ARTICLES

Descartes and Problem-Solving

JUDITH GRABINER
Pitzer College
Claremont, CA 91711

Introduction

What does Descartes have to teach us about solving problems? At first glance it seems easy to reply. Descartes says a lot about problem-solving. So we could just quote what he says in the *Discourse on Method* [12] and in his *Rules for Direction of the Mind* ([2], pp. 9–11). Then we could illustrate these methodological rules from Descartes' major mathematical work, *La Géométrie* [13]. After all, Descartes claimed he did his mathematical work by following his "method." And the most influential works in modern mathematics—calculus textbooks—all contain sets of rules for solving word problems, rather like this:

1. Draw a figure.
2. Identify clearly what you are trying to find.
3. Give each quantity, unknown as well as known, a name (e.g., $x, y, \ldots$).
4. Write down all known relations between these quantities symbolically.
5. Apply various techniques to these relations until you have the unknown(s) in equations that you can solve.

The calculus texts generally owe these schemes to George Pólya's *Mathematical Discovery*, especially Chapter 2, "The Cartesian Pattern," and Pólya himself credits them to Descartes' *Rules for Direction of the Mind* ([32], pp. 22–23, 26–28, 55–59, 129ff). So I studied those philosophical works as I began to write about Descartes and problem-solving. But the more I re-read Descartes' *Geometry*, the more convinced I became that it is from this work that his real lessons in problem-solving come. One could claim that, just as the history of Western philosophy has been viewed as a series of footnotes to Plato, so the past 350 years of mathematics can be viewed as a series of footnotes to Descartes' *Geometry*.

Now Descartes said in the *Discourse on Method* that it didn't matter how smart you were; if you didn't go about things in the right way—with the right method—you would not discover anything. Descartes' *Geometry* certainly demonstrates a successful problem-solving method in action. Accordingly, this article will bring what historians of mathematics know about Descartes' *Geometry* to bear on the question, what can Descartes teach the mathematics community about problem-solving? To answer this question, let us look at the major types of problems addressed in the *Geometry* and at the methods Descartes used to solve them.

A First Look at Descartes' *Geometry*

We have all heard that Descartes' *Geometry* contains his invention of analytic geometry. So when we look at the work, we may be quite surprised at what is *not* there. We do not see Cartesian coordinates. Nor do we see the analytic geometry of the straight line, or of the circle, or of the conic sections. In fact we do not see *any* new curve plotted from its equation. And what curves did Descartes allow? Not, as we might think, any curve that has an equation; that is secondary. He allowed only curves constructible by some mechanical device that draws them according to specified rules. Finally, we do not find the term "analytic geometry," nor the claim that he had invented a new subject—just a new (and revolutionary) method to deal with old problems.

What we *do* see is a work that is problem-driven throughout. Descartes' *Geometry* has a purpose. It is to solve problems. Some are old, some are new; all are hard. For all the lip service in Descartes' *Discourse on Method* to mathematics as logical deduction from self-evident first principles ([**12**], pp, 12–13, 18–19), the *Geometry* is not like that at all. It discovers; it does not present a finished logical structure. The specific purpose of the book is to answer questions like "What is the locus of a point such that a specified condition is satisfied?" And the answer to these questions must be geometric. Not "it is such-and-such a curve," or even "it has this equation," but "it is this curve, it has this equation, and it can be constructed in this way." Everything else in the *Geometry*—and that does include algebra, theory of equations, classifying curves by degree, etc.—are just means to this geometric end. To solve a problem in geometry, one must be able to construct the curve that is its solution.

The Background of Descartes' *Geometry*

To appreciate how much Descartes accomplished, we must first look at some achievements of the ancient Greeks. They solved a range of locus problems, some quite complicated. To find their solutions, they too had "methods." Greek mathematics recognized two especially useful problem-solving strategies: *reduction* and *analysis* ([**25**], pp. 23–24).

First, let us describe the method of reduction [in Greek, *apogōgē*]. Given a problem, we observe that we could solve it if only we could solve a second, simpler problem, and so we attack the second one instead. For instance, consider the famous problem of duplicating the cube. In modern notation, the problem is, given a^3, to find x such that $x^3 = 2a^3$. Hippocrates of Chios showed that this problem could be reduced to the problem of finding two mean proportionals between a and $2a$. That is, again in modern notation, if we can find x and y such that:

$$a/x = x/y = y/2a, \tag{1}$$

then, eliminating y, we obtain $x^3 = 2a^3$ as required ([**25**], p. 23). But more geometric knowledge led to a further reduction ([**25**], p. 61). If we consider just the first two terms of (1),

$$a/x = x/y,$$

we obtain $x^2 = ay$, which represents a parabola. The equation involving the first and third terms in (1) yields

$$a/x = y/2a$$

or $xy = 2a^2$, which represents a hyperbola. Thus the problem of duplicating the cube is reducible to the problem of finding the intersection of a parabola and a hyperbola. This reduction promoted Greek interest in the conic sections.

