Toward a More Complete List of Completeness Axioms

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Abstract. We first discuss the *Cut Axiom*, due to Dedekind, which is one of the many equivalent formulations of the completeness of the real numbers. We point out that the Cut Axiom is equivalent to four "cornerstone theorems" of single-variable Real Analysis, namely, the *Intermediate, Extreme, and Mean Value Theorems*, as well as *Darboux's Theorem*.

We then describe some general properties of ordered fields, in particular the *Archimedean Property* and its consequences, and provide a list of statements that are equivalent to completeness and may thus serve as alternate completeness *axioms*.

1. INTRODUCTION. Various ways of "axiomatizing" the completeness of the real numbers are possible; popular ones (in textbooks on elementary real analysis) include the existence of suprema and the convergence of Cauchy sequences (plus the *Archimedean Property*). Another possibility is the *Cut Axiom,* which was originally formulated by Dedekind [**7**], and which has recently been introduced into elementary analysis as a simple and intuitive way of defining completeness [**1, 13, 14, 22**].

The Cut Axiom is easily seen to be equivalent to the *Intermediate Value Theorem* (IVT) [**22**]. In the first part of this note, we point out that the Cut Axiom, and thus the completeness of the real numbers, is also equivalent to other "cornerstone theorems" of single-variable real analysis, namely, the *Extreme Value Theorem* (EVT), the *Mean Value Theorem* (MVT), and *Darboux's Theorem* (DT). Any of these statements may therefore be used to "axiomatize" the completeness of the real numbers.

In fact, completeness turns out to be equivalent to many other statements of real analysis, which are typically presented as theorems. The second part of this paper is devoted to illustrating this. We first set the stage by broadening the view slightly and describing some general properties of totally ordered fields, in particular the Archimedean Property and its consequences. We then provide a (still very much incomplete) list of possible completeness axioms and assorted proofs.

PART I.

2. THE CUT AXIOM. One possible way of defining the *completeness of the real numbers* is given by the *Cut Axiom* (CA) [**1, 13, 14, 18, 22**]. For any pair of subsets $A, B \neq \emptyset$ satisfying :

$$\forall a \in A, b \in B \qquad a < b \tag{1}$$

and

$$\mathbb{R} = A \cup B, \tag{2}$$

there exists a number $c \in \mathbb{R}$ (called a *cut point for* A, B) such that

$$\forall a \in A, b \in B \qquad a \leq c \leq b. \tag{3}$$

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Remarks. (a) Properties (1) and (3) may be abbreviated by $A < B$ and $A \leq c \leq B$, respectively. (b) A pair of sets $A, B \neq \emptyset$ satisfying (1) and (2) will be called a *cut of* $\mathbb{R}$. (c) It is easy to show that cut points are unique (if two cut points $c_1 \neq c_2$ were to exist, $\frac{c_1+c_2}{2}$ would belong to $A \cap B$, which is impossible in view of (1)). (d) In the first part of this paper, our use of the letter $\mathbb{R}$ for the real numbers is somewhat lax in that we also use it for what might be called "proto-reals", i.e., (potentially) non-complete versions of the real numbers. We will be more careful later in Part II.

Cuts of the *rational numbers* are well known as *Dedekind cuts*, and form the basis of one of the standard *constructions* of the real numbers. However, it is interesting to note that in [**7**] Dedekind actually introduces and uses his cuts for both the *construction* of the real numbers ("the creation of irrational numbers") *and* the *axiomatic definition of completeness* (which he mostly refers to as *continuity*)[1]. After describing the "fact that that every point p of the straight line produces a separation of the same into two portions [...]" he continues

> I find the essence of continuity in the converse, i.e., in the following principle:
> If all points of the straight line fall into two classes such that every point in the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes [...].

Moreover, he leaves no doubt that he considers this "principle" an axiom (i.e., an unprovable *"basic truth"* in the parlance of B. Bolzano [**4**]; see Section 4 below):

> [...] I am utterly unable to adduce any proof of its correctness, nor has any one the power. The assumption of this property of the line is nothing else than an axiom by which we attribute to the line its continuity [...].

It is not too difficult to see that the Cut Axiom is equivalent to the existence of suprema (ES), which is one of the standard ways of formulating completeness and which is, maybe somewhat confusingly, often referred to as *Dedekind's completeness axiom*. Although the proof is available in the literature (see, e.g., [**3**] and [**1**] for a proof of "$\Rightarrow$"), we include it here for completeness.

Proposition 1. *The statements (CA) and (ES) are equivalent.*

Proof.
"$\Rightarrow$" Let $S \subset \mathbb{R}$ be nonempty and bounded above, and define $B := \{b \in \mathbb{R} \mid b \text{ is an upper bound for } S\} \neq \emptyset$, and $A := \mathbb{R} \setminus B \neq \emptyset$. Then (2) and $A < B$ hold by definition. Let $c \in \mathbb{R}$ be the (unique) cut point guaranteed by (CA). To show that c is a (the) least upper bound for S, assume first that $c \in A$. Then, since c is not an upper bound for S, there exists an $s \in S$ such that $c < s$; so letting $a = \frac{c+s}{2}$ implies $a \in A$ ($a < s$ means that a cannot be an upper bound for S), and $a \in B$ (since $a > c$ and c is a cut point), which is impossible. Thus $c \in B$, which means that c *is* an upper bound for S. Moreover, for any $\varepsilon > 0$, we have $c - \varepsilon < c$, hence $c - \varepsilon \in A$, which yields that $c - \varepsilon$ is not an upper bound. This completes the proof of "$\Rightarrow$".

[1] We do not mean to imply, however, that Dedekind made an explicit attempt at an axiomatization of analysis. We agree with one of the anonymous referees of this paper, who feels that "if Dedekind had been asked to provide an axiomatization of analysis, he would have used the Cut Principle as his axiom of continuity, but he did not provide such an axiomatization."

"$\Leftarrow$" Let A, B be a cut. Since $A < B$, the set A is (nonempty and) bounded above and hence $c := \sup(A)$ exists by (ES). To show that c is a (the) cut point for A, B, let $a \in A$ and $b \in B$. Since $\sup(A)$ is an upper bound for A, we immediately get $a \leq c$. Moreover, if b were $< c$, a number $a' \in A$ would exist such that $b < a' < c$, which is obviously impossible, as $A < B$. Thus $b \geq c$, which completes the proof. ■

Propp [**22**] notes that "the Cut Axiom is very similar to Axiom 3 from Tarski's axiomatization of the reals." In fact, Tarski's axiom (TA3) arises by simply dropping condition (2) from CA, and Tarski himself regarded it as a simplified version of Dedekind's: "This axiom—in a slightly more complicated formulation—originates with the German mathematician R. Dedekind [...]" [**29**].

