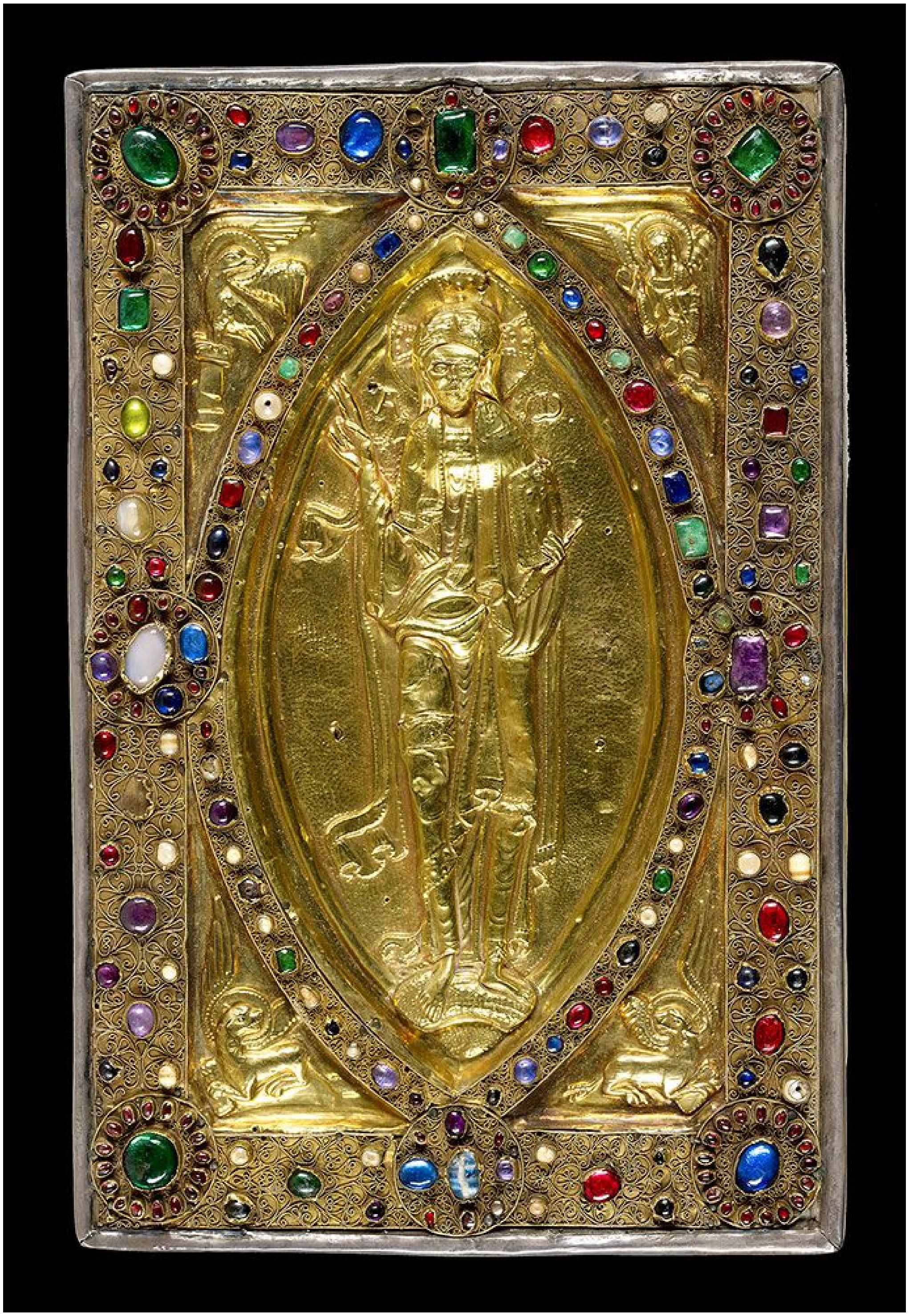
Homework Book



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Introduction

Our homework book is a little different. We (mostly) don't give you problems to solve. We give you ideas to study and ask for you to write questions from them, store those questions in a folder, and several weeks later, take them out and answer all the ones you can. This will show you what you know and we suspect you can be a good teacher.

