Japanese Real Interest Rates over the Life-Cycle*

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Abstract

Japan has faced rapid ageing, persistently low interest rates and deflation for decades. Concurrently, during this period, there has been a gradual convergence in productivity between young and elderly workers. This paper aims to explore the relationship amongst productivity, demographic shifts, and interest rates in Japan during the post-bubble era, using an overlapping generations two-agent New Keynesian (OTANK) life-cycle DSGE model. The narrowing productivity gap between younger and older cohorts puts upward pressure on interest rates. Meanwhile, factors such as longer life expectancy and negative population growth rates exert downward pressure on interest rates. The latter effect dominates. An important policy implication emerges: Enhancing worker productivity across the entire lifespan and bridging the productivity gap between younger and older workers can help offset the decline in interest rates.

JEL Codes: E17, J11
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1 Introduction

In the last few decades Japanese policy makers have faced a unique, but increasingly common dimension of heterogeneity: An ageing society. Lise et al. (2014) document a series of striking cross-sectional and life-cycle facts of the Japanese economy, one of which is the demographic shift that the country has undergone: Between 1981 and 2005 the fraction of household heads aged 33 fell from 4.8% to 2.8% percent while the fraction aged 55 rose from 1.9% to 4.4% percent. This change in the age composition of the Japanese economy has implications on consumption inequality and the natural interest rate.[1] Empirical studies such as Bailey et al. (2022) and Cesa-Bianchi, Harrison, and Sajedi (2023) have documented that the decline in real interest rates globally are due in significant part to demographic ageing in advanced economies.

Theoretical studies such as Fujiwara and Teranishi (2008) and Carvalho, Ferrero, and Nechio (2016) find that decreases in population growth and an increase in life expectancy together could explain the persistent deflation and low interest rates in Japan. They argue that if a central bank fails to take into account these demographic transitions, it may inadvertently compound the problem by setting an overly aggressive monetary policy. We take these transitions as our baseline, and add another demographic change observed in the last decades in Japan: An increase in the relative productivity of elder workers. Wage earnings across different age cohorts have converged, implying that productivity has also converged. The specifics of the demographic transition are in Section 1.1.

This paper, using an overlapping generations two-agent New Keynesian (OTANK) model based on Yaari (1965), Blanchard (1985), and Gertler (1999) calibrated to Japanese data, seeks to explore the effects of converging productivity across different aged cohorts in an ageing society. Using data from the Japanese Ministry of Health, Labor, and Welfare (MHLW), we find that the relative real wage – which we use as a proxy for relative productivity – of elderly workers to young workers was approximately 0.88 in 1990 and rises to 0.94 in 2017 (See Figure 1b). In our model, we assume elder households continue to participate in labour markets, albeit less efficiently than their young worker counterparts. We then introduce three ways to simulate converging productivity across cohorts: First, we increase the relative productivity of elder workers, but not to parity with young workers. Second, we increase the "retirement age", which is equivalent to increasing the period that workers are at full productivity. Third, we combine the first two types of shocks. We find that productivity convergence of the

1. The secular decline in natural interest rates and real interest rates has garnered attention from policy makers (Bernanke, 2015) due to, for example, its implications regarding asset pricing and slower economic growth (Del Negro et al., 2019).

young and the elder workers alleviates the decline of interest rates. When workers are productive for longer, there is less capital accumulation since there is less incentive to save for old age, and therefore an upward pressure on the interest rates. This result may give some credence to the idea that a central bank which does take demographics into account may be setting an overly-aggressive monetary policy.

[KMS: These results are preliminary] Furthermore, we study the effects of ageing demographics when the economy hits the effective lower bound (ELB) on nominal interest rates – a practical concern for Japanese policymakers during the "Lost Decade", and a potential future concern for many other advanced economies. We induce an ELB-binding episode through a negative capital quality shock (Eggertsson, 2012). Our simulations show that ageing demographics extend the duration at which the economy is at the ELB constraint; despite the converging productivity between young and elderly cohorts. Finally, related to the theoretical contributions of Angeletos, Lian, and Wolf (2023), we explore fiscal considerations for the Japanese economy, namely why considerable fiscal expansions did not lead to significant inflation. Under the fiscal theory of the price level (FTPL), sustained large government deficits lead to high inflation in order for government debt to remain sustainable. However, the likely explanation for why this did not occur in Japan is due to the real interest rate being lower than the growth rate of the economy along the balanced growth path.

Relation to the literature. This paper contributes to the heterogeneous-agent literature, and a subset of it with overlapping generations models. The contemporary heterogeneous-agent literature in macroeconomics, established by seminal works such as Gornemann, Kuester, and Nakajima (2016), Bilbiie (2018), Auclert (2019), and Kaplan, Moll, and Violante (2018), focus on how income and asset distributions of agents in an economy matter for the transmission of monetary policy. Simply said, the key dimension of heterogeneity arises from the different consumption-saving decisions that different (typically "rich" and "poor") households make.

This paper, along with Fujiwara and Teranishi (2008), Carvalho and Ferrero (2014), Braun and Ikeda (2021), Braun and Ikeda (2022), and Fujita and Fujiwara (2023), focuses on a different dimension of heterogeneity among agents, which is especially relevant for Japan: the elderly and the young. Building on the original life-cycle model presented by Gertler (1999), this paper contributes to the literature explaining the persistent deflation and low interest rates in Japan. Different to the other papers, this paper focuses on aspects of the demographic transitions in Japan in the past decades might have mitigated the decline in interest rates. More specifically, we find that the convergence in productivity of young and elder workers alleviated the decrease in interest rates. In this way, we find potential policy reforms that put upward pressure on interest rates, such as an increase in elder worker productivity and/or an increase in the retirement age.

The rest of the paper is as follows. Section 1.1 begins by establishing stylised facts in regards to Japanese demographics, productivity, and wages, presenting empirical targets for our quantitative model. Section 2 introduces the OTANK model and explains the model calibration strategy. Section 3 presents the main quantitative experiments conducted in this paper, including our exercises on the ELB and fiscal sustainability. Section 4 concludes the study, and discusses potential future research.

1.1 Japanese demographics

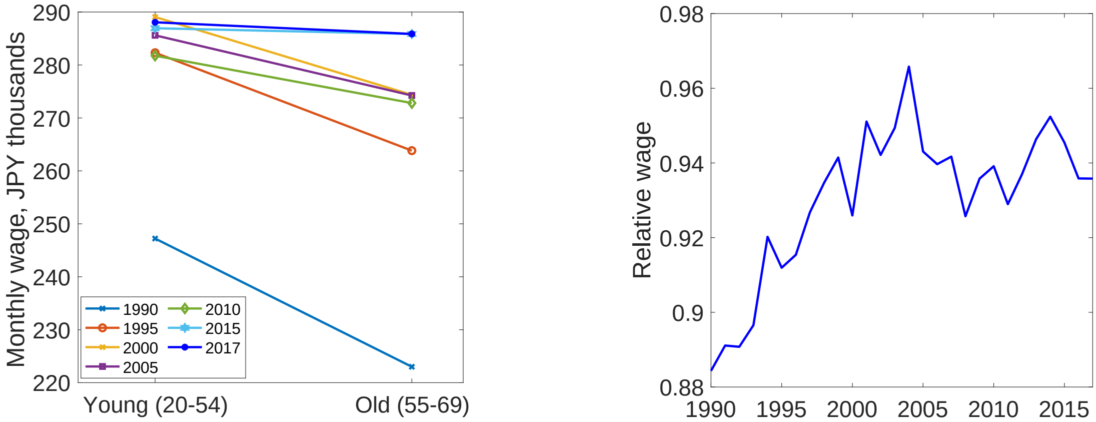
The model-relevant Japanese demographic transitions are the following: (i) a decline in the population growth rate, (ii) an increase in life expectancy, and (iii) an increase in the relative productivity of the elderly to the young. The first two are relatively known stylised facts for Japan as an ageing society. The average life expectancy was 78 years in 1990, and is projected to be 88 years by 2050. Despite the increase in life expectancy, the population of Japan has been declining since the early 2010s due to low fertility rates. Where the Japanese population was still growing at 0.43% per year in 1990, the forecast for the rate is -0.57% for 2050.

A lesser known stylised fact relating to the demographic transition in Japan in the past few decades is the convergence of productivity of the elderly to the young. In the model we assume that wages are reflective of productivity, and specifically the marginal product of labour. We analyse average monthly wage earnings data for both Japanese males and females across different age cohorts between 1990 and 2017 from the Japanese Ministry of Health, Labour, and Welfare (MHLW). The data, plotted in Figure 1a suggests that wage earnings over the lifetime for the average Japanese worker,[2] during the observed period, have become more uniform. For example, in 1990, the average monthly earnings for a young Japanese worker aged between 20 to 54 years old was approximately ¥294,895. For an elderly worker, classified as someone being older than 55 years of age,[3] the monthly average wage in 1990 was approximately ¥225,411. Thus the relative wage of an elderly worker to a young worker in 1990 was approximately 0.88. This relative wage reaches a peak of 0.97 in 2004, as monthly wages

2. The data shows that earnings over the life-cycle has become more uniform for Japanese males. Japanese females' wage earnings over the lifetime were initially quite flat, suggesting incremental productivity increases over the lifetime. We adjust the plotted data for gender and age shares.

3. 55 years of age was selected as the cutoff age between young and elderly workers as after the age of 55, wages decline. This suggests a drop in productivity. Furthermore, MHLW *Basic Survey on Wage Structure* only has wage profiles for workers in specific age brackets, and where he oldest age bracket is 65-69 years of age.

Figure 1: Relative wage of Japanese elder and young workers



(a) Average Japanese wage profiles of young and elderly

(b) Relative wage of Japanese elderly workers to young workers

Source: Japanese Ministry of Health, Labor, and Welfare.

are ¥292,637 and ¥282,627 for young and elderly workers, respectively. The relative wages between 1990 and 2017 are calculated and plotted in Figure 1b. For the purposes of simulations with our quantitative model, we proxy the relative productivity between elderly and young workers with their relative wage.

2 The model

As mentioned previously, the model employed is based on Gertler (1999), and draws elements from Fujiwara and Teranishi (2008) and Ferrero (2010). There are three main blocks in the economy: households, firms, and government.[4] Households are divided into two cohorts, young and old, and choose to supply labour in order to fund their consumption and maximise their lifetime utility. Of the income they save, households may choose to invest in three assets: physical capital, domestic government bonds, and/or shares in firms. Firms consist of intermediate goods producing firms, which produce differentiated goods and are monopolistically competitive, and perfectly competitive final goods producing firms. The government consists of a fiscal authority and a central bank, which distributes income and sets monetary policy, respectively. The fiscal authority funds its consumption through the issuing of debt and its collection of lump-sum taxes, and it also provides pension funds to the elderly.

4. In line with Carvalho and Ferrero (2014), this paper avoids open-economy considerations. However, the model can easily be modified to account for a small-open economy, as shown in Gertler (1999).

2.1 Firms and production

2.1.1 Final good firms

The model features three types of firms which operate in the economy, and their characteristics are standard as in the New Keynesian literature (Christiano, Eichenbaum, and Evans, 2005; Galí, 2015). A continuum of intermediate firms are monopolistically competitive, and combine labor and capital to produce intermediate goods, $Y_t(i)$, where $i \in (0,1)$. Investment goods are produced by perfectly competitive firms. Final good producers, which also operate under perfect competition, produce final goods, Y_t, from intermediate goods, which are then used for consumption, C_t, investment, I_t, and government spending, G_t:

$$Y_t = \left[\int_0^1 Y_t(i)^{\frac{\epsilon-1}{\epsilon}} di\right]^{\frac{\epsilon}{1-\epsilon}} = C_t + I_t + G_t, \tag{1}$$

where $\epsilon > 1$ is a constant elasticity of substitution (CES) parameter for the differentiated intermediate goods. Final good firms maximise their profits by selecting how much of each intermediate good to purchase, and so their problem is:

$$\max_{Y_t(i)} P_t Y_t - \int_0^1 P_t(i) Y_t(i) di.$$

Thus, as in Blanchard and Kiyotaki (1987), following the first-order condition (FOC) of the final good firm problem, intermediate good producers face a downward sloping demand curve for their products:

$$Y_t(i) = \left[\frac{P_t(i)}{P_t}\right]^{-\epsilon} Y_t, \tag{2}$$

where $P_t(i)$ is the price of good i and P_t is the price of the final good and also the price index for the aggregate economy and is defined as:

$$P_t = \left(\int_0^1 P_t(i)^{1-\epsilon} di\right)^{\frac{1}{1-\epsilon}}.$$

2.1.2 Intermediate goods firms

Intermediate firms use aggregate labour, L_t, and capital, K_t, and pay wages, w_t, and rents, r_t^k, as factor payments. Intermediate firm i produces its goods using a constant returns to scale Cobb-Douglas production technology where X_t denotes trend productivity growth, Z_t is a temporary total factor productivity shock, and $\alpha \in (0,1)$ denotes the labour share of output:

$$Y_t(i) = Z_t K_{t-1}(i)^{\alpha} \left[X_t L_t(i)\right]^{1-\alpha}.$$

Wages, rents, and the real marginal cost (the Lagrangian multiplier from the intermediate firm's problem, φ_t) arises from the first order conditions:[5]

$$r_t^k = \alpha\varphi_t(i)K_{t-1}(i)^{\alpha-1}\left[X_t L_t(i)\right]^{1-\alpha},$$
$$w_t = (1-\alpha)\varphi_t(i)K_{t-1}(i)^{\alpha}\left[X_t L_t(i)\right]^{-\alpha},$$

which yields:

$$\varphi_t = \frac{1}{Z_t}\left(\frac{r_t^k}{\alpha}\right)^{\alpha}\left(\frac{w_t}{X_t(1-\alpha)}\right)^{1-\alpha}. \tag{3}$$

Price setting and the NKPC. The next step for the intermediate firm is its dynamic price-setting decision, whereby it solves the following problem:

$$\max_{P_t(i)} \sum_{t=0}^{\infty} \frac{1}{\prod_{s=1}^{t} R_{t+s-1}/\pi_{t+s}} D_t^I(i), \tag{4}$$

subject to (2), where $D_t^I(i)$ is firm i's per-period profits defined as:[6]

$$D_t^I(i) = \left[\frac{P_t(i)}{P_t} - \varphi_t\right] Y_t(i) - \frac{\phi_I}{2}\left[\frac{P_t(i)}{P_{t-1}(i)} - 1\right]^2 Y_t, \tag{5}$$

and where R_t is the gross nominal interest rate, π_t is gross inflation, ϕ_I is a price adjustment parameter, and $\mathcal{M} = \frac{\epsilon}{\epsilon-1}$ is the optimal markup charged by intermediate firms arising from monopolistic competition.

