Chapter 4 Solved Exercises

Question 1. (Exercise 4.1) Prove that the sum of the first n odd natural numbers equals n^2 by induction or strong induction.

Question 2. (Exercise 4.3 part (f)) Use induction or strong induction to prove that $5 \mid (6^n - 1)$ for every $n \in \mathbb{N}$.

Question 3. (Exercise 4.4 parts (e) and (f)) Prove that each of the following hold for every $n \in \mathbb{N}$.

(e) $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$

(f) $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1$

Question 4. (Exercise 4.5 part (e)) Prove that $1 + 2^n \leq 3^n$ for every $n \in \mathbb{N}$. Also, before each proof, pick three specific n-values and verify that the result holds for those values.

Question 5. (Exercise 4.8) The *Fermat number* $\tilde{F}_n$ is defined to be $\tilde{F}_n = 2^{2^n} + 1$ for $n \geq 0$. For example, $\tilde{F}_3 = 2^{2^3} + 1 = 2^8 + 1 = 256 + 1 = 257$. The first five are $\tilde{F}_0 = 3$, $\tilde{F}_1 = 5$, $\tilde{F}_2 = 17$, $\tilde{F}_3 = 257$ and $\tilde{F}_4 = 65537$. These are all prime numbers, which lead Pierre de Fermat to conjecture that all the Fermat numbers are prime. As it turns out, $\tilde{F}_5$ is not prime (as Euler showed), and so far there is no known prime Fermat number after $\tilde{F}_4$. (Oops.)

Prove that, for every $n \in \mathbb{N}_0$,

$$\tilde{F}_0 \cdot \tilde{F}_1 \cdot \tilde{F}_2 \cdot \tilde{F}_3 \cdots \tilde{F}_n = \tilde{F}_{n+1} - 2.$$

Question 6. (Exercise 4.10) Explain the error in the "proof" of Fake Proposition 4.11.

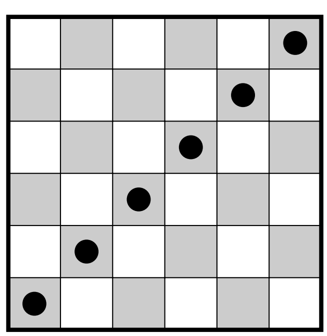
Question 7. (Exercise 4.15) Use induction to prove that if A is a set and $|A| = n$, then $|\mathcal{P}(A)| = 2^n$.

Question 8. (Exercise 4.24) The following conjecture is false. Prove that it is false by finding a counterexample.

<u>Conjecture:</u> For every positive integer n,

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} < 3.$$

Question 9. (Exercise 4.28) In chess, a rook attacks all the squares in its row and column. Consider the problem of placing n non-attacking rooks on an $n \times n$ chessboard; that is, n rooks such that none attack any other. One way to do this is to place the rooks on a single diagonal.



But that's boring. Prove that for every $n \geq 4$, it is possible to place n non-attacking rooks on the $n \times n$ chessboard so that none of the rooks are on either (or both) of the two diagonals.

Question 10. (Exercise 4.31 part (a)) Read the *Introduction to Sequences* following this chapter, and prove that

$$F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$$

for every $n \in \mathbb{N}$.