Theoretical and Mathematical Models for Maximizing Economic Returns: A Review Paper on Non-Renewable Resource Exploitation

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February 26, 2024

Abstract

This paper review delves into the economic and operational facets of exploiting non-renewable resources, guided by the theoretical framework presented in Kane 1988 seminal work, "The Economic Definition of Ore." It explores the intricate balance between economic, technological, and operational considerations that dictate the pace and strategy of resource extraction. Central to our discussion is the concept of the present value (PV) of future cash flows derived from resource exploitation, encapsulated in a mathematical model that seeks to maximize this value over the resource's life. This optimization problem hinges on identifying the optimal set of cut-off grade policies, represented as Ω^*, within a broader strategy space Ω.

The model is structured around the dynamic relationship between the present time, the available resource reserve R, and the present value V, with the latter being influenced by the rate of resource depletion and the chosen exploitation strategy. A key contribution of this work is the formulation of a mathematical model that captures the economic implications of different exploitation strategies over time. By considering the differential impacts of these strategies on the present value and available reserves, the model provides a robust framework for decision-making in the context of non-renewable resource management.

In addition, this study introduces an innovative application of Artificial Intelligence (AI) and Metaheuristics within a framework where Lane acts as a manager of dynamic programming operations. In this scheme, metaheuristics are utilized to solve optimization subproblems, while AI is tasked with learning from all possible solutions or paths, thus optimizing the decision-making process. This multidisciplinary approach allows

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tackling the complexity of the problem from various perspectives, enriching the final solution and facilitating adaptation to changes and new information.

This integration of AI and Metaheuristics under a framework controlled by Lane not only expands the knowledge frontier in the exploitation of non-renewable resources but also provides a powerful tool to address the inherent challenges in predicting future values and costs, thus contributing to the sustainable and economically viable management of resources.This research not only advances the theoretical understanding of non-renewable resource exploitation but also offers practical insights for industry practitioners. By systematically analyzing the interplay between economic factors, resource depletion, and strategic planning, the paper contributes valuable knowledge towards sustainable and economically viable resource management practices.

1 Preliminaries

Kane, in his book *The Economic Definition of Ore* Kane 1988, studies the use of non-renewable resources based on the fact that exploiting a non-renewable resource involves economic and technological components that define its rate of exploitation and considering that each day it is depleted, hence the strategy is not unique. Lane proposes a non-renewable resource where, over the life (n years), there are annual flows $C_1, C_2, C_3, \ldots, C_n$ that, brought to the present with a capital cost δ, we have the following present value V as shown in equation 1:

$$V = \sum_{i=1}^{n} \frac{C_i}{(1+\delta)^i} \tag{1}$$

As the present time advances, reserves are being exploited making the available reserve quantity R decrease, which implies that V depends on R and T as shown in equation 2

$$V = V(T, R) \tag{2}$$

On the other hand, the function V has the following boundary values shown in equation 3

$$V(T, R) = V, \quad V(T, 0) = 0 \tag{3}$$

Also, the function V depends on other variables that define the form of exploitation, which are the set of strategies Ω that can be followed during the time the resource is exploited, so the function V is defined as shown in equation 4.

$$V(T, R, \Omega) \tag{4}$$

2 Mathematical Model

The proposed mathematical model considers as its objective to maximize the function V, which focuses on finding that set of strategies Ω^*, also called cut-off grade policies $g_1, g_2, g_3, \ldots, g_n$ and which are a subset of Ω. Thus, we can define the function V^* as shown in equation 5 that represents V for a particular strategy Ω^*.

$$V^* = V^*(T, R) \tag{5}$$

Then, we want to define what happens at the economic level in the time interval (between T and $T+t$), where each end is associated with a value of V and an available reserve R. We can define that in this period of time, a certain part of the reserves r was exploited, thus having a flow c that depends on the cut-off grade g and the elapsed time t, where g belongs to the cut-off grade strategy set Ω. To express this logic graphically, we can see it in Table 1.