The solution to the price-setting problem in a symmetric equilibrium where $P_t(i) = P_t, \forall i$ yields the following New Keynsian Phillips Curve (NKPC):

$$(\pi_t - 1)\pi_t = \frac{\epsilon - 1}{\phi_I}(\mathcal{M}\varphi_t - 1) + \frac{\pi_{t+1}}{R_t}\frac{Y_{t+1}}{Y_t}(\pi_{t+1} - 1)\pi_{t+1}. \tag{6}$$

Finally, because factor payments are standard across all intermediate firms, in the symmetric equilibrium we get:

$$Y_t = Z_t K_{t-1}^{\alpha}(X_t L_t)^{1-\alpha}, \tag{7}$$

and the following expenditure share:

$$\frac{w_t L_t}{r_t^K K_{t-1}} = \frac{1-\alpha}{\alpha}. \tag{8}$$

5. The cost minimization problem for each intermediate goods producer is:

$$\min_{K_{t-1}(i), L_t(i)} r_t^k K_{t-1}(i) + w_t L_t(i),$$

subject to:

$$Z_t K_{t-1}(i)^{\alpha}\left[X_t L_t(i)\right]^{1-\alpha} \geq Y_t(i) = \left[\frac{P_t(i)}{P_t}\right]^{-\epsilon} Y_t.$$

6. For simplification, we assume that intermediate firms discount profits with the real interest rate.

2.1.3 Capital good firms

Capital goods are produced by perfectly competitive firms. The aggregate capital stock grows according to a standard law of motion:

$$K_t = I_t + (1-\delta)K_{t-1}, \tag{9}$$

where I_t is investment and $\delta \in (0,1)$ is the depreciation rate.

The objective of the capital good producing firm is to choose I_t to maximise revenue, $Q_t I_t$. We assume that capital goods producing firms are subject to investment adjustment costs, $\Phi(\cdot)$, as in Christiano, Eichenbaum, and Evans (2005).[7] Thus, the representative capital good producing firm's objective is:

$$\max_{I_t} \sum_{t=0}^{\infty} \frac{\Pi_t^K}{\prod_{s=1}^{t} R_{t+s-1}/\pi_{t+s}},$$

where per-period profits are:

$$\Pi_t^K \equiv \left\{ Q_t - 1 - \Phi\left(\frac{I_t}{I_{t-1}}\right)\right\} I_t.$$

Solving the optimisation problem yields the following first-order condition for the price of capital:[8]

$$Q_t = 1 + \Phi\left(\frac{I_t}{I_{t-1}}\right) + \Phi'\left(\frac{I_t}{I_{t-1}}\right) - \frac{\pi_{t+1}}{R_t}\Phi'\left(\frac{I_t}{I_{t-1}}\right)\left(\frac{I_t}{I_{t-1}}\right)^2. \tag{10}$$

2.2 Households

As in Gertler (1999) and corresponding overlapping generations models, households have finite lives and go through two stages in life: youth and old age. In order to develop a parsimonious model with tractable consumption/savings behavior and realistic lifetimes, simplifying assumptions are made pertaining to population dynamics, insurance markets, and household preferences.

First, we extend the baseline Gertler model by allowing both young and elderly households to choose the amount of labor to supply, as in Kilponen, Kinnunen, and Ripatti (2006) and Fujiwara and Teranishi (2008). As such, labor supply and the population mass of households differ. Consider the period $t-1$, and let the population mass of the young and old be denoted as N_{t-1}^y and N_{t-1}^e, respectively. Between $t-1$ and t, a

7. We assume the following functional form for $\Phi(\cdot)$:

$$\frac{\kappa_I}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2,$$

with $\Phi(1) = \Phi'(1) = 0$ and $\Phi''(\cdot) > 0$.

8. Note that here we do not adjust for trend. Appendix B provides a full set of de-trended equilibrium conditions.

young household remains young with probability ω_t, and with probability $1-\omega_t$ of becoming old. Thus, the average length of time an individual spends in her youth is given by $\frac{1}{1-\omega_t}$. In addition, an elderly individual survives from $t-1$ to t with probability γ_t, and conversely perishes with probability $1-\gamma_t$. In period t, $(1-\omega_t+n_t)N^y_{t-1}$ new young households are born, giving the following law of motion for the young population:

$$\begin{aligned} N^y_t &= (1-\omega_t+n_t)N^y_{t-1} + \omega_t N^y_{t-1} \\ &= (1+n_t)N^y_{t-1}. \end{aligned} \tag{11}$$

So, n_t is the growth rate of the labor force between periods $t-1$ and t. Meanwhile, the law of motion for the elderly population is given as:

$$N^e_t = (1-\omega_t)N^y_{t-1} + \gamma_t N^e_{t-1}. \tag{12}$$

As will be discussed, the young and elderly possess different consumption, saving, and asset profiles, therefore we can track the dependency ratio, $\Gamma_t = N^e_t/N^y_t$, defined as:

$$\Gamma_t = \frac{(1-\omega_t)N^y_{t-1} + \gamma_t N^e_{t-1}}{(1+n_t)N^y_{t-1}}, \tag{13}$$

with the following law of motion:

$$(1+n_t)\Gamma_t = (1-\omega_t) + \gamma_t \Gamma_{t-1}. \tag{14}$$

For simplification, we assume no aggregate risk. Risk is idiosyncratic for both the young and old: the young may suddenly face a drop in productivity, and the elderly face an uncertain time of death. To counter these uncertainties we assume a perfect insurance market, as in Yaari (1965) and Blanchard (1985). The elderly purchase annuities in each period from a mutual fund which invests on their behalf. In each following period, γ_t fraction of elderly that survive receive all the returns – there is no estate bequest mechanism for the $1-\gamma_t$ fraction of elderly that perish. As such, if R_t is the gross return on investments made by the mutual fund, then the gross return on wealth for the elderly that survive is given by R_t/γ_t.

To account for the loss in productivity (and income) of young households that become old, the model features a special class of recursive preferences (Kreps and Porteus, 1978; Epstein and Zin, 1989) that assumes risk neutrality. There is no insurance market which covers the risk of income loss of a young household. The primary reason for this is to capture life-cycle behavior. The existence of a perfect insurance market would allow an individual to perfectly smooth their income between being young and old. Thus, in the absence of such an insurance market, income earning potential is mostly skewed towards individuals during their working life. This provides a better representation of the life-cycle.

Let V^z_t denote an individual's recursive utility where the superscript $z = \{y, e\}$ indicates whether the individual is young (y) or elderly (e). Individuals maximize utility by choosing consumption, C_t^z, labor supply, L_t^z, and their asset allocations (described below). Let β^z be an individual's subjective discount factor. Preferences are given by:[9]

$$V_t^z = \left\{ \left[(C_t^z)^{\upsilon} (1 - L_t^z)^{1-\upsilon} \right]^{\rho} + \beta_{t+1}^z (V_{t+1}|z)^{\rho} \right\}^{\frac{1}{\rho}}, \tag{15}$$

where:

$$\beta_{t+1}^y = \beta,$$
$$\beta_{t+1}^e = \gamma_{t+1}\beta,$$

as the young and elderly have different discount factors due to the risk of death. Additionally, ρ pins down the intertemporal elasticity of substitution (IES) and υ is an individual's utility weight on consumption.

An individual's future value of utility in (15) also differs between the young and elderly due to the probability of transitioning from young to old:

$$V_{t+1}^y = \omega_{t+1} V_{t+1}^y + (1 - \omega_{t+1}) V_t^e.$$

With this specification of Epstein-Zin (EZ) preferences, the OTANK model is analytically tractable since the transition probabilities, ω_t and γ_t, are independent of age and the age of retirement of an individual. This will greatly assist in the aggregation of households across the two cohorts. Additionally, EZ preferences help provide a better representation of consumption/savings decisions observed in actual data. With standard constant relative risk aversion preferences, individuals would have overly strong precautionary savings motive during their youth due to the risk of a decline in income and death (Farmer, 1990; Gertler, 1999). EZ preferences also allow for more realistic responses to changes in the interest rate as the IES is decoupled from risk aversion.

All households choose to allocate their funds between government bonds, B_t, and shares of intermediate good producing firms, x_t^I. Government bonds pay a gross nominal return, R_t, firm shares are priced at P_t^I and intermediate firms pay shareholders a dividend of D_t^I.

2.2.1 Elder workers

An elderly individual born in period j and enters retirement age in period k chooses consumption, $C_t^e(j,k)$, labor supply, $L_t^e(j,k)$, and assets, $A_t^e(j,k)$, comprising of bonds and shares, to maximize their utility given by:

$$V_t^e(j,k) = \max_{C_t^e(j,k), L_t^e(j,k), A_t^r(j,k)} \left\{ \left[C_t^e(j,k)^{\upsilon} (1 - L_t^e(j,k))^{1-\upsilon} \right]^{\rho} + \beta\gamma_{t+1} V_{t+1}^e(j,k)^{\rho} \right\}^{\frac{1}{\rho}}, \tag{16}$$

9. Note that we omit expectation operators since we only consider a perfect-foresight setting.

subject to

$$\begin{aligned}C^e_t(j,k)+&\frac{B^e_t(j,k)}{P_t}+P^I_t x^I_t(j,k)=\\&\frac{1}{\gamma_t}\left[\frac{R_{t-1}}{\pi_t}\frac{B^e_{t-1}(j,k)}{P_{t-1}}+(P^I_t+D^I_t)x^I_{t-1}(j,k)\right]+\varsigma_t w_t L^e_t(j,k)+E^e_t(j,k),\end{aligned} \tag{17}$$

where $\varsigma_t = [0,1]$ denotes the relative productivity of an elderly worker to a young worker, and $E^e_t(j,k)$ denotes social security transfers. For model tractability we require that an elderly individual's initial asset holdings be equal to its last period asset holdings during youth. In the case of bonds, this is:

$$B^e_k(j,k) = B^y_{k-1}(j). \tag{18}$$

Furthermore, in the absence of aggregate uncertainty, the real return on bonds and shares are equalised, creating the following no-arbitrage condition:[10]

$$\frac{R_t}{\pi_{t+1}} = \frac{P^I_{t+1}+D^I_{t+1}}{P^I_t}. \tag{19}$$

Total real financial assets for an elderly individual are defined as:

$$A^e_t(j,k) = \frac{B^e_t(j,k)}{P_t} + P^I_t x^I_t(j,k). \tag{20}$$

Using (19) and (20), an elderly individual's budget constraint, (17), can be compactly rewritten as:

$$C^e_t(j,k) + A^e_t(j,k) = \frac{1}{\gamma_t}\frac{R_{t-1}}{\pi_t}A^e_{t-1}(j,k) + \varsigma_t w_t L^e_t(j,k) + E^e_t(j,k). \tag{21}$$

From the first order conditions, which can be found in the Appendix A.1.2, a relationship between labor supply and consumption can be derived:

$$L^e_t(j,k) = 1 - \frac{1-\upsilon}{\upsilon}\frac{1}{\varsigma_t w_t}C^e_t(j,k), \tag{22}$$

as can the consumption Euler equation:

$$C^e_{t+1}(j,k) = \left(\beta\frac{R_t}{\pi_{t+1}}\right)^{\sigma}\left(\frac{w_t}{w_{t+1}}\right)^{\rho(1-\upsilon)\sigma} C^e_t(j,k), \tag{23}$$

where $\sigma = 1/(1-\rho)$.

A recursive expression for non-financial wealth of an elderly individual can be written as the following:

$$H^e_t(j,k) = \varsigma_t w_t L^e_t(j,k) + \frac{\pi_{t+1}}{R_t}\gamma_{t+1}H^e_{t+1}(j,k). \tag{24}$$

We can also write the present discounted value of social security benefits as:

$$S^e_t(j,k) = E^e_t(j,k) + \frac{\pi_{t+1}}{R_t}\gamma_{t+1}S^e_{t+1}(j,k). \tag{25}$$

10. For details, please refer to Appendix A.1.1.

From these equations, we can guess that the consumption function for an elderly individual is a fraction of total wealth:

$$C_t^e(j,k) = \xi_t^e \left[\frac{R_{t-1}}{\pi_t} \frac{A_{t-1}^e(j,k)}{\gamma_t} + H_t^e(j,k) + S_t^e(j,k) \right], \tag{26}$$

where ξ_t^e is the marginal propensity to consume (MPC) for an elderly individual which satisfies the following first order non-linear difference equation:

$$\frac{1}{\xi_t^e} = 1 + \gamma_{t+1}\beta^\sigma \left(\frac{R_t}{\pi_{t+1}} \right)^{\sigma-1} \left(\frac{w_t}{w_{t+1}} \right)^{\rho(1-v)\sigma} \frac{1}{\xi_{t+1}^e} \tag{27}$$

Notice that ξ_t^e is independent of any individual elderly worker. This is crucial when it comes to aggregation later. Finally, an expression for the value function satisfying the above equations can be found:

$$V_t^e(j,k) = \left(\xi_t^e\right)^{-\frac{1}{\rho}} C_t^e(j,k) \left(\frac{1-v}{v} \frac{1}{\varsigma_t w_t} \right)^{1-v}. \tag{28}$$

2.2.2 Young workers

A young individual born in period j with no initial assets chooses consumption, $C_t^y(j)$, labor supply, $L_t^y(j)$, and assets, $A_t^y(j)$, to maximize their utility:

$$V_t^y(j) = \max_{C_t^y(j), L_t^y(j), A_t^y(j)} \left\{ \begin{array}{c} \left(C_t^y(j)^v \left[1 - L_t^y(j) \right]^{1-v} \right)^\rho \\ +\beta \left[\omega_{t+1} V_{t+1}^y(j) + (1-\omega_{t+1}) V_{t+1}^e(j, t+1) \right]^\rho \end{array} \right\}^{\frac{1}{\rho}}, \tag{29}$$

subject to

$$\begin{aligned} C_t^y(j) + \frac{B_t^y(j)}{P_t} + P_t^I x_t^I(j) + T_t^y(j) \\ = \frac{R_{t-1}}{\pi_t} \frac{B_{t-1}^y(j)}{P_{t-1}} + w_t L_t^y(j) + (P_t^I + D_t^I) x_{t-1}^I(j), \end{aligned} \tag{30}$$

where $T_t^y(j)$ are pension payments paid by young individuals. As in the case with the elderly, one can rewrite the young individual's budget constraint (30) more compactly as:

$$C_t^y(j) + A_t^y(j) + T_t^y(j) = \frac{R_{t-1}}{\pi_t} A_{t-1}^y(j) + w_t L_t^y(j). \tag{31}$$

From the first order conditions, which can be found in the Appendix A.1.3, the optimal relationship between consumption and labour supply is:

$$L_t^y(j) = 1 - \frac{1-v}{v} \frac{1}{w_t} C_t^y(j). \tag{32}$$

In line with equation (28), we can conjecture a value function for a young individual as:

$$V_t^y(j) = (\xi_t^y)^{-\frac{1}{\rho}} C_t^y(j) \left(\frac{1-v}{v} \frac{1}{w_t} \right)^{1-v}, \tag{33}$$