In some places we will use computer science terminology because it works so well. When we get to tensors we will be explaining a calculation in a way that uses "for" loops.

Counting

We can work with a carton that was sold with 12 eggs.

On any day when you open the carton, if you count the eggs and the empty spots together, you get 12.

If there are ten eggs there will always be two empty spots.

We can write this as (10,2) where the rules are the following:

- Use () to signal you have a group of numbers
- Use a , to separate the numbers
- The first number must always be number of eggs
- The second number must always be the number of empty spots

It is important to strictly follow the rules. Someone else seeing (2,10) will know immediately to request the grocery shopper to buy more eggs.

Play a game called "Count by numbers". Do your counting doing something like what you see in the following:

- 1,2,3,4,5,6,7,8,9,10,11,12...
- 2,4,6,8,10,12…
- 3,6,9,12…
- 4,8,12…

Addition

Addition is the aggregation of counts. You might look at two egg cartons and see five eggs in the first carton and six eggs in the second carton and immediately your mind thinks "11 eggs". In your mind you had memorized "five and six is eleven" and using the memory was faster than counting from 1 to 11.

Your mind will recognize things it sees repeatedly without even trying. It was natural to talk about it and then formalize it.

But Addition goes way beyond the addition of numbers. Addition is one of two important things in Vector Space. Addition is about adding two somethings to make a something. Consider the two vectors $\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. If we add them together the result is a vector of the same type as the first two:

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Multiplication

Multiplication is the addition of likes. Add five seven times (5+5+5+5+5+5+5=35) or add seven five times (7+7+7+7+7=35).

You won't truly see multiplication until you can multiply something that isn't a number. We can take a vector and scale it with a number:

$$2\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

Can you agree that 2 took (3,4) and doubled it? If yes, then we can triple it with the following:

$$3\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \end{bmatrix}$$

Multiplication is about a "something else" and a "something" making a "something".

It was trivial for the something and something else to be the same type--and it was legal to do that, just like it is legal for you to tell me you have a velocity of 0 in a direction when your car is not moving (not moving relative to our agreed upon frame of reference--probably the road).

\begin{bmatrix} \end{bmatrix}

Algebra

Axioms

Below we are writing a few Axioms that we need in future work:

- The sum of two constants is a constant
- The sum of two integers is an integer
- The product of two constants is a constant
- The product of two integers is an integer

Equality

There are three axioms that define equality:

- Equality is Reflexive: a=a
- Equality is Symmetric: if a=b then b=a
- Equality is Transitive: if a=b and b=c then a=c

Substitution Property

For a function f(x) and values ‘a’ and ‘b’, if a=b then F(a)=F(b)

One note of caution with this--it doesn’t work in the other direction.

if $90^{\circ} = \frac{\pi}{2}$,

then $cos(90^{\circ}) = cos(\frac{\pi}{2})$.

Knowing cos(a)=cos(b) does not give us a=b.

Substitution Definition

If a=b then any ‘a’ in an equation can be replaced by ‘b’.

d=ac can be changed to d=bc

Euclid Common Notion One

“If two things are both equal to a third thing then they are equal to each other.”

We can use the above idea to write specifics such as the following:

“If two variables are equal to the same value then the variables are equal.”

Euclid Common Notion Two

If equal quantities are added to equals, the equality is maintained.

Let $f(x)=g(x)$, then $f(x)+k = g(x)+k$.

