A new presentation of categorical syllogisms

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Abstract

One proposes two simple schemes - a graphical and a tree like one - for finding logical conclusions (LCs) from any pair of categorical premises (PCP), via using a set model for the syllogistic terms S,P,M, non-S (S'), non-P (P'), non-M (M'). In this model, any universal, (resp. particular), premise empties, (resp. lays set elements into), two subsets out of the 8-subset partition of the universal set U which models the categorical statements: U=MSP+MS'P+MSP' +MS'P'+M'SP+M'S'P+M'SP' +M'S'P', (where the union of disjoints sets is denoted by a plus sign, S',P',M' are the complementary sets in U of the S,P,M terms respectively, and MSP:= M∩S∩P, etc.). A cylindrical Venn diagram replaces the 8 "irregularly shaped" subsets from the usual 3-circle Venn diagram of U by rectangular shapes drawn on a cylinder and provides a very clear presentation of the two LCs finding schemes. One shows, (without using syllogistic moods and figures, syllogistic axioms and inference rules, nor valid syllogism rules and syllogism reduction), that any LC refers to just one of the 8 subsets of U and that any PCP entailing an LC, (called valid syllogistic argument (VCA), may be recast as, e.g., one of the Barbara, Darapti or Darii valid syllogisms (VSs). The recasting is done via set relabelings instead of the syllogism reduction (used, e.g., by Aristotle and J. Lukasiewicz). One may say that this paper approaches categorical syllogisms in the same spirit as George Boole approached them in The Mathematical Analysis of Logic – but instead of Boole's equations one uses graphic and tree like methods for finding the LCs.

1. The Cylindrical Venn diagram (the Karnaugh map for n=3)

S'P'M	SP'M	SPM	S'PM
S'P'M'	SP'M'	SPM'	S'PM'

Fig. 1

The universal set U is graphed as a rectangle – but the left and right borders of the rectangle are glued together to generate a cylinder, so that S'PM and S'P'M are adjacent, and S'PM' and S'P'M' are adjacent, too – as in the usual 3-circle Venn diagram. Since the 8 subsets of Figure 1 are the "elementary" subsets of U, one

calls them just subsets; no other set will be a "subset". On this "cylindrical Venn diagram", (or Karnaugh map with n=3), all the syllogistic conclusions are graphically obvious, much more so than on a 3-circle Venn diagram. After I thought, in 2017, that I "invented" the cylindrical Venn diagram - I found out that Alan Marquand really invented it in 1881, (a year after John Venn proposed his 3-circle Venn diagram), and then, in 1952 Edward Veitch, and, in 1953 Maurice Karnaugh, used Karnaugh (-Veitch) maps for n=3, n=4, etc., to find the optimal design for digital circuits. These cylindrical or toroidal maps make a standard appearance in engineering books, but, apparently, never made it into, (may I say any?), logic textbooks – which are still using the 3-circle Venn diagrams.

One can see that any universal premise empties two "horizontal subsets" located either on the M or the M' row, and its contradictory particular premise places set elements in at least one of the same two "horizontal subsets" emptied by the universal premise. For example, Barbara's PCP, A(M,P)A(S,M), contains two universal premises, acting one on M and the other on M'. The premises mean MP'= Ø, SM'=Ø, i.e., 4 subsets are emptied on Fig. 1, thus reducing S to S=SPM and P' to P'=S'P'M' – which translate to the LCs A(S,P) and resp. A(P',S') – the latter being equal, via contraposition, to A(S,P). Using existential import (ei) on S, and resp. P' one gets the ei LC I(S,P) – Barbari, and the no name ei LC I(S',P'). The same results are easily found via a "tree like method" which eliminates, (i.e., closes), any subset (i.e., branch), emptied by a universal premise: S=SM+SM'=SM=SPM+SP'M=SPM and P'=P'M+P'M'=P'M'=SP'M'+S'P'M'=S'P'M'. This amounts - for all PCPs containing two universal premises acting one on M and the other on M' - to the general rule of starting two trees, one for each letter other than the middle terms M and M'. Note that after finding the LC in either the graphical or tree like way, the middle term elimination consists in "just do not mention the middle term" – and by so doing, the usual LC wording refers to a column of the cylindrical Venn diagram instead of only one subset of it.