We can then combine this conjectured equation with (28) and the first order conditions yields the following consumption Euler equation for a young individual:

$$
\begin{aligned}
C^y_t(j)\left[\frac{\beta R_t \Omega_{t+1}}{\pi_{t+1}}\left(\frac{w_t}{w_{t+1}}\right)^{\rho(1-\upsilon)}\right]^\sigma \\
= \omega_{t+1} C^y_{t+1}(j) + (1-\omega_{t+1})\Xi_{t+1}^{\frac{\sigma}{1-\sigma}} C^e_{t+1}(j, t+1)\left(\frac{1}{\varsigma_t}\right)^{1-\upsilon},
\end{aligned}
\tag{34}
$$

where Ξ_t is the ratio of MPCs of the old and young:

$$
\Xi_t = \frac{\xi^e_t}{\xi^y_t}, \tag{35}
$$

and the adjustment factor to account for the different MPCs between the old and young is given as:

$$
\Omega_t = \omega_t + (1-\omega_t)\Xi_t^{\frac{1}{1-\sigma}}\left(\frac{1}{\varsigma_t}\right)^{1-\upsilon}. \tag{36}
$$

The present values of non-financial wealth and social security, respectively, are:

$$
\begin{aligned}
H^y_t(j) = w_t L^y_t(j) + \omega_{t+1}\frac{\pi_{t+1}}{R_t \Omega_{t+1}} H^y_{t+1}(j) \\
+ (1-\omega_{t+1})\Xi_{t+1}^{\frac{1}{1-\sigma}}\left(\frac{1}{\varsigma_t}\right)^{1-\upsilon}\frac{\pi_{t+1}}{R_t \Omega_{t+1}} H^e_{t+1}(j, t+1),
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
S^y_t(j) = \omega_{t+1}\frac{\pi_{t+1}}{R_t \Omega_{t+1}} S^y_{t+1}(j) - T^y_t \\
+ (1-\omega_{t+1})\Xi_{t+1}^{\frac{1}{1-\sigma}}\left(\frac{1}{\varsigma_t}\right)^{1-\upsilon}\frac{\pi_{t+1}}{R_t \Omega_{t+1}} S^e_{t+1}(j, t+1).
\end{aligned}
\tag{38}
$$

We then conjecture a consumption function for young workers:

$$
C^y_t = \xi^y_t\left[\frac{R_{t-1}}{\pi_t} A^y_{t-1}(j) + H^y_t(j) + S^y_t(j)\right]. \tag{39}
$$

Then, use the consumption Euler equation (34), conjectured consumption function (39), and the per-period budget constraint (31), to yield an expression for the young worker's MPC:

$$
\frac{1}{\xi^y_t} = 1 + \beta^\sigma\left(\frac{R_t \Omega_{t+1}}{\pi_{t+1}}\right)^{\sigma-1}\left(\frac{w_t}{w_{t+1}}\right)^{\rho(1-\upsilon)\sigma}\frac{1}{\xi^y_{t+1}}. \tag{40}
$$

2.2.3 Household aggregation

As is evident from Equations (27), (36), and (40), the MPCs of young and old workers are independent of individual characteristics. Combine this with the facts that consumption and labor supply decisions are linear, and that taxes and pension benefits are lump sum by nature,[11] and we can write the following aggregate consumption functions:

$$C_t^y = \xi_t^y \left(\frac{R_{t-1}}{\pi_t} A_{t-1}^y + H_t^y + S_t^y \right), \tag{41}$$

$$C_t^e = \xi_t^e \left(\frac{R_{t-1}}{\pi_t} A_{t-1}^e + H_t^e + S_t^e \right), \tag{42}$$

for the young and old workers, respectively. It follows that we can also write aggregate labor supply for the two cohorts as:

$$L_t^y = N_t^y - \frac{1-\upsilon}{\upsilon} \frac{1}{w_t} C_t^y, \tag{43}$$

$$L_t^e = \Gamma_t N_t^y - \frac{1-\upsilon}{\upsilon} \frac{1}{\varsigma_t w_t} C_t^e. \tag{44}$$

Aggregate non-financial wealth for the two cohorts needs to be adjusted to account for the population growth rate. Thus, they can be expressed as (with some slight algebraic simplification):

$$H_t^y = w_t L_t^y + \frac{\omega_{t+1}}{(1+n_{t+1})} \frac{\pi_{t+1}}{R_t \Omega_{t+1}} H_{t+1}^y + \frac{(1-\omega_{t+1})}{(1+n_{t+1})} \Xi_{t+1}^{\frac{1}{1-\sigma}} \left(\frac{1}{\varsigma_t} \right)^{1-\upsilon} \frac{\pi_{t+1}}{R_t \Omega_{t+1}} H_{t+1}^e, \tag{45}$$

$$H_t^e = \varsigma_t w_t L_t^e + \frac{\gamma_{t+1}}{(1+n_{t+1})} \frac{\pi_{t+1}}{R_t} H_{t+1}^e. \tag{46}$$

Likewise, aggregate social security for the young and elderly can be expressed respectively as:

$$S_t^y = \frac{\omega_{t+1}}{(1+n_{t+1})} \frac{\pi_{t+1}}{R_t \Omega_{t+1}} S_{t+1}^y + \frac{(1-\omega_{t+1})}{(1+n_{t+1})} \Xi_{t+1}^{\frac{1}{1-\sigma}} \left(\frac{1}{\varsigma_t} \right)^{1-\upsilon} \frac{\pi_{t+1}}{R_t \Omega_{t+1}} S_{t+1}^e - T_t^y, \tag{47}$$

$$S_t^e = E_t^e + \frac{\gamma_{t+1}}{(1+n_{t+1})} \frac{\pi_{t+1}}{R_t} S_{t+1}^e. \tag{48}$$

To account for the heterogeneity across the two cohorts, an additional state variable is needed to account for the proportion of asset holdings held by either cohort. To this end, let Ψ_t be the proportion of assets held by the elderly:

$$\Psi_t = \frac{A_t^e}{A_t}. \tag{49}$$

The aggregate consumption is then given by:

$$C_t = \xi_t^y \left[(1-\Psi_{t-1}) \frac{R_{t-1} A_{t-1}}{\pi_t} + H_t^y + S_t^y \right] + \xi_t^e \left[\Psi_{t-1} \frac{R_{t-1} A_{t-1}}{\pi_t} + H_t^e + S_t^e \right]. \tag{50}$$

We can also characterise the law of motion of aggregate asset holdings across the two

11. In other words, taxes paid by young workers and the pension benefits for the elderly are independent of an individual's demographic characteristics.

cohorts as:

$$A_t^y = \omega_{t+1}\left(\frac{R_{t-1}}{\pi_t}A_{t-1}^y + w_t L_t^y - C_t^y - T_t^y\right), \tag{51}$$

$$\begin{aligned} A_t^e = & \frac{R_{t-1}}{\pi_t}A_{t-1}^e + \varsigma_t w_t L_t^e + E_t^e - C_t^e \\ & + (1-\omega_{t+1})\left(\frac{R_{t-1}}{\pi_t}A_{t-1}^y + w_t L_t^y - C_t^y - T_t^y\right). \end{aligned} \tag{52}$$

Households hold the entirety of all assets in the economy:

$$A_t \equiv A_t^y + A_t^e, \tag{53}$$

and thus, by substituting (42) and (51) into (52), the law of motion of the distribution of financial wealth, Ψ_t, evolves according to:

$$\begin{aligned} & [\Psi_t - (1-\omega_{t+1})]\,A_t = \\ & \omega_{t+1}\left[(1-\xi_t^e)\frac{R_{t-1}}{\pi_t}\Psi_{t-1}A_{t-1} + \varsigma_t w_t L_t^e + E_t^e - \xi_t^e\left(H_t^e + S_t^e\right)\right]. \end{aligned} \tag{54}$$

2.3 Fiscal and monetary policy

Fiscal and monetary policy is undertaken by the government and central bank, respectively. The government collects lump sum taxes from the young population and issues nominal debt, $P_t B_t$, to finance government spending, G_t, and pension expenses. The budget government constraint, in real terms, is given as:

$$\frac{R_{t-1}}{\pi_t}\frac{B_{t-1}}{P_{t-1}} + E_t + G_t = \frac{B_t}{P_t} + T_t. \tag{55}$$

For simplicity, the government is assumed to follow the following rules pertaining to issuing debt and expenditures:

$$\frac{G_t}{Y_t} = S_t^g, \tag{56}$$

$$\frac{B_t/P_t}{Y_t} = S_t^b, \tag{57}$$

$$E_t = E_t^e, \tag{58}$$

$$T_t = T_t^y. \tag{59}$$

As in Carvalho, Ferrero, and Nechio (2016), the government offers social security benefits to elderly workers:

$$E_t = \varrho_t\left(w_t L_t^y - T_t\right) \tag{60}$$

where $\varrho_t \in [0,1]$ is the net replacement rate.

Monetary policy is set according to the following inflation-targeting rule:

$$R_t = \bar{R}^{\phi_R}(R_t^*)^{1-\phi_R}\pi_t^{\phi_\pi}, \tag{61}$$

where $\bar{R}$ is the steady state nominal interest rate, R_t^* is the flexible price equilibrium real interest rate, $\phi_R \in [0, 1]$, and $\phi_\pi > 1$.

2.4 Market clearing

Market clearing conditions are provided by the following equations pertaining to the intermediate goods market, final goods market, capital market, asset market, and labor market. Begin by the economy wide resource constraint:

$$Y_t = C_t + \left[1 + \Phi\left(\frac{I_t}{I_{t-1}}\right)\right] I_t + G_t + \frac{\phi_I}{2}(\pi_t - 1)^2 Y_t. \tag{62}$$

Returns from government bonds, investing in intermediate goods firms, and returns on capital are equalized under the following no-arbitrage condition:

$$\frac{R_{t-1}}{\pi_t} = \frac{P_{t+1}^I + D_t^I}{P_t^I} = r_t^K + 1 - \delta. \tag{63}$$

In aggregate, financial assets in the economy must equal the sum of capital, bonds, and total shares in intermediate firms:[12]

$$A_t = K_t + B_t + P_t^I. \tag{64}$$

Finally, aggregate labour is the sum of the variable labour supplied by both the young and old:

$$L_t = L_t^y + \varsigma_t L_t^e. \tag{65}$$

2.5 Equilibrium

An equilibrium system of equations is characterised when: final good producers maximise profits subject to their resource constraint (1); intermediate good producers maximise profits (4) subject their resource constraint (2), taking demand for their differentiated goods as given; households, both young and old, maximise utility ((16) and (29)) subject to their budget constraints ((17) and (30)), taking prices and wages as given; when the government chooses debt and taxes to satisfy its budget constraint (55); when the central bank sets nominal interest rates based on its interest rate rule (61); and when the market for goods and labor clears ((62) and (65)).

Data based on real world observations and projections are input into the model for it to have realistic demographic transitions. In particular, shocks fed into the model are based on changes in ω, γ, n, and ς, and they drive the model's dynamics. Although initial shocks are unanticipated, all agents have perfect foresight of all future steady state paths. Furthermore, all model variables are adjusted for productivity and popu-

12. Total net supply of shares in intermediate firms is set to one (i.e., $x_t^I = 1$).

lation growth, and are defined in per-capita terms. This is to assist in the discovery of a well defined steady state. The full system of equations are located in Appendix B.

2.6 Parameterisation

Parameter values for the baseline model are given in Table 1. Each period in the model is one quarter, and so parameters are given in quarterly values unless otherwise stated. As the paper is primarily focused on describing data facts since the collapse of the asset price bubble, the model targets parameter values corresponding from 1990 through to 2017.

An individual enters the model economy as a 20 year old young worker, and on average transitions to an elderly worker at the age of 65,[13] corresponding to an average duration of being young of $1/(1-\omega)$ periods. In line with data and forecasts from the United Nations (UN) *World Population Prospects: The 2017 Revision*, in 1990 individuals are expected to have an average lifetime of 78 years and the population annual growth rate is 0.43%. By 2050, an average Japanese individual is expected to live until the age of 88, and the population growth rate is forecast to be -0.57% per annum. These figures discipline our choices for n and γ, and recall that γ is the survival probability once an individual transitions from young to old. Thus they live for $1/(1-\gamma)$ periods as an elder worker. Thus, under the baseline calibration and in 1990, an individual enters the workforce at 20 years of age, works until they are 65, retires, and has an expected 13 years of life as an elder worker.

We calibrate ς, the relative productivity between young and elderly workers to 0.80 in 1990 for the baseline simulation of the model. For Simulations 2 and 4, ς increases following a concave, monotonically increasing process to 0.9 in 2015, based on real wage data from the MHLW. We assume that ς is then fixed at 0.9 until the end of simulation period.

The remaining parameters are set according to common values in the literature (Christiano, Eichenbaum, and Evans, 2005; Smets and Wouters, 2007). The growth rate of labour augmenting productivity, x_t, is an updated version of Hayashi and Prescott (2002). This corresponds to a quarterly growth rate of 0.14% (or 0.56% per annum). The labour share of income, α, is set to 0.377 as per Braun et al. (2006). The capital depreciation rate, δ, is set to 2.5% per quarter, and the investment adjustment cost parameter κ_I is set to 2/3 reflecting the empirical findings of Eberly (1997).

13. According to OECD Pensions Indicators, in the 1990-2017 period, Japanese men and women have an average effective retirement age of 70.1 and 66.8 years, respectively. However, we make two simplifying assumptions: i) we make no distinction between males and females; and ii) we use the official optional retirement age of 65 to pin down ω. Furthermore information is at: https://www.oecd.org/els/public-pensions/PAG2023-country-profile-Japan.pdf.

Table 1: **Parameter values and exogenous variables**

Parameter	Value	Description
ω	$[0.9944, 0.9950]$	Transition from young to old
γ	$[0.9815, 0.9891]$	Survival probability
n	$[0.001075, -0.001425]$	Population growth rate
ς_t	$[0.8, 0.9]$	Young-old relative productivity
x	0.0014	Productivity growth rate
α	0.377	Labour share of income
δ	0.025	Depreciation rate
κ_I	2/3	Investment adjust cost term
ϵ	12	Price elasticity of demand
ϕ_I	132	Rotemberg pricing parameter
ρ	-1	Pins down σ
σ	$(1-\rho)^{-1}$	Intertemporal elasticity of substitution
υ	0.65	Preference weight of consumption over leisure
g	0.17	Government spending (% of GDP)
b	2.08	Government debt (% of quarterly GDP)
ϱ	0.4	Net replacement rate
ϕ_π	2	Monetary policy response to inflation
ϕ_R	0.2	Monetary policy weight
β	1.0033	Discount factor (real interest rate target)

The elasticity of substitution between intermediate goods, κ, is set to 12 in order to generate a steady state markup of 9% (Høj et al., 2007). The Rotemberg pricing parameter, ϕ_I, is set to 132 so that prices have an average duration of four quarters, which broadly fits Japanese data according to Higo and Saita (2007).[14] The IES, σ, is set to 0.5, slightly higher than the baseline 0.25 setting in Gertler (1999), and in accordance to empirical evidence from Hall (1988) and Yogo (2004). We also assume a more lax value for household preferences of consumption over leisure, $\upsilon = 0.65$, as compared to $\upsilon = 0.4$ in Gertler (1999) and Cooley and Prescott (1995). This more accurately reflects the findings in Boppart and Krusell (2020).