Proposition 2. *The statements (CA) and (TA3) are equivalent.*

Proof.
"$\Leftarrow$" is clear.
"$\Rightarrow$" In view of Proposition 1, it is sufficient to show that (ES) implies (TA3). However, the proof of this statement is identical to the proof of "$\Leftarrow$" above (property (2) is never used in that proof). ■

Finally, it is also well-known that the completeness of the real numbers is equivalent to their connectedness (see, e.g., [**20**, Fact 1.1]). Here we will content ourselves with proving just one implication:

Proposition 3. *If (CA) does not hold, then there exists a cut of open sets.*

Proof. Since CA does not hold, there exists a cut A, B that does not possess a cut point. We will show that both sets A and B are open. First choose $c \in A$; then $(-\infty, c] \subset A$. If there was no $a \in A$ such that $a > c$, then $a \leq c$ for all $a \in A$ and $c < b$ for all $b \in B$ (since $c \in A$), so c would be a cut point. As a result, there must be an $a \in A$ such that $a > c$, which implies $[c, a] \subset A$ and so $c \in (-\infty, a) \subset A$. This shows that A is open. The argument for B is similar. ■

3. FOUR "PILLARS" OF CALCULUS. By this we mean the following theorems of the standard analysis curriculum.

In the following, let $f : [a, b] \to \mathbb{R}$ be an arbitrary continuous function (where $a, b \in \mathbb{R}$, $a < b$).

1. *Intermediate Value Theorem* (IVT). f has the intermediate value property (i.e., for any number $y \in \mathbb{R}$ such that $f(a) \leq y \leq f(b)$ or $f(b) \leq y \leq f(a)$, there exists a number $c \in [a, b]$ such that $f(c) = y$).
2. *Mean Value Theorem* (MVT). If f is differentiable on (a, b), then there exists a number $c \in (a, b)$ such that $f'(c) = (f(b) - f(a))/(b - a)$.
3. *Extreme Value Theorem* (EVT). There exists a number $c \in [a, b]$ such that $f(x) \leq f(c)$ for all $x \in [a, b]$.
4. *Darboux's Theorem* (DT). If f is differentiable on $[a, b]$, then f' has the intermediate value property.

The Cut Axiom (and hence completeness per se) turns out to be equivalent to each of the statements above. This fact is based on the following simple corollary of Proposition 3.

Lemma 1. *Assume that (CA) does not hold and let A, B be a cut of open sets. Then, if $g, h : \mathbb{R} \to \mathbb{R}$ are continuous (resp. differentiable) functions, the function*

$$f(x) = \begin{cases} g(x), & x \in A \\ h(x), & x \in B \end{cases}$$

is also continuous (resp. differentiable).

To prove that the statements (CA) and (#) are equivalent, where # ∈ {IVT, MVT, EVT, DT}, it will be sufficient to prove "⇐"; the implications "⇒" are part of the standard analysis curriculum, once the completeness of the reals has been established. In each case, the proof of "⇐" will be accomplished by a suitable choice of functions g, h such that f provides a counterexample for the statement in question (assuming that the Cut Axiom does not hold).

In each of the arguments below, assume that the Cut Axiom does not hold; let A, B be a cut such that both sets A and B are open and $a \in A$, $b \in B$.

3.1. The IVT does not hold, if (CA) is not satisfied. We choose $g \equiv 0$, $h \equiv 1$; the function f, defined by

$$f(x) = \begin{cases} 0, & x \in A \cap [a, b] \\ 1, & x \in B \cap [a, b], \end{cases}$$

is continuous, but it does *not* have the intermediate value property.

3.2. The MVT does not hold, if (CA) is not satisfied. We use the same function as in 3.1; f is differentiable on (a, b) and satisfies $f' \equiv 0$. However,

$$\frac{f(b) - f(a)}{b - a} = \frac{1}{b - a} > 0,$$

which completes the argument.

3.3. The EVT does not hold, if (CA) is not satisfied. Let $g(x) = x$ and $h(x) \equiv a - 1$; i.e.,

$$f(x) = \begin{cases} x, & x \in A \cap [a, b] \\ a - 1, & x \in B \cap [a, b]. \end{cases}$$

This function is continuous but does not attain its (global) maximum in $[a, b]$. Assume it does at $c \in [a, b]$. Clearly, c cannot be in $B \cap [a, b]$, since $f(x) = x \geq a > a - 1 = f(c)$ for all $c \in B \cap [a, b]$ and $x \in A \cap [a, b]$; so $c \in A \cap [a, b]$. However, since A is open, there exists an $a' \in A \cap [a, b]$ such that $a' > c$, which implies $f(a') = a' > c = f(c)$. This yields a contradiction.

3.4. DT does not hold, if (CA) is not satisfied. We use the same function as in 3.3; f is differentiable on $[a, b]$. However, since $f'([a, b]) = \{0, 1\}$, the derivative clearly does not have the intermediate value property.

3.5. Counterexamples in the rationals. If the underlying (incomplete) field is assumed to be the rationals $\mathbb{Q}$, additional and/or more concrete counterexamples can be given [**8**, Chapter 1, Examples 11 a–d]:

(a) the sets $A, B \subset \mathbb{Q}$ in 3.1 may be specialized to $A = (-\infty, \sqrt{2})$ and $B = (\sqrt{2}, \infty)$;

(b) another function failing to have the intermediate value property is given by $f : \mathbb{Q} \cap [1, 2] \to \mathbb{Q}$, $f(x) = x^2$ (the value 2 between $1 = f(1)$ and $4 = f(2)$ is not assumed);

(c) an unbounded continuous function $f : \mathbb{Q} \cap [1, 2] \to \mathbb{Q}$ is given by $f(x) = 1/(x^2 - 2)$;

(d) a *bounded* continuous (even *uniformly* continuous) function $f : \mathbb{Q} \cap [0, 1] \to \mathbb{Q}$ not attaining its maximum in $\mathbb{Q} \cap [0, 1]$ is given by $f(x) = x - x^3$ (as a function $\mathbb{R} \to \mathbb{R}$ the functions assumes its maximum at $x = 1/\sqrt{3} \in \mathbb{R} \setminus \mathbb{Q}$).

4. DISCUSSION. The four "Pillars of Calculus" of this section (IVT, MVT, EVT, DT) are *existence theorems*, which can be difficult to teach to beginning students. Students often have a very hard time appreciating the content/meaning of these statements, or even seeing any point in stating them at all. In fact, one first-year student recently told me that her high school teacher had called these statements[2] *"silly theorems"*, presumably because they are so obvious/self-evident that only "silly" mathematicians would make any fuss about them.

The fact that these "silly theorems" turn out to be equivalent to the completeness of the real numbers may offer a philosophical "explanation" as to why they almost seem to be too obvious. After all, the completeness of the reals itself appears completely self-evident and only becomes "a problem" when somebody (like B. Bolzano [**4**]) sets out to prove self-evident facts such as the *Intermediate Value Theorem*.