Association of Addition

$(a+b)+c = a+(b+c)$

Association of Multiplication

$(ab)c = a(bc)$

Commutation of Addition

$a + b = b + a$

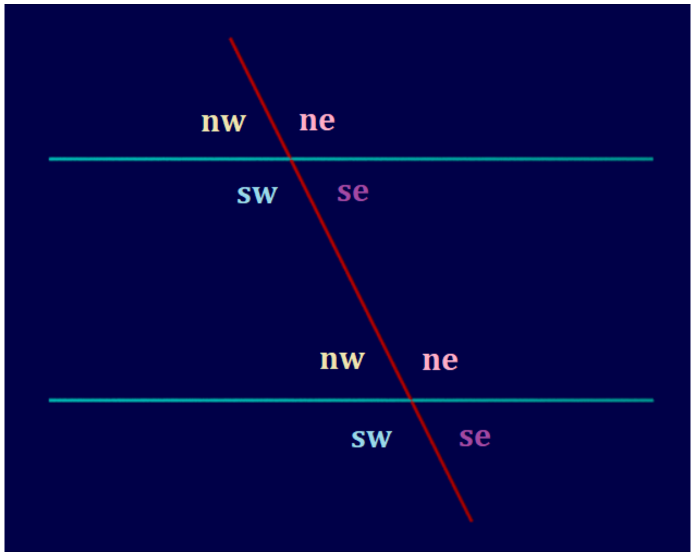
Commutation of Multiplication

$ab = ba$

Geometry

Corresponding Angles

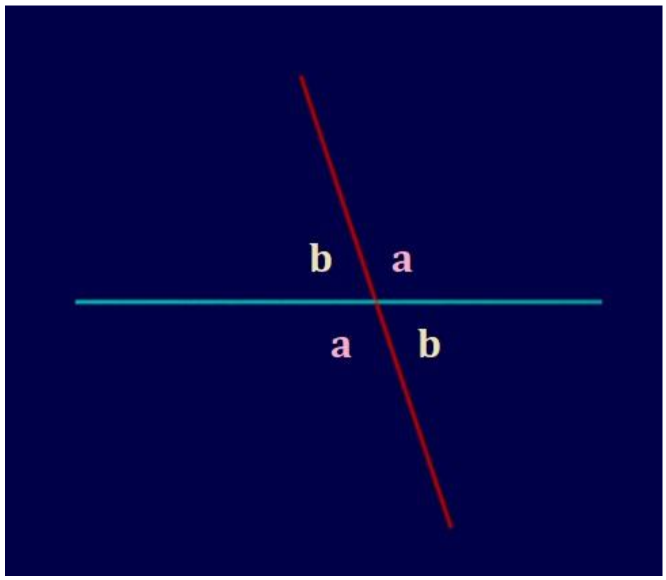
When a transversal cuts two parallel lines, it creates eight angles. These angles can be grouped into two sets, a set at each intersection. Corresponding angles between the two sets are equal. In the illustration below the corresponding angles are labeled the same: “nw” is the corresponding angle to “nw” in the other set.


nw
ne
sw
se
nw
ne
sw
se

Opposite Angles

Opposite angles are equal. This will be shown in the diagram below. For the illustration above with {ne,nw,sw,se}, there are only two angles. After you are comfortable with finding opposite angles, you are invited to go back and divide those angles into two groups. You will need to come up with your own way of distinguishing ‘ne’ on top from ‘ne’ on the bottom.

Below, the two angles marked ‘a’ are opposite angles, and opposite angles are equal. The two angles marked ‘b’ are also opposite angles, and they are equal.

