Contents

1 The Reals **1**
1.1 Zeno's Paradoxes . . . 1
1.2 Basic Set Theory Definitions . . . 4
1.3 What is a Number? . . . 8
1.4 Ordered Fields . . . 12
1.5 The Completeness Axiom . . . 19
1.6 Working with Sups and Infs . . . 24
1.7 The Archimedean Principle . . . 28
Exercises . . . 35

2 Cardinality **43**
2.1 Bijections and Cardinality . . . 43
2.2 Counting Infinities . . . 45
2.3 How Many Infinities Are There? . . . 56
Exercises . . . 60

3 Sequences **65**
3.1 Basic Sequence Definitions . . . 65
3.2 Bounded Sequences . . . 66
3.3 Convergent Sequences . . . 69
3.4 Divergent Sequences . . . 77
3.5 Limit Laws . . . 85
3.6 The Monotone Convergence Theorem . . . 90
3.7 Subsequences . . . 94
3.8 The Bolzano-Weierstrass Theorem . . . 99
3.9 The Cauchy Criterion . . . 102
Exercises . . . 110

4 Series **117**
4.1 Sequences of Partial Sums . . . 117
4.2 Series Convergence Tests . . . 121
4.3 Absolute Convergence . . . 130
4.4 Rearrangements . . . 134
Exercises . . . 140

5 The Topology of $\mathbb{R}$ **149**
5.1 Open Sets . . . 149
5.2 Closed Sets . . . 153
5.3 Open Covers . . . 157
5.4 The Greatest Definition in Mathematics . . . 158
Exercises . . . 164

6 Continuity **171**
6.1 Approaching Continuity . . . 171
6.2 Weird Examples . . . 172
6.3 Functional Limits . . . 177
6.4 Properties of Functional Limits . . . 183
6.5 Continuity . . . 186
6.6 Topological Continuity . . . 190
6.7 The Extreme Value Theorem . . . 195
6.8 The Intermediate Value Theorem . . . 198
6.9 Uniform Continuity . . . 202
Exercises . . . 208

7 Differentiation **217**
7.1 Graphical Interpretations of Velocity . . . 218
7.2 The Derivative . . . 222
7.3 Continuity and Differentiability . . . 224
7.4 Differentiability Rules . . . 227
7.5 Topologist's Sine Curve Examples . . . 232
7.6 Local Minimums and Maximums . . . 235
7.7 The Mean Value Theorems . . . 238
7.8 L'Hôpital's Rule . . . 244
Exercises . . . 250

8 Integration **257**
8.1 The Area of a Circle . . . 258
8.2 Simplistic Approach . . . 264
8.3 The Darboux Integral . . . 269
8.4 Integrability . . . 274
8.5 Integrability Criteria . . . 279
8.6 Integrability of Continuous Functions . . . 281
8.7 Integrability of Discontinuous Functions . . . 284
8.8 The Measure Zero Integrability Criterion . . . 293
8.9 Linearity Properties of the Integral . . . 294
8.10 More Properties of the Integral . . . 297
8.11 The Fundamental Theorem of Calculus . . . 301
8.12 Integration Rules . . . 306
Exercises . . . 311

9 Sequences and Series of Functions **323**
9.1 Introduction to Pointwise Convergence . . . 323
9.2 Continuity and Functional Convergence . . . 326
9.3 Other Properties with Functional Convergence . . . 331
9.4 Convergence of Derivatives and Integrals . . . 334
9.5 Series of Functions . . . 342
9.6 Power Series . . . 346
9.7 Properties of Power Series . . . 349
9.8 New Power Series from Old . . . 351
9.9 Taylor and Maclaurin series . . . 353
9.10 A beautiful application . . . 359
Exercises . . . 362

Appendices **369**

A Construction of $\mathbb{R}$ **371**
A.1 Axioms of Set Theory . . . 372
A.2 Constructing $\mathbb{N}$ . . . 373
A.3 Constructing $\mathbb{Z}$ . . . 375
A.4 Constructing $\mathbb{Q}$ . . . 376
A.5 Constructing $\mathbb{R}$ . . . 377

B Peculiar and Pathological Examples **379**
B.1 The Curious Case of the Cantor Set . . . 381
B.2 Doubled Digits of Diametrical Degrees . . . 384
B.3 Structuring Stuff from its Shadows' Shapes . . . 386
B.4 Menger's Matterless Material . . . 388
B.5 Obtainable Outrageousness in an Orderly Overhang . . . 391
B.6 A Composition Conundrum . . . 396
B.7 Turning the Tables on your Teetering Troubles . . . 398
B.8 A Devilish De-Descent . . . 400
B.9 A Pack of Pretty Proofs by Picture . . . 404
B.10 An Abundant Addition Aboundingly Ascends . . . 407
B.11 A Smooth and Spiky Solution . . . 408
B.12 Finding ϕ for First-Place Finishes . . . 412
B.13 Peculiar and Pathological Perimeters . . . 414
B.14 Fractal Functions Filling Foursquare Frames . . . 416
B.15 A Prestigious Proof of a Primal Puzzle . . . 419
B.16 A Topological Treatment with a Tremendous Twist . . . 422
B.17 Modern Measuring's Misfit Member . . . 424
B.18 Tarski's Terrific Talents Times Two . . . 426

Index **429**