The other problem-solving strategy is what the Greeks called "analysis"—literally, "solution backwards" (*àrnápalin lýsin* [**20**], Vol. ii, p. 400; [**25**], p. 9; cp. pp. 354–360). The Greek "analysis" works like this. Suppose we want to learn how to construct an angle bisector, and suppose that we already know how to bisect a line segment. We proceed by first assuming that we have the problem solved. Then, from the assumed existence of that angle bisector, we work backward until we reach something we do know. In FIGURE 1, take the angle A, and draw AK bisecting it. Then, mark off any length AB on one side of the angle, and an equal length AC on the other side. Connect B and C with the straight line BC, as in FIGURE 2. Now let M be the intersection of the angle bisector with the line BC. Since angle $BAM =$ angle MAC, $AB = AC$, and $AM = AM$, triangle ABM is congruent to triangle ACM. Thus M bisects BC. But wait. Recall that we already know how to bisect a line segment. Thus, we can find such an M. Now we can construct the angle bisector by reversing the process we just went through. That is, suppose we are given an angle A. To construct the angle bisector, construct $AB = AC$, construct the line BC, bisect it at M, and connect the points A and M. AM bisects the angle. This method—assuming that we have the thing we are looking for and working backwards from that assumption until we reach something we do know—was well-named "solution backwards." Pappus of Alexandria, in the early fourth century C.E., compiled a "treasury of analysis" in which he gave the classic definition of "analysis" as "solution backwards"; described 33 works, now mostly lost, by Euclid, Apollonius, Aristaeus, and Eratosthenes, which included substantial problems solvable by the method of analysis; and provided some lemmas that illustrate problem-solving by analysis ([**20**], Vol. ii, pp. 399–427).

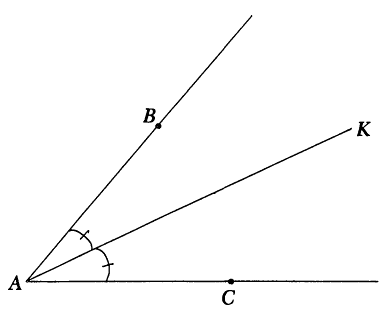

FIGURE 1

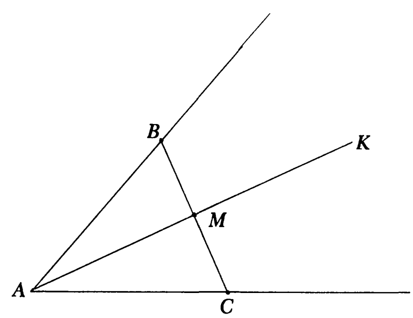

FIGURE 2

In our example of bisecting an angle, the mathematical knowledge needed was minimal. But the Greeks knew all sorts of properties of other geometric figures, notably the conic sections, and so had an extensive set of theorems to draw on in using "analysis" to solve problems in geometry ([**6**], pp. 21–39; [**10**], pp. 43–58; [**20**], passim; [**25**]). (The best and fullest account is that of Knorr [**25**].)

Thus we see that Descartes, though he championed these techniques, clearly did not invent the method of analysis and the method of reduction. Descartes' ideas on problem-solving, moreover, have other antecedents besides the Greek mathematical tradition. First, a preoccupation with finding a universal "method" to find truth appears in the work of earlier philosophers, including the thirteenth-century Raymond Lull, whose method was to list all possible truths and select the right one, the sixteenth-century Petrus Ramus, who saw method as the key to effective teaching and to allowing learners to make their own discoveries ([**29**], pp. 148–9), and the seventeenth-century philosopher of science Francis Bacon, whose method to empirically discover natural laws was one of systematic induction and testing [**1**]. All of these seekers for method suggested that intellectual progress, unimpressive earlier in history, could be achieved once the right method for finding truth was employed. Descartes shared this view.

A second, more specific antecedent of Descartes' work was the invention of symbolic algebra as a problem-solving tool, a tool that was explicitly recognized as a kind of "analysis" in the Greek sense by its discoverer, Vieta, in 1591 ([**6**], p. 65; cp pp. 23, 157–173). To say "let $x =$ " the unknown, and then calculate with x—square it, add it to itself, etc., *as if it were known*—is a powerful technique when applied to word problems both in and outside of geometry. Vieta recognized that naming the unknown and then treating it as if it were known was an example of what the Greeks called "analysis," so he called algebra "the analytic art." Incidentally, Vieta's use of this term is the origin of the way we use the word "analysis" in mathematics. In the seventeenth and early eighteenth centuries, the term "analysis" was often used interchangeably with the term "algebra," until by the mid-eighteenth century "analysis" became used for the algebra of infinite processes as opposed to that of finite ones [**4**].

Descartes was quite impressed with the power of symbolic algebra. But, although he had all these predecessors, Descartes combined, extended, and then exploited these earlier ideas in an unprecedented way. To see how his new method worked, we need to look at a specific problem.

Descartes' Method in Action

We begin with the first important problem Descartes described solving with his new method ([**13**], pp. 309–314, 324–335). The problem is taken from Pappus, who said in turn that it came from Euclid and Apollonius ([**13**], p. 304). The problem is illustrated in FIGURE 3 (from [**13**], p. 309).