Table 1: Description of the time period

	Start	**End**
Present Time	T	$T+t$
Available Resource	R	$R-r$
Present Value	$V(T,R)$	$V(T+t,R-r)$
Flow	$c=c(g,t)$	

If we take as an economic model the flows discounted to the present time based on a discount rate δ we can relate the functions $V(T,R)$ and $V(T+t,R-r)$ so that carrying the flows $V(T+t,R-r)$ and c to time t we have the following equality as shown in equation 6.

$$V(T,R) = r * c + \frac{V(T+t,R-r)}{(1+\delta)^t} \tag{6}$$

In order to make the calculations simpler, the approximation shown in equation 7 is made:

$$\frac{1}{(1+\delta)^t} \approx 1 - \delta t \tag{7}$$

This way, we transform equation 6 into a simpler form to work with as shown in equation 8.

$$V(T,R) = r * c + V(T+t,R-r) * (1 - \delta t) \tag{8}$$

If we operate on the set of strategies $\Omega^{'}$ that maximizes the value of V being this $V^{'}$, that is when $g \in \Omega^{'}$. Given this, we can define the function $V^{'}$ as shown in equation 9.

$$V^{'}(T,R) = \max_{g\in\Omega^{'}}\{r * c + V^{'}(T+t,R-r) * (1-\delta t)\} \tag{9}$$

On the other hand, to solve equation 9, we must consider the differentials of t and r and we will obtain the total derivative of V with respect to the variables T and R as shown in equation 10, which would allow us to relate $V^{'}(T,R)$ and $V^{'}(T+t,R-r)$.

$$V^{'}(T+t,R-r) = V^{'}(T,R) + \frac{\partial V^{'}}{\partial T}t - \frac{\partial V^{'}}{\partial R}r \tag{10}$$

Now, if we multiply by $(1-\delta t)$ equation 10 transforms into equation 11.

$$V^{'}(T+t,R-r)*(1-\delta t) = V^{'}(T,R)*(1-\delta t) + (\frac{\partial V^{'}}{\partial T}t - \frac{\partial V^{'}}{\partial R}r)*(1-\delta t) \tag{11}$$

If we multiply the factors we obtain equation 12.

$$V^{'}(T+t,R-r)*(1-\delta t) = V^{'}(T,R)*(1-\delta t) + (\frac{\partial V^{'}}{\partial T}t - \frac{\partial V^{'}}{\partial R}r) - \frac{\partial V^{'}}{\partial T}\delta t^2 + \frac{\partial V^{'}}{\partial R}\delta t * r \tag{12}$$

If we consider the multiplication of differentials that tend to zero, then we can write equation 12 as shown in equation 13.

$$V^{'}(T+t,R-r)*(1-\delta t) = V^{'}(T,R)*(1-\delta t) + (\frac{\partial V^{'}}{\partial T}t - \frac{\partial V^{'}}{\partial R}r) \tag{13}$$

Now, if we combine equation 9 with equation 13 we have equation 14 which would be the equation to evaluate.

$$V^{'}(T,R) = \max_{g\in\Omega^{'}}\{r*c + V^{'}(T,R)*(1-\delta t) + (\frac{\partial V^{'}}{\partial T}t - \frac{\partial V^{'}}{\partial R}r)\} \tag{14}$$

Given that $V^{'}(T,R)$ is independent of the policy g, we can express the equation as shown in equation 15.

$$0 = \max_{g\in\Omega^{'}}\{r*c - V^{'}(T,R)*\delta t + (\frac{\partial V^{'}}{\partial T}t - \frac{\partial V^{'}}{\partial R}r)\} \tag{15}$$

It is interesting to note that the function that maximizes does not depend on the derivative with respect to R so we can remove that part with the equation shown in equation 16.

$$\frac{\partial V^{'}}{\partial R}r = \max_{g\in\Omega^{'}}\{r*c - V^{'}(T,R)*\delta t + \frac{\partial V^{'}}{\partial T}t\} \tag{16}$$