Government consumption as a percentage of GDP, g, is set to 0.17, an averaged value between 1990 and 2016 according to the World Bank and OECD national accounts. It should be noted that we use net debt as a parameter target, and that government debt shows large differences between net and gross debt. However, due to concerns over overvalued assets on government balance sheets, a simple average of both gross and net government debt as a fraction of GDP from the IMF World Economic database is taken as the parameter target for b. The net replacement, ϱ, which determines social security benefits in the model is set to 0.4. This is based on evidence from Yashiro and Oshio (2008) and the OECD.

14. With this specification, the linearised NKPC slope is equivalent to a staggered pricing model à la Calvo with an average price change every four quarters.

For the central bank monetary policy rule, we assume $\phi_\pi = 2$ and $\phi_R = 0.2$. Finally, the discount factor β is calibrated to 1.0033 to target the initial real interest rate in 1990 of approximately 7.13% per annum.

3 Quantitative simulations

In this section we show the results of our numerical analysis with the model as described above. First we show the effect of the demographic transition in Japan on the real interest rate. We show that a decrease in population growth rates and an increase in life expectancy can explain the decrease in real interest rates in Japan. We also perform counterfactual policy exercises to counteract the decline in real interest rates. Second, we present our analysis on the interaction of the demographic transition and the ELB. Last, we also link fiscal sustainability to our analysis.

3.1 The effects of demographic transition

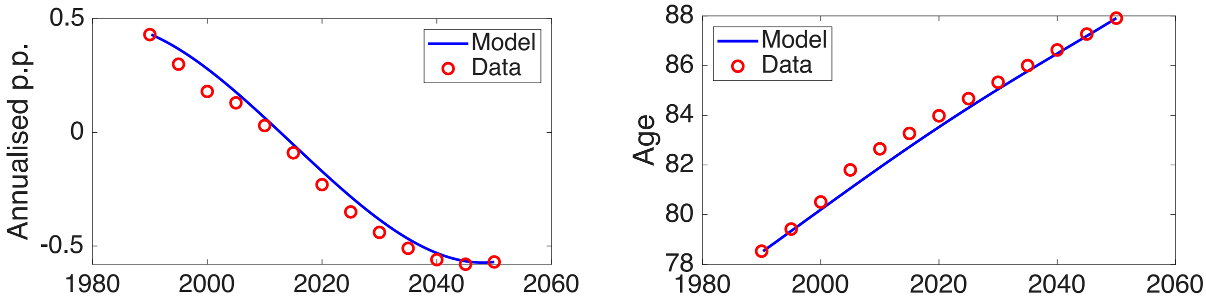
There are four types of demographic shocks that our paper focuses on: A decrease in the population growth rate, an increase in life expectancy, an increase in elder worker productivity, and an increase in the retirement age. Our baseline scenario (Simulation 1) uses the first two shocks and shows that these exert downward pressure on the real interest rate. Then, we add the increase in elder worker productivity and delay in retirement age, first separately (Simulation 2 and 3) and then together (Simulation 4), to show that those alleviate the downward pressure exerted on real interest rates by the demographic processes from the baseline scenario.

For these demographic transition simulations, we assume that $\phi_R = 0$, and we note that since the inflation gap is always closed, the nominal interest rate that the central back sets coincides with the natural interest rate.

Simulation 1: Baseline – A decrease in population growth rate and an increase in life expectancy. We feed in the decrease in population growth rates and an increase in life expectancy using a smoothing process according to data and projections as shown in Figure 2. The population growth rate was about 0.4% in 1990 and is projected to be almost -0.6% in 2050. The life expectancy was around 78 years in 1990 and is projected to be 88 years in 2050.

We simulate the model by feeding these shocks into the model, and plot the time path of key model variables in Figure 3. The decrease in population growth and the increase in life expectancy have the following effect on the economy: Because there are

Figure 2: Fitted shocks to population growth rate and life expectancy



(a) Population growth rates

Source: UN *World Population Prospects: The 2017 Revision*. Data points after 2015 are forecasts.

(b) Life expectancy

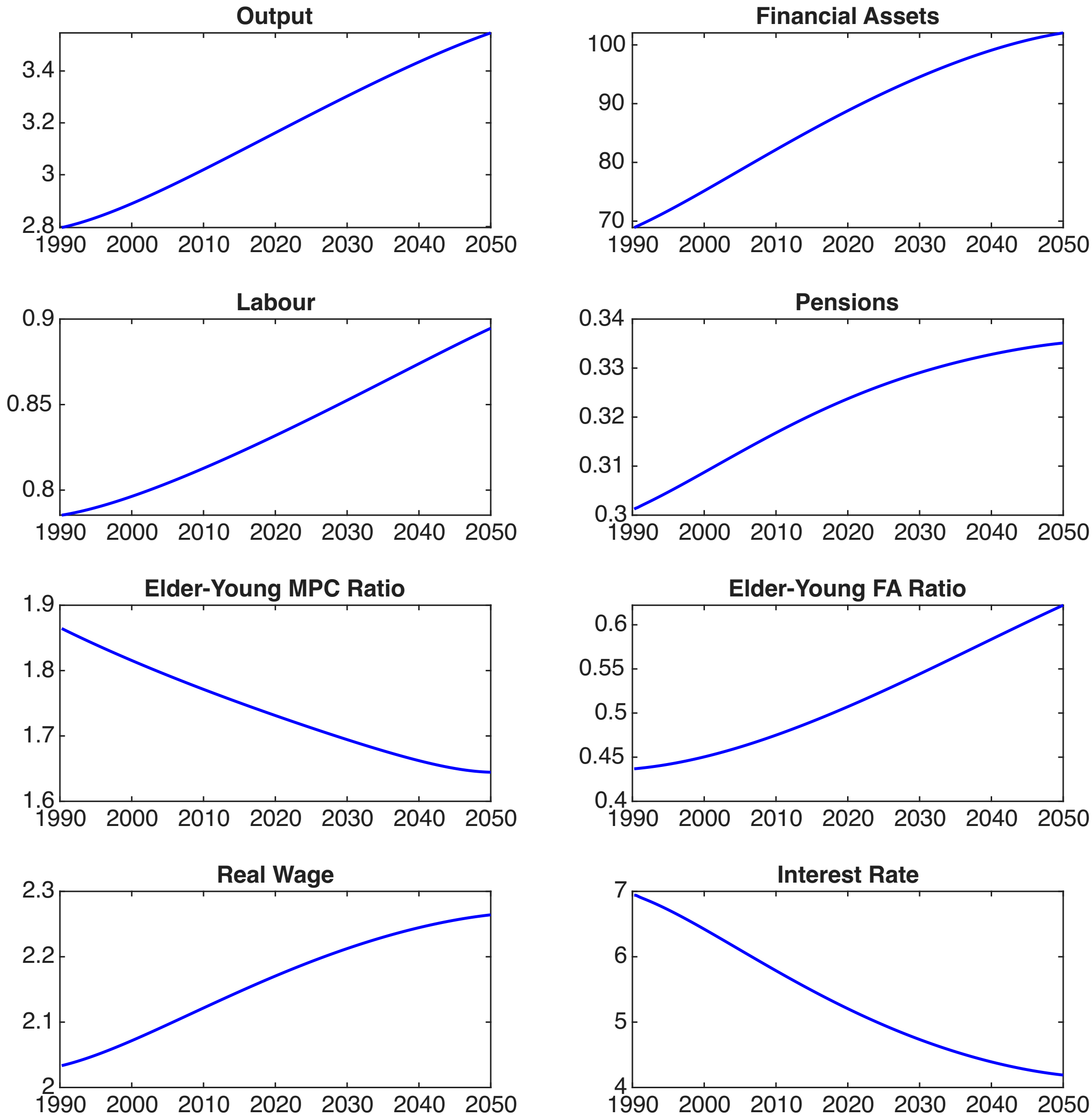
Source: UN *World Population Prospects: The 2017 Revision*. Data points after 2015 are forecasts.

more elder workers in the economy, pensions expenditures increase. Moreover, financial assets increase through higher aggregate savings because agents anticipate living longer and funding consumption during retirement in which their productivity – and thus wage – is relatively low compared to their youth. The financial assets ratio between the elderly and the young increases too: Firstly, there are more elderly workers due to the higher life expectancy, and they also accumulate financial assets. Secondly, the dependency ratio increases (relatively less young people) from the decline in population growth. The economy-wide increase in financial assets decreases (real) interest rates, as we see in the case for Japan. In Figure 4, we show how the model fits the downward trend of the Japanese real interest rates relatively well.

Aggregate labour supply increases as a result of the two shocks, which indicates that the effect of the increase in labour from a higher life expectancy exceeds the effect of a decrease in population growth rate. Agents also work more hours because they engage in precautionary savings for their retirement. Because labour supply and capital accumulation increases, output also increases. The marginal propensity to consume (MPC) is inversely related to the interest rates. So, the decline in interest rates from the demographic trends exerts upward pressure on the MPC. Though, we can see in Figure 3 that this force is weaker for the elder households (hence the elder-young MPC ratio declines), because of the increase in life expectancy and therefore the need to save. Finally, real wages in the economy increase, since the marginal product of labour is higher from the increased capital accumulation.

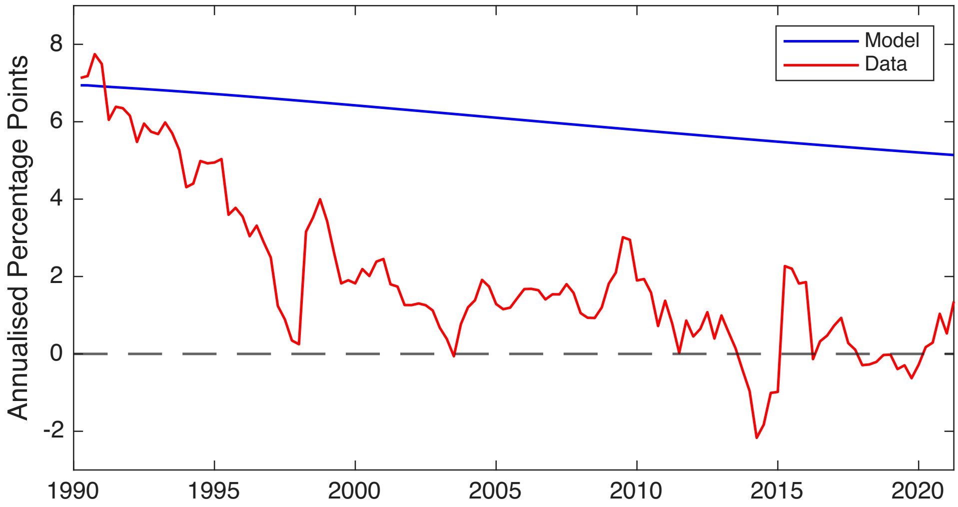
Simulation 2: An increase in elder worker productivity. Now, we add a third demographic shock observed between 1990 and 2017: the relative productivity of elderly workers to young workers (0.8 to 0.9 in the calibration as in Table 1). The time paths for key variables are plotted in Figure 5. The shock's effect on financial assets in the econ-

Figure 3: Simulation 1: Baseline results



Note: Figure plots response of variables as n transitions from 0.0011 to −0.0014 and γ transitions from 0.9815 to 9891. Interest rate is expressed in net annualised percentage points.

Figure 4: Demographics and the real interest rate



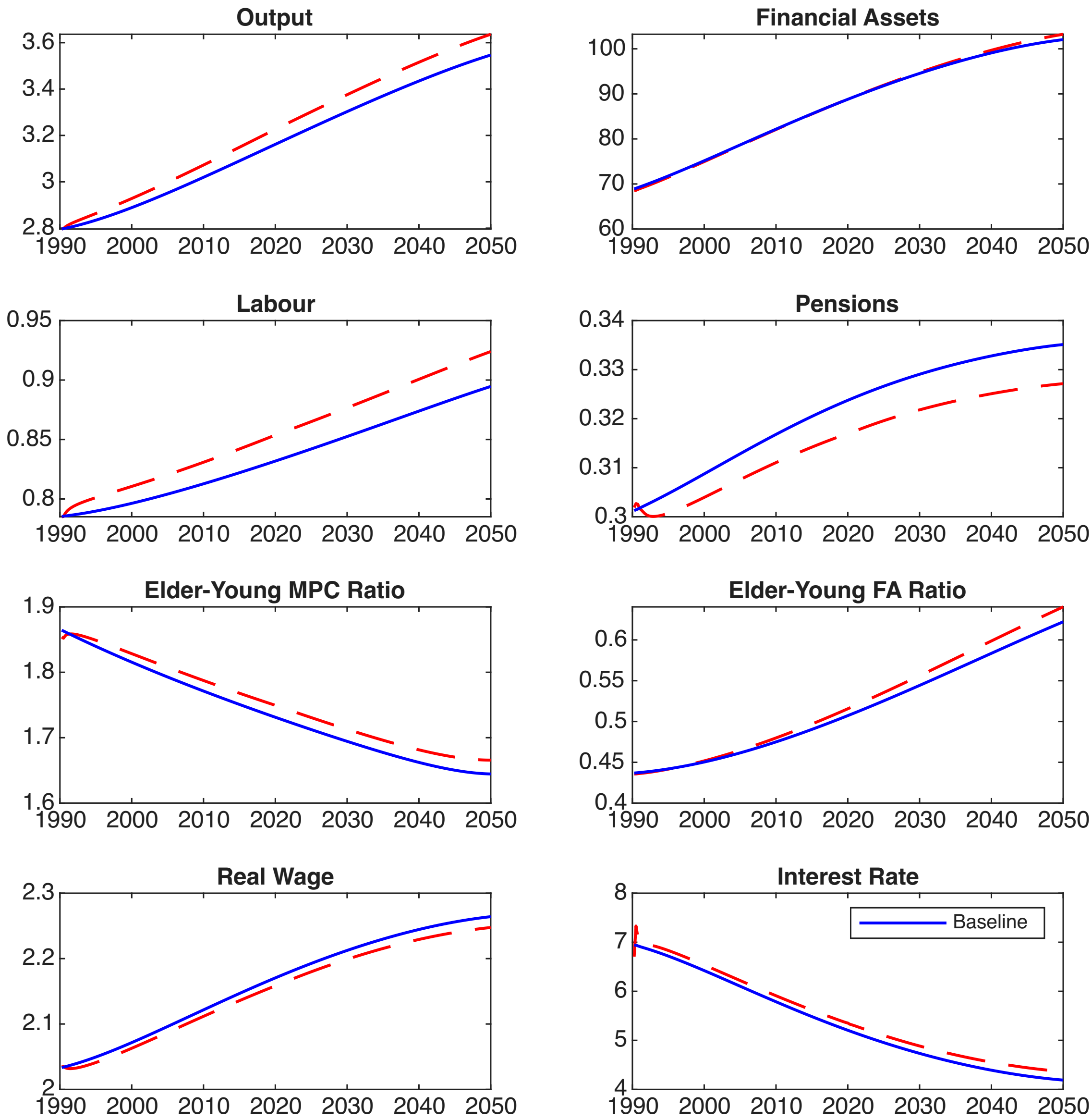
Note: The real interest rate is constructed using the 10-year government bond yield net of annual inflation using the GDP deflator. Data sourced from FRED and the OECD.

omy are small, but important for the effect on the real interest rate. The ratio of financial assets held by the elderly increases as a result of higher productivity for elder workers. There are two channels in place: First, through the income effect of higher real wages, elder workers can accumulate more assets. Second, young workers hold less assets because saving for old age is less rewarding (there is a substitution effect). The young workers holding less assets leads to higher interest rates relative to the baseline case.

