Chapter 1 Solutions to Selected Exercises

Notes:

- The questions are in a separate PDF on LongFormMath.com.
- For most problems there are many correct solutions, so the below are not the only correct ways to solve the problems.
- If you spot an error, please email it to me at LongFormMath@gmail.com. Thanks!

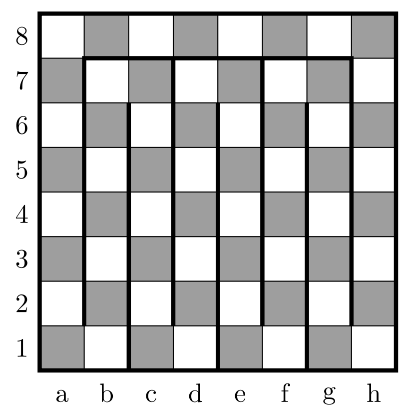
Solution to Question 1. The error is moving from the third line to the fourth. We had assumed that $x = y$, which means that $x - y = 0$. It's true that

$$(x + y) \cdot 0 = y \cdot 0,$$

no matter what $x + y$ and y are. But you are never allowed to divide by zero, which is what's done to move to the the next step. So indeed, $x + y$ and y could be anything at all in line 3, and certainly do not have to be equal to each other as asserted in line 4.

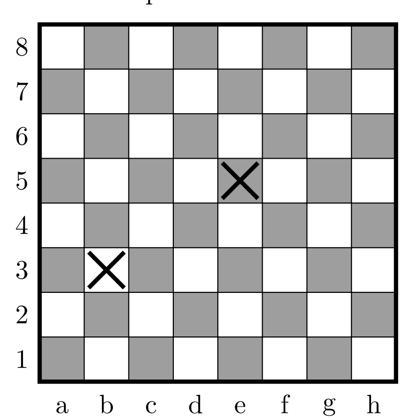
Note: There could also be a problem in the final step. If $2y = y$, then you can not necessarily cancel a y from each side; again, what if $y = 0$? In fact, the only way that $2y = y$ is possible is if y does equal 0!

Solution to Question 2. Yes, if you remove two squares of different colors, then the result must have a perfect cover. To see this, observe that one can divide up the 8×8 chessboard like this:

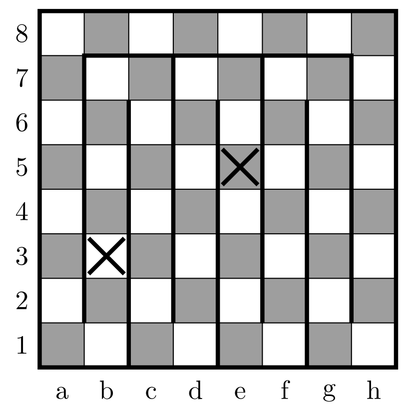


Notice that adding these lines produces a single path through the squares of the chessboard. Then, if you cross off any two squares of opposite color, there is an even number of squares between them in this path, and hence can be tiled with dominoes.

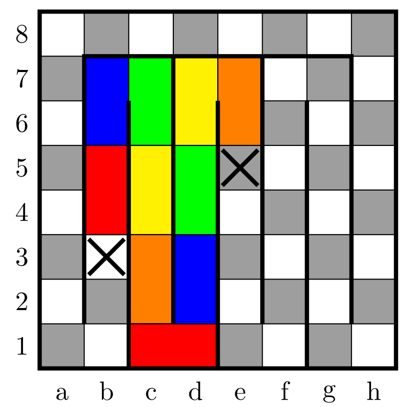
As an example, if you crossed off these two squares:



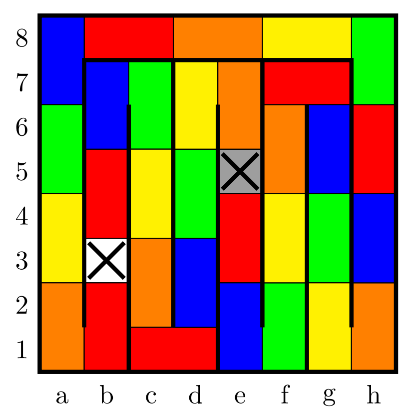
Then by adding these lines back onto the board:



We can then follow the path to tile the squares between the first X and the second X:

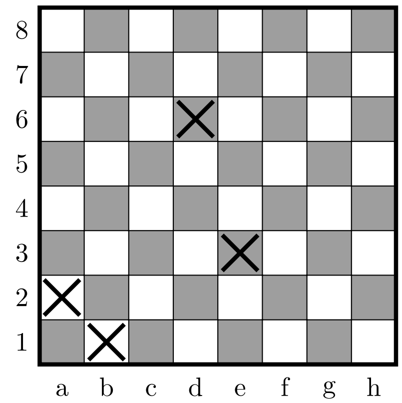


We are guaranteed that the tiles will fill up this path perfectly because the squares are opposite colors. For the same reason, we can fill the path from the second X to the first X in the same way.



In this way, given any two squares of opposite colors, if you remove those squares from the chessboard, the resulting chessboard has a perfect color.

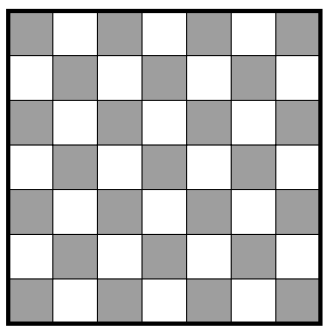
Solution to Question 3. No, the resulting chessboard need not have a perfect cover. As a counterexample to the claim, consider the following four squares:



This chessboard can not be perfectly covered because no tile can cover a1 without also covering one of the crossed-out squares a2 or b1.

Solution to Question 4.

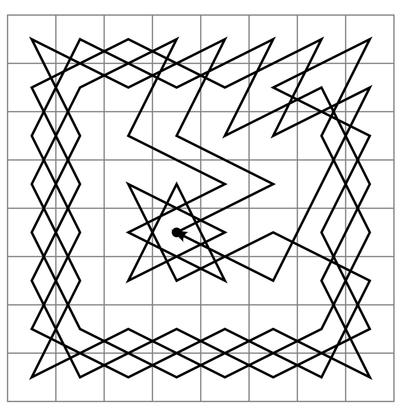
Part (a). No, it is not possible. First, note that a 7×7 chessboard has 49 squares, and hence has an unequal number of white and black squares. In the below, it has 25 black squares and 24 white squares.



So if a knight were on every square of this chessboard, then there are 25 knights on black squares and 24 knights on white squares.

Observe that a knight always moves from a black square to a white square, or a white square to a black square. Thus, after these 49 knights all move, there must be 25 knights on white squares and 24 knights on black squares. But since this 7×7 chessboard does not have 25 white squares and 24 black squares, such a simultaneous move must be impossible.

Part (b). Yes, it is possible. For example, here is one way:



In this image is a single route that touches each square once and moves from one square to the next using a knight-move. Thus, if every knight moves one jump in the same direction along this path, then every square will have a knight land on it after the simultaneous move.

Solution to Question 5.

- From the set $\{1, 2, 3, 4, 5, 6\}$, I chose 1, 3, 4 and 5. Among these four numbers, 3 and 4 sum to 7.
- From the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$, I chose 1, 3, 4, 5 and 6. Among these numbers, 4 and 5 sum to 9.
- From the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, I chose 1, 2, 5, 7, 8 and 9. Among these numbers, 2 and 9 sum to 11.

Now we will prove the general statement.

Proof. We will use the pigeonhole principle. Let our objects be the chosen $n+1$ numbers from $\{1, 2, 3, \dots, 2n\}$. Let our boxes be the sets $\{1, 2n\}, \{2, 2n-1\}, \{3, 2n-2\}, \dots, \{n, n+1\}$. Notice that we have n boxes, and the two numbers in each box sum to $2n+1$. We are placing $n+1$ objects into n boxes, and so by the pigeonhole principle at least one of the boxes will contain (at least) two numbers. These two numbers sum to $2n+1$. $\square$

Solution to Question 6. The numbers 101, 102, 103, 104, …, 200 is a selection of 100 numbers such that no number divides any other.