If we divide by r in equation 16, it would be expressed as shown in equation 17.

$$\frac{\partial V^{'}}{\partial R} = \max_{g\in\Omega^{'}}\{c - (V^{'}*\delta - \frac{\partial V^{'}}{\partial T})*\frac{t}{r}\} \tag{17}$$

We can define $\tau = \frac{t}{r}$ as the time per unit of resource and the opportunity cost $F = \delta V^{'} - \frac{\partial V^{'}}{\partial T}$, we can rewrite equation 17 as equation 18.

$$\frac{\partial V'}{\partial R} = \max_{g \in \Omega'} \{c - F * \tau\} \tag{18}$$

If we integrate in the range of 0 to R the integral would be as shown in equation 19.

$$\int_0^R \frac{\partial V'}{\partial R}\, dR = \int_0^R \max_{g \in \Omega'} \{c - F * \tau\}\, dR \tag{19}$$

Then the maximum value we would obtain by exploiting the resource R would be given by equation 20.

$$V' = \int_0^R \max_{g \in \Omega'} \{c - F * \tau\}\, dR \tag{20}$$

The application of this equation for different industries whose resource is non-renewable would imply knowing at each t the following:

1. τ: Is the time per unit of resource r exploited.
2. F: Is the opportunity cost of the resource $R - r$ given that r has already been exploited.
3. c: Is the cash flow obtained by exploiting r in a time t.

The real application is complex if we want to solve the differential equations given that the function F depends on a derivative with respect to time, something we do not know, but its form evidences that the best value of a stage $i + 1$ depends on the stage i which suggests a typical behavior of Dynamic Programming where each stage is the best possible value obtained before continuing with the next stage. Then if we manage to obtain the combination of policies throughout the exploitation of R we could obtain the maximum value of V.

Then one way to solve it would be given a Δt given, we could know a Δr resource extracted and therefore we would obtain for a set of strategies Ω different cash flows c. Then we can define V_t in the last extraction period as shown in equation 21.

$$V_t = \frac{c_t}{(1 + \delta)^t} \tag{21}$$

Then we could define the partial derivative of ΔV_t with respect to T as the differentials between t and $t + \Delta t$ as shown in equation 22.

$$\frac{\partial V_t}{\partial T} = V_t - \frac{V_{(t+\Delta t)}}{(1 + \delta)^{t+\Delta t}} \tag{22}$$

Then we can reformulate equation 17 using equation 22 as shown in equation 23.

$$\Delta V_{r_t} = \max_{g \in \Omega'} \{c_t - (V_t * \delta - \frac{V_{(t+\Delta t)}}{(1 + \delta)^{t+\Delta t}}) * \frac{\Delta t}{\Delta r}\} \tag{23}$$

As mentioned before, the central problem to solve the optimization problem lies in the unknown $V_{(t+\Delta t)}$, but in the last differential n, that is in $t + \Delta t$ its value is zero since there are no more resources and the value of V_t can be estimated with equation 21, with which we can rewrite equation 23 as follows

$$\Delta V_{r_t} = \max_{g \in \Omega'} \{ c_t * (1 + \frac{(1-\delta)}{(1+\delta)^t} * \frac{t}{r}) \} \tag{24}$$

Then in the last differential n the search for the best strategy Ω_n, depends only on the function c and finding the maximum of equation 24 would be reduced to searching for the cut-off grade policy that maximizes c. Having found that value, which would represent the best value, following equation 23 we could find the best value for the differential $n-1$ and successively we could find all values up to the first differential. Using this methodology we can estimate the function V^* that goes through the n differential but since we start by assuming an n we do not know exactly its value, so we must iterate with a global function that analyzes how much it evolves for each iterative search and at some point, the function reaches an optimum which would be when it has obtained the optimal cut-off grade policy Ω that maximizes V.