In fact, Bolzano appears to have been the first to recognize the connection between the Intermediate Value Theorem and completeness. He also regarded the Intermediate Value Theorem as so self-evident that its mere *confirmation* would not have warranted a proof [**4**]:

> [...] while the geometrical truth to which we refer here is [...] extremely *evident*, and therefore needs no proof in the sense of confirmation, it nonetheless needs *justification*. For its component concepts are obviously so combined that one cannot hesitate for a moment to say that it is not one of those *simple* truths which are called basic propositions, or *basic* truths [...]. It is a *theorem* or *consequent-truth* [...] and, therefore, [...] it must be proved by derivation from these other truths.

Ironically, the Intermediate Value Theorem in the end turns out to be a *simple* or *basic truth* (i.e., an axiom) after all, being equivalent to the completeness of the reals.

PART II.

We now adopt a slightly more general point of view than the one taken in Part I, in that we consider *(totally) ordered fields* in general, rather than focussing on the real numbers specifically. The two most interesting and far-reaching properties of ordered fields are the *Archimedean Property* and *(Dedekind) Completeness*.

[2] Most likely, referring only to the first three, as Darboux's Theorem is almost never mentioned in high school.

5. ORDERED FIELDS. Most of the material in this section is available in standard texts such as [**3, 10, 11, 15, 17, 30**]. A recent introduction to ordered fields emphasizing their completeness properties is [**9**].

5.1. Definitions and Notation. A *(totally) ordered field* $\mathbb{F}$ is a (commutative) field with an *order relation* $<$ such that the known rules for operating on inequalities are satisfied. Every ordered field has an *absolute value* $|.|$ given by $|x| = \max\{x, -x\}$, which satisfies the usual properties such as the triangle inequality. It is easy to see that the field $\mathbb{Q}$ is *canonically imbedded* into $\mathbb{F}$; so we consider $\mathbb{Q}$ an *ordered subfield* of $\mathbb{F}$.

We use the usual notation for intervals such as $[a, b] = \{x \in \mathbb{F} \mid a \leq x \leq b\}$, $(-\infty, b) = \{x \in\mid x < b\}$, etc., as well as the definitions $\mathbb{F}_+ := \{x \in \mathbb{F} \mid x > 0\}$, $I(\mathbb{F}) := \{x \in \mathbb{F} \mid \forall n \in \mathbb{N} \quad |x| < \frac{1}{n}\}$, and $L(\mathbb{F}) := \{x \in \mathbb{F} \mid \forall n \in \mathbb{N} \quad |x| > n\}$. The elements in $I(\mathbb{F})$ and $L(\mathbb{F})$ are called *infinitesimals*[3] and *infinitely large*, respectively. Numbers in $\mathbb{F} \setminus (I(\mathbb{F}) \cup L(\mathbb{F}))$ are called *finite.* Note that $x \in I(\mathbb{F}) \setminus \{0\}$ if and only if $1/x \in L(\mathbb{F})$.

A subset $S \subset \mathbb{F}$ is called *bounded above* if there exists a number $M \in \mathbb{F}$ such that $x \leq M$ for all $x \in S$ (similarly, *bounded below* and *bounded "per se"*).

Every ordered field $\mathbb{F}$ is equipped with the *order topology*, a basis of which is given by the open intervals (a, b), $a, b \in \mathbb{F}$ with $a \leq b$ (the closed intervals $[a, b]$ also form a basis). All topological properties such as continuity, convergence, and connectedness will always be understood with respect to the order topology of $\mathbb{F}$.

As in Part I, we define a *cut of* $\mathbb{F}$ as a pair of nonempty subsets $A, B \subset \mathbb{F}$ such that (1) and (2) are satisfied (with $\mathbb{R}$ replaced by $\mathbb{F}$). A cut is called a *gap* if it does not possess a cut point. The sets A and B of a gap are both open (see the proof of Proposition 3). A cut A, B is called *regular* if

$$\forall \varepsilon \in \mathbb{F}_+ \, \exists a \in A, b \in B \qquad b - a < \varepsilon.$$

A cut that is not regular is called *irregular.* Obviously, an irregular cut A, B satisfies $B - A \geq \delta$ for some $\delta \in \mathbb{F}_+$ (where $B - A = \{b - a \mid a \in A, b \in B\}$). From this it easily follows that every irregular cut is a gap.

An ordered field is called *complete* if it satisfies the *Cut Axiom*, which may now be succinctly expressed as the absence of gaps.

5.2. Examples. The rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$ are ordered fields; the *complex numbers* $\mathbb{C}$ and the *p-adic numbers* $\mathbb{Q}_p$ are not ordered fields.

The fields $\mathbb{Q}$ and $\mathbb{R}$ are both *Archimedean* (see Section 6 below). The simplest example of a *non*-Archimedean ordered field containing $\mathbb{R}$ as a subfield is the field $\mathbb{R}(x)$ of *rational functions* in the variable x with coefficients in $\mathbb{R}$. This is the *fraction field* of the *ordered domain* $\mathbb{R}[x]$ *of polynomials over* $\mathbb{R}$. A rational function $\frac{P(x)}{Q(x)} \in \mathbb{R}(x)$ is positive if the product of the trailing coefficients of the polynomials P and Q are positive.[4] With respect to this order, the function $f(x) = x$ is a positive infinitesimal

[3] According to the convention adopted here, zero *is* an infinitesimal.

[4] This is the unique order for $\mathbb{R}(x)$ such that the imbedding $\mathbb{R}[x] \hookrightarrow \mathbb{R}(x)$ is an order-morphism, if the integral domain $\mathbb{R}[x]$ is equipped with the order given by the *trailing* coefficients (i.e., $\sum_{k=m}^{n} a_k x^k > 0$ if and only if $a_m > 0$). Typically, $\mathbb{R}[x]$ is considered with the order defined by the *leading* coefficients ($\sum_{k=m}^{n} a_k x^k > 0$ if and only if $a_n > 0$). However, the convention adopted here has the advantage that $\mathbb{R}[x]$ is an ordered subring of $\mathbb{R}[\![x]\!]$. For information on ordered domains, their polynomial rings, and (ordered) fraction fields, see, e.g., [**17**]; the fact that $\mathbb{R}(x)$ (with the "traditional" order) is an ordered field is also the topic of [**12**, Exercise 1.31].

(showing that $\mathbb{R}(x)$ is not Archimedean; see Proposition 4 (ii) following); indeed, for any $n \in \mathbb{N}$, the trailing coefficient of $x - \frac{1}{n}$ is $-\frac{1}{n} < 0$, which implies $x < \frac{1}{n}$.

The fraction field of the ordered domain $\mathbb{R}[\![x]\!]$ of *formal power series over* $\mathbb{R}$ is the field of *formal Laurent series* $\mathbb{R}((x)) = \left\{\sum_{k=m}^{\infty} a_k x^k \mid a_k \in \mathbb{R}, m \in \mathbb{Z}\right\}$, where $\sum_{k=m}^{\infty} a_k x^k > 0$ if and only if $a_m > 0$ [**17**]. This field is non-Archimedean (series $\sum_{k=m}^{\infty} a_k x^k$ whose order m is ≥ 1 are infinitesimals) and contains $\mathbb{R}$. In addtion, it is *Cauchy complete*; i.e., it has the property that *every Cauchy sequence converges* (for a proof of the Cauchy completeness of $\mathbb{R}((x))$, see, e.g., [**16**]).

