Symmetry and the Circular Geometry of Debt as an explanation of Biblical Sabbatical and Debt Laws

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Abstract

I show that the economic activities of buying and selling (the real side of an economy) and lending and borrowing (the financial side) can be represented in a single geometric model displayed in Euclidean two dimensional space. I separate the economy into two groups: debtors and creditors. I show that the transactions that occur due to agents interacting with other agents within in their own group do not affect the aggregated debt level. In contrast, transaction that occur between agents of different groups do affect the aggregate debt level. I show that the geometric representation of these types of transactions is, in expectation, reflective of the combination of biblical laws related to the weekly sabbatical cycle and a seven year debt cycle.

JEL-Classification: E191, C0

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1 Introduction

It is assumed that modern analysis of the economy has little, if anything, to do with religion. But biblical scripture does have economic content. However, modern economics would not accept those principles as authoritative merely because they derive from a divine source. TThis paper provides a link between biblical economics and mathematical economics. Secondly, and of importance to modern macroeconomics, the model developed provides a framework in which the real side of the economy and financial side are modelled in one geometric representation. Finally, it provides an application of Pythagoras' theorem to the macroeconomics of debt. Symmetry is used throughout to develop a geometric representation of an economy with debt. Two geometric representations are derived and compared: an economy that does not permit debt and an economy that does permit debt. The rest of the paper is broadly structured as follows: I describe the model and its geometric representation before showing that the model is reflective two biblical laws that have economic content.

This paper further develops a my own paper that was written for the 'Discussion Paper Series' of the Association of Christian Economists (Morgan, 2013). There I consider the relationship between the Sabbath and debt laws. Unlike my 2013 paper, this paper abstracts from a particular monetary system, thereby making it more general. It also and also has a firmer geometrical explanation. I firstly discuss key texts from the bible. In the second 'book of the bible', Exodus, in the 20th chapter of that book it reads (in verses 8 -11): 'Remember the Sabbath day, to keep it holy. Six days you shall labor, and do all your work, but the seventh day is a Sabbath to the Lord your God. On it you shall not do any work, you, or your son, or your daughter, your male servant, or your female servant, or your livestock, or the sojourner who is within your gates. For in six days the Lord made heaven and earth, the sea, and all that is in them, and rested on the seventh day. Therefore the Lord blessed the Sabbath day and made it holy.' The Ancient Israelites were thereby commanded to work for six days a week and then rest on the seventh. This version of the commandment makes particular reference to the Creation story, where God is told to have made the world in six days and rested on the seventh. There is clear economic content here, as it outlines a seven repeated cycle of work and rest. In the fifteenth chapter of Deuteronomy it reads: "At the end of every seven years you shall grant a release. And this is the manner of the release: every creditor shall release what he has lent to his neighbour. He shall not exact it of his neighbour, his brother, because the Lord's release has been proclaimed." This verse dictates that the debt cycle was to occur over seven years but at the end of the seventh year the debt cycle was to cease. Concurrently, in this time period, six years of work (of buying and selling) will have occurred if the observance of the Sabbath command was simultaneously practised. This implies a ratio of six-sevenths between two activities: that of buying and selling (and associated productive effort) and lending and borrowing.

In effect, the model in this paper presents two groups that act as if they are in a two country world in a two-period current account model. I consider, what are, in effect, transactions that occur within the borders of a country and transactions that occur across borders, where the first type of transaction does not affect the current account whilst the second type of transaction does. The symmetry of the budget constraints is akin to the fact that the current accounts of a two country world sum to zero by definition.

Stiglitz and Gallegati (2011) provides a useful backdrop to why the approach taken in this paper is necessary to analyze debt. They argue that the representative agent approach of mainstream economics rules out key macroeconomic interactions by assumption. The explain that the representative agent approach 'has (by construction) nothing to say about the network aspects of lending'. They go on to say that 'In [a representative agent] model, there are no lenders, no borrowers, and therefore no credit markets.' They argue that 'Financial markets (lending and borrowing) are...central, and these can only be studied within a framework with heterogenous agents.' The heterogeneity I use in this paper is, at any given point in time, agents will have different levels of incomes and expenditures and therefore different credit balances. This is a realistic and necessary assumption to analyze financial markets.

