathcal{I}} p_i \log p_i}{\chi_\mu},$$

where χ_μ is the Lyapunov exponent of Φ with respect to μ defined by

$$\chi_\mu := -\int \sum_{i=1}^{N} p_i \log|\phi_i'(x)| d\mu(x).$$

The result is valid for non-necessarily constant weight functions (replacing $\sum_{i\in\mathcal{I}} p_i \log p_i$ by $h_\mu := \int \sum_{i\in\mathcal{I}} p_i(x) \log p_i(x) d\mu(x)$).

Example 2

The n-th moment of a Borel measure μ on $\mathbb{R}$ is defined by

$$\mu(x^n) := \int_{-\infty}^{\infty} x^n \mathrm{d}\mu(x), n = 0, 1, \ldots.$$

Theorem (Probabilistic method of moments)

Let μ_n be a sequence of probabilities measures with moments of all orders. Suppose that for each k, $\mu_n(x^k)$ converges to a number μ_k. Then, there is a measure μ with $\mu(x^k) = \mu_k$. If μ is uniquely determined by its moments, then for every bounded continuous function $\int f d\mu_n \to \int f d\mu$.

For $m \in [1, \infty)$, the Wasserstein distance of order m between two probability measures μ and ν is defined by

$$W_m(\mu, \nu) = \inf \left\{ [\mathbb{E}|X - Y|^m]^{\frac{1}{m}} : \text{law}(X) = \mu, \text{law}(Y) = \nu \right\}.$$

Theorem (Fraser, Pollicott and C., C., Lichtenegger and Niedzialomski)

If $\Phi = \{\phi_i\}_{i \in I}$ consists of positive Lipschitz contractions such that $\phi_i(0,1) \cap \phi_j(0,1) = \emptyset$ for every $i \neq j$ and $\mathbf{p}, \mathbf{q}$ are probability vectors such that $\sum_{i=1}^{m}(p_i - q_i) \geqslant 0$ (or $\leqslant 0$) for every $m = 1, 2, \ldots, |\mathcal{I}|$, then

$$W_1\left(\mu^{(\Phi,\mathbf{p})}, \mu^{(\Phi,\mathbf{q})}\right) = \left|\mu^{(\Phi,\mathbf{p})}(x) - \mu^{(\Phi,\mathbf{q})}(x)\right|.$$

Complex contracting IFS

Definition (Bandtlow and Jenkinson)

A family of maps $\Phi = \{\phi_i\}_{i \in \mathcal{I}}$, $\phi_i : [0,1] \to [0,1]$ is called complex contracting IFS if there exists a non-empty, bounded, connected and open set $\mathcal{D} \subset \mathbb{C}$ such that $[0,1] \subset \mathcal{D}$ and

1. each ϕ_i extends holomorphically to $\mathcal{D}$
2. ϕ_i' is continuous on the boundary of $\mathcal{D}$
3. $\sup_{i \in \mathcal{I}} \sup_{z \in \mathcal{D}} |\phi_i'(z)| < 1$.

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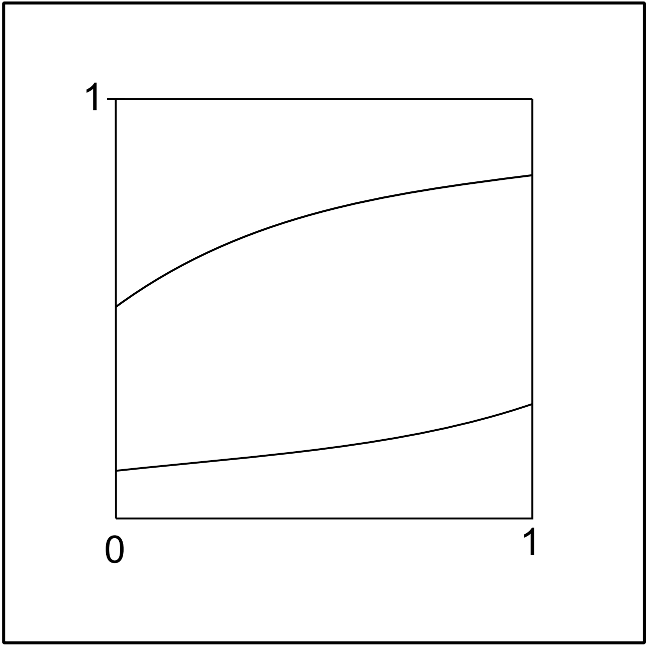
if Φ is a complex contracting family of maps then there exists some complex domain $D \subset \mathcal{D}$ such that $[0,1] \subset D$, each ϕ_i extends holomorphically to D and $\overline{\phi_i(D)} \subset D$. In this case we say that D is an admissable domain for Φ.