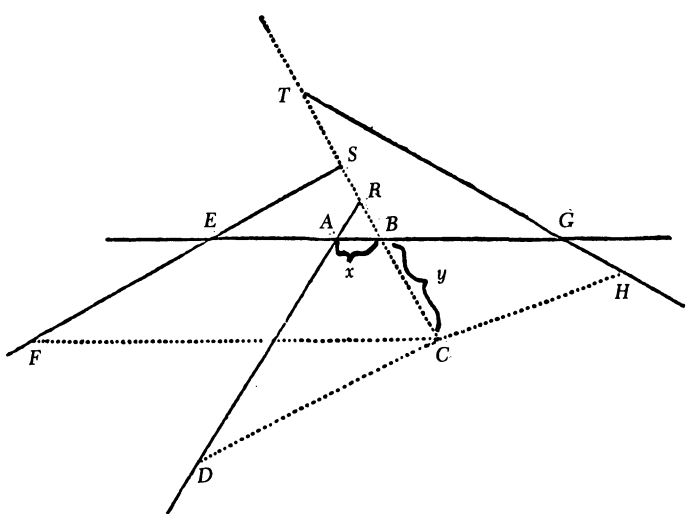

FIGURE 3

Given four lines in a plane, and given four angles. Take an arbitrary point C. Consider now the distances (dotted lines) from C to the various given lines, where the distances are measured along lines making the given angles with the given lines. (For instance, the distance CD makes the given angle CDA with the given line AD.) A further condition on C is that the four distances CD, CF, CB, and CH satisfy

$$(CD \cdot CF)/(CB \cdot CH) = \text{a given constant}. \tag{2}$$

The problem is to find the locus of all such points C. For Descartes, that means to discover what curve it is, and then to construct that curve. (At this time, any reader who does not already know the answer is encouraged to conjecture what kind of curve it is—or to imagine constructing even *one* such point C.)

Here is how Descartes attacked this problem. First assume, as we must in order to draw FIGURE 3, that we already have one point on the curve. We will then work backwards, by the method of analysis. Draw the point C, and draw the distances. Label the distance from C to the line EG as y, and the line segment between that distance and the given line DA as x. Given these labels x and y, we use them and look for other relationships that can be derived in terms of them. For instance, independent of the choice of C, the angles in the triangle ABR are all known (since angle CBG is one of the given angles in the problem, we have angle ABR by vertical angles; angle RAB is determined by the position of the two given lines that include the segments DR and GE). Thus the shape of triangle ABR is determined, so the side RB is a fixed multiple of x. Descartes therefore called that side $(b/z)\cdot x$, where he took b/z to be a known ratio. Thus $CR = y + (b/z)\cdot x$ ([13], p. 310). Using his knowledge of geometry in this fashion, Descartes found many more such relationships, and was able to express each of the distances CD, CF, CB, and CH as a different linear function of the line segments x and y. For the case where $(CD\cdot CF)/(CB\cdot CH) = 1$, those expressions let him derive an equation between the unknowns x and y and various constants he called m, n, z, o, and p:

$$y = m - (n/z)\cdot x + \sqrt{\{m^2 + ox + (p/m)\cdot x^2\}} \tag{3}$$

([13], p. 326). Now perhaps the modern reader can guess what type of curve that equation represents. So could Descartes. From his studies of Greek geometry, Descartes knew quite a lot about the conic sections, so he said, though he did not explain, that if the coefficient of the x^2 term is zero, the points C lie on a parabola; if that coefficient is positive, on a hyperbola; if negative, on an ellipse; etc. The positions, diameters, axes, centers, of these curves can be determined also, and he briefly discussed how to do this ([13], p. 329–332).

The reader will have observed that there is no fixed coordinate system here. Descartes labeled as x and y the lengths of line segments that arose in this particular situation. Let us also make a comment about his choice of notation. Vieta had used uppercase vowels for the unknowns, consonants for knowns. Since matters of notation are relatively arbitrary, the fact that we use Descartes' lowercase x and y, rather than Vieta's A and E, testifies to the great influence of Descartes' work on our algebra and geometry. Further, though Descartes himself wrote *mm* and *xx* rather than m^2 and x^2 ([13], p. 326), he did use raised numbers, exponents, for integer powers greater than two (e.g. [13], p. 337, p. 344). Today we follow Descartes here too, using exponential notation for all powers.

The Greeks already knew that the Pappus four-line locus was a conic section. Nonetheless, the way Descartes derived this result is impressive. In line with our overall purpose, let us reflect on the method Descartes used. Why is "let x equal the first unknown" so powerful here? Because the technique of "reduction" was used by Descartes to effectively reduce a problem in geometry to a problem in algebra. Once he had done this, he could use the algorithmic power and generality of algebra to solve a formerly difficult problem with relative ease. It is an old problem-solving method, to reduce a problem to a simpler one, but because the simpler one is algebraic, Descartes had something different in kind from what had been done before. Algebra puts muscles on the problem-solving methods of analysis and reduction.

Beyond the Greeks

To fully exploit the power of algebra—to go beyond the Greeks—Descartes had to make a major break with the past. The earlier symbolic algebra of Vieta was based on the theory of geometric magnitudes inherited from the Greeks. Because of this geometric basis, the product of three magnitudes was spoken of as a volume. This created a problem: What might the product of five magnitudes be? Also, Greek geometry presupposed the Archimedean axiom: Quantities cannot be compared unless some multiple of one can exceed the other, so one cannot add a point to a line, or an area to a solid. How then could one write $x^2 + x$ ([**6**], p. 61, p. 84)? Descartes, like his predecessors, did not envision pure numbers, but only geometrical magnitudes. He too felt constrained to interpret all algebraic operations in geometric terms. But he invented a new geometric interpretation for algebraic equations that freed algebraists from crippling restrictions like being unable to write x^5 or $x^2 + x$. He freed himself, and therefore freed his successors, including us. Here is how he did it.