2 The model

There are J agents who buy and sell and lend and borrow to one another over two periods of equal length. Each good has a fixed price. Each agent has a credit (debt) balance. An agent's credit balance at the end of each period is the difference between the sum of their incomes and expenditures over that period. An agent is a creditor in the first period if the sum of their first period income exceeds the sum of their first period expenditure. Similarly, a debtor is an agent whose expenditure exceeds their income over a given period. From here on I consider incomes and expenditures to depict the summation of incomes and expenditures over an entire period. The only information that determines an agents credit balance is the agents income and expenditure. This encapsulates the idea of property rights: any agents income and expenditure is 'owned' uniquely by that agent; and only counts against that agent's credit balance.

Any excess income over and above expenditure of a creditor agent is instantaneously translated into loan to a debtor: a type of forced gift. Each debtor uses these loaned funds to purchase goods from the creditor group. It is assumed that all first period debts are repaid by the end of the second period. This means that each agent in the creditor group in the first period will be in the debtor group in the second period and vice versa. As such, this analysis is about an idealized economy where all debt contracts are fulfilled. Models such as Alvarez and Jermann (2000) and Kehoe and Levine (1993) have endogenous debt limits in the form of individual rationality constraints to ensure that agents cannot enter into a contract in which they would have an incentive to default in some state. This is one method that is used in the existing literature to prevent prevent defaults occurring in a model. I assume that all agents have a finite amount of debt or credit, but that this can be made arbitrarily close to zero. The aggregate debt level can tend to zero but only in the limit. This ensures that the two groups are always defined. No aggregate savings are possible and loans are made at zero interest rates.

I assume that there is an equal probability of a unit of expenditure being spent on either a debtor or creditor agent, when considering the entire two periods as a whole. This is an 'independence assumption': the decisions regarding purchases are independent to the debt position of the seller. Underlying this 'independence assumption' is the previous assumption that all debts are repaid over the entire two periods.

3 Generating a geometry

3.1 Orthogonality and Symmetry

I capture the stock of debt levels and the accumulated end of period incomes and expenditures two dimensions. This serves to integrate the real and financial sides of the economy. Changes to income and expenditure are modelled as comparative static change of the aggregated end of period accumulated incomes and expenditures. The x and y axes measure the debtor and creditor group's incomes and expenditures respectively. The positive part of the axis measures income and the negative part measures expenditure. The axes are perpendicular because the respective incomes and expenditures of each group do not directly feed into the credit balances of the other group. For instance, the creditor group's incomes and expenditures do not directly feed into the debtor's credit (debt) balance. The aggregate amount of debt in the economy is the sum of the debtor groups income less the sum of their aggregated expenditures. One can consider there there to be orthogonal information sets: the only information required to determine a given agent's credit balance are that individual's income and expenditure. There is a need for at least two perpendicular axes as incomes and expenditures of each group (of debtors and creditors) are permitted to differ.

There are two central symmetries in this model. Firstly, the total amount of expenditure in the economy equals the total amount of expenditure. Secondly, the difference between the creditor group's incomes and expenditures is equal to the aggregate credit (debt) balance. By symmetry, this is also equal to the absolute difference between the debtor group's income and expenditure.

3.2 The space of potential transactions

A transaction occurs when a unit of income is paired with a unit of expenditure. A central concept I introduce to generate the results is that of a 'space of potential economic transactions'. This is the product of potential pairings of units of expenditure and income. This is similar to an action space in Game Theory. I consider the total number of potential pairs of units (or sub-units) or expenditure and income. For examples, in the UK, one could label each pound of income and expenditure for each individual. Moreover, each individual is either a net debtor or net creditor. So one can consider all the potential pairings of a pound of expenditure and a pound of income. If there is a re-ordering of the different combinations of buyers and sellers this would mean that the different flows of income and expenditure would be between different buyers and sellers. It is the product of different permutations pairwise matches (tuples) of units of income and expenditure.