Finally, we mention the *Levi–Civita field* $\mathcal{R}$, which can be defined as the set of all functions $f : \mathbb{Q} \to \mathbb{R}$ such that $\{q \in \mathbb{Q} \mid f(q) \neq 0\} \cap (-\infty, r)$ is finite for every $r \in \mathbb{Q}$, with $f > 0$ if and only if $f(\min\{q \in \mathbb{Q} \mid f(q) \neq 0\}) > 0$. This field is non-Archimedean, contains $\mathbb{R}$, and is Cauchy complete. In addition, it is *"real-closed"*, which means that $\mathcal{R}_+ = \{f^2 \mid f \in \mathcal{R} \setminus \{0\}\}$ and every odd-degree polynomial with coefficients in $\mathcal{R}$ has a root, or, equivalently, that every polynomial has the intermediate value property. For proofs and more information on $\mathcal{R}$, see [**2**].

6. THE ARCHIMEDEAN PROPERTY.

Definition. An ordered field $\mathbb{F}$ satisfies the *Archimedean Property* (AP) if and only if

$$\forall x \in \mathbb{F}\, \exists n \in \mathbb{N} \qquad |x| < n.$$

A field that satisfies (AP) is called *Archimedean*; otherwise it is called *non-Archimedean*.

An ordered field $\mathbb{F}$ is said to have *countable cofinality* if there exists a countable subset $L \subset \mathbb{F}$ that is *"cofinal"*, i.e., that satisfies

$$\forall x \in \mathbb{F}\, \exists \lambda \in L \qquad |x| < \lambda.$$

Clearly, every Archimedean ordered field has countable cofinality, but the converse is not true. In fact, all the fields in Section 5.2 have countable cofinality (for instance, the set $L = \left\{\left(\frac{1}{x}\right)^n \mid n \in \mathbb{N}\right\} \subset \mathbb{R}((x))$ is countable and cofinal), but $\mathbb{R}(x)$, $\mathbb{R}((x))$, and $\mathcal{R}$ are all non-Archimedean.

Standing Assumption. For the remainder of our work, we will always assume that the ordered fields under consideration have countable cofinality.[5]

Proposition 4. *For any ordered field* $\mathbb{F}$ *the following statements are equivalent.*

(i) $\mathbb{F}$ *is Archimedean.*
(ii) $I(\mathbb{F}) = \{0\}$.
(iii) $L(\mathbb{F}) = \emptyset$.
(iv) $\mathbb{Q}$ *is dense in* $\mathbb{F}$ *(i.e., between any two distinct numbers lies a rational number).*
(v) $\mathbb{F}$ *can be imbedded into* $\mathbb{R}$ *(i.e., considered an ordered subfield of* $\mathbb{R}$*).*
(vi) *The sequence* $\left(\frac{1}{n}\right)_{n \geq 1}$ *is convergent.*
(vii) *Every cut of* $\mathbb{F}$ *is regular.*
(viii) *Every increasing sequence that is bounded above is a Cauchy sequence.*

[5]This assumption is not always necessary. However, in order to avoid having to distinguish cases, we make it "globally" for the remainder of this paper.

(ix) *Every uniformly continuous function maps bounded sets to bounded sets.*[6]

(x) *Every monotone function is Darboux integrable.*[7]

Proof. The fact that the statements (i)–(vi) are equivalent is standard material in the theory of ordered fields (see, e.g., [**6, 9, 11**]). So we focus on statements (vii)–(x).

(i) $\Rightarrow$ (vii). This can be shown by using a classical bisection idea; let A, B be a cut and $a \in A$, $b \in B$. We define sequences $(a_n) \subset A$, $(b_n) \subset B$, and $(c_n) \subset \mathbb{R}$ recursively by $a_0 = a$, $b_0 = b$; $c_n = (a_n + b_n)/2$; $a_{n+1} = c_n$ if $c_n \in A$; $a_{n+1} = a_n$ if $c_n \notin A$; $b_{n+1} = b_n$ if $c_n \in A$; $b_{n+1} = c_n$ if $c_n \notin A$. Then $b_n - a_n = 2^{-n}(b-a)$ by construction. So, given $\varepsilon \in \mathbb{F}_+$, by (AP) and the elementary estimate $2^{-n} < 1/n$, there exists $n \in \mathbb{N}$ such that $b_n - a_n = 2^{-n}(b-a) < \varepsilon$.

$\neg$(i) $\Rightarrow$ $\neg$(vii). See Lemma A in Appendix A1.

(i) $\Rightarrow$ (viii). This follows by standard elementary analysis.

$\neg$(i) $\Rightarrow$ $\neg$(viii). If $\mathbb{F}$ is not Archimedean, it possesses a positive infinitesimal $\delta \in I(\mathbb{F}) \cap \mathbb{F}_+$. Let $x_n := n\delta$. Then (x_n) is (strictly) increasing and bounded (by 1). However, $x_{n+1} - x_n = \delta > 0$ for all $n \in \mathbb{N}$, so (x_n) is not Cauchy.

(i) $\Rightarrow$ (ix). The proof is standard fare in elementary analysis. We nevertheless present it here to emphasize the fact that it does *not* require completeness. Let $I \subset \mathbb{F}$ be bounded, $f : I \to \mathbb{F}$ uniformly continuous, and $\delta \in \mathbb{F}_+$ such that $|f(x) - f(y)| < 1$ for all $x, y \in I, |x-y| < \delta$. Assume that $f(I)$ is not bounded. Then there exists a sequence $(x_n) \subset I$ such that $|f(x_n)| \geq n$ for all $n \in \mathbb{N}$. By selecting a monotonic subsequence if necessary, we may assume that (x_n) is monotonic. Since it is bounded, by (AP) and (viii), it is a Cauchy sequence, so there exists an $n \in \mathbb{N}$ such that $|x_{n+k} - x_n| < \delta$ for all $k \in \mathbb{N}$. Thus

$$\forall k \in \mathbb{N} \quad n + k \leq |f(x_{n+k})| \leq |f(x_n)| + |f(x_{n+k}) - f(x_n)| < |f(x_n)| + 1,$$

which is a contradiction to (AP).

$\neg$(i) $\Rightarrow$ $\neg$(ix). Assume that $\mathbb{F}$ is not Archimedean. We will show that we can construct a function that is

uniformly continuous, defined on a bounded set, but <u>un</u>bounded.