Since elderly workers enjoy higher productivity, labour supply increases compared to the baseline case, and thus output expands. The MPC ratio for elderly and young workers is higher than in the baseline case, because the elder workers have higher wage income and are willing to consume more output. However, relative to the baseline simulation slightly higher real interest rate imply that in this simulation there is less capital in the economy, and therefore a lower overall marginal product of labour (despite the increase in productivity for elder workers). Hence, the headline aggregate real wage is lower than the baseline case. Since real wages determine pension payments, the pensions are also lower than the baseline case.

To emphasise the effect on interest rates, a decrease in population growth and an increase in life expectancy (baseline) decrease the real interest rate, as we see in the data. An increase in elder worker productivity puts upward pressure on interest rates, but not nearly enough to counteract the first two effects. As Japan has experienced all three of these shocks in the past few decades, we can deduce that the decrease in population growth and the increase in life expectancy played a bigger role in the determination of real interest rates. However, policy to increase the productivity of elder workers could alleviate the downward secular trend in real interest rates.

Figure 5: Simulation 2: Increase in elder worker productivity



Note: Figure plots response of variables as n transitions from 0.0011 to −0.0014, γ transitions from 0.9815 to 9891, and ς transitions from 0.8 to 0.9. Interest rate is expressed in net annualised percentage points.

Simulation 3: An increase in the retirement age. Here, we introduce a policy counterfactual: An increase in the retirement age from 65 to 70 years of age. However, since we assume that elder households keep working, but with lower productivity, we can also rephrase it as a shock for which workers are productive for longer. Simulation results for the time path of variables are displayed in Figure 6.

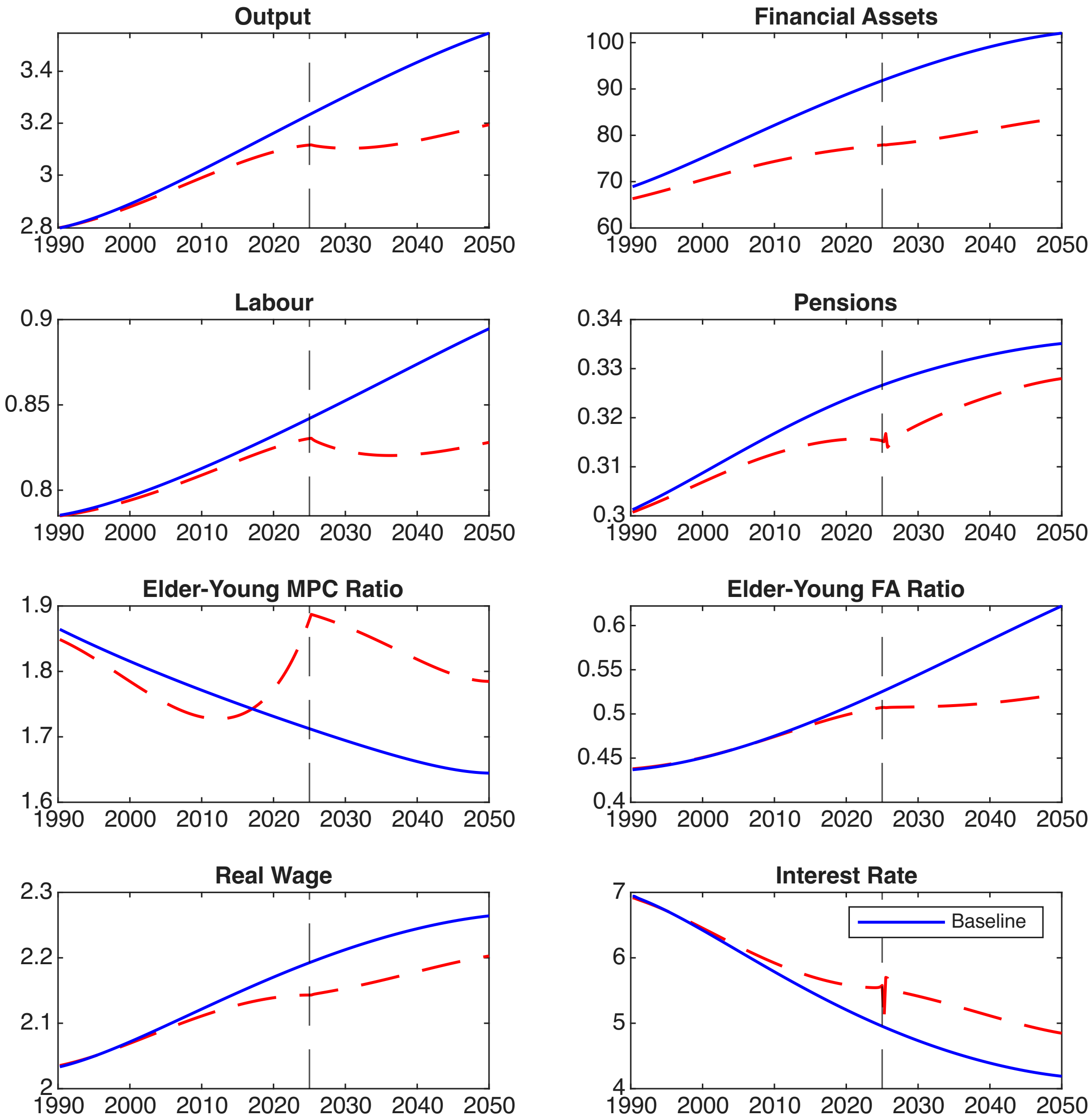
In the simulation, we assume that agents know from the start that the retirement age is going to increase in the year 2025, where the vertical dashed line is. The increase in retirement age is similar to the previous shock, as both increase the productivity of the marginal elder worker. With an increase in the retirement age, the marginal elder worker obtains the same productivity as a young worker. However, with the increase in productivity of the elder worker, all elder workers obtain a higher productivity, but not as high as the young.

With an increase in the retirement age, the increase in financial assets is less than in the baseline case because agents need to save less. This is because, all else being equal, a deferral in the retirement age also coincides to an decline in the survival probability of an elderly worker – in other words, γ contemporaneously declines for a given life expectancy. There is also a lower increase in the financial assets ratio of the elderly because there are less elder workers in the economy compared to the baseline simulation. The smaller asset accumulation through a lower supply of savings results into a higher interest rate and a lower level of capital in the economy. Hence, the marginal productivity of labour is lower, which cause real wages and labour to increase by much less than the baseline scenario. Output follows the labour path closely and also hardly increases after the increase in retirement age. The ratio of the elderly-young MPC is higher after the shock materialises, because agents are willing to increaase their consumption in anticipation of remaining productive for longer during their youth.

All in all, an increase in the retirement age, which makes workers more productive for longer, puts an upward pressure on real interest rates because the level of capital is lower than the baseline case. So, even though a decrease in population growth and an increase in life expectancy (baseline) decreases real interest rates in the economy, making the marginal elder worker more productive alleviates the downward pressure on interest rates, as we saw in the previous exercise. Comparing the two exercises, we can conclude that making the marginal elder worker as productive as the young (increasing retirement age) has a bigger alleviating effect than to make all elder workers more productive (increasing relative elder worker productivity). The difference in capital accumulation between these two scenarios drives these results. With an increase in the retirement age, the financial assets are a lot less than the baseline case, whereas for the increase in elder worker productivity it does not make much of a difference.

Simulation 4: Increase in retirement age and elder worker productivity. In this last exercise, we combine the previous two simulations: as in, elderly worker productivity increase (increase in ς) and the government defers the retirement age to 70 years of

Figure 6: Simulation 3: Increase in retirement age



Note: Figure plots response of variables as n transitions from 0.0011 to -0.0014 and γ transitions from 0.9815 to 9860. ω changes from 0.9944 to 0.9950 in 2025. Interest rate is expressed in net annualised percentage points.

age in the year 2025. Results are shown in Figure 7.

As discussed, the shocks are similar in the sense that they both increase the productivity of the marginal elder worker. Combining the shocks amplifies the effect on real interest rates and alleviates its secular decline. For the rest of the economy, the effect of the increase in retirement age is bigger than the increase in elder worker productivity, which makes sense when looking at the previous two exercises and the time path of model variables.

Thus, a decrease in population growth rate together with an increase in life expectancy (baseline) indeed decreases real interest rates, as observed in Japan in the past few decades. Our results show that the increased productivity of the elderly, marginal or as a whole, most likely did alleviate the decline, and that policy to make elder workers even more productive is beneficial to steeper declines in real interest rates.

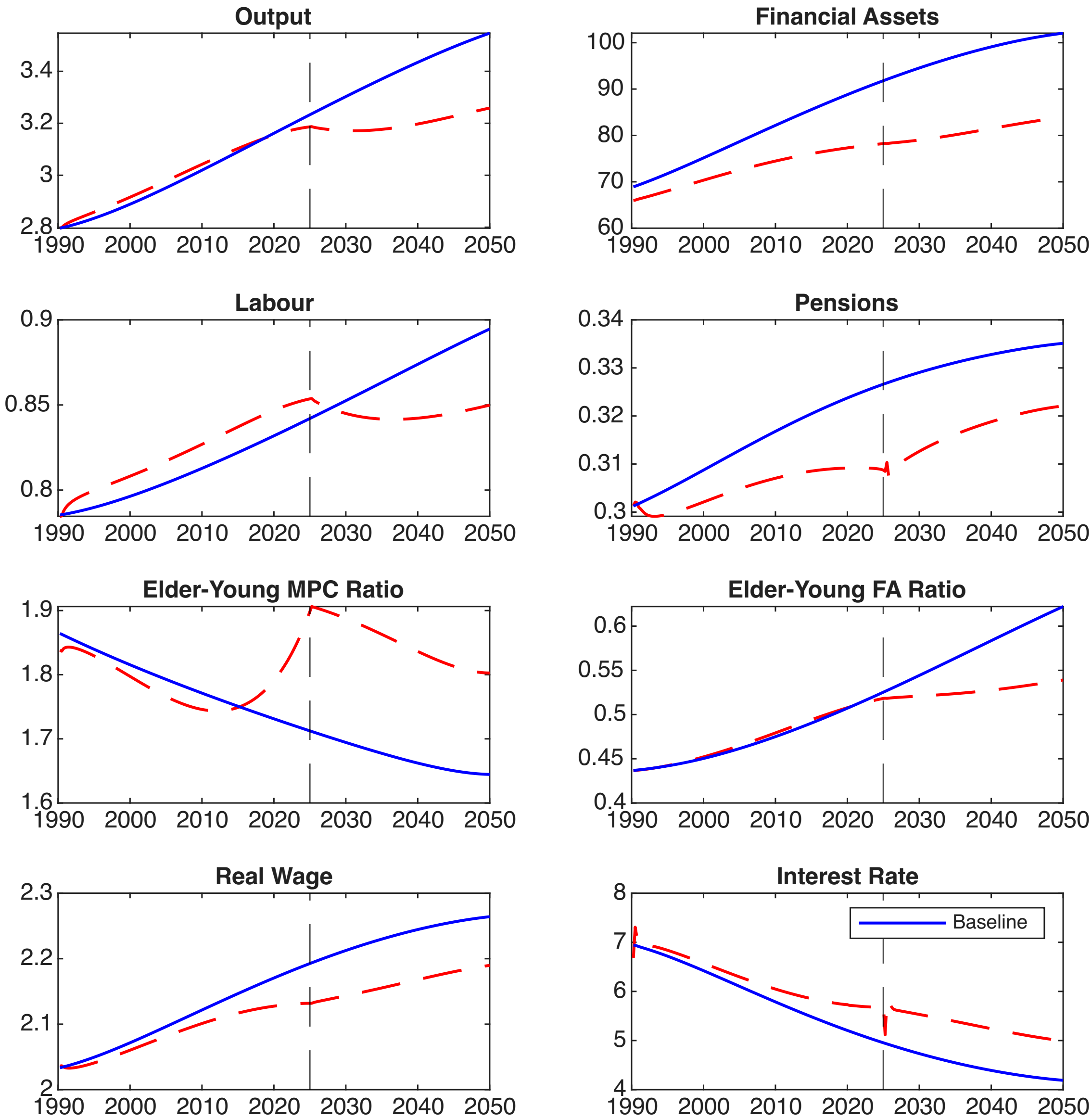
3.2 Demographics and the ELB

(in progress)

3.3 Demographics and fiscal sustainability

(in progress)

Figure 7: Simulation 4: Increase in retirement age and elder worker productivity



Note: Figure plots response of variables as n transitions from 0.0011 to −0.0014, γ transitions from 0.9815 to 9860, and ς_t transitions from 0.8 to 0.9. ω changes from 0.9944 to 0.9950 in 2025. Interest rate is expressed in net annualised percentage points.

4 Conclusion

This paper uses an overlapping generations two-agent New Keynesian (OTANK) life-cycle model based on Gertler (1999), complete with social security and variable labour, and Japanese demographic and wage data to assess the effects of: i) a decrease in the productivity gap between young and old workers; and, ii) deferring the effective age of retirement. We use our OTANK model to conduct policy counterfactuals to investigate the extent to which higher productivity of the marginal elderly worker alleviates the downward pressure on real interest rates due to demographic ageing.