The figure below shows the potential transaction space of 5 units of expenditure and income. Where the first number represents a unit of income and the second number the associated unit of expenditure. One potential realization is along the diagonal where each labelled unit is matched to its equivalent counterpart.

Figure 1: Cartesian product of Space of potential transactions

5,1	5,2	5,3	5,4	5,5
4,1	4,2	4,3	4,4	4,5
3,1	3,2	3,3	3,4	3,5
2,1	2,2	2,3	2,4	2,5
1,1	1,2	1,3	1,4	1,5

3.3 Stock Flow Consistency principles and Within-group and Between-group transactions

The derived geometry encapsulates the four main principles of SFC models, which are summarised by Nikoforos and Zezza (2017). These principles are consistent with the geometric representation described above. The first principle is that of Horizontal consistency: Everything comes from somewhere and goes somewhere: there are no black holes. For instance, income for somebody is a payment from somebody else. This principle is captured by the first symmetry of this model: that total income equals total expenditure. The second principles is that of Vertical consistency: every transaction involves at least two entries within each unit. For example, a consumer's expenditure implies, say, a reduction in the consumers credit balance. The third principle is that every flow implies the change in one or more stocks, for instance, a change in a debt stock. The fourth principle is that of Stocks consistency: The financial liabilities of an agent or sector are the financial assets of some other agent or sector. For example, a loan is a liability for a household and an asset for a bank; a Treasury bond is a liability for the government and an asset for its holder. As a result, the net financial wealth of the system as a whole is zero. This leads to the Quadruple entry accounting system. Principles two to four are captured because the each axis capture both expenditures and incomes as well as this aggregate debt balance of the economy. The difference between the incomes and expenditures of both groups measure the aggregate debt balance.

In this model there are four types of transaction. Firstly, a creditor agent spending a unit of money that is paired with unit of income of a another creditor agent. Secondly, a debtor agent spending a unit of money that is paired with a unit of income for another debtor agent. These first two types are the two examples of within-group transactions. Neither of these transactions will alter the aggregate debt level in an economy as expenditures flow back into the respective group's as income. On the axes of a two dimensional geometric representation of the model, this is be represented by an increase in the positive part of the axis associated with an equal increase of the negative part. These changes cancel out. This leaves the absolute difference between the two, the aggregate debt (credit) balance unchanged. The integration of income-expenditure balances and debt-credit balances, summarized by the second symmetry of my model, encapsulates the second, third and fourth principles of the SFC model. The third and fourth types of transaction are between-group transactions. The first of these is when a creditor agent spends a unit of money that is paired with a unit of income for a debtor agent. This type of transaction will decrease the overall level of debt. This is because it increases the credit balance of the debtor group whilst simultaneously decreasing the credit balance of the creditor group.

The second type of between-group transaction is when a debtor agents spends a unit of money that is paired with a unit of income for a creditor agent. This type of transaction will increase the overall level of debt. This is because any particular transaction represents an increase in income of one group and an equal increase in expenditure of the other group. This will either increase or decrease the aggregate level of debt in the economy.

I assume that there is an equal likelihood of a unit of expenditure flowing within-groups as opposed to between-groups. This is an 'independence assumption' with regard to the market for incomes and expenditures (the goods market) and the market for lending and borrowing (the financial market).