3 Applied Model in Mining

The purpose of the economic model is to construct the cash flow c of the system based on the following variables:

- **Cut-off grade**: represents the mineral grade that defines which materials should be sent to the process and which should not. Generally, it acts as a policy.
- **Throughput**: the operation's performance in material processed at each stage per unit of time.
- **Prices** and **Costs**: The associated costs for each process and the price of the product generated in the market.

There are **three processes** in a mining operation, related to the *Throughput*, which I describe below in Table 2

Table 2: Processes and activities in the mining operation

Process	**Subprocess**
Mining (Open Pit, Underground)	Ore, Waste
Treatment	Crushing, Grinding, Flotation
Commercialization	Transport, Refinery

Then, to define the interrelation between the different processes, their capacities, and variable costs, we can use the definitions shown in Table 3

Table 3: Summary of mining processes and activities

Component	**Type**	**Quantity**	**Variable Cost**	**Capacity**
Mining	Material	1	m	M
Treatment	Ore	x	h	H
Commercial	Metal	$xy\hat{g}$	k	K

We also have the following variables:

$$\begin{aligned}
f &: \text{Fixed cost per year} \\
p &: \text{Price of metal} \\
c &: \text{Cash flow per unit of mineralized material} \\
x &: \text{Is the proportion of material classified as ore} \\
y &: \text{Metallurgical recovery of the ore} \\
\hat{g} &: \text{The applied cut-off grade that defines the material's destination} \\
\tau &: \text{Time it takes to treat 1 unit of mineralized material} \\
M &: \text{Is the annual Mining capacity of the system} \\
H &: \text{Is the annual Treatment capacity of the system} \\
K &: \text{Is the annual Commercialization capacity of the system}
\end{aligned}$$

Finally, the cash flow c is determined by equation 25.

$$c = (p - k)xy\hat{g} - xh - m - f * \tau \tag{25}$$

Then we can rewrite equation 20 as shown in equation 26.

$$V' = \int_0^R \max_{g \in \Omega'} \{x((p - k)y\hat{g} - h) - m - f * \tau - F * \tau\} \, dR \tag{26}$$

3.1 Mine Limits the System

When the mine limits the system, it implies that τ, which is the time to process a unit of mineralizable material, is equivalent to equation 27.

$$\tau = 1/M \tag{27}$$

Then, finding the maximum of the internal component of equation 26, taking into account the result of equation 27, implies deriving with respect to g and setting the system of equations to zero. If we derive with respect to g and set it to 0, we obtain the following function as shown in equation 28.

$$\frac{\partial x}{\partial g} * ((p-k)y\hat{g} - h) + x * ((p-k)y\frac{\partial \hat{g}}{\partial g}) = 0 \tag{28}$$

If we appropriately group, we can obtain a separable differential equation as shown in equation 29.

$$\frac{\partial \hat{g}}{\frac{h}{y(p-k)} - \hat{g}} = \frac{\partial x}{x} \tag{29}$$

If we integrate x in the range $[0, x]$ and $\hat{g}$ in the range of $[g_{max}, \hat{g}]$, we can obtain equation 30 that represents the cut-off grade when the mine limits the system.

$$\hat{g_m} = \frac{h}{y(p-k)} \tag{30}$$

This result implies that when the mine limits the system, that is, we are at the maximum capacity of the mining process and still have not met the plant's requirement, the optimal grade that defines the maximum of function 26 is given by equation 30.

3.2 Plant Limits the System

When the plant limits the system, it implies that τ, the time to process a unit of mineralizable material, is equivalent to equation 31.

$$\tau = x/H \tag{31}$$

Then, finding the maximum of the internal component of equation 26, taking into account the result of equation 31, implies deriving with respect to g and setting the system of equations to zero. If we derive with respect to g and set it to 0, we obtain the following function as shown in equation 32.

$$\frac{\partial x}{\partial g} * ((p-k)y\hat{g} - h - \frac{(f+F)}{H}) + x * ((p-k)y\frac{\partial \hat{g}}{\partial g}) = 0 \tag{32}$$