To this end, let $x_n = 1 - \frac{1}{n}$ $(n \geq 1)$. Define

$$I := \{x \in [0,1) \mid \exists n \in \mathbb{N} \quad x \leq x_n\} = \bigcup_{n \geq 1} [x_n, x_{n+1}] \subset [0,1)$$

with $y_n = \frac{x_{n+1}+x_n}{2} \in (x_n, x_{n+1})$, and choose a positive infinitesimal δ. Moreover, for each $n \in \mathbb{N}$, let $\tilde{A}_n, \tilde{B}_n$ be an irregular gap satisfying

$$y_n \in \tilde{A}_n, \tag{4a}$$

$$\forall a \in \tilde{A}_n, b \in \tilde{B}_n \; b - a \geq \delta, \qquad \text{and} \tag{4b}$$

$$\forall y \in [y_n, \infty) \qquad y - y_n \text{ finite implies that } y \in \tilde{B}_n \tag{4c}$$

(such cuts exist by Lemma A in Appendix A1). Note that (4a) and (4c) imply $A_n := \tilde{A}_n \cap [x_n, x_{n+1}] \neq \emptyset$ and $B_n := \tilde{B}_n \cap [x_n, x_{n+1}] \neq \emptyset$, respectively, since $y_n \in (x_n, x_{n+1})$ and $x_{n+1} - y_n$ is finite. Finally, let $(\lambda_n)_{n \in \mathbb{N}} \subset \mathbb{F}$ be a cofinal sequence (which exists,

[6]The countable cofinality of $\mathbb{F}$ is only used in the proof of the implication "(ix) $\Rightarrow$ (i)".

[7]For information on integration in ordered fields, see Appendix A2.

since $\mathbb{F}$ is assumed to have countable cofinality). Now define $f : I \to \mathbb{F}$ by

$$\forall n \in \mathbb{N}, x \in [x_n, x_{n+1}] \qquad f(x) = \begin{cases} \lambda_n, & x \in A_n \\ \lambda_{n+1}, & x \in B_n. \end{cases}$$

Clearly, f is (well-defined and) unbounded. It remains to be shown that it is uniformly continuous; this will certainly follow if we can show that $f(x) - f(y) = 0$ for all $x, y \in I$ such that $|x - y| < \delta$. Assume that $x \in [x_n, x_{n+1}]$ and, without loss of generality, $x < y$. Then, by (4b), $x \in A_n$ and $y \in B_m$ is not possible for any $m \geq n$; so either $x, y \in A_n$ or $x, y \in B_n$ or $x \in B_n$, $y \in A_{n+1}$. In the first case we have $f(x_n) = f(y_n) = \lambda_n$; in the second and third case, $f(x_n) = f(y_n) = \lambda_{n+1}$.

(i) $\Rightarrow$ (x). The proof is standard again, but we want to demonstrate where the Archimedean Property comes into play. Assume without loss of generality that $f : [a, b] \to \mathbb{F}$ is increasing. For $n \in \mathbb{N}$, let $\Delta x = \frac{b-a}{n}$, $x_j = j\Delta x$ $(j = 0, \ldots, n)$, $I_j = [x_{j-1}, x_j)$ $(j = 1, \ldots, n-1)$, and $I_n = [x_{n-1}, x_n]$. Then $f(x_{j-1}) \leq f(x) \leq f(x_j)$ for $x \in I_j$ and the functions $g(x) = \sum_{j=1}^{n} f(x_{j-1})\chi_{I_j}(x)$ and $h(x) = \sum_{j=1}^{n} f(x_j)\chi_{I_j}(x)$ are step functions satisfying $g(x) \leq f(x) \leq h(x)$. (Here χ_S denotes the *characteristic function* of a set $S \subset \mathbb{F}$.) Moreover, $\int g = \Delta x \sum_{j=1}^{n} f(x_{j-1})$ and $\int h = \Delta x \sum_{j=1}^{n} f(x_j)$, hence $\int h - \int g = \frac{b-a}{n}(f(b) - f(a)) \longrightarrow 0$ (as $n \to \infty$), where the limit property follows from (AP) and (vi).

$\neg$(i) $\Rightarrow$ $\neg$(x). Since $\mathbb{F}$ possesses an irregular gap by part (vii), this implication is a direct consequence of Lemma D (ii) in Appendix A2. ■

Remarks. (a) Note that statement (x) is false if "Darboux" is replaced with "Riemann." For instance, the function $f : \mathbb{Q} \cap [1, 2] \to \mathbb{Q}$,

$$f(x) = \begin{cases} 1, & x < \sqrt{2} \\ 0, & x > \sqrt{2} \end{cases}$$

is fairly easily seen not to be Riemann integrable (see Lemma D (i)), although $\mathbb{Q}$ is Archimedean. This is also an example of a function that is Darboux integrable but not Riemann integrable.

(b) It is often proved in elementary real analysis that (ES) implies (AP). In view of Proposition 1, this obviously implies "(CA) $\Rightarrow$ (AP)." However, a direct proof of this implication is fairly straightforward and can be given to undergraduate analysis students as an exercise [**22**].

Define $B = \{r \in \mathbb{F} \mid \exists n \in \mathbb{N} \quad 1/n < r\}$, $A = \mathbb{F} \setminus B$, and let c be the (unique) cut point. It is easy to see that it is sufficient to show that $c = 0$. It is also easy to see that $c < 0$ is impossible. Assume that $c > 0$. Then $0 < c/2 < c < 2c$, which implies $c/2 \in A$ and $2c \in B$. So there exists $n \in \mathbb{N}$ such that $1/n < 2c$. However, this means that $1/(4n) < c/2$; i.e., $c/2 \in A \cap B$, which yields a contradiction.

7. COMPLETENESS.

7.1. The List. Each of the following properties may equivalently be used to "axiomatize" the completeness of ordered fields (with countable cofinality[8]):[9]

[8]This assumption is only used in the proof that statements 14, 15, and 16 each imply CA.

[9]The list could be expanded in a trivial manner by adding "technical variations" of some of the statements given here, such as the compactness of closed and bounded intervals (variation on the Bolzano–Weierstrass property), the statement that the only connected subsets of the reals are the intervals (connectedness), or the fact that continuous functions map intervals to intervals (IVT).

1. Cut Axiom,
2. Tarski's Axiom 3,
3. Connectedness,
4. Principle of Real Induction [**5**, Theorem 4],
5. Existence of suprema/infima of bounded nonempty sets,
6. Convergence of bounded monotonic sequences,
7. Bolzano–Weierstrass Theorem,
8. Nested Interval Property + Archimedean Property,
9. Intermediate Value Theorem,
10. Mean Value Theorem,
11. Extreme Value Theorem,
12. Darboux's Theorem,
13. Continuous functions defined on closed and bounded intervals that are one-to-one (injective) are open [**20**, Corollary 1.3] (see also [**19**]),
14. Continuous functions defined on closed and bounded intervals are bounded,
15. Continuous functions defined on closed and bounded intervals are uniformly continuous,
16. Continuous functions defined on closed and bounded intervals are Riemann integrable,
17. Cauchy–Completeness (CC) + Archimedean Property,
18. Every Darboux integrable function is Riemann integrable + Archimedean Property.