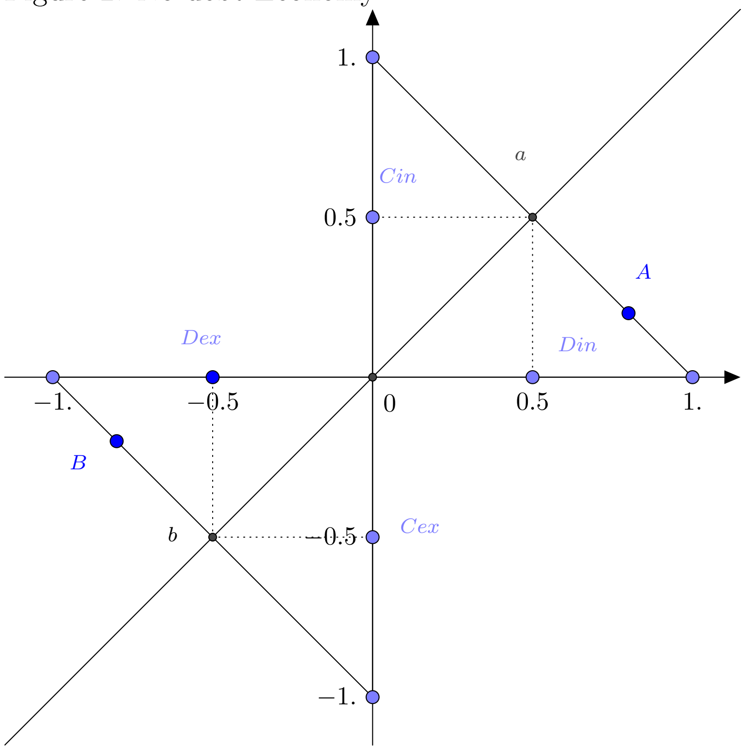
3.4 No-debt economy: within-group transactions

Figure 2, as all figures in this paper, captures only the first of two periods. Also, in each case, a second period debt repayment outcome can be found by a reflection of the geometric representation in the line $y = -x$.

In an extreme case where the creditor group receives all the income, normalized to the unit. This is the baseline normalization to compare different alternative hypothetical realizations of the aggregate flows of incomes and expenditures. Figure 2 represents this by two points: $(0, 1)$, the creditor group's income, and $(0, -1)$, the creditor group's aggregate expenditure. This shows the creditor groups expenditure is one unit and income also one unit. This means that that group accounts for 100 percent of the economies within-group incomes and expenditures. Likewise, the other polar example is where the debtor group accounts for 100 percent of the within-group activity. This is represented by points $(1, 0)$ (the debtor group's income) and $(-1, 0)$, the debtor group's expenditure.

The total joint 'space' is a one-dimensional length of one unit that can be 'distributed' between either group. However, potential distributions will leave the total lengths (incomes) equal. This length (one-dimensional space) that is kept invariant along the lines $y + x = 1$ for incomes and $y + x = -1$ for expenditures. To find a particular combination of incomes an arbitrary point along the line $y = 1 - x$ is selected. This point is then projected onto onto the axes. The projection onto the y axis gives the creditor group's normalized income level and the projection onto the x axis gives the debtor group's normalized income level.

Figure 2: No-debt Economy



At any randomly chosen point in time, the two groups have an equal chance of receiving an arbitrary unit of income. This means that the expected distribution of total incomes, when all transactions are within-groups, is at point $(0.5, 0.5)$. Similarly, the expected distribution of expenditures will be at point $(-0.5, -0.5)$.

3.5 Debt-economy: between-group transactions

The baseline case I use for between-group transactions is again when the creditor group receives 100 percent of the income. This allows it to be comparable to the within-group scenario. It means that both the no-debt (within-group) and debt (between-group) representations have a unit of creditor group income as the baseline for comparing alternative distributions of income and expenditure. In this baseline case there is a key difference in the geometry when compared to the within-group example. In the between-group example, 100 percent of the expenditure must come from the debtor group. In contrast, the within-group example requires 100 percent of the expenditure to come from the creditor group. In the case of between-group transactions, two dimensions are required to represent the space of potential transactions: two square spaces of potential pairwise matches of income and expenditure. The first squared area represents the Cartesian product of labelled units of the debtor group's income with the creditor group's expenditure. The second square represents the c Cartesian product of the debtor group's expenditure and the creditor group's income. These are all the potential ways of pairing the labelled units of these incomes and expenditures. This space is kept invariant when considering different distributions of economic activity with and between the debtor and credit groups. If the pairing is between two debtors or two creditors, then they belong to a one-dimensional space.