He took a line that he called "unity," of length one, which could be chosen arbitrarily. This let him interpret the symbol x as the area of a rectangle with one side of length x, the other of length one. He could now write $x^2 + x$ with a clear conscience, since it could be thought of as the sum of two areas. Even more important, he interpreted products as lengths of lines, so that he could interpret any arbitrary power as the length of a line. That is, the product of the line segments a and b for Descartes did not have to be the area ab, but could be another length such that $ab/a = b/1$. And the length ab could be constructed, as in FIGURE 4 ([**13**], p. 298).

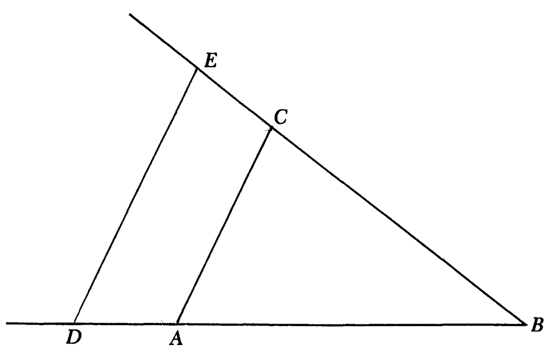

FIGURE 4

In this example, the product of the lines BD and BC is constructed, given a unit line AB. Let the line segments AB and BD be laid off on the same line originating at B, and let the segment BC be laid off on a line intersecting BD. Extending BC and constructing ED parallel to AC yields the proportion $BE/BD = BC/1$, since $AB = 1$. Thus BE is the required product $BD \cdot BC$. Of course this is an easy construction, but he had to give it explicitly. Descartes' philosophy of geometry did not let him merely assert that there was a length equal to the product of the two lines; he had to construct it. Now there was no problem in writing such expressions as x^5. This was just the length such that $x^5/x^3 = x^2/1$.

By showing that all the basic algebraic operations had geometric counterparts, Descartes could use them later at will. Furthermore, he had made a major advance in writing general algebraic expressions. Because of Descartes' innovations, later mathematicians came to consider algebra as a science of numbers, not geometric magnitudes, even though Descartes himself did not explicitly take this step. Descartes took his notational step in the service of solving geometric problems, in order to legitimize the algebraic manipulations needed to solve these geometric problems. What became a major conceptual breakthrough, then, was in the service of Descartes' problem-solving.

Descartes could now go beyond the Greeks, extending the Pappus four-line problem to five, six, 12, 13, or arbitrarily many lines. With these more elaborate problems, he still followed the same method: Label line segments, work out equations. But when he found the final equation and it was not recognizable as the equation of a conic, what then? To answer this, let me give the simple example he gave, a special case of the five-line problem. He considered four parallel lines separated by a constant distance, with the fifth line perpendicular to the other four ([**13**], pp. 336–337). (See FIGURE 5.) What, he asked, is the locus of all points C such that

$$CF \cdot CD \cdot CH = CB \cdot CM \cdot AI, \tag{4}$$

where AI is the constant distance between the equally spaced parallel lines and where the distances are all measured at right angles?

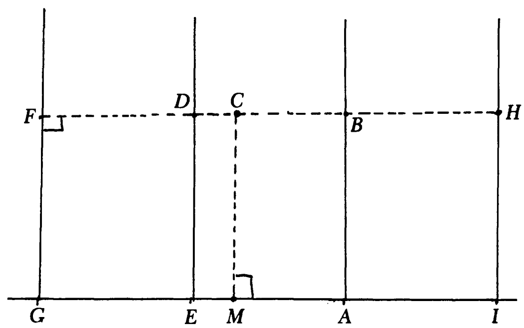

FIGURE 5

Again, Descartes proceeded by analysis. Assuming that he had such a point C, he labelled the appropriate line segments x and y $(x = CM, y = CB)$, designated the known distance AI as a, and wrote down algebraic counterparts of all known geometric relationships. For this problem they are simple ones. For instance, $CD = a - y$ and $CF = a + (a - y) = 2a - y$. Thus condition (4) becomes

$$(2a - y)(a - y)(y + a) = y \cdot x \cdot a,$$

which, multiplied out, yields the equation

$$y^3 - 2ay^2 - a^2y + 2a^3 = axy \quad ([\mathbf{13}], \text{p. } 337). \tag{5}$$

This is not a conic (it is now often called the cubical parabola of Descartes), so the next question must be, can the curve this represents be constructed? That is, given x, can we find the corresponding value of y and thus construct any point C on the curve? Until these questions are answered affirmatively, Descartes would not consider the five-line problem solved, because, for him, it is a problem in geometry. The algebraic equation was just a means to the end for Descartes; it was not in itself the solution.

So precisely what does "constructible" mean for Descartes? Can the curve represented by that cubic equation be constructed, and, if so, how?