We find that since the collapse of the asset price bubble in the early 1990s, average wage earning profiles across the life-cycle for Japanese workers has flattened. In other words, the wages gap across different aged cohorts since the early 1990s has shrunk. If one assumes that wages accurately reflect productivity, then the flattening of earnings profiles across the life-cycle can be interpreted as a convergence of productivity across different aged workers. Through the lens of our model, the increase in the relative productivity of elderly workers exerts upward pressure on the interest rate. As elderly workers become more productive, the earnings potential of an individual worker undergo a smaller decline when they transition from young to old. As such, the incentive to save during their youth is diminished, decreasing savings and the capital stock, and therefore increasing interest rates. We find similar, and even stronger effects when we increase the retirement age.

Despite the increase in the relative productivity of the elderly between 1990 and 2017, when simulating the model calibrated to Japanese data, this study finds downward pressure on interest rates stemming from the ageing society overpower the upward pressures coming from changes in relative productivity. This is consistent with the real world experience in Japan (Lise et al., 2014) and studies on the secular decline in natural interest rates (Bailey et al., 2022; Cesa-Bianchi, Harrison, and Sajedi, 2023).

The findings in this paper have strong implications for policy makers, particularly in Japan which faces challenging demographic transitions and ballooning public pension expenses. Unsurprisingly, deferring the age of effective retirement (from 65 to 70 in our counterfactual simulations) is only a temporary fix. While it sets an asymptotic old age dependency ratio of 0.40, an improvement over the baseline rate of 0.58, it is relatively ineffective in combating the deflationary pressures discussed here and in other studies such as Carvalho, Ferrero, and Nechio (2016). A policy mix of deferring the effective age of retirement and encouraging higher relative productivity of elderly workers is a first best option. [work in progress: need to add results from ELB and fiscal sustainability sections.]

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A Appendix

A.1 Model solutions

A.1.1 The no-arbitrage condition

An elderly individual wishes to maximize (16) subject to (17). Their problem can be written recursively, where λ_t is the Lagrangian multiplier:

$$V_t^e(j,k) = \max_{C_t^e(j,k), L_t^e(j,k), A_t^r(j,k)} \left\{ \left(C_t^e(j,k)^v \left[1 - L_t^e(j,k)\right]^{1-v} \right)^\rho + \beta \gamma_{t+1} V_{t+1}^e(j,k)^\rho \right\}^{\frac{1}{\rho}}$$
$$+ \lambda_t \left\{ \begin{array}{l} \frac{1}{\gamma_t} \left[\frac{R_{t-1}}{\pi_t} \frac{B_{t-1}^e(j,k)}{P_{t-1}} + (P_t^I + D_t^I) x_{t-1}^I(j,k) \right] + \varsigma_t w_t L_t^e(j,k) \\ + E_t^e(j,k) - C_t^e(j,k) - \frac{B_t^e(j,k)}{P_t} - P_t^I x_t^I(j,k) - \mathcal{T}_t \end{array} \right\}.$$

The first-order conditions with respect to real government bonds and intermediate firm shares are:

$$\frac{\partial V_t^e(j,k)}{\partial (B_t^e(j,k)/P_t)} = \beta \frac{\lambda_{t+1}}{\gamma_{t+1}} \frac{R_t}{\pi_{t+1}} - \lambda_t = 0,$$
$$\frac{\partial V_t^e(j,k)}{\partial x_t^I(j,k)} = \beta \frac{\lambda_{t+1}}{\gamma_{t+1}} \left(P_{t+1}^I + D_{t+1}^I \right) - \lambda_t P_t^I = 0.$$

Note that these first-order conditions are not indexed per individual. After some simple algebraic manipulation and setting these two first-order conditions equal to one another we get:

$$\frac{R_t}{\pi_{t+1}} = \frac{P_{t+1}^I + D_{t+1}^I}{P_t^I}, \tag{66}$$

which is the no-arbitrage condition (19).

A.1.2 The elder worker problem

Using the definition of financial assets, A_t, an elderly individual wishes to maximize (16) subject to (21). Their problem can be written recursively as:

$$V_t^e(j,k) = \max_{C_t^e(j,k), L_t^e(j,k), A_t^r(j,k)} \left\{ \left(C_t^e(j,k)^v \left[1 - L_t^e(j,k)\right]^{1-v} \right)^\rho + \beta \gamma_{t+1} V_{t+1}^e(j,k)^\rho \right\}^{\frac{1}{\rho}}$$
$$+ \lambda_t \left\{ \frac{R_{t-1}}{\gamma_t \pi_t} A_{t-1}^e(j,k) + \varsigma_t w_t L_t^e(j,k) + E_t^e(j,k) - C_t^e(j,k) - A_t^e(j,k) \right\}.$$

The first-order conditions with respect to consumption, labour, and financial assets are:

$$\frac{\partial V_t^e(j,k)}{\partial C_t^e(j,k)} = \frac{1}{\rho} \underbrace{\left\{ \left(C_t^e(j,k)^{\upsilon} \left[1 - L_t^e(j,k)\right]^{1-\upsilon} \right)^{\rho} + \beta \gamma_{t+1} V_{t+1}^e(j,k)^{\rho} \right\}^{\frac{1}{\rho}-1}}_{V_t^e(j,k)^{1-\rho}}$$

$$\times \rho \left(C_t^e(j,k)^{\upsilon} \left[1 - L_t^e(j,k)\right]^{1-\upsilon} \right)^{\rho-1} \upsilon C_t^e(j,k)^{\upsilon-1} \left[1 - L_t^e(j,k)\right]^{1-\upsilon}$$

$$- \lambda_t = 0$$

$$\Leftrightarrow \lambda_t = \upsilon V_t^e(j,k)^{1-\rho} C_t^e(j,k)^{\upsilon\rho-1} \left[1 - L_t^e(j,k)\right]^{\rho(1-\upsilon)}, \tag{67}$$

$$\frac{\partial V_t^e(j,k)}{\partial L_t^e(j,k)} = \frac{1}{\rho} V_t^e(j,k)^{1-\rho} \rho \left(C_t^e(j,k)^{\upsilon} \left[1 - L_t^e(j,k)\right]^{1-\upsilon} \right)^{\rho-1}$$

$$\times (1-\upsilon) C_t^e(j,k)^{\upsilon} \left[1 - L_t^e(j,k)\right]^{1-\upsilon-1} (-1) + \lambda_t \varsigma_t w_t = 0$$

$$\Leftrightarrow \lambda_t \varsigma_t w_t = (1-\upsilon) V_t^e(j,k)^{1-\rho} C_t^e(j,k)^{\upsilon\rho} \left[1 - L_t^e(j,k)\right]^{\rho(1-\upsilon)-1}, \tag{68}$$

$$\frac{\partial V_t^e(j,k)}{\partial A_t^e(j,k)} = \frac{1}{\rho} V_t^e(j,k)^{1-\rho} \rho \beta \gamma_{t+1} V_{t+1}^e(j,k)^{\rho-1} \left[\frac{\partial V_{t+1}^e(j,k)}{\partial A_t^e(j,k)} \right] - \lambda_t$$

$$\Leftrightarrow \lambda_t = V_t^e(j,k)^{1-\rho} \beta \gamma_{t+1} V_{t+1}^e(j,k)^{\rho-1} \left[\frac{\partial V_{t+1}^e(j,k)}{\partial A_t^e(j,k)} \right]. \tag{69}$$

Combine Equations (67) and (68) to get the intratemporal Euler equation (22) in the text:

$$1 - L_t^e(j,k) = \frac{1-\upsilon}{\upsilon} \frac{C_t^e(j,k)}{\varsigma_t w_t}.$$

Then use (67) and (69) to get:

$$\upsilon V_t^e(j,k)^{1-\rho} C_t^e(j,k)^{\upsilon\rho-1} (1 - L_t^e(j,k))^{\rho(1-\upsilon)}$$
$$= V_t^e(j,k)^{1-\rho} \beta \gamma_{t+1} V_{t+1}^e(j,k)^{\rho-1} \left[\frac{\partial V_{t+1}^e(j,k)}{\partial A_t^e(j,k)} \right] \tag{70}$$

By the Envelope Theorem we have:

$$\frac{\partial V_t^e(j,k)}{\partial A_{t-1}^e(j,k)} = \frac{\lambda_t}{\gamma_t} \frac{R_{t-1}}{\pi_t},$$

and then use (67) to write:

$$\frac{\partial V_t^e(j,k)}{\partial A_{t-1}^e(j,k)} = \upsilon V_t^e(j,k)^{1-\rho} C_t^e(j,k)^{\upsilon\rho-1} \left[1 - L_t^e(j,k)\right]^{\rho(1-\upsilon)} \frac{R_{t-1}}{\gamma_t \pi_t}$$

and roll forward by one period to get:

$$\frac{\partial V_{t+1}^e(j,k)}{\partial A_t^e(j,k)} = \upsilon V_{t+1}^e(j,k)^{1-\rho} C_{t+1}^e(j,k)^{\upsilon\rho-1} \left[1 - L_{t+1}^e(j,k)\right]^{\rho(1-\upsilon)} \frac{R_t}{\gamma_{t+1} \pi_{t+1}}.$$

Substitute this value back into (70) and do some simplification to yield an intertemporal consumption Euler equation:

$$1 = \beta \frac{R_t}{\pi_{t+1}} \left[\frac{C^e_{t+1}(j,k)}{C^e_t(j,k)} \right]^{\upsilon\rho - 1} \left[\frac{1 - L^e_{t+1}(j,k)}{1 - L^e_t(j,k)} \right]^{\rho(1-\upsilon)}. \tag{71}$$

Then use (22) to get:

$$C^e_{t+1}(j,k) = \left(\beta \frac{R_t}{\pi_{t+1}} \right)^{\sigma} \left(\frac{w_t}{w_{t+1}} \right)^{\rho(1-\upsilon)\sigma} C^e_t(j,k),$$

which is Equation (23) in the text and where $\sigma = 1/(1-\rho)$.

To get the law of motion for the MPC of the elderly, start by substituting the guessed consumption function (26) into the intertemporal Euler equation (23):

$$\begin{aligned} &\xi^e_{t+1} \left[\frac{R_t}{\gamma_{t+1}\pi_{t+1}} A^e_t(j,k) + H^e_{t+1}(j,k) + S^e_{t+1}(j,k) \right] \\ &= \left(\beta \frac{R_t}{\pi_{t+1}} \right)^{\sigma} \left(\frac{w_t}{w_{t+1}} \right)^{\rho(1-\upsilon)\sigma} \xi^e_t \left[\frac{R_{t-1}}{\gamma_t \pi_t} A^e_{t-1}(j,k) + H^e_t(j,k) + S^e_t(j,k) \right], \end{aligned} \tag{72}$$

Then substitute the guessed consumption function (26) into the budget constraint (21) to get an expression for the dynamics of financial assets of a retiree:

$$A^e_t(j,k) = (1 - \xi^e_t) \left[\frac{R_{t-1}}{\gamma_t \pi_t} A^e_{t-1}(j,k) + H^e_t(j,k) + S^e_t(j,k) - \mathcal{T}_t \right]. \tag{73}$$

Use this expression, and Equations (24) and (25), and substitute into (72), and then rearrange to write:

$$\begin{aligned} \xi^e_{t+1} \frac{A^e_{t+1}(j,k)}{1 - \xi^e_{t+1}} &= \left(\frac{\beta R_t}{\pi_{t+1}} \right)^{\sigma} \left(\frac{w_t}{w_{t+1}} \right)^{\rho(1-\upsilon)\sigma} \xi^e_t \frac{A_t}{1 - \xi^e_t} \\ \Leftrightarrow \frac{1}{\xi^e_t} &= 1 + \left(\frac{\beta R_t}{\pi_{t+1}} \right)^{\sigma} \left(\frac{w_t}{w_{t+1}} \right)^{\rho(1-\upsilon)\sigma} \frac{1}{\xi^e_{t+1}} \\ &\quad \times \frac{A^e_t(j,k)}{\frac{R_t}{\gamma_{t+1}\pi_{t+1}} A^e_t(j,k) + H^e_{t+1}(j,k) + S^e_{t+1}(j,k)} \\ \Leftrightarrow \frac{1}{\xi^e_t} &= 1 + \gamma_{t+1} \beta^{\sigma} \left(\frac{R_t}{\Pi_{t+1}} \right)^{\sigma - 1} \left(\frac{w_t}{w_{t+1}} \right)^{\rho(1-\upsilon)\sigma} \frac{1}{\xi^e_{t+1}} \\ &\quad \times \frac{A^e_t(j,k)}{A^e_t(j,k) + H^e_t(j,k) - \varsigma_t w_t L^e_t(j,k) + S^e_t(j,k) - E^e_t(j,k)}, \end{aligned}$$

and since

$$\varsigma_t w_t L^e_t(j,k) + E^e_t(j,k) = C^e_t(j,k) + A^e_t(j,k) - \frac{R_{t-1}}{\gamma_t \pi_t} A^e_{t-1}(j,k),$$

the denominator of the last term in the $1/\xi^e_t$ expression can be written as:

$$H^e_t(j,k) + S^e_t(j,k) - C^e_t(j,k) + \frac{R_{t-1}}{\gamma_t \pi_t} A^e_{t-1}(j,k).$$

Then use (26) to write this expression as:

$$\frac{R_{t-1}}{\gamma_t \pi_t} A^e_{t-1}(j,k) + H^e_t(j,k) + S^e_t(j,k) - \xi^e_t \left[\frac{R_{t-1}}{\gamma_t \pi_t} A^e_{t-1}(j,k) + H^e_t(j,k) + S^e_t(j,k) \right],$$

and then following (73) the above expression is merely $A^e_t(j,k)$, which allows us to write:

$$\frac{1}{\xi^e_t} = 1 + \gamma_{t+1}\beta^\sigma \left(\frac{R_t}{\pi_{t+1}}\right)^{\sigma-1} \left(\frac{w_t}{w_{t+1}}\right)^{\rho(1-\upsilon)\sigma} \frac{1}{\xi^e_{t+1}},$$

which is Equation (27) in the text.

Next, guess that the value function is linear in consumption and leisure:

$$\begin{aligned} V^e_t(j,k) &= \Lambda^e_t C^e_t(j,k)^\upsilon (1 - L^e_t(j,k))^{1-\upsilon} \\ &= \Lambda^e_t C^e_t(j,k) \left(\frac{1-\upsilon}{\upsilon}\frac{1}{\varsigma_t w_t}\right)^{1-\upsilon}, \end{aligned} \tag{74}$$

where we used (22) for the second line. From the value function (16), we can then write:

$$\begin{aligned} \left[\Lambda^e_t C^e_t(j,k)\left(\frac{1}{\varsigma_t w_t}\right)^{1-\upsilon}\right]^\rho &= \left[C^e_t(j,k)\left(\frac{1}{\varsigma_t w_t}\right)^{1-\upsilon}\right]^\rho \\ &\quad + \beta\gamma_{t+1}\left[\Lambda^e_{t+1} C^e_{t+1}(j,k)\left(\frac{1}{\varsigma_t w_{t+1}}\right)^{1-\upsilon}\right]^\rho. \end{aligned}$$

Use the expression for C_{t+1} from the intertemporal consumption Euler equation (71) and then simplify to get:

$$(\Lambda^e_t)^\rho = 1 + \beta^\sigma \gamma_{t+1} \left(\frac{R_t}{\pi_{t+1}}\right)^{\sigma-1} \left(\frac{w_t}{w_{t+1}}\right)^{\rho(1-\upsilon)\sigma} (\Lambda^e_{t+1})^\rho.$$

From (27) we can then deduce that:

$$\Lambda^e_t = (\xi^e_t)^{\frac{\sigma}{1-\sigma}}, \tag{75}$$

and hence we can get Equation (28).