[2] Oscar Bandtlow and Oliver Jenkinson (2008), On the Ruelle eigenvalue sequence, Ergodic Theory and Dynamical Systems, 28(6), 1701-1711.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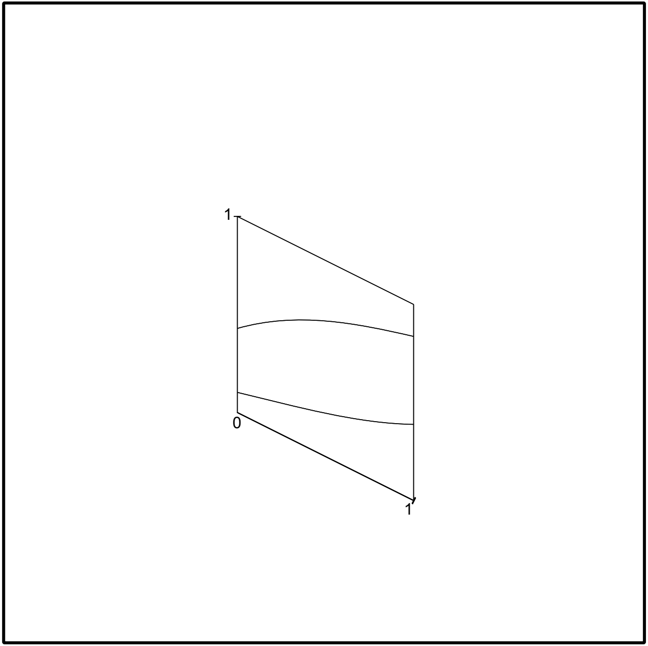
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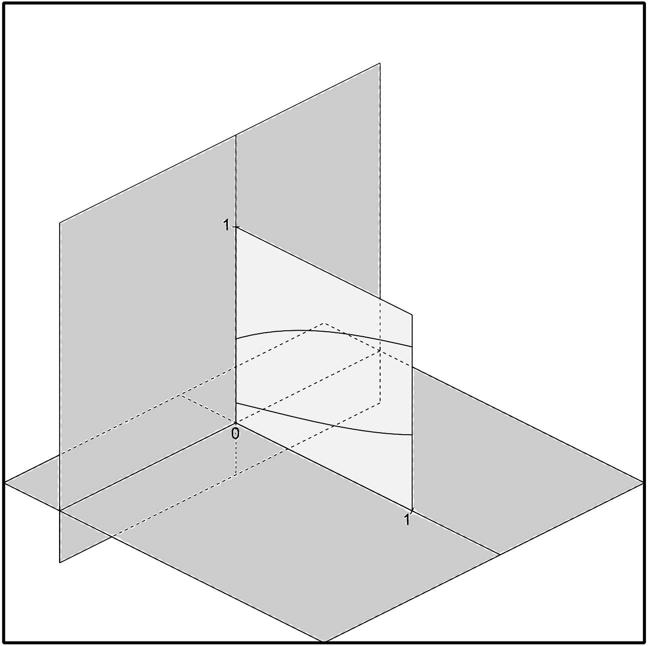
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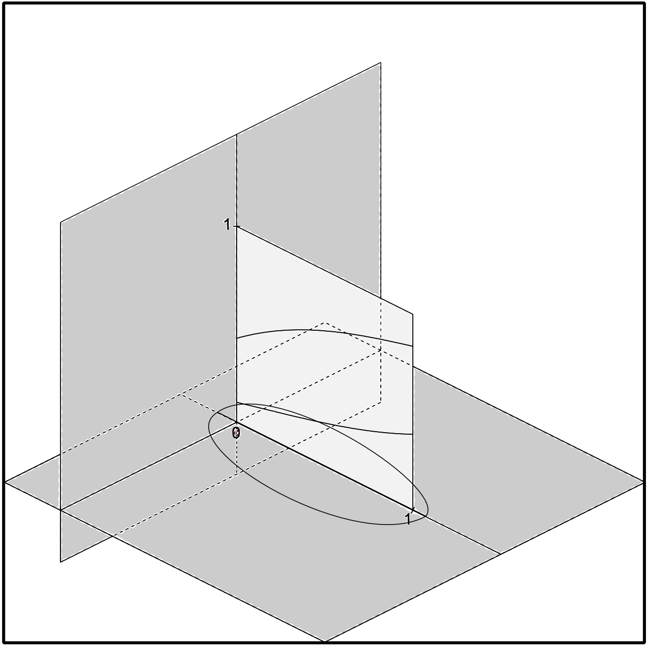
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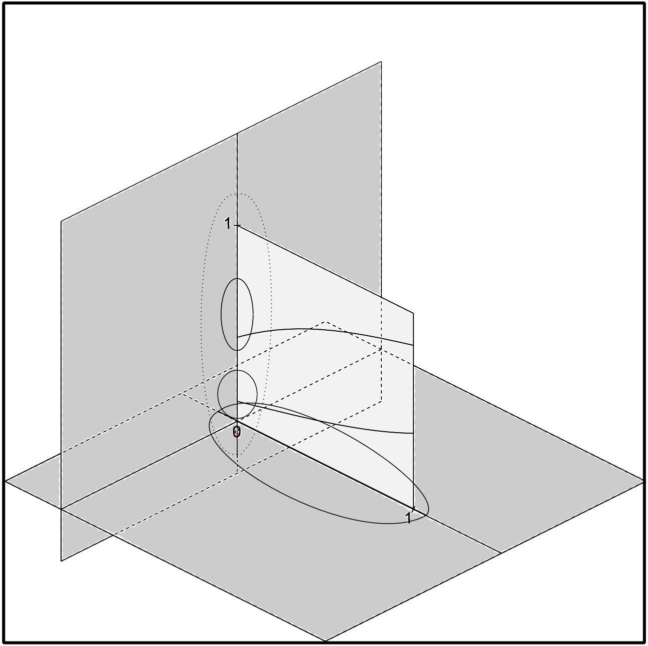
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Subclass of continuous functions to be integrated