The problem doesn't ask for an explanation, but here is one anyways:

The reason that no number can divide another is that if a number n divides a number m, then either $m = 2n$, or $m = 3n$, or $m = 4n$, etc. This tells us that $m \geq 2n$. But since none of the 100 numbers is twice as big as another, there is no way that one divides another.

Indeed, taking a smaller number divided by a larger number, the possible ratios are as small as

$$\frac{200}{199} = 1.005$$

and as large as

$$\frac{200}{101} = 1.9801.$$

Thus, dividing a smaller number from a larger one is guaranteed to leave a decimal, as the ratio will land between 1.005 and 1.9801.

Solution to Question 7. We will use the pigeonhole principle. Let the objects be the seven chosen integers. Let the boxes be the sets $\{1, 9\}$, $\{2, 8\}$, $\{3, 7\}$, $\{4, 6\}$, $\{5\}$ and $\{0\}$. We will place a number in a box based on the last digit of the number. For example, 438 would be placed in box $\{2, 8\}$, and -679 would be placed in box $\{1, 9\}$, and 80 would be placed in box $\{0\}$.

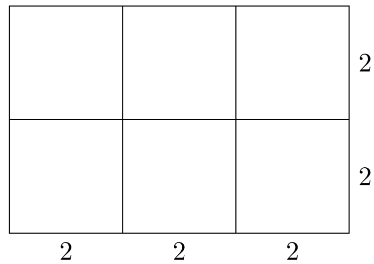
Notice that we have 7 numbers being placed into 6 boxes, and hence by the pigeonhole principle 2 numbers must be in the same box. Suppose m and n are the two numbers. We will consider two cases: either m and n have the same final digit, or different final digits.

If these two numbers have the same final digit, then $m - n$ will be a number with 0 as its final digit—meaning that the number is divisible by 10.

If these two numbers do not have the same final digit, then since they are in the same box these final digits must either be 1 and 9, or 2 and 8, or 3 and 7, or 4 and 6. In all of these cases, the final digits of m and n will sum to 10, and hence $m + n$ will be a number with 0 of its final digit—meaning that the number is divisible by 10.

In both cases, we have proven that either the sum or difference of the two numbers will be divisible by 10, completing the proof. $\square$

Solution to Question 8. We will use the pigeonhole principle. Let us take the 6×4 rectangle and split it into six 2×2 parts.



Notice that each square has an area of 4. Consider these squares to be our boxes and consider the 19 points to be our objects. When you choose 19 points, place them into the box in which they land (if a point lands on the line between two boxes, place the point in either box). By the general form of the pigeonhole principle, placing 19 objects into 6 boxes guarantees that at least one box will contain at least 4 points objects. That is, we are guaranteed to find 4 points in the same 2×2 box.

Since no three points form a straight line, these four points must form a quadrilateral, and since this quadrilateral is contained inside a square of area 4, its area must be at most 4. This proves the result. $\square$

Solution to Question 9.

(a) Solutions will include a positive and negative number. For example, $x = -2$ and $y = 3$.
(b) Solutions will include an x between -1 and 1. For example, $x = 0.5$.
(c) Solutions will include $x = 0$. For example, $x = 0$ and $y = 2$.

Solution to Question 10. Notice that if you may only use the numbers -1, 0 or 1, then the sum of n numbers can be as large as n and as small as $-n$, and must be an integer. That is, the row, column, and diagonals must sum to a number in the set $\{-n, -n+1, -n+2, \dots, -3, -2, -1, 0, 1, 2, 3, \dots, n-2, n-1, n\}$. Notice that there are $2n+1$ numbers in this set.

We will now use the pigeonhole principle. Let the row, column and diagonal sums of an $n \times n$ matrix be the objects; note that there are n rows, n columns and 2 diagonals, totaling $2n+2$ objects. And let the sets $\{-n\}$, $\{-n+1\}$, $\{-n+2\}$, $\dots$, $\{n\}$ be the boxes, totaling $2n+1$ boxes.

We are placing $2n+2$ objects into $2n+1$ boxes, and so by the pigeonhole principle there must be at least one box with at least two numbers in it. Being in the same box, this means that two of the sums are the same, which means we are guaranteed that the matrix is not an antimagic square. $\square$