Chapter 1 Solved Exercises

Question 1. (Exercise 1.2) Explain the error in the following "proof" that $2 = 1$.

Let $x = y$. Then

$$\begin{aligned} x^2 &= xy \\ x^2 - y^2 &= xy - y^2 \\ (x+y)(x-y) &= y(x-y) \\ x + y &= y \\ 2y &= y \\ 2 &= 1. \end{aligned}$$

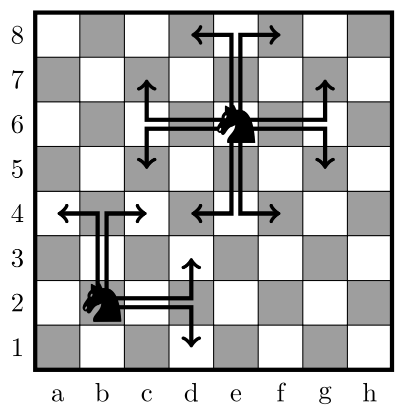
Question 2. (Exercise 1.5) If I remove two squares of different colors from an 8×8 chessboard, must the result have a perfect cover?

Question 3. (Exercise 1.6) If I remove four squares—two black, two white—from an 8×8 chessboard, must the result have a perfect cover?

→ If you believe a perfect cover must exist, justify why.

→ If you believe a perfect cover does not need to exist, give an example of four squares that you could remove for which the result does not have a perfect cover.

Question 4. (Exercise 1.7) In chess, a *knight* is a piece that can move two squares vertically and one square horizontally, or two squares horizontally and one square vertically. Below are two examples of knights, and the squares to which they could move.



A knight can legally move to any one of these squares, provided there is not another piece on that same square.

(a) Suppose there is a knight on every square of a 7×7 chessboard. Is it possible for every one of these knights to simultaneously make a legal move?

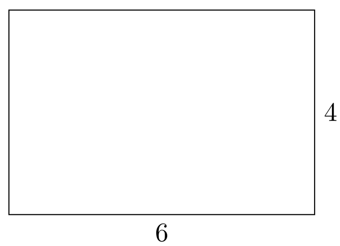
(b) Suppose there is a knight on every square of a 8×8 chessboard. Is it possible for every one of these knights to simultaneously make a legal move?

Question 5. (Exercise 1.9) Assume that n is a positive integer. Prove that if one selects any $n+1$ numbers from the set $\{1, 2, 3, \dots, 2n\}$, then two of the selected numbers will sum to $2n+1$. Also, before your proof, write down 4 numbers from $\{1, 2, 3, 4, 5, 6\}$ and locate two of them which sum to 7. Then, repeat for 5 numbers from $\{1, 2, \dots, 8\}$ and 6 numbers from $\{1, 2, \dots, 10\}$.

Question 6. (Exercise 1.16) Give an example of 100 numbers from $\{1, 2, 3, \dots, 200\}$ such that none of your selected numbers divides any of the others. By doing so, this proves that Proposition 1.11 is optimal.

Question 7. (Exercise 1.17) Prove that any set of seven integers contains a pair whose sum or difference is divisible by 10. Also, before your proof, write down three different sets of seven integers, and for each set locate a pair whose sum or difference is divisible by 10. Have your sets contain a diverse collection of integers — some bigger, some smaller, some positive, some negative.

Question 8. (Exercise 1.18) Prove that if one chooses any 19 points from the interior of a 6×4 rectangle and no three of them form a straight line, then there must exist four of these points which form a quadrilateral of area at most 4.



Note: a quadrilateral is a four-sided shape, and the condition that no three form a straight line is simply to guarantee that any four points form a quadrilateral.

Question 9. (Exercise 1.22) The following conjectures are all false. Prove that they are false by finding a counterexample to each.

(a) Conjecture 1: If x and y are real numbers, then $|x+y| = |x| + |y|$.

(b) Conjecture 2: If x is a real number, then $x^2 < x^4$.

(c) Conjecture 3: Suppose x and y are real numbers. If $|x+y| = |x-y|$, then $y = 0$.

Question 10. (Exercise 1.27) A *magic square* is an $n \times n$ matrix where the sum of the entries in each row, column and diagonal equal the same value. For example,

8	1	6
3	5	7
4	9	2

is a 3×3 matrix whose three rows, three columns, and two diagonals each sum to 15. Thus, this is a magic square.

An *antimagic square* is an $n \times n$ matrix where each row, column and diagonal sums to a distinct value. For example,

9	4	5
10	3	-2
6	9	7

is a 3×3 matrix whose rows sum to 18, 11 and 22, columns sum to 25, 16 and 10, and diagonals sum to 19 and 14. Notice that all eight of these numbers is different than the rest, showing that this is an antimagic square.

Prove that, for every $n \geq 2$, there does not exist an $n \times n$ antimagic square where each entry is -1, 0 or 1.