Chapter 3 Solved Exercises

Question 1. (Nearly Exercise 3.3 parts (b),(c),(j) and (k)) Rewrite each of the following sets by listing their elements between braces. If it is an infinite set, write out enough elements for the reader to see the pattern, then use ellipses.

(b) $\{n \in \mathbb{N} : -7 \leq n < 6\}$

(c) $\{4n : n \in \mathbb{Z} \text{ and } |2n| < 7\}$

(j) $\{\frac{m}{n} \in \mathbb{Q} : \left|\frac{m}{n}\right| < 1 \text{ and } 1 \leq n \leq 4\}$

(k) $\{x \in \mathbb{R} : x^2 - 1 = 0\}$

Question 2. (Nearly Exercise 3.5) Rewrite each of the following sets in set-builder notation.

(a) $\{2, 5, 8, 11, 14, \dots\}$

(b) $\left\{\dots, -\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \dots\right\}$

(c) $\{-4, -3, -2, -1, 0, 1\}$

(d) $\left\{\dots, -\frac{3125}{243}, -\frac{125}{27}, -\frac{5}{3}, 1, \frac{25}{9}, \frac{625}{81}, \dots\right\}$

Question 3. (Exercise 3.8) Determine whether each of the following is true or false.[1]

(a) $1 \in \{1, \{1\}\}$

(b) $1 \subseteq \{1, \{1\}\}$

(c) $1 \in \mathcal{P}(\{1, \{1\}\})$

(d) $\{1\} \in \{1, \{1\}\}$

(e) $\{1\} \subseteq \{1, \{1\}\}$

(f) $\{1\} \in \mathcal{P}(\{1, \{1\}\})$

(g) $\{\{1\}\} \in \{1, \{1\}\}$

(h) $\{\{1\}\} \subseteq \{1, \{1\}\}$

(i) $\{\{1\}\} \in \mathcal{P}(\{1, \{1\}\})$

(j) $\emptyset \in \mathbb{N}$

(k) $\emptyset \subseteq \mathbb{N}$

(l) $\emptyset \in \mathcal{P}(\mathbb{N})$

(m) $\mathbb{Q} \times \mathbb{Q} \subseteq \mathbb{R} \times \mathbb{R}$

(n) $\mathbb{R}^2 \subseteq \mathbb{R}^3$

(o) $\emptyset \subseteq \{1, 2, 3\} \times \{a, b\}$.

Question 4. (Nearly Exercise 3.10) The set $\{7a + 2b : a, b \in \mathbb{Z}\}$ is equal to a familiar set. By examining which elements are possible, determine the familiar set.

Question 5. (Exercise 3.15) Suppose A and B are sets. Prove that

$$\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B).$$

Also, before your proof, write down an example of sets A and B, find their union and the powersets of all three, and check that this property holds in your example.

Question 6. (Exercise 3.23) Prove that

$$\{n \in \mathbb{Z} : n \equiv 1 \pmod 4\} \not\subseteq \{n \in \mathbb{Z} : n \equiv 1 \pmod 8\}.$$

Question 7. (Exercise 3.26) Suppose someone conjectured that $A \cup (B \cap C) = (A \cup B) \cap C$ for any sets A, B and C. Find three sets which are a counterexample to this conjecture. Also, draw Venn diagrams for each to see why they do not align. (Note: This shows that parentheses matter for these set operations! Writing "$A \cup B \cap C$" is ambiguous.)

[1] For part (n), note that $\mathbb{R}^2$ is defined to be $\mathbb{R} \times \mathbb{R}$, which is the set of ordered pairs of real numbers. Meanwhile, $\mathbb{R}^3$ is the set of ordered triples of real numbers.

Question 8. (Exercise 3.27) Suppose someone conjectured that, for any sets A and B which contain finitely many elements, we have

$$|A \cup B| = |A| + |B|.$$

Determine whether this conjecture is true or false. If it is true, prove it. If it is false, give a counterexample demonstrating that it is false, and then conjecture a different formula for $|A \cup B|$.

Question 9. (Exercise 3.31 part (a)) Suppose A, B and C are sets. Prove that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Question 10. (Exercise 3.40 part (a)) Let C be any set. Prove that there is a unique set $A \in \mathcal{P}(C)$ such that for every $B \in \mathcal{P}(C)$ we have $A \cup B = B$.