Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

1. Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2. Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3. Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4. Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5. Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6. Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7. Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8. Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y – 2)/(x – 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x – 1)(x + 1)$, $(x – 1)(x^2 + x + 1)$, and $(x – 1)(x^3 + x2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word “understand” are often especially good opportunities to connect the practices to the content. Students who lack understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential “points of intersection” between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

Mathematical Practices		Students	Teachers
Overarching habits of mind of a productive math thinker	**1. Make sense of problems and perseveres in solving them**	☐ Understand the meaning of the problem and look for entry points to its solution ☐ Analyze information (givens, constrains, relationships, goals) ☐ Make conjectures and plan a solution pathway ☐ Monitor and evaluate the progress and change course as necessary ☐ Check answers to problems and ask, "Does this make sense?"	☐ Involve students in rich problem-based tasks that encourage them to persevere to reach a solution ☐ Provide opportunities for student to solve problems that have multiple solutions. ☐ Encourage students to represent their thinking while problem solving
	6. Attend to precision	☐ Communicate precisely using clear definitions ☐ State the meaning of symbols, carefully specifying units of measure, and providing accurate labels ☐ Calculate accurately and efficiently, expressing numerical answers with a degree of precision ☐ Provide carefully formulated explanations ☐ Label accurately when measuring and graphing	☐ Emphasize the importance of precise communication by encouraging students to focus on clarity of the definitions, notation, and vocabulary to convey their meaning. ☐ Encourage accuracy and efficiency in computation and problem-based solutions, expressing numerical answers, data and/or measurements with a degree of precision appropriate for the context of the problem.
Reasoning and Explaining	**2. Reason abstractly and quantitatively**	☐ Make sense of quantities and relationships in problem situations ☐ Represent abstract situations symbolically and understand the meaning of quantities ☐ Create a coherent representation of the problem at hand ☐ Consider the units involved ☐ Flexibly use properties of operations	☐ Facilitate opportunities for students to discuss or use representations to make sense of quantities and their relationships. ☐ Encourage the flexible use of operations, objects, and solution strategies when solving problems. ☐ Provide opportunities for student to decontextualize (abstract a situation) and/or contextualize (identify referents for symbols involved) the mathematics they are learning.
	3. Construct viable arguments and critique the reasoning of others	☐ Use definitions and previously established causes/effects (results) in constructing arguments ☐ Make conjectures and use counterexamples to build a logical progression of statements to explore and support their ideas ☐ Communicate and defend mathematical reasoning using objects, drawings, diagrams, actions ☐ Listen to or read the arguments of others ☐ Decide if the arguments of others make sense and ask probing questions to clarify or improve the arguments	☐ Provide and orchestrate opportunities for students to list to the solution strategies of others, discuss alternate solutions, and defend their ideas. ☐ Ask higher-order questions that encourage students to defend their ideas. ☐ Provide prompts that encourage students to think critically about the mathematics they are learning.

Modeling and Using Tools	**4. Model with mathematics**	☐ Apply prior knowledge to solve real world problems ☐ Identify important quantities and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas ☐ Make assumptions and approximations to make a problem simpler ☐ Check to see if an answer makes sense within the context of a situation and change a model when necessary	☐ Use mathematical models appropriate for the lesson ☐ Encourage student use of developmentally and content-appropriate mathematical models (e.g., variables, equations, coordinate grids.) ☐ Remind students that a mathematical model used to represent a problem's solution is a work in progress, and may be revised as needed.
	5. Use appropriate tools strategically.	☐ Make sound decisions about the use of specific tools. Examples might include: ☐ Calculator ☐ Concrete models ☐ Digital Technology ☐ Pencil/paper ☐ Ruler, compass, protractor ☐ Use technological tools to visualize the results of assumptions, explore consequences and compare predications with data ☐ Identify relevant external math resources (digital content on a website) and use them to pose or solve problems ☐ Use technological tools to explore and deepen understanding of concepts	☐ Use appropriate physical and/or digital tools to represent, explore, and deepen student understanding ☐ Help students make sound decisions concerning the use of specific tools appropriate for the grade-level and content focus of the lesson ☐ Provide access to materials, models, tools, and/or technology-based resources that assist student in making conjecture necessary for solving problems.
Seeing structure and generalizing	**7. Look for and make use of structure**	☐ Look for patterns or structure, recognizing that quantities can be represented in different ways ☐ Recognize the significance in concepts and models and use the patterns or structure for solving related problems ☐ View complicated quantities both as single objects or compositions of several objects and use operations to make sense of problems	☐ Engage students in discussions emphasizing relationships between particular topics within a contend domain or across content domains ☐ Recognize that the quantitative relationships modeled by operations and their properties remain important regardless of the operational focus of a lesson ☐ Provide activities in which students demonstrate their flexibility in representing mathematics in a number of ways, e.g., 76=(7 x 10) + 6; discussing types of quadrilaterals, and so on
	8. Look for and express regularity in repeated reasoning	☐ Notice repeated calculations and look for general methods and shortcuts ☐ Continually evaluate the reasonableness of intermediate results (comparing estimates) while attending to details and make generalizations based on findings	☐ Engage students in discussion related to repeated reasoning that may occur in a problem's solution. ☐ Draw attention to the prerequisite steps necessary to consider when solving a problem ☐ Urge students to continually evaluate the reasonableness of their results.