The second type of PCP that entails an LC contains two universal premises acting on the same row. Darapti's PCP, A(M,P)A(M,S), meaning MP'=Ø and MS'=Ø, is an example of such a PCP. From Fig. 1 it is clear that the LC is M=SPM, which, via ei on M produces the ei LC I(S,P). Alternatively, one should start the tree with M=MP+MP'=MP=MPS+MPS'=SPM.

Finally, Darii's PCP, A(M,P)I(M,S), contains one universal premise plus one particular premise, both acting on the same row (either M or M'). The LC is SPM≠Ø, and results either from Fig. 1, since I(M,S) places set elements on either SP'M or/and SPM, but A(M,P), by emptying S'P'M and SP'M "forces" I(M,S) to definitely place its element(s) **only** on SPM. Or, one may start a (very short) tree with the non-empty set specified by the particular premise: MS=MSP+MSP'=SPM. Let me also observe that, (as I just found out), Boole 1847, on pages 35 to 41, introduces and discusses four classes of PCPs and their LCs. The first three of George Boole's classes can be viewed as similar to the PCP sets described above – whose representatives respectively are Barbara, Darapti and Darii. Boole divided his fourth class into two subclasses of PCPs – both these subclasses do not entail any LCs. The cylindrical Venn diagram makes self evident the well known fact that PCPs containing two particular premises do not entail any LCs. Similarly, no LCs

may be drawn from PCPs containing one universal premise acting on one row plus one particular premise acting on the other row. As more fully described in the next section, these are the only five types of PCPs possible. They have been figured out, by and large, by Gerge Boole since 1847 – although his discoveries are not used in the present day logic textbooks. Note also that Boole 1847, pp.34-35, embraces the existence of VCAs: "The Aristotelian canons, however, beside restricting the *order* of the terms of a conclusion, limit their nature also;—and this limitation is of more consequence than the former. We may, by a change of figure, replace the particular conclusion of *bramantip,* by the general conclusion of *barbara;* but we cannot thus reduce to rule such inferences," (aka LCs) "as Some not-Xs are not Ys. Yet there are cases in which such inferences may lawfully be drawn, and in unrestricted argument they are of frequent occurrence. Now if an inference of this, or of any other kind, is lawful in itself, it will be exhibited in the results of our method." Then, on pages 35 to 41, using equations to eliminate the middle term, Boole notices and discusses 4 classes of PCPs and their LCs or lack thereof.

2. SETS AND CATEGORICAL PREMISES

By definition a categorical syllogism is made of a PCP to which one tacks a 3rd statement, an (S,P)-conclusion, i.e., one of the categorical operators, (or quantifiers), A,O,E,I applied to the ordered pair (S,P). If the conclusion is truly entailed by the PCP one has a VS, otherwise the syllogism is invalid. There are 36 distinct PCPs expressed via only the S,P,M terms and the categorical operators A,O,E,I applied to pairs of these 3 terms. Modeling such premises on a cylindrical Venn diagram or Karnaugh map with n=3, where the middle term M occupies one row and M', its complementary set, occupies a 2nd row, one sees that the PCPs do not act symmetrically on the M and M' rows: four P-premises (resp. S-premises) out of six act on M and only two act on M'. The case of 36 PCPs was treated in Radulescu 2017. But if one wants to have a set of PCPs which remains unchanged under the relabelings M↔M', S↔S' and P↔P', and thus transforms the set relabelings from Section 3 into an 8-element group - one has to add two more P-premises, E(M',P')= A(M',P):="All M' is P" (shorthand A') and I(M',P')=O(M',P):="Some M' is not P" (shorthand O'), to the standard six P-premises, A(M,P)=E(M,P'), O(M,P)=I(M,P'), E(M,P), I(M,P), A(P,M)=E(M',P), O(P,M)=I(M',P), and, similarly one has to add two more S-premises to the standard six S-premises. This gives 64 pairs of categorical premises (PCPs) instead of the usual 36 PCPs gotten from A,O,E,I operators applied to only the S,P,M terms. The usual conversions, obversions and contrapositions apply, and are directly justified as set operations. Thus, e.g., A(M,P)=A(P',M')=E(M,P') (shorthand A) acts on the M row, and A(P,M)= A(M',P')=E(M',P) (shorthand E') acts on the M' row, in a similar way as E(M,P) (shorthand E) acts on the M row. In the LC finding recipes below all PCPs are expressed using only the E and I operators, and since they are symmetric, for uniformity, the middle term (M or M') is written first.