Here another of Descartes' methodological commitments helped him solve this problem: his commitment to generality. The ancients allowed the construction of straight lines and circles, said Descartes, but classified more complex curves as "mechanical, rather than geometrical" ([13], p. 315). Presumably this was because instruments were needed to construct them. (For instance, Nicomedes had generated the conchoid by the motion of a linkage of rulers ([25], pp. 219–220), and then used the curve in duplicating the cube and trisecting the angle.) But even the ruler and compass are machines, said Descartes, so why should one exclude other instruments ([13], p. 315; tr., p. 43)? Descartes decided to add to Euclid's construction postulates that "two or more lines can be moved, one upon the other, determining by their intersections other curves" ([13], p. 316). The curves must be generated according to a definite rule. And for Descartes, such a rule, at least in principle, was given by the use of a mechanical device that generated a continuous motion. Exactly what this means is complex—for instance, the machine is not allowed to convert an arc length to a straight line—but Bos has provided an enlightening discussion ([3], pp. 304–322, esp. p. 314).

FIGURE 6 reproduces one of Descartes' curve-constructing devices ([13], p. 320). The first curve he generated using it was produced by the intersection of moving straight lines. The straight line KN (extended as necessary) is at a fixed distance KL from a ruler GL. The ruler is attached to the point G, around which it can rotate. The point L can slide along the ruler GL. The segment KL moves up the fixed line AB (extended as needed). As KL moves up, the ruler, which has a "sleeve" attached to L, rotates about G. Note that KL, KN, and the angle between them are all fixed. Then the point at which the ruler GL intersects the straight line KN extended, namely C, will be a point on the curve generated by this device.

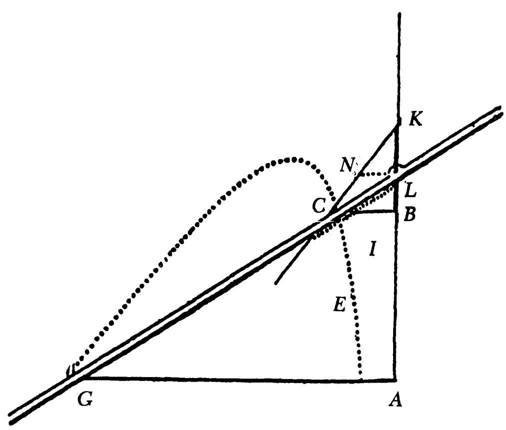

FIGURE 6

To help the reader understand the operation of this device, I show, in FIGURE 7, the construction of a second point C' by this device. KL has moved up; KN thus has a new position; the ruler has rotated to a new position. Where the ruler and KN extended now intersect is another point C' on the curve. If one continues moving KL up and down, the points C, C', etc., trace a new curve.

But what kind of curve is it? Descartes solved this problem in his usual way. He labelled the key line segments (he let the unknowns $y = CB$ and $x = BA$, and the knowns $a = AG$, $b = KL$, and $c = NL$), and algebraically represented the geometric relationships between them. He then showed that if KNC is, as it is in our diagram, a straight line, the new curve generated by the points C, C', etc., is a hyperbola ([13], p. 322). (In fact AB is one of the hyperbola's asymptotes, and the other asymptote is parallel to KN, as was shown by Jan van Schooten in his Latin edition of Descartes'

Geometry ([13], p. 55n).) If instead of the straight line *KNC*, one uses a parabola whose axis is the straight line *KB*, the new curve constructed by the device can again be identified once its equation is found. In this case, Descartes showed by his usual method that the curve produced was precisely the cubic curve of (5) that he got for the simple five-line problem! ([13], p. 322.)

This coincidence must have suggested to Descartes that his construction method could obtain any desired curve. Also, using algebra, Descartes showed that his device would produce curves of successively higher degrees ([13], p. 321–323). For instance, when *KN* was a straight line, it produced a curve represented by a quadratic; when *KN* was a parabola, it produced a third-degree curve. Descartes, struck by the generality of these results, said that any algebraic curve could be defined as a Pappus n-line locus ([13], p. 324), but here he went too far. (For a proof that this is incorrect, see [3], pp. 332–338; incidentally, Newton was the first to try to prove that Descartes was in error on this point ([3], p. 338).) Descartes also seems to have believed that any curve with an algebraic equation could be constructed by one of his devices. And here he was right, as was shown in the nineteenth century by A. B. Kempe ([22], cited in [3], p. 324). Thus Descartes' methods really did yield results of the generality he sought. We can now understand and appreciate the claim with which Descartes' *Geometry* begins: "Every problem in geometry can easily [!] be reduced to such terms that a knowledge of the lengths of certain straight lines is sufficient for its construction." (See [13], p. 297.)