Definition

Given a complex contracting IFS $\Phi = \{\phi_i\}_{i \in \mathcal{I}}$ on the unit interval, we define the subclass of continuous functions $g : [0,1] \to \mathbb{R}$

$$\mathcal{C}_\Phi := \bigcup_D \{g \text{ has an ext. to some bdd. holomor. funct. on } D\},$$

where the union is taken over all admissable domains for Φ.

Notation I

Let $\mathcal{I}^n = \{i_1 \cdots i_n : i_j \in \mathcal{I}\}$ and $\phi_{i_1 \ldots i_n} := \phi_{i_1} \circ \cdots \circ \phi_{i_n}$ for $n \in \mathbb{N}$. Define $\mathcal{I}^* = \bigcup_{n\in\mathbb{N}} \mathcal{I}^n$. For $\mathbf{i} = i_1 \ldots i_n \in \mathcal{I}^*$ and $1 \leqslant k \leqslant n-1$ we define

$$\sigma^k \mathbf{i} := i_{k+1} \ldots i_n i_1 \ldots i_k,$$

$$\tilde{\sigma}^k \mathbf{i} := i_{k+1} \ldots i_n \text{ and}$$

$$p_\mathbf{i} := \prod_{i=1}^{n} p_i.$$

Define $\Pi : \mathcal{I}^\mathbb{N} \to \Lambda$ by

$$\Pi(i_1 i_2 \ldots) := \lim_{n\to\infty} \phi_{i_1} \circ \cdots \circ \phi_{i_n}([0,1]).$$

For each $\mathbf{i} \in \mathcal{I}^*$ let $(\mathbf{i})^\infty$ denote the periodic point $(\mathbf{i})^\infty = \mathbf{iii}... \in \mathcal{I}^\mathbb{N}$ and let $z_\mathbf{i}$ denote the fixed point $z_\mathbf{i} = \Pi((\mathbf{i})^\infty)$ of $\phi_\mathbf{i}$.

Notation II

Define

$$t_m := \sum_{\mathbf{i}\in\mathcal{I}^m} p_{\mathbf{i}} \frac{1}{1-\phi'_{\mathbf{i}}(z_{\mathbf{i}})}$$

and

$$\tau_m := \sum_{\mathbf{i}\in\mathcal{I}^m} p_{\mathbf{i}} \frac{g(z_{\mathbf{i}}) + g(z_{\sigma\mathbf{i}}) + \ldots + g(z_{\sigma^m\mathbf{i}})}{1-\phi'_{\mathbf{i}}(z_{\mathbf{i}})}.$$

Also define

$$\alpha_n := \sum_{l=1}^{n} \frac{(-1)^l}{l!} \sum_{n_1+\ldots+n_l=n} \sum_{j=1}^{l} \frac{\tau_{n_j}}{n_j} \prod_{1\leqslant m\leqslant l, m\neq j} \frac{t_{n_m}}{n_m}$$

and

$$a_n := \sum_{l=1}^{n} \frac{(-1)^l}{l!} \sum_{n_1+\ldots+n_l=n} \prod_{i=1}^{l} \frac{t_{n_i}}{n_i}.$$

Main result

Theorem (Jurga and C.)