The eight P-premises, written using only the E and I operators are "lumped" into two squares of opposition, the one acting on M, (I)={E(M, P*), I(M, P*)}, P*∈{P', P}, (also denoted as A,O,E,I) plus (II)={E(M', P*), I(M', P*)}, P*∈{P', P}, (also

denoted as A',O',E',I') - acting on M'. Similarly the eight S-premises are "lumped" into another two squares of opposition, (III)={E(M, S*), I(M, S*)}, S*∈{S', S}, (also denoted as A,O,E,I), and (IV)={E(M', S*), I(M', S*)} , S*∈{S', S}, (also denoted as A',O',E',I'). A universal premise, e.g., E(M, P*) means "No M is P*" or MP*:=M∩P*=Ø and its contradictory, I(M, P*) means "Some M is P*" or MP*≠Ø, P*∈{P', P}. Using the convention of always listing the P-premise first, makes it clear that, e.g., AE'=A(M,P)E(M',S)=A(M,P) A(S,M) is the PCP for Barbara. And, A'E=A(M',P)E(M,S) is the PCP for Barbara' – an M' version of Barbara whose LCs are S=SPM', P'=S'P'M, A(S,P)=A(P',S'), and the ei LCs, I(S,P) when S≠Ø, Barbari'; I(S',P') when P'≠Ø, no name. Note also that modulo a slight change in notations, the eight P-premises and the eight S-premises coincide with the cube of opposition from Figure 4 of Dubois, 2015, p.2936. The 64, (as well as the 36), distinct PCPs are naturally partitioned into 5 subsets: two subsets whose PCPs do not entail any LC, and three subsets whose PCPs do each entail at least one LC and thus generate valid categorical arguments (VCAs), which are also split into three VCA classes, each class containing some of the VS. Any VS is a VCA with a special LC: E(S,P*) or I(S,P*) with P*∈{P, P'}. The VCA\VS set has LCs of one of the formats: I(S',P'), E(S',P)=A(P,S) or I(S',P)=O(P,S). From a set theoretical point of view the difference between the VS and the VCA\VS sets is not significant since any VCA\VS may be recast as a VS via a set relabeling, cf. Radulescu 2017. The five PCP subsets are characterized as: **#1PCPs).** Two universal premises, acting one on M and one on M'. There are 8 such PCPs, (obtained by combining the universal premises from (I) and (IV), plus the similar ones from (II) and (III). The result of each such PCP is the emptying of 4 subsets of U - two on M and two on M' – with two empty subsets out of the empty 4 being located on the same column, and the other two empty ones being located on each side of that column but on different rows (one on the M row, one on the M' row). Consequently, the LCs of such a PCP are: each set from one of the pairs of sets, (S,P), (S',P'), (S,P'), (S',P) - whose intersection is a column of U - is reduced to one subset out of four. For example, Barbara's PCP empties the column SP' and leaves S=SPM and P'=S'P'M', (see the proof in the previous section), which leads to the following four LCs: A(S,P), A(P',S'), and via ei on S and on P', one gets I(S,P) and I(S',P'). The two "universal LCs" are not independent since A(S,P)=A(P',S')=E(S,P'), cf. contraposition or set definitions, but I(S,P) and I(S',P') are independent ei LCs. **#2PCPs).** Two universal premises both acting on either M or M'. There are 8 such PCPs, (gotten by combining the universal premises in (I) and (III) and adding to them the universal PCPs extracted from (II) and (IV)), with each LC being that either M or M' is reduced to just one subset out of its 4 subsets. Then one needs existential import (ei) on either M or M' in order to express the LC without any reference to M (or M'). (According to Aristotle, M or M' appearing in the LC would mean repeating the premises' content instead of stating the "new knowledge" the VCA or VS "should" bring (Striker 2009: p. 20). But in Aristotle's time, set theory was not invented yet – thus, in the present paper, the middle term will be very much part of any LC, before one removes it, with some loss of information, in order to write the LCs in their traditional format.) **#3PCPs).** One universal premise and one particular premise both acting on either M or M'. There are 16 such PCPs, (8 are obtained by