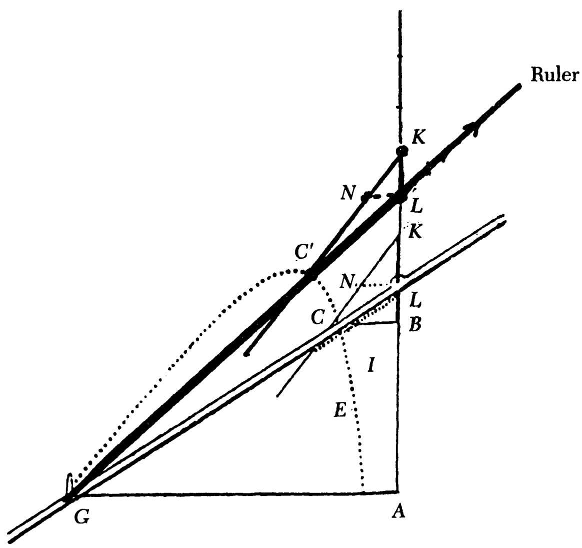

FIGURE 7

The Power of Descartes' Methods: Tangents and Equations

Descartes held that curves were admissible in geometry only if they could be constructed, but of course he also had equations for them. Thus the study of the curves, and of many of their properties, could be advanced by the study of the corresponding equations. Let us briefly consider one example where Descartes did this.

All properties of geometric curves he had not yet discussed, he said, depend on the angles curves make with other curves ([**13**], pp. 341–342). This problem could be completely solved, he continued, if the normal to a curve at a given point could be found. The reader will recognize that this is an example of the reduction of one problem to another. And how does one find the normal to a curve? Again, by a reduction. It is easy to find the normal to a circle, so we can find the normal to a curve at a point by finding the normal to the circle tangent to the curve at the same point. Thus we must find such a tangent circle. And how did Descartes begin his search for that circle? By yet another reduction, this time to algebra: He sought an algebraic equation for the circle tangent to the given curve at the given point.

He did this by starting with a circle that hit the curve at two points, and then letting the two points get closer and closer together. This required, first, writing an algebraic equation for a circle that hit the curve twice. The equation for the points of intersection of that circle and the original curve would have two solutions. But "the nearer together the points... are taken, the less difference there is between the roots; and when the points coincide, the roots are equal" —that is, the equation has only one solution when the points coincide, and thus has only one solution when the intersecting circle becomes the tangent circle ([**13**], pp. 346–7). To find when the two solutions of the algebraic equation became one, Descartes in effect set the discriminant equal to zero, providing another demonstration of the power of algebraic methods to solve geometric problems. Thus, the algebraic equation let him find the tangent circle. Finally, the normal to that circle at the point of tangency gave him the normal to the curve ([**6**], pp. 94–95). Quite a triumph for the method of reduction!

Descartes applied this technique to find normals to several curves. For instance, he did it for the so-called ovals of Descartes ([**13**], pp. 360–2), whose properties, including normals, he used in optics. He also discussed finding the normal to the cubical parabola whose equation is (5) ([**13**], pp. 343–4). Descartes' method was the first treatment of a tangent as the limiting position of a secant to appear in print ([**6**], p. 95). Thus his method of normals was a step in the direction of the calculus, as was Fermat's contemporary, independent, simpler, and more elegant method of tangents ([**6**], pp. 80, 94–5; [**30**], pp. 165–169; [**5**], pp. 166–169, 157–8).

There is one more important class of problems taken up in Descartes' *Geometry*, the solution of algebraic equations. As we have mentioned, classical problems like duplicating the cube required solving equations. So did constructing arbitrary points on the curve that solved a locus problem. Descartes said in fact that "all geometric problems reduce to a single type, namely the question of finding the roots of an equation." (See [**13**], p. 401.) Since this process was so important, if one were given an equation, it would be good to learn as much about the solutions as possible before trying to construct them geometrically.

In the last section of the *Geometry*, Descartes tried to do just this, by developing a great deal of what is now called the theory of equations. One example will suffice to illustrate his approach:

$$(x-2)(x-3)(x-4)(x+5)=0. \tag{6}$$

Using this numerical example and multiplying it out, he obtained

$$x^4-4x^3-19x^2+106x-120=0. \tag{7}$$

Descartes pointed out that one can see from the way the polynomial in (7) is generated from (6) that it has three positive roots and one negative one, and that the number of positive roots is given by the number of changes of sign of the coefficients (this is the principal case of what is now called Descartes' Rule of Signs). Also, a polynomial with several roots is divisible by x minus any root, and it can have as many distinct roots as its degree ([13], pp. 372–4). Descartes was not the first to have pointed out these things, but his presentation was systematic and influential, and the context made clear the importance of the results. The algebra was not an end in itself; it was all done to solve geometric problems.

The last major class of problems addressed in the *Geometry* was constructing the roots of equations of degree higher than two. Going beyond the Greek example of a cubic solved by intersecting conics, Descartes solved fifth- and sixth-degree equations. Why? They come up, he said, in geometry, if one tries to divide an angle into five equal parts ([13], pp. 412), or if one tries to solve the Pappus 12-line problem ([13], p. 324). To illustrate his solution method, he solved a sixth-degree equation with six positive roots by using intersecting cubic curves. The curve he used was not $y = x^3$, which we might think of as simple, but the cubics he had defined as the intersections of moving conic sections and lines. In FIGURE 8, the diagram for one such solution is shown ([13], p. 404). The cubic curve, a portion of which is shown as NCQ, intersects the circle QNC at the points that solve the sixth-degree equation. The cubic curve involved in this construction, generated by the motion of the parabola CDF, is the cubic curve (5) once again.

Descartes said that he could construct the solution to every problem in geometry. We can now see why he thought that!