A.1.3 The young worker problem

The derivation for the young worker's problem closely follows that of the elder worker in Section A.1.2. A young worker wishes to maximize (29) subject to (31):

$$\begin{aligned} V^y_t(j) = \max_{C^y_t(j), L^y_t(j), A^y_t(j)} &\left\{ \begin{matrix} \left(C^y_t(j)^\upsilon (1 - L^y_t(j))^{1-\upsilon}\right)^\rho \\ +\beta[\omega_{t+1} V^y_{t+1}(j) + (1-\omega_{t+1}) V^e_{t+1}(j, t+1)]^\rho \end{matrix} \right\}^{\frac{1}{\rho}} \\ &+ \lambda_t \left\{ \frac{R_{t-1}}{\pi_t} A^y_{t-1}(j) + w_t L^y_t(j) - C^y_t(j) - A^y_t(j) - T^y_t(j) \right\}. \end{aligned}$$

The first-order conditions with respect to consumption, labour, and financial assets are:

$$\frac{\partial V_t^y(j)}{\partial C_t^y(j)} = \upsilon V_t^y(j)^{1-\rho} C_t^y(j)^{\upsilon\rho-1} \left[1 - L_t^y(j)\right]^{\rho(1-\upsilon)} - \lambda_t = 0, \tag{76}$$

$$\frac{\partial V_t^y(j)}{\partial L_t^y(j)} = -(1-\upsilon) V_t^y(j)^{1-\rho} C_t^y(j)^{\upsilon\rho} (1 - L_t^y(j))^{\rho(1-\upsilon)-1} + \lambda_t w_t = 0, \tag{77}$$

$$\begin{aligned}\frac{\partial V_t^y(j)}{\partial A_t^y(j)} &= V_t^y(j)^{1-\rho} \beta \left[\omega_{t+1} V_{t+1}^y(j) + (1-\omega_{t+1}) V_{t+1}^e(j, t+1)\right]^{\rho-1} \\ &\times \left[\omega_{t+1} \frac{\partial V_{t+1}^y(j)}{\partial A_t^y(j)} + (1-\omega_{t+1}) \frac{\partial V_{t+1}^e(j,t+1)}{\partial A_t^y(j)}\right] - \lambda_t = 0. \end{aligned} \tag{78}$$

Combine (76) and (77) to get the intratemporal Euler equation:

$$1 - L_t^y(j) = \frac{1-\upsilon}{\upsilon} \frac{C_t^y(j)}{w_t},$$

which is Equation (32) in the text.

Then use (76) and (78) to write:

$$\begin{aligned}\upsilon V_t^y(j)^{1-\rho} C_t^y(j)^{\upsilon\rho-1} (1 - L_t^y(j))^{\rho(1-\upsilon} &= V_t^y(j)^{1-\rho} \beta \left[\omega_{t+1} V_{t+1}^y(j) + (1-\omega_{t+1}) V_{t+1}^e(j,t+1)\right]^{\rho-1} \\ &\times \left[\omega_{t+1} \frac{\partial V_{t+1}^y(j)}{\partial A_t^y(j)} + (1-\omega_{t+1}) \frac{\partial V_{t+1}^e(j,t+1)}{\partial A_t^y(j)}\right].\end{aligned}$$

The envelope conditions are:

$$\frac{\partial V_{t+1}^y(j)}{\partial A_t^y(j)} = \upsilon V_{t+1}^y(j)^{1-\rho} C_{t+1}^y(j)^{\upsilon\rho-1} (1 - L_{t+1}^y(j))^{\rho(1-\upsilon)} \frac{R_t}{\pi_{t+1}},$$

$$\frac{\partial V_{t+1}^e(j,t+1)}{\partial A_t^y(j)} = \upsilon V_{t+1}^e(j,t+1)^{1-\rho} C_{t+1}^e(j,t+1)^{\upsilon\rho-1} (1 - L_{t+1}^e(j,t+1))^{\rho(1-\upsilon)} \frac{R_t}{\pi_{t+1}}.$$

Combining the above envelope conditions with (76) and (78) yields the following:

$$C_t^y(j)^{\upsilon\rho-1} \left[1 - L_t^y(j)\right]^{\rho(1-\upsilon)} = \frac{\beta R_t}{\pi_{t+1}} \left\{ \begin{array}{c} \left[\omega_{t+1} V_{t+1}^y(j) + (1-\omega_{t+1}) V_{t+1}^e(j,t+1)\right]^{\rho-1} \\ \times \omega_{t+1} V_{t+1}^y(j)^{1-\rho} C_{t+1}^y(j)^{\upsilon\rho-1} \left[1 - L_{t+1}^y(j)\right]^{\rho(1-\upsilon)} \\ + \left[\omega_{t+1} V_{t+1}^y(j) + (1-\omega_{t+1}) V_{t+1}^e(j,t+1)\right]^{\rho-1} \\ \times (1-\omega_{t+1}) V_{t+1}^e(j,t+1)^{1-\rho} \\ \times C_{t+1}^e(j,t+1)^{\upsilon\rho-1} \left[1 - L_{t+1}^e(j,t+1)\right]^{\rho(1-\upsilon)} \end{array} \right\}.$$

Then, based on (74), (22), and (32), conjecture that the value functions are linear in consumption and leisure:

$$V_t^y(j) = \Lambda_t^y C_t^y(j) \left(\frac{1-\upsilon}{\upsilon} \frac{1}{w_t}\right)^{1-\upsilon},$$

$$V_t^e(j,k) = \Lambda_t^e C_t^e(j,k) \left(\frac{1-\upsilon}{\upsilon} \frac{1}{\varsigma_t w_t}\right)^{1-\upsilon},$$

and then make the appropriate substitutions and simplify to obtain an expression for the intertemporal Euler equation:

$$C_t^y(j)\left[\frac{\beta R_t \Omega_{t+1}}{\pi_{t+1}}\left(\frac{w_t}{w_{t+1}}\right)^{\rho(1-\upsilon)}\right]^{\sigma} = \omega_{t+1} C_{t+1}^y(j) + (1-\omega_{t+1})\frac{\Lambda_{t+1}^e}{\Lambda_{t+1}^y} C_{t+1}^e(j,t+1)\left(\frac{1}{\varsigma_t}\right)^{1-\upsilon}, \tag{79}$$

where the adjustment term, Ω_t, is defined as:

$$\Omega_t = \omega_t + (1-\omega_t)\left(\frac{\Lambda_t^e}{\Lambda_t^y}\right)^{1-\rho}\left(\frac{1}{\varsigma_t}\right)^{1-\upsilon}.$$

Use the conjectured consumption functions, (26) and (39), and substitute these into the consumption Euler equation. Note that we use the fact that an elderly worker born in period j, who just retired at the start of period t, has the following consumption function:

$$C_t^e(j,t) = \xi_t^e\left[\frac{R_{t-1}}{\pi_t} A_{t-1}^y(j,t) + H_t^e(j,t) + S_t^e(j,t)\right].$$

After algebraic rearranging we can write the consumption Euler equation as:

$$\begin{aligned}&\omega_{t+1}\left(A_t^y(j) + \frac{H_{t+1}^y(j) + S_{t+1}^y(j)}{R_t/\pi_{t+1}}\right)\\ &\quad + (1-\omega_{t+1})\left(\frac{\Lambda_{t+1}^e}{\Lambda_{t+1}^y}\right)\Xi_{t+1}\left(A_t^y(j) + \frac{H_{t+1}^e(j,t+1) + S_{t+1}^e(j,t+1)}{R_t/\pi_{t+1}}\right)\left(\frac{1}{\varsigma_t}\right)^{1-\upsilon}\\ &\quad = \frac{\xi_t^y}{\xi_{t+1}^y}\left(\frac{R_{t-1}}{\pi_t} A_{t-1}^y(j) + H_t^y(j) + S_t^y(j)\right)\left(\frac{R_t}{\pi_{t+1}}\right)^{\sigma-1}\left[\beta\Omega_{t+1}\left(\frac{w_t}{w_{t+1}}\right)^{\rho(1-\upsilon)}\right]^{\sigma},\end{aligned}$$

where $\Xi_t = \xi_t^e/\xi_t^y$. Using the definition of Ω_t we can simplify the above expression as:

$$\begin{aligned}&A_t^y(j) + \omega_{t+1}\frac{H_{t+1}^y(j) + S_{t+1}^y(j)}{\Omega_{t+1} R_t/\Pi_{t+1}}\\ &\quad + (1-\omega_{t+1})\left(\frac{\Lambda_{t+1}^e}{\Lambda_{t+1}^y}\right)^{1-\rho}\left(\frac{1}{\varsigma_t}\right)^{1-\upsilon}\frac{H_{t+1}^e(j,t+1) + S_{t+1}^e(j,t+1)}{\Omega_{t+1} R_t/\pi_{t+1}}\\ &\quad = \frac{\xi_t^y}{\xi_{t+1}^y}\left(\frac{R_{t-1}}{\pi_t} A_{t-1}^y(j) + H_t^y(j) + S_t^y(j)\right)\beta^{\sigma}\left(\frac{R_t\Omega_{t+1}}{\pi_{t+1}}\right)^{\sigma-1}\left(\frac{w_t}{w_{t+1}}\right)^{\rho(1-\upsilon)\sigma},\end{aligned} \tag{80}$$

if $\Xi_{t+1} = (\Lambda_{t+1}^e/\Lambda_{t+1}^y)^{-\rho}$. From (75) we have that $\xi_{t+1}^e = (\Lambda_{t+1}^e)^{-\rho}$.

It remains that we need to verify $\xi_{t+1}^y = (\Lambda_{t+1}^y)^{-\rho}$. Begin by using the budget constraint, (31), and the guessed consumption function of a young worker, (39), to write:

$$\xi_t^y\left[\frac{R_{t-1}}{\pi_t} A_{t-1}^y(j) + H_t^y(j) + S_t^y(j)\right] + A_t^y(j) + T_t^y(j) = \frac{R_{t-1}}{\pi_t} A_{t-1}^y(j) + w_t L_t^y(j).$$

Then use the definitions for the present values of a young worker's non-financial assets and social security, (37) and (38), to write the above expression as:

$$A_t^y(j) + \omega_{t+1}\frac{H_{t+1}^y(j) + S_{t+1}^y(j)}{\Omega_{t+1}R_t/\Pi_{t+1}} + (1-\omega_{t+1})\left(\frac{\Lambda_{t+1}^e}{\Lambda_{t+1}^y}\right)^{1-\rho}\left(\frac{1}{\varsigma_t}\right)^{1-\upsilon}\frac{H_{t+1}^e(j,t+1) + S_{t+1}^e(j,t+1)}{\Omega_{t+1}R_t/\pi_{t+1}} = (1-\xi_t^y)\left(\frac{R_{t-1}}{\pi_t}A_{t-1}^y + H_t^y(j) + S_t^y(j)\right). \tag{81}$$

which is the law of motion of assets for a young worker. Substitute this expression into the intertemporal Euler equation, (80), to then get the MPC of young workers:

$$\frac{1}{\xi_t^y} = 1 + \beta^\sigma\left(\frac{R_t\Omega_{t+1}}{\Pi_{t+1}}\right)^{\sigma-1}\left(\frac{w_t}{w_{t+1}}\right)^{\rho(1-\upsilon)\sigma}\frac{1}{\xi_{t+1}^y}.$$

Then check the validity of the value function, (33), by writing:

$$\left[\Lambda_t^y C_t^y(j)\left(\frac{1-\upsilon}{\upsilon}\frac{1}{w_t}\right)^{1-\upsilon}\right]^\rho = \left[C_t^y(j)\left(\frac{1-\upsilon}{\upsilon}\frac{1}{w_t}\right)^{1-\upsilon}\right]^\rho + \beta\left[\omega_{t+1}\Lambda_{t+1}^y C_{t+1}^y(j)\left(\frac{1-\upsilon}{\upsilon}\frac{1}{w_{t+1}}\right)^{1-\upsilon} + (1-\omega_{t+1})\Lambda_{t+1}^e C_{t+1}^e(j,t+1)\left(\frac{1-\upsilon}{\upsilon}\frac{1}{\varsigma_t w_{t+1}}\right)^{1-\upsilon}\right]^\rho.$$

Combine this expression with the intertemporal Euler equation, (79), to yield:

$$(\Lambda_t^y)^\rho = 1 + \beta^\sigma\left(\frac{R_t\Omega_{t+1}}{\pi_{t+1}}\right)^{\sigma-1}\left(\frac{w_t}{w_{t+1}}\right)^{\rho(1-\upsilon)\sigma}(\Lambda_{t+1}^y)^\rho.$$

This expression, as in the case of elderly workers, implies that:

$$\Lambda_t^y = (\xi_t^y)^{\frac{\sigma}{1-\sigma}}. \tag{82}$$

This concludes our verification of Equation (80), and we can also write Ω_t as:

$$\Omega_t = \omega_t + (1-\omega_t)\Xi_t^{\frac{1}{1-\sigma}}\left(\frac{1}{\varsigma_t}\right)^{1-\upsilon}.$$

B Equilibrium conditions

The competitive equilibrium is a sequence of 13 aggregate quantities $\{Y_t, C_t, I_t, K_t, L_t, D_t^I, H_t, A_t, B_t, G_t, E_t, T_t\}$; eight prices $\{w_t, r_t^K, P_t^I, \varphi_t, R_t, \pi_t, R_t^K, Q_t\}$; nine adjustment factors $\{n_t, \gamma_t, \omega_t, \Gamma_t, \Psi_t, \varsigma_t, \Xi_t, \Omega_t, \varrho_t\}$; seven variables for elderly workers $\{\xi_t^e, C_t^e, A_t^e, S_t^e, L_t^e, H_t^e, E_t^e\}$; and seven variables for young workers $\{\xi_t^y, C_t^y, A_t^y, S_t^y, L_t^y, H_t^y, T_t^y\}$. We specify the equilibrium conditions below.