combining the universal premises in (I) with the particular premises in (III) and vice versa, and the other 8 are obtained in the same manner from (II) and (IV)); each of their LCs is of the type: one subset of U is $\neq\varnothing$. Since there are 16 such PCPs and only 8 subsets of U, each of the four M, (and M'), subsets appears in an LC twice as being $\neq\varnothing$: for example, SP'M$\neq\varnothing$, meaning O(S,P), is the LC of E(M,P)I(M,S), Ferio/Festino/Ferison/Fresison, and also the LC of I(M,P')E(M,S'), Bocardo. **#4PCPs).** Both premises are particular. Obviously there are 16 such PCPs and they entail no LC. **#5PCPs).** One premise is universal and one particular, and they act one on M, and the other one on M', (8 are obtained by combining a universal, (resp. particular), premise in (I) with a particular (resp. universal) premise in (IV), and the other 8 are obtained from (II) and (III) in a similar fashion). These are 16 more PCPs not entailing any LC. Thus out of 64 distinct PCPs, 32 do not entail any LC, 16 PCPs from set #3 entail each one unique LC, the 8 PCPs from set #2 entail each one unique ei LC and the 8 PCPs from set #1, entail each one LCs plus two ei LCs. This way one gets to a total total of 48 distinct VCAs out of which 24 are ei VCAs.

To prove the above claims about the shape of the LCs one may notice that each VCA (and VS) LC is easily found either via a "tree like method" (which eliminates, (i.e., closes), any subset (i.e., branch), emptied by a universal premise), or, directly by looking at the cylindrical Venn diagram in Fig. 1. For any PCP in subset #3 one starts the (very short) tree with the non-empty intersection of the two sets appearing in the particular premise: in Ferio/Festino/Ferison/Fresison's case, $\varnothing\neq$MS:=M$\cap$S= MSP+MSP' =MSP' since the premise E(M,P) says MP= $\varnothing$. Thus the LC is MSP'$\neq\varnothing$ or O(S,P). For any PCP in subset #2, the unique subset the LC is referring to is found by "starting a tree" with either M or M' – the set which appears in both universal premises. It will result that M (or M') equals its intersection with the complements of the other two sets appearing in the two **#2PCPs** universal premises. (For example, in Darapti's PCP case, A(M,P) A(M,S)=E(M,P') E(M,S'), write M=MP+MP'=MP=MPS+ MPS'=MPS.) For any PCP in subset #1 each of the two universal "LC subsets" can be found via two short trees, each starting with one of the "letter sets" other than M and M' and continuing by eliminating its subsets emptied by the two universal premises. For example, in the case of Barbara's PCP, A(M,P)A(S,M)=E(M,P') E(M',S) start with S=SM'+SM=SM=SPM+SP'M=SPM and with P'=MP'+M'P'= M'P' =S'P'M'+SP'M'=S'P'M'. The LCs are A(S,P)=A(P',S') (Barbara), and via ei on S, I(S,P) (Barbari), plus, via ei on P', I(S',P'). This explains why there are 48 distinct VCAs, generated by only 32 PCPs. Out of the 48 VCAs, only 8 are distinct VS, and 6 are distinct ei VS. (If syllogistic figures are used, then one counts 15 VS and 9 ei VS, but this means that, e.g., the same content VS, Ferio/Festino/Ferison/Fresison, receives four different names and counts as 4 distinct VS, when in reality one deals with one PCP, E(M,P)I(M,S), and one LC: O(S,P). The VCA\VS subset contains 16 non ei VCAs and 18 ei VCAs.