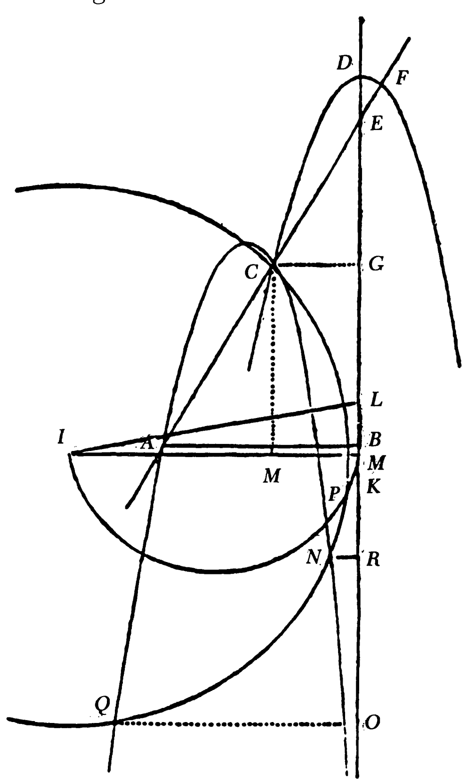

FIGURE 8

Conclusion

Now that we have seen Descartes in action, let us assess his influence on problem-solving. First, consider the mathematics that we now call "analysis." Descartes' *Geometry* solved hard problems by novel methods. There was, as an additional aid for his successors, the simultaneous and analogous work of Fermat; though Fermat's work on analytic geometry, tangents, and areas was not printed until the 1670s, it was circulated among mathematicians in the 1630s and 1640s and exerted great influence. *Geometry* itself attracted many followers. Continental mathematicians, especially Frans van Schooten and Florimond Debeaune, wrote commentaries and added explanations for Descartes' often cryptic statements. They also extended Descartes' methods to construct other loci. The second edition of Schooten's commentary on Descartes' *Geometry* (with a Latin translation) was published in 1659–1661 together with several other influential works based on Descartes. One was Jan de Witt's *Elements of Curves*, which systematized analytic geometry, including a discussion of constructing conic sections from their equations ([**6**], p. 115–116); another was Hendrik van Heuraet's work on finding arc lengths. Schooten's collection helped inspire both John Wallis and Isaac Newton. Wallis "seized upon the methods and aims of Cartesian geometry" ([**6**], p. 109) and went even further in replacing geometric concepts by algebraic or arithmetic ones. Many mid-seventeenth-century mathematicians, including Wallis, James Gregory, and Christopher Wren, influenced both by Descartes and by Fermat, used algebraic methods to make further progress on the problem of tangents, and—as Descartes had suggested, but did not do—to find areas. Men like van Heuraet, William Neil, and Wren also found arc lengths for some curves this way ([**5**], p. 162), which Descartes, who couldn't do it, had said couldn't be done ([**13**], p. 340). Wallis also extended the algebraic approach of Descartes to infinitesimals. In the 1660s, Isaac Newton carefully studied Schooten's edition of Descartes, using it (together with the work of men like Barrow, Wallis, and Gregory) as a key starting point in his invention of the calculus ([**35**], pp. 106–111, 128–130). In 1674, less than two years before his own invention of the calculus, Gottfried Wilhelm Leibniz worked his way through Descartes' *Geometry*; he was especially interested in the algebraic ideas ([**21**], p. 143). He later even examined some of Descartes' unpublished manuscripts ([**21**], p. 182–183).

Some scholars have credited Descartes with bringing about a revolution in analysis ([**7**], pp. 157–159, 506; [**3**], p. 304; for dissenting views, see [**31**], pp. 110–111; [**21**], pp. 202–210; [**18**], p. 55; [**19**], p. 164). But at the very least we may say of the *Geometry* what Thomas Kuhn once said about Copernicus' *On the Revolution of the Celestial Orbs* ([**26**], p. 134); though it may not have been revolutionary, it was "a revolution-making text." The problem-solving methods introduced in Descartes' *Geometry* and developed in the commentaries on it were clearly seminal throughout the seventeenth century, influencing both Newton and Leibniz, whether or not Descartes was the first inventor of these techniques. And such influence continued through the eighteenth century and beyond ([**17**], pp. 156–158, 505–507).

Incidentally, the systematic approach to analytic geometry we all learned in school is not in either Descartes or Fermat (though Fermat, unlike Descartes, did plot elementary curves from their equations), but dates from various eighteenth-century textbooks, especially those from the hands of Euler, Monge, Lagrange, and Lacroix ([**16**], pp. 192–224). Descartes, though, was not a textbook writer, but a problem-solver. The essence of his influence was in his new approach and his self-consciousness about method. These highlight his achievement as a problem-solver.