If we appropriately group, we can obtain a separable differential equation as shown in equation 33.

$$\frac{\partial \hat{g}}{\frac{h+\frac{(f+F)}{H}}{y(p-k)} - \hat{g}} = \frac{\partial x}{x} \tag{33}$$

If we integrate x in the range $[0, x]$ and $\hat{g}$ in the range of $[g_{max}, \hat{g}]$, we can obtain equation 34 that represents the cut-off grade when the plant limits the system.

$$\hat{g_p} = \frac{h + \frac{(f+F)}{H}}{y(p-k)} \tag{34}$$

This result implies that when the plant limits the system, that is, we are at the maximum capacity of the treatment process, the optimal grade that defines the maximum of function 26 is given by equation 34. This equation is interesting because it includes the opportunity cost F in the determination of the optimal cut-off grade, evidencing its connection throughout the exploitation of the deposit and during the early years based on the equation the optimal cut-off grade will be high and as the deposit is exploited, it will reduce since the opportunity cost decreases due to the financial nature of the discounted cash flows.

3.3 Market Limits the System

When the market limits the system, it implies that τ, the time to process a unit of mineralizable material, is equivalent to equation 35.

$$\tau = \frac{xy\hat{g}}{K} \tag{35}$$

Then, finding the maximum of the internal component of equation 26, taking into account the result of equation 35, implies deriving with respect to g and setting the system of equations to zero. If we derive with respect to g and set it to 0, we obtain the following function as shown in equation 36.

$$\frac{\partial x}{\partial g} * ((p-k)y\hat{g} - h - \frac{(f+F)}{K}y\hat{g}) + x * ((p-k-\frac{(f+F)}{K})y\frac{\partial \hat{g}}{\partial g}) = 0 \tag{36}$$

If we appropriately group, we can obtain a separable differential equation as shown in equation 37.

$$\frac{\partial \hat{g}}{\frac{h}{y(p-k-\frac{(f+F)}{K})} - \hat{g}} = \frac{\partial x}{x} \tag{37}$$

If we integrate x in the range $[0, x]$ and $\hat{g}$ in the range of $[g_{max}, \hat{g}]$, we can obtain equation 38 that represents the cut-off grade when the market limits the system.

$$\hat{g_k} = \frac{h}{y(p-k-\frac{(f+F)}{K})} \tag{38}$$

This result implies that when the market limits the system, the optimal grade that defines the maximum of function 26 is given by equation 38.

4 Integrated Planning

The mining system consists of three fundamental components: Mining, Plant, and Market. Each represents a key phase in the operational cycle:

1. Mining: This is the extraction phase, where mineral resources are obtained from the ground. It is the first step in the mining value chain.

2. Plant: Here occurs the transformation. The extracted minerals are processed and refined to turn them into usable or marketable products. This step is crucial to add value to the extracted materials.

3. Market: Finally, the commercialization phase represents the sale or distribution of the transformed mining products. Here is where the products reach consumers and generate revenue for the mining operation.

At some point in their operation, each of these components may face limitations due to two types of factors:

1. Own Operational Factors: These are challenges or constraints inherent to the mining operations themselves, such as technical problems in extraction or processing, or logistical challenges in commercialization.

2. External Factors: These are external influences that can affect mining operations, such as changes in market demand, price fluctuations, environmental or policy regulations, and global economic conditions.

Each of these components and the associated challenges are interdependent and crucial for the success and sustainability of mining operations, and the theory of cut-off grades defines the behavior depending on which cut-off grade is limiting the system. Therefore, effective management in mining involves not only optimizing each of these steps individually but also ensuring their alignment and coordination to face and overcome both operational and external challenges.

5 Conclutions

The exploration of non-renewable resource exploitation within this paper underscores the critical interplay between economic, technological, and operational parameters in determining the optimal extraction strategy. Through the lens of Kane's theoretical insights on ore economics, the study elucidates the nuanced considerations that underpin the sustainable management of finite resources. The mathematical model developed herein serves as a pivotal tool for maximizing the present value (PV) of resource exploitation, highlighting the significance of strategic cut-off grade policies (Ω^*) within the broader strategy space (Ω).