Thus the VCA are partitioned into three classes, each class being generated by the PCPs from the subsets #1 to #3 above. In the next section one will show that inside each of the three VCA classes, any VCA may be recast or reformulated, via a relabeling transformation of the sets S,P,M,S',P',M', as any other VCA in the same class, which makes all VCAs equivalent with three representatives chosen one per

VCA class. For example the Darapti, Darii and Barbara representatives may be chosen. In particular, the VCA\VS set whose LCs have one of the formats A(P,S), O(P,S) or I(S',P') may be recast, via a set relabeling as VSs.

3. VCAs EQUIVALENCIES VIA SET RELABELINGS

One may group the 32 PCPs which entail at least one LC and thus generate VCAs, into 8 subsets made of 4 PCPs per subset (and generating 6 VCAs per subset). The first PCP in such a group of 4 PCPs belongs to the **#3PCPs**, the following two belong to **#2PCPs** and the last one to the **#1PCPs**. We'll say that each of such a 4 PCPs group is "bound to" a same subset of U: the one **#3PCP**, the following two **#2PCPs** and the one **#1PCP** do not act at all on the subset of U on which they are all "bound", but act on some of its "neighbors" in the cylindrical Venn diagram. Thus to each of the 8 subsets of U one "attaches" a group of four PCPs "bound" to it:

1. VCAs "bound to" the subset S'P'M:
EE=E(M,P)E(M,S)   M=S'P'M. If M≠∅: I(S',P'), No name
IE=I(M,P)E(M,S)   S'PM≠∅ or O(P,S), No name
EI=E(M,P)I(M,S)   SP'M≠∅ or O(S,P), Ferio/Festino/Ferison/Fresison
EE'=E(M,P)E(M',S)   S=SP'M, P=S'PM', E(S,P),   Celarent/Cesare
                    O(S,P) if S≠∅, Celaront/Cesaro; O(P,S) if P≠∅, No name

2. VCAs bound to the subset SP'M:
EA=E(M,P)E(M,S')   M=SP'M. If M≠∅: O(S,P), Felapton/Fesapo
IA=I(M,P)E(M,S')   SPM≠∅ or I(S,P), Disamis/Dimaris
EO=E(M,P)I(M,S')   S'P'M≠∅ or I(S',P'), No name
EA'=E(M,P)E(M',S')   S'=S'P'M, P=SPM', A(P,S)=A(S',P'),
                    I(S,P) if P≠∅, Bramantip'; I(S',P') if S'≠∅, No name

3. VCAs bound to the subset S'PM:
AE=E(M,P')E(M,S)   M=S'PM. If M≠∅: O(P,S), No name
OE=I(M,P')E(M,S)   S'P'M≠∅ or I(S',P'), No name
AI=E(M,P')I(M,S)   SPM≠∅ or I(S,P), Darii/Datisi
AE'=E(M,P')E(M',S)   S=SPM, P'=S'P'M', A(S,P), Barbara
                    I(S,P) if S≠∅, Barbari; I(S',P') if P'≠∅, No name

4. VCAs bound to the subset SPM:
AA=E(M,P')E(M,S')   M=SPM. If M≠∅: I(S,P), Darapti
OA=I(M,P')E(M,S')   SP'M ≠∅ or O(S,P), Bocardo
AO=E(M,P')I(M,S')   S'PM≠∅ or O(P,S), No name
AA'=E(M,P')E(M',S')   S'=S'PM, P'=SP'M', E(S',P'), No name
                    O(P,S) if S'≠∅, No name; O(S,P) if P'≠∅, No name

M' row VCAs:
5. VCAs bound to the subset S'P'M':
E'E=E(M',P)E(M',S)   M'=S'P'M'. If M'≠∅: I(S',P'), No name

I'E'=I(M',P)E(M',S)   S'PM'≠Ø or O(P,S), No name
E'I'=E(M',P)I(M',S)   SP'M'≠Ø or O(S,P), Baroco
E'E=E(M',P)E(M,S)   S=SP'M', P=S'PM, E(S,P), Camestres/Camenes
          O(S,P) if S≠Ø, Camestros/Camenos; O(P,S) if P≠Ø, No name