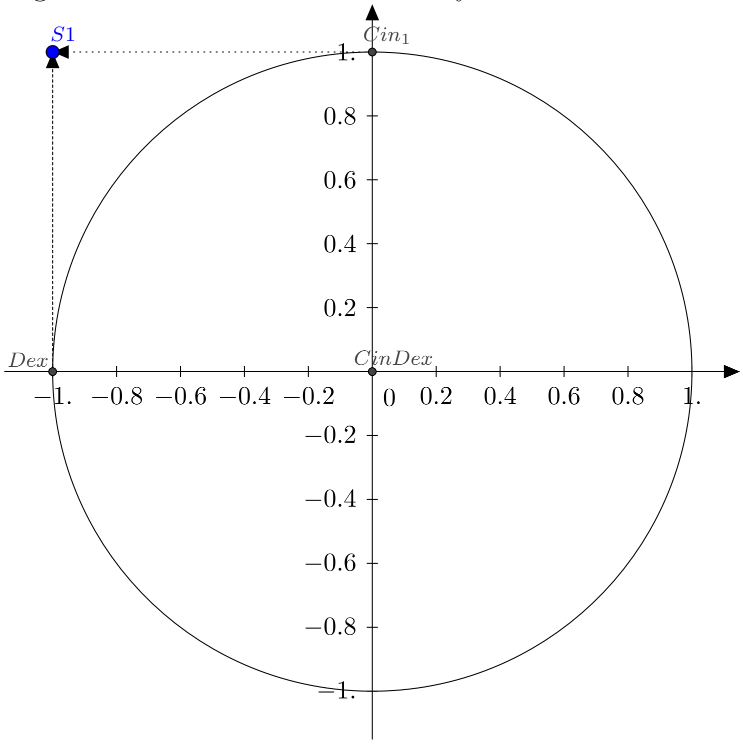
3.6 Examples of between-group geometric representations

As I move between the three examples (of an infinite number of potential examples) the total two dimensional space of the squares in quadrants two and four of the unit circle have a constant total aggregated area of the squared unit. This makes sense intuitively as I am isolating the geometric characteristics of debt. I want to be able to consider what the economy looks like with differing amounts of debt. However, I cannot simultaneously constrain the level of output to be equal in all differing amounts of debt. This is because this would imply that there is no deep underlying relationship between debt and income (or output). The potential space of economic transactions is also a more of a primitive concept than output. This is because there must first be a space of potential transactions before output is realized.

3.7 Geometric representation variant 1: Maximum Debt

As the level of debt reaches a maximum I obtain figure 3 below. This shows a case where the creditor group has zero expenditure and the debtor group have zero income. With 'maximum debt', the creditor group contributes one hundred percent of output and receives 100 percent of the income. The coordinates of the lengths of the income levels of both group's incomes is $(0, 1)$ and $(-1, 0)$ respectively. This point lies on the unit circle.