Key conclusions drawn from this study include:

1. **Strategic Optimization**: The paper successfully demonstrates the utility of a mathematical model in optimizing the extraction of non-renewable resources. By focusing on maximizing the present value of cash flows derived from resource exploitation, it provides a strategic framework for decision-making that balances economic benefits against the rate of resource depletion.

2. **Dynamic Programming Approach**: The adoption of a dynamic programming approach to resolve the optimization problem emphasizes the paper's innovative methodology. This approach allows for the iterative examination of exploitation policies, facilitating the identification of an optimal set of strategies that maximize the economic return over the resource's lifespan.

3. **Complexity of Real-World Application**: The study candidly addresses the complexities involved in applying the proposed model to real-world scenarios. It acknowledges the challenges in forecasting future values and costs, underscoring the need for robust analytical tools and methodologies in the strategic management of non-renewable resources.

4. **Sustainability and Economic Viability**: The research highlights the delicate balance between resource sustainability and economic viability. It contributes to the discourse on sustainable resource management by providing insights into how strategic planning and optimization techniques can enhance the economic outcomes of non-renewable resource exploitation while mindful of conservation principles.

5. **Future Research Directions**: The paper identifies avenues for future research, particularly in refining the model to better accommodate the unpredictability of market conditions and technological advancements. It calls for continued exploration into more adaptive and resilient strategic planning tools that can navigate the uncertainties inherent in non-renewable resource management.

In conclusion, this paper enriches the field of resource economics by offering a comprehensive model for the strategic exploitation of non-renewable resources. It bridges theoretical concepts with practical applications, offering valuable perspectives for both researchers and industry practitioners aimed at optimizing the economic returns of resource exploitation within sustainable and environmentally conscious frameworks.

6 Disclaimer

It's important to highlight that 100% of the formulas presented in this research have been developed by the author based in Kane 1988 research. However, it's pertinent to mention that the responsibility for the correct formulation and development of these equations rests solely with the author. Lane provides an initial framework but does not bear direct responsibility for the validity of the developed formulas. The author has independently solved the equations, assuming full responsibility for their accuracy and applicability within the context of the research.

7 Acknowledgments

This journey of research and discovery would not have been possible without the support and faith of a remarkable group of individuals and leaders. I am profoundly grateful to each one of them for their unique contributions to my work and personal growth.

First and foremost, I extend my deepest gratitude to our CEO, Melissa Amado, for her visionary leadership and unwavering support. Her commitment to fostering research in mathematics, natural resources, and computer science has not only paved the way for groundbreaking discoveries but has also inspired a culture of innovation and excellence within our organization. Melissa's belief in the transformative power of research has been a guiding light throughout this project, and for that, I am eternally thankful.

I would also like to express my heartfelt appreciation to my mentors in Strategic Mining Planning, Henry Brañes and Marco Maulen. Their expertise, dedication, and insightful guidance have been instrumental in shaping my approach to complex problem-solving and strategic thinking. Henry and Marco's mentorship has been a cornerstone of my development, providing me with the tools and confidence needed to navigate the intricate landscape of mining and resource management.

Furthermore, my journey would not have been the same without the academic guidance and intellectual generosity of my MSc Computer Science advisor, PhD Ernesto Cuadros. Ernesto's profound knowledge, meticulous attention to detail, and encouragement have been pivotal in my research endeavors. His ability to challenge and push the boundaries of my understanding has been invaluable, fostering a deeper appreciation for the nuances of computer science and its application to real-world challenges.

To Melissa, Henry, Marco, and Ernesto: your collective wisdom, support, and encouragement have been the bedrock of this research. I am immensely grateful for your contributions and for believing in the potential of this work. Thank you for your unwavering faith, guidance, and for being a source of inspiration and strength.

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