Let $\Phi = \{\phi_i\}_{i\in\mathcal{I}}$ be a complex contracting IFS on the unit interval and $\mathbf{p}$ a probability vector in $[0,1]^{|\mathcal{I}|}$. Assume that $g \in \mathcal{C}_\Phi$. Then for

$$\mu_k(g) := \frac{\sum_{n=0}^{k} \alpha_n}{\sum_{n=0}^{k} n a_n}$$

and μ the stationary probability measure associated to Φ and $\mathbf{p}$,

$$|\mu(g) - \mu_k(g)| < C \exp(-\lambda k^2) \qquad (2)$$

for some constants $C, \lambda > 0$ which are independent of k.

Main tool used in the proof

Spectral properties of the Ruelle operator $\mathcal{L}_s$

$$\mathcal{L}_s f(z) := \sum_{i=1}^{k} p_i \exp(sg(\phi_i(z)))f(\phi_i(z))$$

on a suitable Banach space (studied in more generality by Bandtlow and Jenkinson [3] [4]). In particular, we prove that it is a trace-class operator with decreasing sequence of eigenvalues $\{\lambda_n(s)\}_{n\in\mathbb{N}}$, $\lambda_1(0) = 1$ is a simple eigenvalue of $\mathcal{L}_0$ of maximum modulus, $\lambda_1(s)$ is analytic in s in a neighbourhood of 0 and

$$\mu(g) = \frac{\mathrm{d}}{\mathrm{d}s}\lambda_1(s)\bigg|_{s=0}.$$

[3] Oscar Bandtlow and Oliver Jenkinson (2008), On the Ruelle eigenvalue sequence, Ergodic Theory and Dynamical Systems, 28(6), 1701-1711.

[4] Oscar Bandtlow and Oliver Jenkinson (2008), Explicit eigenvalue estimates for transfer operators acting on spaces of holomorphic functions, Advances in Mathematics 218.3 (2008): 902-925.

Remarks

- Approximations are non-effective.
- When $\phi_i(x) = a_i x + b_i$ for some constants a_i, b_i (for every $i \in \mathcal{I}$), the constants C, λ can be bounded, similarly to Jurga and Morris [5] and Jenkinson and Pollicott [6].

[5] Natalia Jurga and Ian Morris (2019), Effective estimates on the top Lyapunov exponent for random matrix products, arXiv preprint arXiv:1901.10944.

[6] Oliver Jenkinson and Mark Pollicott (2018), Rigorous effective bounds on the Hausdorff dimension of continued fraction Cantor sets: a hundred decimal digits for the dimension of E2, Advances in Mathematics 325: 87-115.

Numerical experiment: Hausdorff moments

Let $\Phi = \{\frac{1}{3}x, \frac{1}{3}x + \frac{2}{3}\}$ and $p = \left(\frac{1}{3}, \frac{2}{3}\right)$. Comparison of the approximate values for the moments with the exact values.

Table: Approximate values for the moments ($k = 14$).

Order n	Error
0	0
1	0
2	$< 10^{-48}$
3	$< 10^{-48}$
4	$< 10^{-47}$
5	$< 10^{-48}$
6	$< 10^{-46}$

Numerical experiment: Wasserstein distances

Let $\Phi = \left\{\frac{1}{3}x, \frac{1}{2}x + \frac{1}{2}\right\}$ and $p = \left(\frac{1}{3}, \frac{2}{3}\right)$, $q = \left(\frac{3}{4}, \frac{1}{4}\right)$. Comparison of the approximate value of the Wasserstein distance for $k = 12, \ldots, 16$, with the exact value $W_1\left(\mu^{(\Phi,\mathbf{p})}, \mu^{(\Phi,\mathbf{q})}\right) = \frac{2}{5}$.

Table: Approximate values for the Wasserstein distances.