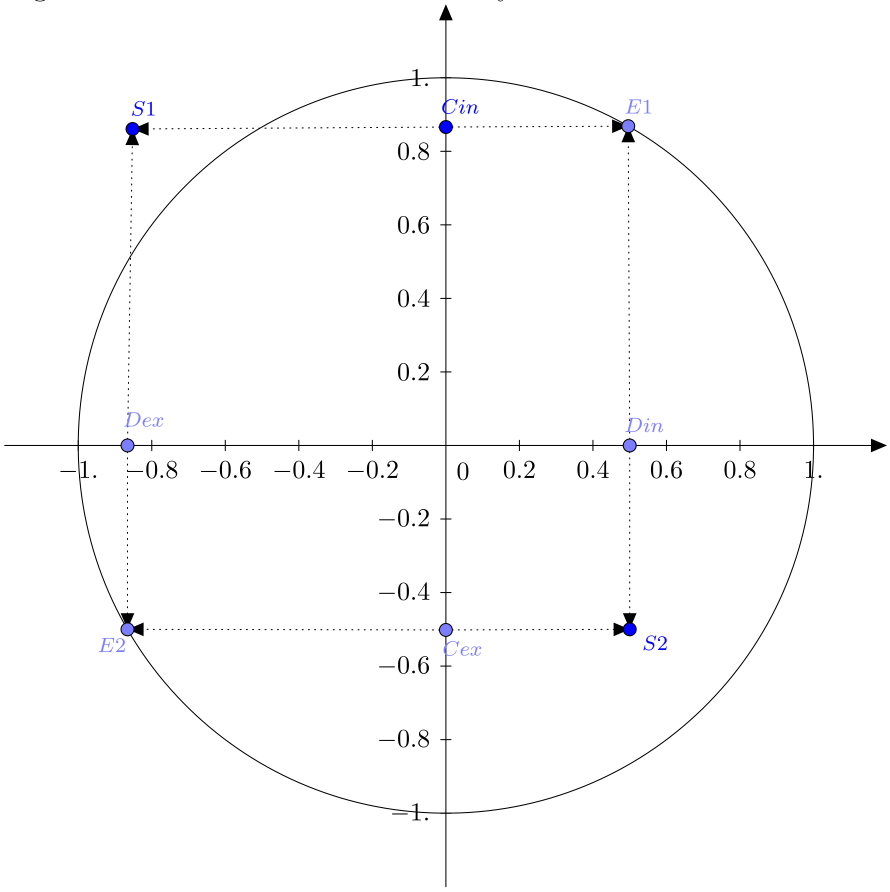
Figure 3: Maximum Debt-economy



3.8 Geometric representation variant 3: Intermediate case

Figure 4 below shows an example of an intermediate case. In this case the respective income level of either group are unequal. The creditor group is receiving more income than the debtor group. The symmetry that holds is that the absolute difference in these quantities, assuming only between group flows, are equal. Once again, points E1 and E2 lie on the unit circle.

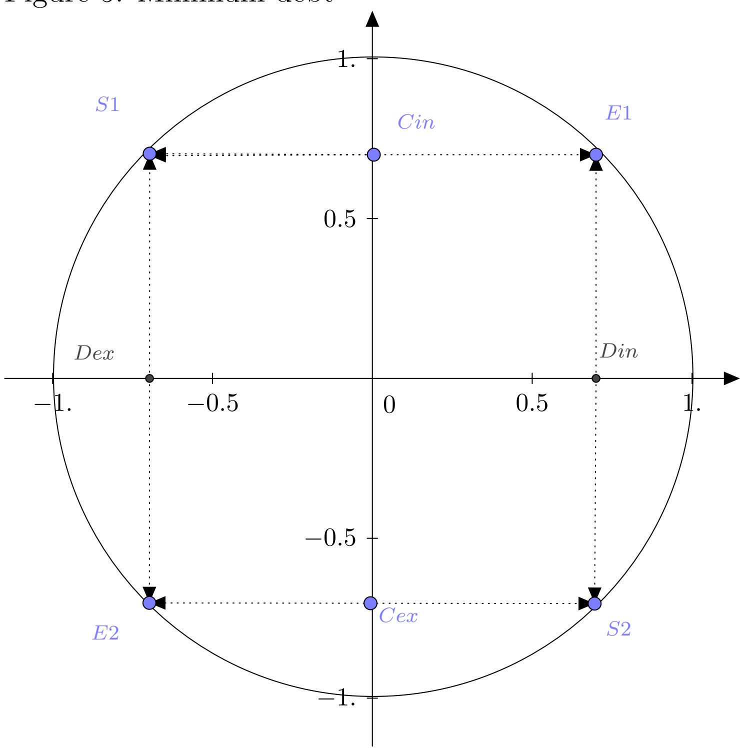
Figure 4: Intermediate Debt Economy



3.9 Zero Debt-economy

Figure 5, below shows the limiting case of the economy as debt tends to zero. Debtor group's income (Din) must equal creditor group's expenditure (Cex). The point $S2$ is the intersection of the where the projection of the creditor group's expenditure (Cex) value, along the x axis, and the debtor group's income value (Din) value, along the y axis, meet. The length of each is $\frac{1}{\sqrt{2}}$. Likewise the square $S1$ is formed as debtor expenditure (Dex) must equal creditor income (Cin). At the limit as the level of debt tends to zero, if income equals expenditure for all individuals and the squares are of equal area of 0.5 squared units. The projection of the debtor group's income level (along the y axis) and the projection of the creditor groups income level (along the x axis) meet on the unit circle.