7.2. Assorted Proofs. The equivalences 1–3 and 5–12 are either standard knowledge or dealt with in Part I of this paper. For 4 and 13, we refer the reader to the cited literature [**5, 20, 19**]. So we focus on items 14–18.

As in Part I, it often suffices to construct counterexamples (assuming that (CA) does not hold), since the implications "(completeness) $\Rightarrow$ (#)" (where $\# \in \{15, \dots, 18\}$) are common knowledge from a first course in real analysis.

7.2.1. A continuous function on a closed and bounded interval that is unbounded and not uniformly continuous (proves "$\neg$(CA) $\Rightarrow$ $\neg$(14)" and "$\neg$(CA) $\Rightarrow$ $\neg$(15)"). Let A, B be a gap and $(a_n) \subset A$ a strictly increasing sequence as in Lemma B of Appendix A1, $a = a_0$, and $b \in B$. Moreover, let $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{F}_+$ be a cofinal sequence (which exists by the countable cofinality of $\mathbb{F}$). Without loss of generality, we may assume that (λ_n) is increasing and that the sequence $\lambda_{n+1} - \lambda_n$ is also unbounded. (This can be accomplished easily by considering the sequence $\sum_{k=1}^{n} \lambda_k$ of partial sums if necessary.) Now define

$$\forall n \in \mathbb{N}, x \in [a_n, a_{n+1}] \qquad g(x) = \lambda_n + \frac{\lambda_{n+1} - \lambda_n}{a_{n+1} - a_n}(x - a_n),$$

$h \equiv 0$, and

$$f(x) = \begin{cases} g(x), & x \in [a, b] \cap A \\ h(x), & x \in [a, b] \cap B. \end{cases}$$

Note that $g : [a, b] \cap A \to \mathbb{F}$ is well defined because of property (5d) in Appendix A1. To see that f is continuous, it is sufficient to verify that g is continuous, since A, B

is a gap (cf. Lemma 1 in Section 3). But this directly follows from the fact that g is piecewise linear and continuous at the breakpoints a_n.

Since f is unbounded by construction, it remains to be shown that f is not uniformly continuous. If $\mathbb{F}$ is Archimedean, this is a direct consequence of Prop. 4 (ix). So assume that $\mathbb{F}$ is not Archimedean and let δ be a positive infinitesimal such that $a_{n+1} - a_n = \delta$ (this is possible by Lemma B in Appendix A1). Then, for any $\delta_1 \in (0, \delta]$, $a_n + \delta_1 \in (a_n, a_{n+1}]$ and so

$$f(a_n + \delta_1) - f(a_n) = \frac{(\lambda_{n+1} - \lambda_n)\delta_1}{a_{n+1} - a_n} = (\lambda_{n+1} - \lambda_n)\frac{\delta_1}{\delta},$$

which is unbounded, so f is not uniformly continuous.

7.2.2. Statement (16) does not hold, if (CA) is not satisfied. The previous example also proves this implication, as Riemann (as well as Darboux) integrable functions are bounded. However, there are even functions that are bounded, uniformly continuous, and monotone that fail to be Riemann integrable. See Lemma D in Appendix A2 and Section 7.3 below.

7.2.3. (CA) is equivalent to (17). *(CA)* $\Rightarrow$ *(CC)* + (17). $\mathbb{F}$ is Archimedean by Remark (b) at the end of Section 6. Let (x_n) be a Cauchy sequence. Since it is sufficient to show that some subsequence converges, we may assume without loss of generality that (x_n) is monotonic. Assume it is increasing, and define $A = \{a \in \mathbb{F} \mid \exists n \in \mathbb{N} \quad a < x_n\}$ and $B = \mathbb{F} \setminus A$. By (CA), the cut A, B has a cut point c; we claim that $\lim_{n\to\infty} x_n = c$. Let $\varepsilon \in \mathbb{F}_+$. Since $\mathbb{F}$ is Archimedean, the cut A, B is regular (Prop. 4 (vii)). So there exist $a \in A$ and $b \in B$ such that $b - a < \varepsilon$ and further an $n \in \mathbb{N}$ such that $a < x_n \leq c \leq b$, which implies that $0 \leq c - x_m \leq c - x_n \leq b - a < \varepsilon$ for all $m \geq n$.

$\neg$(CA) + (AP) $\Rightarrow$ $\neg$(CC). See Lemma B in Appendix A1.

The fact that (AP) is a necessary condition follows from the existence of Cauchy-complete non-Archimedean fields; $\mathbb{R}((x))$ and $\mathcal{R}$ in 5.2 are such fields. (Note that, in view of Remark (b) in Section 6, non-Archimedean fields cannot be complete.)

7.2.4. (CA) is equivalent to (18). See Lemma F in Appendix A3. (AP) is necessary because the Scott-completion (see Lemma G) of a non-Archimedean ordered field is a non-Archimedean field in which every Darboux integrable function is Riemann integrable (cf. Lemma E).

7.3. More (counter)examples. The function $f : \mathbb{Q} \cap [1, 2] \to \mathbb{Q}$ in Section 3.5 (c) is continuous, unbounded, and therefore *not* uniformly continuous (since $\mathbb{Q}$ is Archimedean; see Proposition 4 (ix)).

Additional interesting (counter)examples in *non*-Archimedean fields may be found in [**26, 28**].

In $\mathbb{Q}$, examples of uniformly continuous, bounded and monotone functions (defined on closed and bounded intervals) that are not Riemann integrable may be constructed by using rational functions; one such example is the function $f : \mathbb{Q} \cap [0, 1] \to \mathbb{Q}$, $f(x) = 1/(1 + x^2)$. As a function, $\mathbb{R} \to \mathbb{R}$ f *is* integrable, with $\int f = \frac{\pi}{4} \in \mathbb{R} \setminus \mathbb{Q}$. However, assuming that f is integrable as a function $\mathbb{Q} \to \mathbb{Q}$ implies $\int f \in \mathbb{Q}$, a contradiction. Note, however, that this function *is* Darboux integrable, since it is decreasing (Proposition 4 (x)).

One final remark concerns integration in *non*-Archimedean fields. As defined here (see Appendix A2), it might actually be of limited use, as, for instance, not even the identity function $f(x) = x$ on the interval $[0, 1]$ is integrable. This is what motivates Shamseddine and Berz [**27**] to develop measure and integration theory for non-Archimedean fields in a different way.

APPENDICES.

A1. Three technical lemmas.