Iteration k	Error
12	$< 10^{-23}$
13	$< 10^{-27}$
14	$< 10^{-31}$
15	$< 10^{-36}$
16	$< 10^{-41}$

Numerical experiment: Lyapunov exponents

Let $\Phi = \{\frac{\sin(\pi x/4)}{6} + \frac{1}{4}, \frac{\sin(\pi x/4)}{3} + \frac{2}{3}\}$ and $\mathbf{p} = \left(\frac{1}{3}, \frac{2}{3}\right)$. We compute the approximate values to the Lyapunov exponent of Φ with respect to μ.

Table: Approximate values for Lyapunov exponents.

Iter. k	**Approx. Lyapunov exponent** w_k
10	1.73672081473731987719335669096051377335988973317409087754512488812667028438133069194736931
11	1.73672081473731987719335669096051377336020590601023577825778088227884147013275372835954447
12	1.73672081473731987719335669096051377336020590600637607989863098312523352089178183012079770
13	1.73672081473731987719335669096051377336020590600637607991887362479762663771464637789116844
14	1.73672081473731987719335669096051377336020590600637607991887362479193249845331374280452940
15	1.73672081473731987719335669096051377336020590600637607991887362479193249845555716884110153
16	1.73672081473731987719335669096051377336020590600637607991887362479193249845555716884109104

Numerical experiment: Hausdorff dimension of SPM

Let $\phi_1(x) = \frac{\sin(\pi x/4)}{6} + \frac{1}{4}$, $\phi_2(x) = \frac{\sin(\pi x/4)}{3} + \frac{2}{3}$ and $\mathbf{p} = \left(\frac{1}{3}, \frac{2}{3}\right)$.

$$\begin{aligned}\dim_H \mu = {} & 0.366503448852305020253935069048618621 \\ & 0891863283880547944186323868301178324 \\ & 7404103748034456446522958810960603\ldots .\end{aligned}$$

Applications

Consider fixed weights $\mathbf{p}$ and a one-parameter dependent family of IFS

$$L \mapsto \Phi^{(L)} = \left\{ \phi_i^{(L)} \right\}_{i \in \mathcal{I}}.$$

In applications it may be important to understand the maps

$$L \mapsto \dim_H \left(\mu^{(\Phi^{(L)}, \mathbf{p})} \right)$$

$$L \mapsto \mu^{(\Phi^{(L)}, \mathbf{p})} \left(x^k \right)$$

$$L \mapsto W_1 \left(\mu^{(\Phi^{(L)}, \mathbf{p})}, \mu^{(\Phi^{(0)}, \mathbf{p})} \right)$$

Numerical Example

Consider the one-parameter dependent iterated function system

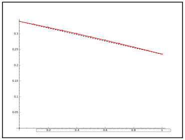
$$[0,1] \ni L \mapsto \Phi^{(L)} = \left\{ \phi_1^{(L)}, \phi_2^{(L)} \right\},$$

where

$$\phi_1^{(L)} = \frac{x}{3} + \frac{L}{6} \sin\left(\frac{x\pi}{2}\right) \text{ and } \phi_2^{(L)} = \frac{x}{3} + \frac{1}{2} + \frac{1-L}{6} \sin\left(\frac{x\pi}{2}\right).$$

Given the weights $\mathbf{p} = \left(\frac{2}{3}, \frac{1}{3}\right)$ and $\mathbf{q} = \left(\frac{1}{3}, \frac{2}{3}\right)$, we plot the graph of

$$[0,1] \ni L \mapsto W_1\left(\mu^{(\Phi^{(L)},p)}, \mu^{(\Phi^{(L)},q)}\right).$$



What else?

Non constant weight functions. It is possible to extend the main result to certain probability measures associated to *non-constant* weight functions. In particular, suppose there exists an admissable domain D for Φ and bounded holomorphic functions $\mathbf{p} = \{p_i\}_{i \in \mathcal{I}}$, $p_i : D \to \mathbb{C}$ with the property that

1. $\sum_{i \in \mathcal{I}} p_i(z) = 1$ for all $z \in D$,
2. $p_i([0,1]) \subset (0,1)$ for all $i \in \mathcal{I}$.

An analogue of the main theorem can be obtained for the unique probability measure $\mu = \mu^{(\Phi, \mathbf{p})}$ supported on the attractor of Φ with the property that

$$\int \varphi(x) \mathrm{d}\mu(x) = \sum_{i \in \mathcal{I}} \int p_i(x) \varphi \circ \phi_i(x) \mathrm{d}\mu(x) \qquad (3)$$

for every continuous function $\varphi : [0,1] \to \mathbb{R}$.