Additionally, as the model features trend technology and population growth, we detrend endogenous variables as follows. The variables $Y, C, I, K, P^I, D^I, H, A, B, G, E$, and T are detrended by XN; L by N; and w by X.

Households. Law of motion for dependency ratio:

$$(1+n_t)\Gamma_t = (1-\omega_t) + \gamma_t \Gamma_{t-1}. \tag{B1}$$

Distribution of wealth:

$$\begin{aligned}&[\Psi_t - (1-\omega_{t+1})]\, a_t = \\ &\omega_{t+1}\left[(1-\xi_t^e)\frac{R_{t-1}}{\pi_t}\frac{\Psi_{t-1}a_{t-1}}{(1+n_t)(1+x_t)} + \varsigma_t w_t l_t^e + e_t^e - \xi_t^e\left(h_t^e + s_t^e\right)\right].\end{aligned} \tag{B2}$$

Ratio of MPCs:

$$\Xi_t = \frac{\xi_t^e}{\xi_t^y}. \tag{B3}$$

Young worker adjustment factor:

$$\Omega_t = \omega_t + (1-\omega_t)\Xi_t^{\frac{1}{1-\sigma}}\left(\frac{1}{\varsigma_t}\right)^{1-\upsilon}. \tag{B4}$$

Elder worker MPC:

$$\frac{1}{\xi_t^e} = 1 + \gamma_{t+1}\beta^\sigma\left(\frac{R_t}{\pi_{t+1}}\right)^{\sigma-1}\left[\frac{\varsigma_t w_t}{(1+x_t)\varsigma_{t+1}w_{t+1}}\right]^{\rho(1-\upsilon)\sigma}\frac{1}{\xi_{t+1}^e}, \tag{B5}$$

Elder worker consumption:

$$c_t^e = \xi_t^e\left[\frac{R_{t-1}}{\pi_t}\frac{a_{t-1}^e}{\gamma_t(1+n_t)(1+x_t)} + h_t^e + s_t^e\right]. \tag{B6}$$

Elder worker asset proportion:

$$\Psi_t = \frac{a_t^e}{a_t}. \tag{B7}$$

Elder worker pension receipts:

$$s_t^e = e_t^e + \frac{\pi_{t+1}}{R_t}\gamma_{t+1}s_{t+1}^e. \tag{B8}$$

Elder worker capitalised human wealth:

$$h_t^e = \varsigma_t w_t l_t^e + (1+n_t)(1+x_t)\frac{\pi_{t+1}}{R_t}\gamma_{t+1}h_{t+1}^e. \tag{B9}$$

Elder worker labour supply:

$$l_t^e = \Gamma_t - \frac{1-\upsilon}{\upsilon}\frac{1}{\varsigma_t w_t}c_t^e. \tag{B10}$$

Young worker MPC:

$$\frac{1}{\xi_t^y} = 1 + \beta^\sigma \left(\frac{R_t\Omega_{t+1}}{\pi_{t+1}}\right)^{\sigma-1}\left[\frac{w_t}{(1+x_{t+1})w_{t+1}}\right]^{\rho(1-\upsilon)\sigma}\frac{1}{\xi_{t+1}^y}. \tag{B11}$$

Young worker consumption:

$$c_t^y = \xi_t^y\left(\frac{R_{t-1}}{\pi_t}\frac{a_{t-1}^y}{(1+n_t)(1+x_t)} + h_t^y + s_t^y\right). \tag{B12}$$

Young worker asset ratio:

$$1-\Psi_t = \frac{a_t^y}{a_t}. \tag{B13}$$

Young worker pension payments:

$$s_t^y = \frac{\omega_{t+1}}{(1+n_{t+1})}\frac{\pi_{t+1}}{R_t\Omega_{t+1}}s_{t+1}^y + \frac{(1-\omega_{t+1})}{(1+n_{t+1})}\Xi_{t+1}^{\frac{1}{1-\sigma}}\left(\frac{1}{\varsigma_t}\right)^{1-\upsilon}\frac{\pi_{t+1}}{R_t\Omega_{t+1}}s_{t+1}^e - t_t^y. \tag{B14}$$

Young worker capitalised human wealth:

$$h_t^y = w_t l_t^y + \frac{\omega_{t+1}}{(1+n_{t+1})}\frac{\pi_{t+1}}{R_t\Omega_{t+1}}h_{t+1}^y + \frac{(1-\omega_{t+1})}{(1+n_{t+1})}\Xi_{t+1}^{\frac{1}{1-\sigma}}\left(\frac{1}{\varsigma_t}\right)^{1-\upsilon}\frac{\pi_{t+1}}{R_t\Omega_{t+1}}h_{t+1}^e. \tag{B15}$$

Young worker labour supply:

$$l_t^y = 1 - \frac{1-\upsilon}{\upsilon}\frac{1}{w_t}c_t^y. \tag{B16}$$

Firms and production. Aggregate output:

$$y_t = Z_t\left[\frac{k_{t-1}}{(1+x_t)(1+n_t)}\right]^\alpha l_t^{1-\alpha}. \tag{B17}$$

Capital-labour ratio:

$$\frac{(1+x_t)(1+n_t)w_t l_t}{r_t^K k_{t-1}} = \frac{1-\alpha}{\alpha}. \tag{B18}$$

Marginal cost:

$$\varphi_t = \frac{1}{Z_t}\left(\frac{r_t^K}{\alpha}\right)^\alpha\left(\frac{w_t}{1-\alpha}\right)^{1-\alpha}. \tag{B19}$$

NKPC:

$$(\pi_t - 1)\pi_t = \frac{\epsilon-1}{\phi_I}\left(\mathcal{M}\varphi_t + \tau^s - 1\right) + \frac{\pi_{t+1}}{R_t}\frac{(1+n_t)(1+x_t)y_{t+1}}{y_t}(\pi_{t+1}-1)\pi_{t+1}. \tag{B20}$$

Law of motion for capital:

$$k_t = \frac{(1-\delta)k_{t-1}}{(1+n_t)(1+x_t)} + i_t. \tag{B21}$$

Price of equity:

$$Q_t = 1 + \frac{\kappa_I}{2}\left[\frac{(1+n_t)(1+x_t)i_t}{i_{t-1}} - (1+n_t)(1+x_t)\right]^2$$
$$+ \kappa_I\left[\frac{(1+n_t)(1+x_t)i_t}{i_{t-1}} - (1+n_t)(1+x_t)\right]\frac{(1+n_t)(1+x_t)i_t}{i_{t-1}}$$
$$- \kappa_I\frac{\pi_{t+1}}{R_t}\left[\frac{(1+n_{t+1})(1+x_{t+1})i_{t+1}}{i_t} - (1+n_{t+1})(1+x_{t+1})\right]\left[\frac{(1+n_{t+1})(1+x_{t+1})i_{t+1}}{i_t}\right]^2. \tag{B22}$$

Profits of intermediate goods producers:

$$d_t^I = [\mathcal{M} - \varphi_t]\, y_t - \frac{\phi_I}{2}(\pi_t - 1)^2\, y_t. \tag{B23}$$

Fiscal and monetary policy. Taylor rule:

$$R_t = \bar{R}^{\phi_R}(R_t^*)^{1-\phi_R}\pi_t^{\phi_\pi}. \tag{B24}$$

Government budget constraint:

$$\frac{R_{t-1}}{\pi_t}\frac{b_{t-1}}{(1+n_t)(1+x_t)} + e_t + g_t = b_t + t_t. \tag{B25}$$

Government spending:

$$\frac{g_t}{y_t} = s_t^g. \tag{B26}$$

Government debt issuance:

$$\frac{b_t}{y_t} = s_t^b. \tag{B27}$$

Aggregate pension expenditure:

$$e_t = e_t^e. \tag{B28}$$

Aggregate pension contributions:

$$t_t = t_t^y. \tag{B29}$$

Aggregate pension payments to elderly:

$$e_t = \varrho_t\left(w_t l_t^y - t_t\right), \tag{B30}$$

with the net replacement rate:

$$\varrho_t = \bar{\varrho}.$$

Equilibrium and aggregation. Resource constraint

$$y_t = c_t + \left[1 + \Phi\left(\frac{i_t}{i_{t-1}}\right)\right] i_t + g_t + \frac{\phi_I}{2}(\pi_t - 1)^2 y_t. \tag{B31}$$

Aggregate consumption:

$$c_t = c_t^y + c_t^e. \tag{B32}$$

Aggregate capitalised human wealth:

$$h_t = h_t^y + h_t^e. \tag{B33}$$

Aggregate labour supply:

$$l_t = l_t^y + \varsigma_t l_t^e. \tag{B34}$$

Aggregate financial assets:

$$a_t = k_t + b_t + p_t^I. \tag{B35}$$

No-arbitrage condition capital goods markets:

$$\frac{R_t}{\pi_{t+1}} = R_{t+1}^K. \tag{B36}$$

No-arbitrage in intermediate goods stocks:

$$\frac{R_t}{\pi_{t+1}} = \frac{(1 + n_{t+1})\left(1 + x_{t+1})(p_{t+1}^I + d_{t+1}^I\right)}{p_t^I}. \tag{B37}$$

Gross return on capital:

$$R_t^k = \frac{r_t^k + (1 - \delta)Q_t}{Q_{t-1}}. \tag{B38}$$

C Model steady state

In this appendix we analytically solve for the model deterministic steady state. Steady state values of a variable, say X_t, are denoted as simply X.

Absent of trend inflation, we have $\pi = Q = Z = 1$. Through individual Euler equations, $R = \beta^{-1} = R^k$. This yields r^k:

$$r^k = R^k - 1 + \delta.$$

Marginal cost, from (B20), is $\varphi = \mathcal{M}^{-1}$. This allows us to pin down w through (B19):

$$w = \frac{(1-\alpha)\varphi^{\frac{1}{1-\alpha}}}{\left(\frac{r^k}{\alpha}\right)^{\frac{\alpha}{1-\alpha}}}.$$

From the law of motion of capital (B21) we get:

$$\frac{i}{k} = 1 - \frac{1-\delta}{(1+n)(1+x)}.$$

Output to capital ratio can be obtained through the aggregate output condition (B17):

$$\frac{y}{k} = \left(\frac{k}{l}\right)^{\alpha-1} \frac{1}{[(1+n)(1+x)]^{\alpha}}.$$

The capital-labour ratio from (B18) is given by:

$$\frac{k}{l} = \frac{\alpha(1+n)(1+x)}{(1-\alpha)} \frac{w}{r^k}. \tag{C1}$$

Then, use the resource constraint (B31) to get c/k:

$$\begin{aligned} \frac{y}{k} &= \frac{c}{k} + \frac{i}{k} + s^g \frac{y}{k} \\ \implies \frac{c}{k} &= (1-s^g)\frac{y}{k} - \frac{i}{k}. \end{aligned} \tag{C2}$$

One can then get a relationship between aggregate labour supply and consumption using (B1), (B34), (B16), and (B10):

$$l = 1 + \varsigma\Gamma - c\frac{1-\upsilon}{\upsilon}\frac{1}{w},$$

with Γ given by

$$\Gamma = \frac{1-\omega}{1+n-\gamma} \tag{C3}$$

One can then get k using (C2) and (C1)

$$k = \frac{1+\varsigma\Gamma}{\frac{l}{k} + \frac{c}{k}\frac{1-\upsilon}{\upsilon}\frac{1}{w}}. \tag{C4}$$

With k at hand, one can get y, l, c, and i given the ratios outlined above.

From (B5), one can get ξ^e:

$$\xi^e = 1 - \gamma\beta^{\sigma}R^{\sigma-1}(1+x)^{-\rho(1-\upsilon)\sigma}. \tag{C5}$$

(B3), (B4), and (B11) constitute a system of 3 equations in 3 unknowns, Ω, Ξ, and

ξ^y:

$$\Xi = \frac{\xi^e}{\xi^y},$$
$$\Omega = \omega + (1-\omega)\Xi^{\frac{1}{1-\sigma}}\left(\frac{1}{\varsigma}\right)^{1-v},$$
$$\xi^y = 1 - \beta^\sigma (R\Omega)^{\sigma-1}(1+x)^{-\rho(1-v)\sigma},$$

of which, one variable must be numerically computed as there is no closed-form solution for ξ^y.

Assume that l^y is known. This immediately yields l^e through (B34):

$$l^e = \frac{l - l^y}{\varsigma}. \tag{C6}$$

Then get c^e and c^y through labour supply conditions for each agent:

$$c^e = \frac{v\varsigma}{1-v}(\Gamma - l^e)w, \tag{C7}$$
$$c^y = \frac{v}{1-v}(\Gamma - l^y)w. \tag{C8}$$

Use (B25), (B26), (B27), and (B28) to pin down e and t. Start with the government budget constraint:

$$t = \left[\frac{R}{(1+n)(1+x)} - 1\right] b + e + g,$$

then use:

$$g = s^g y, \tag{C9}$$
$$b = s^b y, \tag{C10}$$
$$e = \bar{\varrho}\left(wl^y - t\right).$$

So we then get:

$$e = \frac{\bar{\varrho}}{1+\bar{\varrho}}\left[wl^y - \left(\frac{R}{(1+n)(1+x)} - 1\right) b - g\right]. \tag{C11}$$

With $e = e^e$, get s^e from (B8)

$$s^e = \frac{e^e}{1 - \frac{\gamma}{R}}. \tag{C12}$$

Since l^e is known, get h^e from (B9):

$$h^e = \frac{\varsigma w l^e}{1 - \frac{\gamma(1+n)(1+x)}{R}}. \tag{C13}$$

Then, we can get Ψ from (B2):

$$\Psi = \frac{\omega\left[\varsigma w l^e + e^e - \xi^e(h^e + s^e)\right]}{a\left[1 - (1-\omega) - \frac{\omega(1-\xi^e)R}{(1+n)(1+x)}\right]}. \tag{C14}$$

Dividends are:

$$d^I = (\mathcal{M} - \varphi)y, \tag{C15}$$

hence, price of shares are:

$$p^I = \frac{d^I}{R-1}. \tag{C16}$$

Thence, total assets are given by:

$$a = b + p^I + k. \tag{C17}$$

Then we can get a^e from (B7):

$$a^e = \Psi a, \tag{C18}$$

a^y from (B13):

$$a^y = (1-\Psi)a, \tag{C19}$$

s^y from (B14):

$$s^y = \frac{(1-\omega)\Xi^{\frac{1}{1-\sigma}}\left(\frac{1}{\varsigma}\right)^{1-\upsilon} s^e - (1+n)R\Omega t^y}{(1+n)R\Omega - \omega}, \tag{C20}$$

and h^y from (B15):

$$h^y = \frac{(1-\omega)\Xi^{\frac{1}{1-\sigma}}\left(\frac{1}{\varsigma}\right)^{1-\upsilon} h^e + (1+n)R\Omega w l^y}{(1+n)R\Omega - \omega}. \tag{C21}$$