Lemma A. *Let $\mathbb{F}$ be a non-Archimedean ordered field. Then, for any $x_0 \in \mathbb{F}$ and any positive infinitesimal $\delta \in I(\mathbb{F}) \cap \mathbb{F}_+$, there exists an irregular gap A, B satisfying*

$$x_0 \in A, \tag{5a}$$

$$\forall a \in A, b \in B \; b - a \geq \delta, \qquad \textit{and} \tag{5b}$$

$$\forall y \in [x_0, \infty) \qquad y - x_0 \textit{ finite implies that } y \in B. \tag{5c}$$

Proof. Let $a_n = x_0 + n\delta$, and let A and B be defined by

$$A := \{x \in \mathbb{F} \mid \exists n \in \mathbb{N} \quad x < a_n\} \quad \text{and} \quad B := \mathbb{F} \setminus A = \{y \in \mathbb{F} \mid \forall n \in \mathbb{N} \quad y \geq a_n\},$$

respectively. Clearly, $\mathbb{F} = A \cup B$ and $A \cap B = \emptyset$. Moreover, if $a \in A$ and $b \in B$, then there exists an $n \in \mathbb{N}$ such that $a < a_n = x_0 + n\delta < x_0 + (n+1)\delta \leq b$, which implies that $b - a > \delta$. If $c \in \mathbb{F}$ is a cut point for A, B, then it belongs to either A or B. Assume the former implies $b - c > \delta$ for all $b \in B$; since $b = c + \frac{\delta}{2} \in B$ (c is cut point), this gives $\frac{\delta}{2} > \delta$, which is absurd. Assuming $c \in B$ will result in a contradiction as well, so no cut point can exist and A, B is a gap.

Since $x_0 \in A$ is obvious, it remains to be shown that (5c) holds. To this end, assume that $y - x_0$ is positive and finite and that $y \in A$. Then there exists $n \in \mathbb{N}$ such that $y < x_0 + n\delta$ and, since $y - x_0$ is finite, there is a $k \in \mathbb{N}$ such that $\frac{1}{k} < y - x_0$. Thus $\frac{1}{k} < y - x_0 < n\delta \Rightarrow \frac{1}{kn} < \delta$, which is impossible, since δ is an infinitesimal. ■

Remark. The proof also guarantees the existence of a strictly increasing, non-convergent sequence $(a_n) \subset A$ such that

$$\forall x \in A \cap (a_0, \infty) \exists! n \in \mathbb{N} \quad \text{with} \quad x \in (a_n, a_{n+1}]. \tag{5d}$$

Lemma B. *Let $\mathbb{F}$ be an incomplete ordered field. Then there exists a gap A, B and a strictly increasing, non-convergent sequence $(a_n) \subset A$ satisfying (5d). Moreover, if $\mathbb{F}$ is Archimedean, (a_n) is a Cauchy sequence. If $\mathbb{F}$ is not Archimedean, (a_n) can be assumed to satisfy $a_{n+1} - a_n = \delta$ for a positive infinitesimal $\delta \in I(\mathbb{F}) \cap \mathbb{F}_+$.*

Proof. In view of the previous lemma and remark, only the Archimedean case needs to be considered. We will show that a sequence $(a_n)_{n \in \mathbb{N}}$ with the specified properties exists in fact for *every* gap A, B. Since $\mathbb{F}$ is Archimedean, A, B is regular (cf. Prop. 4 (vii)); thus, by Lemma C below, there exists a strictly increasing sequence $(a_n) \subset A$ and a strictly decreasing sequence $(b_n) \subset B$ such that $\lim_{n\to\infty} b_n - a_n = 0$. The sequence (a_n) is a Cauchy sequence by Prop. 4 (viii). To verify (5d), assume that there is an $x \in A$ such that $x > a_n$ for all $n \in \mathbb{N}$. Then $x \in (a_n, b_n)$ for all $n \in \mathbb{N}$. However, this is impossible, as it would imply that x is a (the) cut point for A, B. To see this, let $a \in A$ and assume that $\varepsilon := a - x > 0$. Then there exists $n \in \mathbb{N}$ such that $b_n - a_n < \varepsilon$, which implies that $a = x + \varepsilon > b_n \in B$, yielding a contradiction. Since the assumption $x - b > 0$ for $b \in B$ is equally impossible, we obtain $A \leq x \leq B$. ■

Lemma C. *Let $\mathbb{F}$ be an ordered field (with countable cofinality) and A, B a regular gap. Then there exists a strictly increasing sequence $(a_n) \subset A$ and a strictly decreasing sequence $(b_n) \subset B$ such that* $\lim_{n\to\infty} b_n - a_n = 0$.

Proof. Let $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{F}_+$ be an increasing cofinal sequence and $\varepsilon_n := 1/\lambda_n$. Then $\lim_{n\to\infty} \varepsilon_n = 0$. Let $a_0 \in A$, $b_0 \in B$ such that $b_0 - a_0 < \varepsilon_0$ and let $a_{n+1} \in (a_n, \infty) \cap A$, $b_{n+1} \in (-\infty, b_n) \cap B$ such that $b_{n+1} - a_{n+1} < \varepsilon_{n+1}$. (Note that $(a_n, \infty) \cap A$ and $(-\infty, b_n) \cap B$ are non-empty, since A and B are open.) This recursively defines sequences (a_n) and (b_n) with the desired properties. ■

A2. Integration in Ordered Fields. Let $\mathbb{F}$ be an ordered field (not necessarily Archimedean or complete), $a, b \in \mathbb{F}$, $a < b$, and $f : [a, b] \to \mathbb{F}$ a function.

We follow the definitions of J.M.H. Olmsted [**21**], but modify his terminology. We call f *Darboux integrable*[10] if, for any $\varepsilon \in \mathbb{F}_+$, there exists $g \in \mathcal{L}_f$ and $h \in \mathcal{U}_f$ such that

$$\int h - \int g < \varepsilon,$$

where $\mathcal{L}_f$ and $\mathcal{U}_f$ are the sets of step functions g and h satisfying $g(x) \leq f(x)$ and $h(x) \geq f(x)$, respectively, and the integrals have the obvious interpretation.

Furthermore, we say that f is *Riemann integrable*[11] if $\mathcal{L}_f \neq \emptyset$, $\mathcal{U}_f \neq \emptyset$ and there exists a unique number $J \in \mathbb{F}$ such that

$$\forall g \in \mathcal{L}_f, h \in \mathcal{U}_f \qquad \int g \leq J \leq \int h.$$

In this case, we set $\int f = J$.

Olmsted shows that every Riemann integrable function is Darboux integrable [**21**, Theorem 4]. However, the opposite is not true. In fact, it is not too difficult to verify that the function

$$f(x) = \begin{cases} 1, & x \in [a, b] \cap A \\ 0, & x \in [a, b] \cap B \end{cases} \tag{6}$$

is Darboux but *not* Riemann integrable, if A, B is a regular gap (see Lemma D (i) below). In fact, there are continuous—even uniformly continuous—bounded and monotone functions that are not Riemann integrable.

Lemma D. *Let $\mathbb{F}$ be an ordered field (with countable cofinality), A, B a gap, $a \in A$, $b \in B$, and let $f : [a, b] \to \mathbb{F}$ the function given by* (6).