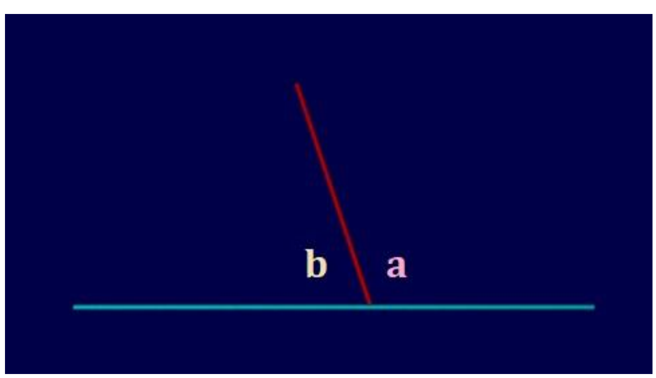


a=a, b=b

Supplementary Angles

The angles ‘a’ and ‘b’ add up to 180 degrees.

a + b = 180



Sum of Three Angles of a Triangle

We have an opportunity here to invoke Euclid Common Notion One. If two things are equal to a third thing then they are equal to each other. Look at the top ‘a’ then look down at the corresponding angle, and for now call it “corresponding angle”. The second ‘a’ is opposite to ‘corresponding angle’. With both ‘a’ angles equal to the same angle, they must be equal to each other. The same strategy will make the two ‘b’ angles equal.

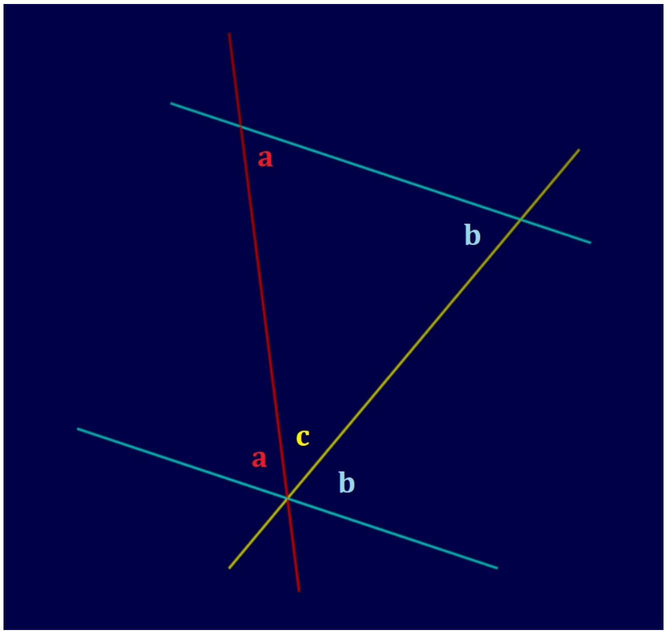
We have two ways to prove that $a+b+c = 180_{\circ}$.

The sum of two angles is an angle. If we manufacture an angle from a+c this new angle will be a supplementary angle to b and ‘new angle’ + b = $180_{\circ}$.

$(a+c) + b = 180_{\circ}$

$a + c + b = 180_{\circ}$

Commutation of Addition lets us change that to $a + b + c = 180_{\circ}$.



Physics

Various equations in physics get a second number by taking a first number and doing a multiplication.

It's typical to see an equation in the form F=ma but you may want to change that to the following:

$$ma = F$$

One thing about math. The equation is often[1] blind to what is the cause and what is the effect. Try to talk through the equation as you see it, both ways:

- F=ma {the force required is calculated by the mass multiplied by the desired acceleration}
- ma=F {mass acting on the desired acceleration yields the required force}

Let's have some fun with a story that has seven variables:

ABC = DEFG

Let's say we pick B to be the independent variable and F to be the dependent variable. That means we want to hold A,C,D,E,G constant in the experiment. We then have the following:

$$F = \frac{AC}{DEG}B$$

$$F = kB$$

$$k = \frac{AC}{DEG}$$

Vector Space

A vector space is built by taking a set of vectors, choosing a field for scalars and then casting magic spells to make addition and scalar multiplication true in a way making several rules (often called axioms) true. We will get to them shortly but first we want to give some perspective.

The rules give you the peace of knowing the difference between adjacent numbers (2,3) or (5,6) or (11,12) is always the same.