Figure 5: Minimum debt



The maximum joint total lengths occur in the limiting case where each individual length tends to, but does not equal $\frac{1}{\sqrt{2}}$. This is the example represented in Figure 5. $\frac{1}{\sqrt{2}}$ is the level because when $\frac{1}{\sqrt{2}}$ is squared one has $\frac{1}{2}$. The sum of two identical squares of size $\frac{1}{2}$ is the square unit. As such, the area is the same as when the creditor received 100 percent of income. When lending and borrowing are permitted, the between-group flows economy has an expected income level of approximately 0.7 for each group: a length of approximately 0.7 for each group. This is shown in figure 5.

For debt related economic activity (between-group transactions), points E1 and E2, in all potential distributions of income, lie on the unit circle, but only in the first and third quadrants of the unit circle. This is where where both x and y are simultaneously positive or negative. These points lie on the unit circle because the aggregate area of the two squares in the second and fourth quadrant are normalized to the squared unit. If all possible distributions of income between the two groups are considered, points E1 and E2, taken from all possible distributions of income and hence debt levels, will map out the part of the unit circle in the first and third quadrants.

4 Between-group transactions and Pythagoras' theorem

Between-group transactions, which relate to debt, can be linked to the unit circle. For between-group transactions, the positive part of the x-axis measures the debtor group's first period income. By symmetry, this is equal to the creditor group's first period expenditure, which is measured along the negative part of the y-axis. These form the sides of one square. Similarly, the negative part of the x-axis measures the debtor group's expenditure, which is equal to the creditor group's income, which is measured along the positive part of the y-axis. The total joint area of the two squares are restricted to the unit square, and so Pythagoras' theorem holds: $x^2 + y^2 = 1$. This Pythagoras' theorem result requires that the total area of the space of potential pairings of incomes and expenditure remains invariant between examples. This constraint is relatively more relaxed than the non-debt constraint that $x + y = 1$. The intuitive explanation of why this is so, is that, in the presence of debt instruments, agents and groups of agents are no longer required to ensure that their expenditure is equalized to their incomes. In contrast, they only require that there be a counterpart agent (or group of agents) that can provide matching incomes (expenditures) that are equal to their expenditures (incomes).

There are three central factors that generate the Pythagoras' theorem result. Firstly, the credit balances of the two groups meet orthogonally. This is because the incomes and expenditures of each group do not register against the credit balance of the other group. Secondly, there is a symmetry that holds. The extent to which the creditor group's first period income exceeds their expenditure is identical to the absolute measure of the extent to which the debtor group's first period expenditure exceeds their income. This is shown by the squares S1 and S2 in Figure 5. Finally, when comparing different distributions of income and expenditure, distributed between the two groups, which represents different debt levels, the squares S1 and S2 are constrained to represent the same amount of the space of potential economic transactions: the Cartesian product of between-group incomes and expenditures.