(i) *If A, B is regular, f is Darboux integrable but not Riemann integrable.*

(ii) *If A, B is irregular, f is not Darboux integrable (and hence not Riemann integrable).*

Proof.

(i) Select sequences $(a_n) \subset A$ and $(b_n) \subset B$ as in Lemma C and define $g_n(x) = \chi_{[a,a_n)}(x) \in \mathcal{L}_f$ and $h_n(x) = \chi_{[a,b_n)}(x) \in \mathcal{U}_f$ (where χ_S denotes the *characteristic*

[10] Olmsted calls this *Riemann integrable*.
[11] *“Has a Riemann integral”* in Olmsted’s terminology.

function of a set $S \subset \mathbb{F}$). Then $\int h_n - \int g_n = b_n - a_n \longrightarrow 0$ (as $n \to \infty$), which shows that f is Darboux integrable. However, if $J \in \mathbb{F}$ were the Riemann integral of f, it would satisfy $J \in (a_n, b_n)$ for all $n \in \mathbb{N}$, which, as in the proof of Lemma B above (with x replaced by J), would imply that J would be the cut point for A, B. This yields a contradiction.

(ii) Let $g \in \mathcal{L}_f$ and $h \in \mathcal{U}_f$. Without loss of generality, g and h are based on the same partition $\{x_0, x_1, \dots, x_m\}$ of $[a, b]$; i.e., $g(x) = \sum_{j=1}^m g_j \chi_{(x_{j-1}, x_j)}(x)$ and $h(x) = \sum_{j=1}^m h_j \chi_{(x_{j-1}, x_j)}(x)$ for all $x \in [a, b] \setminus \{x_0, x_1, \dots, x_m\}$ (the values of f and g at the points $x_0, \dots x_m$ are irrelevant). Now let $k \in \{1, \dots, m\}$ be the unique index such that $x_{k-1} \in A$ and $x_k \in B$, and let $\delta \in \mathbb{F}_+$ be a number such that $B - A \geq \delta$. Then $g_j \leq 1 \leq h_j$ for $j \leq k - 1$, $g_j \leq 0 \leq h_j$ for $j \geq k + 1$ and $g_k \leq 0$, $h_k \geq 1$, since $g(x) \leq f(x) \leq h(x)$. Thus

$$\int h - \int g = \sum_{j \neq k}^{m} \underbrace{(h_j - g_j)}_{\geq 0}(x_j - x_{j-1}) + \underbrace{(h_k - g_k)}_{\geq 1}\underbrace{(x_k - x_{k-1})}_{\geq \delta} \geq \delta.$$

This shows that f is not Darboux integrable. ■

A3. Scott Completion.

Definition. An ordered field $\mathbb{G}$ is called *Scott complete* if it does not have any regular gaps.

Lemma E. *Let $\mathbb{F}$ be an ordered field (with countable cofinality). Then the following statements are equivalent.*

(i) *$\mathbb{F}$ is Scott-complete.*

(ii) *Every Darboux integrable function is Riemann integrable.*

Proof.

(i) $\Rightarrow$ (ii) (see [**21**, Theorem 9]). Let $f : [a, b] \to \mathbb{F}$ be a Darboux integrable function, $A := \{x \in \mathbb{F} \mid \exists g \in \mathcal{L}_f \quad x \leq \int g\}$, and $B := \{x \in \mathbb{F} \mid \quad \exists h \in \mathcal{U}_f \quad x \geq \int h\}$. If there is an $x \in \mathbb{F} \setminus (A \cup B)$, then $\int g < x < \int h$ for all $g \in \mathcal{L}_f$ and $h \in \mathcal{U}_f$ and $x = J$ is the Riemann integral (such an x is unique, since f is Darboux integrable). So assume that $\mathbb{F} = A \cup B$. If there is an $x \in A \cap B$, then there exists $g_0 \in \mathcal{L}_f$ and $h_0 \in \mathcal{U}_f$ such that $\int g \leq \int h_0 \leq x \leq \int g_0 \leq \int h$ for all $g \in \mathcal{L}_f$ and $h \in \mathcal{U}_f$, which again implies that $x = J$ is the Riemann integral. So we may assume that $\mathbb{F} = A \cup B$ with $A \cap B = \emptyset$, which means that A, B is a cut ($A \leq B$ holds by definition). A, B is regular, since (by the Darboux integrability of f), for any $\varepsilon \in \mathbb{F}_+$, there exist $g \in \mathcal{L}_f$ and $h \in \mathcal{U}_f$ such that $\int h - \int g < \varepsilon$ (note that $\int g \in A$ and $\int h \in B$). So A, B cannot be a gap, since $\mathbb{F}$ is Scott-complete; the cut point, J, for A, B satisfies $\int g \leq J \leq \int h$ for all $g \in \mathcal{L}_f$ and $h \in \mathcal{U}_f$, and it is also unique by the Darboux-integrability of f.

$\neg$(i) $\Rightarrow$ $\neg$(ii). See Lemma D (i). ■

Since, by Proposition 4 (vii), a Scott-complete *Archimedean* field cannot have any gaps at all and is thus complete, the following statement immediately follows.

Lemma F. [**21**, Theorem 5] *Let $\mathbb{F}$ be an Archimedean ordered field. Then the property that every Darboux integrable function is Riemann integrable is equivalent to the completeness of $\mathbb{F}$.*

Lemma G. [**25**, Theorem 1] *Let* $\mathbb{F}$ *be an ordered field. Then there exists an ordered field* $\mathbb{G}$ *such that*

(i) $\mathbb{F}$ *is a dense subfield of* $\mathbb{G}$*, and*

(ii) $\mathbb{G}$ *is Scott-complete.*

(The construction of the *Scott-completion* $\mathbb{G}$ may also be found in [**21**], where "Scott-complete" is called "quasi-complete".)

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Added in Proof: After completion and acceptance of this paper, the author became aware of related works by Propp [**23**] (to be published in this JOURNAL and Riemenschneider [**24**] (in German), which also provide various characterizations of completeness. Among the 37 statements given in [**24**] are the ones listed in Section 7.1, with the exception of items 2, 4, 12, 13, and 18 (and property 15 is only listed for Archimedean fields). Moreover, while Riemenschneider's list appears longer partially because "technical variations" (in the sense of footnote 9 on page 107) are listed separately,[12] it also contains interesting additional formulations of completeness. The statements are organized in thematic groups, which Riemenschneider calls "circles". Within each group equivalence is established by a "circle[13] argument" of the form (A) $\Rightarrow$ (B) $\Rightarrow \cdots \Rightarrow$ (A). By contrast, the organizing principle of the present paper is more reminiscent of a tree in that the various formulations of completeness are traced back to a single "root" property, the *Cut Axiom*.

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[12] For instance, his list contains the weak *and* strong nested interval principles; the usual and *and* "generalized" MVTs; etc.

[13] But not circular!

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