Second, then, let us look at his influence on problem-solving in general. The problem-solving methods we teach our students are the direct descendants of Descartes' methods. This is not because he passed them down to us in a set of rules (although he did). Nor is it because his methods work for the problems in elementary textbooks (although they do). It is because his methods solved many outstanding problems of his day. Descartes saw himself as a problem-solver because he had a method. He saw himself also as a teacher of problem-solving. One can see this even in the way he left hard questions as exercises to the reader, as he put it at the end of the *Geometry*, "to leave for others the pleasures of discovery." (See [**13**], p. 413.) His *Geometry* teaches us how to solve problems because it contains a set of solved problems whose successful solutions validate his methods. We may not care about the Pappus four-line problem, but we certainly prize the problem-solving power of a generalized algebra. Descartes' methods have come to us indirectly—who reads the *Geometry* nowadays?—but they have come to us because they are embedded in the work of his successors: In algebraic notation and equation theory, in analytic geometry, in calculus, in Lagrange's view that algebra is the study of general systems of operations, and in the more abstract and general subjects built upon these achievements. Because of his influence on later mathematicians, Descartes' methods are embedded also in the way we teach mathematics, in the standard collections of problems and solutions. In fact, for routine problems, the task of applying Descartes' analytic methods is, as he intended, fairly mechanical. Some of the *Rules for Direction of the Mind* explicitly parallel the method of the *Geometry*, ([**2**], pp. 177–178) and Pólya is thus right to have made such rules explicit for modern students. Inventing new mathematical methods—say, like analytic geometry— is, however, not a routine task. Even here, for Descartes, "method" is crucial.

Third, then, for those of us who want to invent great and new things like analytic geometry, to teachers and students of mathematics, Descartes has something else he wants us to learn, and that is his emphasis on method in general. Here he, together with his great contemporary Sir Francis Bacon, have inspired many. For instance, Leibniz saw his differential calculus as a problem-solving method, explicitly comparing it with analytic geometry, saying "From [my differential calculus] flow all the admirable theorems and problems of this kind with such ease that there is no more need to teach and retain them than for him who knows our present algebra to memorize many theorems of ordinary geometry" ([**27**], excerpted in [**34**], p. 281). Or, in our century, there is Pólya's sophisticated emphasis on teaching about method. Let me put Descartes' lesson this way: Raise problem-solving techniques to consciousness. Reflect on the methods that are successful and on their strengths and weaknesses. Then apply them systematically in attacking new problems. That is how Descartes himself invented analytic geometry, as he said in the *Discourse on Method*: "I took the best traits of geometrical analysis and algebra, and corrected the faults of one by the other." (See [**12**], p. 13, 20.)

Fourth and last, let us briefly consider a key point in Descartes' philosophy: that the methods of mathematics could solve the problems of science. Here, Descartes the philosopher learned from Descartes the mathematician that method was important, that the right method could solve previously intractable problems. He used the ideas of reduction and analysis in his philosophy of science. For instance, he argued that all macroscopic phenomena could be explained by analyzing nature into its component parts, bits of matter in motion. (See [**14**], pp. 409–414) and ([**36**], pp. 32–38). Descartes came to believe that the most powerful methods were both general and mathematical. His *Principles of Philosophy* (1644) attempted to deduce all the laws of nature from self-evident first principles; his principles XXXVII and XXXIX are equivalent to Newton's First Law of Motion (1687) ([**8**], pp. 182–183).) In fact, Descartes went so far as to state that everything that could be known could be found by a method modelled on that of mathematics. He wrote,

> Those long chains of reasoning, so simple and easy, which enabled the geometers to reach the most difficult demonstrations, had made me

> wonder whether all things knowable to man might not fall into a similar logical sequence. If so, we need only refrain from accepting as true that which is not true, and carefully follow the order necessary to deduce each one from the others, and there cannot be any propositions so abstruse that we cannot prove them, not so recondite that we cannot discover them ([**12**], pp. 12–13; 19).

Descartes' vision is clearly echoed by what Leibniz wrote in 1677 about his own search for a general symbolic method of finding truth: "If we could find characters or signs appropriate for expressing all our thoughts as definitely and as exactly as arithmetic expresses numbers or geometric analysis expresses lines, we could in all subjects in so far as they are amenable to reasoning accomplish what is done in Arithmetic and Geometry." (See [**28**], p. 15.) Again, consider the prediction of the great prophet of progress of the Enlightenment, the Marquis de Condorcet, that Descartes' methods could solve all problems. Although the "method" of algebra "is by itself only an instrument pertaining to the science of quantities," Condorcet wrote, "it contains within it the principles of a universal instrument, applicable to all combinations of ideas." This could make the progress of "every subject embraced by human intelligence... as sure as that of mathematics." (See [**9**], pp. 238, 278–279; quoted in [**17**], p. 222.)

Descartes has been attacked as a methodological imperialist and a reductionist, and lauded as an intellectual liberator and one of the founders of modern thought (e.g., [**11**], [**18**], [**24**], [**33**]). For good or ill, the power of Descartes' vision has shaped Western thought since the seventeenth century, and his mathematical work helped inspire his philosophy. But whatever our assessment of Descartes the philosopher may be, his importance for the mathematician is clear. The history of the past 350 years of mathematics can fruitfully be viewed as the story of the triumph of Descartes' methods of problem-solving.

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Math Bite: $\Sigma a_k b_k \leqslant (\Sigma a_k^2)^{1/2}(\Sigma b_k^2)^{1/2}$.

$$\frac{\Sigma a_k b_k}{\left(\Sigma a_k^2\right)^{1/2}\left(\Sigma b_k^2\right)^{1/2}} = 1 - \frac{1}{2}\Sigma\left(\frac{a_k}{\left(\Sigma a_k^2\right)^{1/2}} - \frac{b_k}{\left(\Sigma b_k^2\right)^{1/2}}\right)^2.$$

—PETER SZÜSZ
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