5 The application to biblical laws

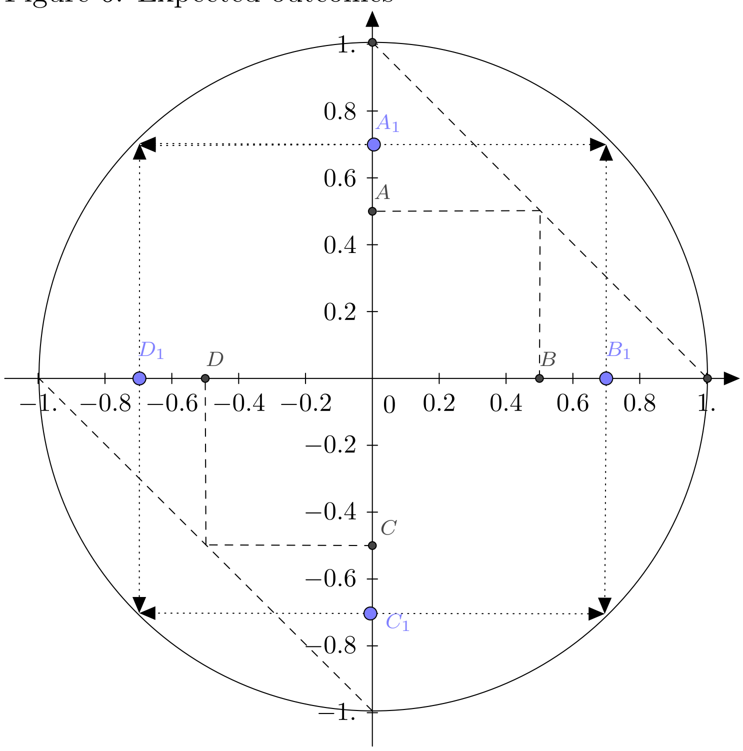
I now relate the results outlined above to the biblical laws considered. The biblical laws dictate that the temporal space of credit cycle is seven years. Meanwhile, the Sabbath commandment stipulates one day of rest out of seven each week. When these laws are observed simultaneously this means that only six of these seven years involve work. Over seven years this would amount to six years or work in total. This also means that there were six years buying and selling with one of the years spent in rest; which is equivalent to cessation from work.

To facilitate the between-group flows one group is lending to the other. The creditor group that does the lending is undertaking work to facilitate the loan, whilst the debtor group is, in effect, 'ceasing' from work in relation to the amount the creditor group lends. The debtor group does not work for those purchases made due to a loan and so it can be viewed as that group 'ceasing from work' in relation to that part of economic activity that is associated with debt. This 'cessation' from work needs to occur to facilitate debt occurring. This means that over the two equal periods, the total group of agents rests (ceases from work) on average for half of the period in so far as the activity relates to debt. This is because the first period debtors rest in the first period whilst the first period creditors rest in the second period.

The no-debt (within-group transaction) geometry has an expected income level of both groups at point $(0.5, 0.5)$ points A and B in Figure 5 below. In contrast the debt (between-group) output equilibrium is satisfied at point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, points A_1 and B_1 in figure 5 below. The intuition for why the expected output and income realizations of an economy with debt exceeds that of an economy without debt, is related to the equations of the circle in comparison to that of the straight line of $y = 1 - x$. These two constraints represent trade-offs that respectively depict no-debt and debt economies. The debt economy constraint of the circle formula is a weaker constraint than the linear constraint of the non-debt economy. There is an economic intuition underlying this. In a non-debt economy, the income levels of both groups (once normalized to the unit) cannot diverge from their respective expenditure levels. With debt, however, there are further possibilities permitted. With debt, income and expenditure levels can diverge as long as the amount that the creditor group's income exceeds its expenditure is equal and opposite to the amount the the debtor group's expenditure exceeds its income. In the case of a no-debt economy, none of these additional potential transaction opportunities are realizable. Instead, there could only be transactions up to, but not exceeding, the output level of the debtor group in each period.

Figure 6 below shows the expected outcomes of the within-group (no-debt transactions) and the between-group (debt transactions) are viewed together.

Figure 6: Expected outcomes



Expected one period debt related activity is at about 0.7 and non-debt activity at about 0.5. However, debt related, between group economic activity is also generated by buying and selling, but just in two as opposed to one dimension. There is an equal likelihood of within-group or between-group activity by the 'independence assumption'. As such, on average buying and selling occurs requires a space that is the average of 0.5 and $\frac{1}{\sqrt{2}}$, which is approximately 0.6. However, debt-related activity requires an expected space of 0.7. This yields a ratio between the two of approximately six-sevenths.This space can be thought of as time, as time is required to facilitate economic activity. Hence, if each unit of space is considered to be a year, a debt related activity requires seven years within which to operate compared to the activity of buying and selling, which requires six years in expectation.

6 Conclusions

The simultaneous observance of two laws: the Sabbath commandment and the law dictating a seven year credit cycle yields a ratio between the real and financial cycle of six-sevenths. This is approximately equal the the ratio generated by the model developed in this paper. This result is derived using a geometric approach.

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