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[1] We say "often" because you may have something set up where it is obvious what is the cause and what is the effect.

Or that x + 3 - 2 will always give you the number you get from x + 1.

As you work through them, it will probably seem that the rules are so sensible--you might have the feeling they are always true. Well, most of the time they are, and we can say there are no monsters. But watch the 1979 and 1986 movies with Sigourney Weaver. Yes, there are bad things out there.

And we don't have to go to outer space to find horror. There is a distortion on a flat paper map of the world (because the world is a globe) and if you travel to Mexico and use a scale that was true in Greenland, you will miscalculate the distance between where you are to a point of safety--the actually difference will be considerably larger than what you calculate.

In fact, someone aware of the curvature might panic when they see you put a ruler on the map and they scream "Don't ever do that again!"

But back to vector space, where things are as simple as a map of a city or a county where you are allowed to use a ruler because the scale of real feet to an inch on the map is the same everywhere on the map. (on the world map the scale changes as we move from point to point)

Addition of Vectors

For any two vectors iv V, u and v, we can add them together and the result is a vector and that vector is also in V.

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$$

Commutation of Addition

u + v = v + u

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix} = \begin{bmatrix} c+a \\ d+b \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

Association of Addition

(u + v) + w = w + (v + w)

$$\begin{bmatrix} a+c \\ b+d \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c+e \\ d+f \end{bmatrix}$$

Linear Combination

Assume a set with (e_1, e_2).

Something is a linear combination of this set if it is a summation of each element times a scalar.

$v = 3(e_1) + 4(e_2)$

Linearity and Multilinearity

Linearity

- Let T be the linear transformation
- let {u,v,w,x,y} be objects from a set of interest
- let {a,b,c,d,e} be scalars for scalar multiplication

We aren't going to tell you what T is:

T(u + v) = T(u) + T(v)	T(av) = aT(v)
Let u=3 and let v=4 and thus u+v=7 T(7)=14 T(3)=6, T(4)=8 14 = 6 + 8	Let a=2 and thus av=8 T(8) = 16 2(8) = 16 16 = 16

For multilinearity, we can say that the relationship is linear in each argument. This will be illustrated by showing bilinearity and then showing multilinearity with four variables (tetralinearity?).

Bilinearity

T(u+v,w) = T(u,w) + T(v,w)
T(au,v) = aT(u,v)

T(u,v+w) = T(u,v) + T(u,w)
T(u,av) = aT(u,v)

Multilinearity

$T(u+v,w,x,y) = T(u,w,x,y) + T(v,w,x,y)$
$T(au,v,w,x) = aT(u,v,w,x)$

$T(u,v+w,x,y) = T(u,v,x,y) + T(u,w,x,y)$
$T(u,av,w,x) = aT(u,v,w,x)$

$T(u,v,w+x,y) = T(u,v,w,y) + T(u,v,x,y)$
$T(u,v,aw,x) = aT(u,v,w,x)$

$T(u,v,w,x+y) = T(u,v,w,x) + T(u,v,w,y)$
$T(u,v,w,ax) = aT(u,v,w,x)$

Appendix A

It is up to you -- if you are a parent and a young person wants to work with you, there are probably more things we could add to the Counting section.

One goal in the counting section was to get you to do things that would make certain math principles start to have a natural feel to them. Later it would be easy, or easier, to grasp the idea. For example, after playing our game of drawing a vector on a piece of glass and holding up pieces of paper with either inch coordinates or centimeter coordinates (and getting different numbers) would prepare you for when you read “mathspeak” about doing transformations where the numbers change but the object or objects didn’t change.

It might seem as easy as believing that a length doesn’t change when you go from 10 inches to 25.4 centimeters. You might laugh and say “all you did was change the ruler”.

Or you might imagine a starship in “Wrath of Khan” flying through a difficult story and changing coordinates several times in order to make the calculations quick enough to stay ahead of the Klingons.