6. VCAs bound to the subset SP'M':
E'A'=E(M',P)E(M',S')   M'=SP'M'. If M'≠Ø: O(S,P), Felapton'/Fesapo'
I'A'=I(M',P)E(M',S')   SPM'≠Ø or I(S,P), Disamis'/Dimaris'
E'O=E(M',P)I(M',S')   S'P'M≠Ø or I(S',P'), No name
E'A=E(M',P)E(M,S')   S'=S'P'M, P=SPM, E(S',P)=A(P,S), No name
          I(S,P) if P≠Ø, Bramantip, I(S',P') if S'≠Ø, No name

7. VCAs bound to the subset S'PM':
A'E=E(M',P')E(M',S)   M'=S'PM'. If M'≠Ø: O(P,S), No name
O'E=I(M',P')E(M',S)   S'P'M'≠Ø or I(S',P'), No name
A' I'=E(M',P')I(M',S)   SPM'≠Ø or I(S,P), Darii'/Datisi'
A'E=E(M',P')E(M,S)   S=SPM', P'=S'P'M, A(S,P)=A(P',S'), Barbara'
          I(S,P) if S≠Ø, Barbari'; I(S',P') if P'≠Ø, No name

8. VCAs bound to the subset SPM':
A'A'=E(M',P')E(M',S')   M'=SPM'. If M'≠Ø: I(S,P), Darapti'
O'A'=I(M',P')E(M',S')   SP'M' ≠Ø or O(S,P), Bocardo'
A'O=E(M',P')I(M',S')   S'PM≠Ø or O(P,S), No name
A'A=E(M',P')E(M,S')   S'=S'PM, P'=SP'M, E(S',P'), No name
          O(P,S) if S'≠Ø, No name; O(S,P) if P'≠Ø, No name

One can now define a "relabeling group" acting on the above VCAs subsets 1,2,...,
8. Let p:= P↔P', s:=S↔S', m:=M↔M'. One can see that compositions of s,p,m
generate a commutative group G with eight distinct elements:1,s,p,m,sp,sm,pm,
spm. Obviously 1=s^2=p^2=m^2= $(spm)^2$ = $(ms)^2$ =$(ps)^2$ = $(pm)^2$. This group acts on the
above VCAs subsets 1,2,...,8, as follows:
p(1)=3, p(2)=4, p(5)=7, p(6)=8; s(1)=2, s(3)=4, s(5)=6, s(7)=8; m(1)=5, m(2)=6,
m(3)=7, m(4)=8.
One can check that {G(1)}= {G(2)}= …={G(8)}= {1,2,3,...,8}. This shows that any
VCA from any of the three VCA classes can be recast as any other VCA in the same
class. For example, spm(E'E')=AA=A(M,P) A(M,S)=E(M,P')E(M,S') which are
Darapti's premises. This means that E'E'=E(M',P)E(M',S)=A(P,M)A(S,M) become
Darapti's premises after an spm relabeling.

4.   Conclusions

Discarding the syllogistic moods and figures, syllogistic axioms and inference rules,
and valid syllogism rules, in favour of a pure set modelling of the syllogistic terms,
greatly simplifies the categorical syllogisms' presentation. Compare, e.g., with other
expositions: Alvarez and Correia 2012, Mineshima, Okada, Takemura 2012,
Avarez-Fontecilla 2016, (or Lukasiewicz 1957). One has shown that the middle

term M always appears in any LC – since the LC always refers to just one subset out of the eight U subsets. Only by losing some information one may recast the LC in the traditional way as referring to a two subset column. Possible LC examples from each of the three VCA classes are S=SPM, SPM≠Ø, M=SPM, as LCs for Barbara, Darii and Darapti, respectively. With some loss of information they translate into the usual A(S,P), I(S,P), and, via existential import on M, I(S,P). Also, using just set relabelings, instead of syllogism reduction, one has shown that the VCAs from the same class are equivalent: any VCA (or VS) can be recast (or reformulated) as any other VCA from the same class. Thus, e.g., Barbara, Darii and Darapti may be chosen as representatives of the three VCA classes. Finally, one may use the "old style" indirect reduction to show that Darii and Darapti are not logically independent of Barbara.

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