Chapter 3 Solutions to Selected Exercises

Notes:

- The questions are in a separate PDF on LongFormMath.com.
- For most problems there are many correct solutions, so the below are not the only correct ways to solve the problems.
- If you spot an error, please email it to me at LongFormMath@gmail.com. Thanks!

Solution to Question 1.
Part b: $\{-7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$
Part c: $\{-12, -8, -4, 0, 4, 8, 12\}$
Part j: $\{\frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, -\frac{1}{2}, -\frac{1}{3}, -\frac{2}{3}, -\frac{1}{4}, -\frac{3}{4}\}$
Part k: $\{-1, 1\}$

Solution to Question 2. Answers vary.

(a) $\{3n - 1 : n \in \mathbb{N}\}$

(b) $\{\frac{k\pi}{2} : k \in \mathbb{Z}\}$

(c) $\{n \in \mathbb{Z} : -4 \leq n \leq 1\}$

(d) $\left\{\frac{(-5)^n}{3^n} : n \in \mathbb{N}\right\}$

Solution to Question 3.

(a) True

(b) False

(c) False

(d) True

(e) True

(f) True

(g) False

(h) True

(i) True

(j) False

(k) True

(l) True

(m) True

(n) False

(o) True

Solution to Question 4. This is the set $\mathbb{Z}$.

Solution to Question 5. As for your chosen example, answers vary. Now for the proof:

Suppose A and B are sets. Assume $X \in \mathcal{P}(A) \cup \mathcal{P}(B)$. By the defunition of a union, this implies $X \in \mathcal{P}(A)$ or $X \in \mathcal{P}(B)$. By the definition of a powerset, this implies $X \subseteq A$ or $X \subseteq B$. Again by the definition of the union, this in turn means that $X \subseteq (A \cup B)$, and so by the definition of the powerset this means that $X \in \mathcal{P}(A \cup B)$.

We have shown that if $X \in \mathcal{P}(A) \cup \mathcal{P}(B)$, then $X \in \mathcal{P}(A \cup B)$. Thus, $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. □

Solution to Question 6. To show that one set is not a subset of another, it suffices to fine a single element in the first set that is not an element of the second set.

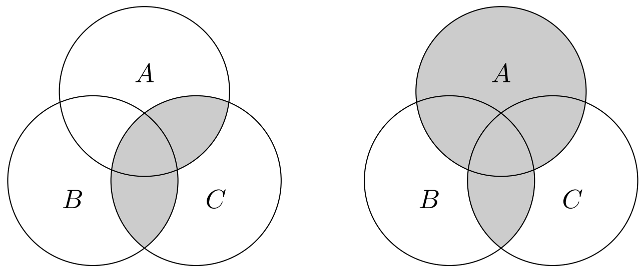
Observe that $5 \in \{n \in \mathbb{Z} : n \equiv 1 \pmod 4\}$ because $5 \in \mathbb{Z}$ and $5 \equiv 1 \pmod 4$; the latter is true because $4 \mid (5 - 1)$.

Next, note that $5 \notin \{n \in \mathbb{Z} : n \equiv 1 \pmod 8\}$. This is because $5 \not\equiv 1 \pmod 8$, since $8 \nmid (5 - 1)$.

We have shown that $5 \in \{n \in \mathbb{Z} : n \equiv 1 \pmod 4\}$, while $5 \notin \{n \in \mathbb{Z} : n \equiv 1 \pmod 8\}$. This proves that $\{n \in \mathbb{Z} : n \equiv 1 \pmod 4\} \not\subseteq \{n \in \mathbb{Z} : n \equiv 1 \pmod 8\}$, as desired. □

Solution to Question 7. Let $A = \{1, 3\}$, $B = \{\pi, 3, 5\}$, and $C = \{2, 4, 3, 5\}$.

Then $A \cup (B \cap C) = \{1, 3, 5\}$ and $(A \cup B) \cap C = \{3, 5\}$. Thus, $A \cup (B \cap C) \neq (A \cup B) \cap C$.



$\square$

Solution to Question 8. This is false, so we will present a counterexample.

Suppose $A = \{1, 2, 3\}$ and $B = \{3, 4\}$. Then $A \cup B = \{1, 2, 3, 4\}$.

Notice that $|A| = 3$ and $|B| = 2$, so $|A| + |B| = 5$. Meanwhile, $|A \cup B| = 4$. Thus, $|A \cup B| \neq |A| + |B|$.

Formula: $|A \cup B| = |A| + |B| - |A \cap B|$. $\square$

Solution to Question 9. To prove that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$

we will prove that

$$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$$

and

$$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C).$$

Part 1: Assume that $x \in A \cup (B \cap C)$. We aim to show that $x \in (A \cup B) \cap (A \cup C)$.

Since $x \in A \cup (B \cap C)$, by the definition of a union, $x \in A$ or $x \in B \cap C$. Let's consider these two cases separately.

Case 1: If $x \in A$, then certainly $x \in A \cup B$ and $x \in A \cup C$, and hence $x \in (A \cup B) \cap (A \cup C)$.

Case 2: If $x \in B \cap C$, then by the definition of an intersection $x \in B$ and $x \in C$. Thus, certainly $x \in A \cup B$ and $x \in A \cup C$, and hence $x \in (A \cup B) \cap (A \cup C)$.

In both cases, if $x \in A \cup (B \cap C)$, then $x \in (A \cup B) \cap (A \cup C)$. This proves that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$, as desired.

Part 2: Assume that $x \in (A \cup B) \cap (A \cup C)$. We aim to show that $x \in A \cup (B \cap C)$.

Since $x \in (A \cup B) \cap (A \cup C)$, by the definition of the intersection, $x \in (A \cup B)$ and $x \in (A \cup C)$. Since $x \in (A \cup B)$, either $x \in A$ or $x \in B$. And since $x \in (A \cup C)$, either $x \in A$ or $x \in C$. Let's consider two cases.

Case 1: If $x \in A$, then certainly $x \in A \cup (B \cap C)$.

Case 2: If $x \notin A$, then according to the end of the second paragraph above, we must have $x \in B$ and $x \in C$, and hence $x \in (B \cap C)$. And since this is the case, certainly $x \in A \cup (B \cap C)$.

In both cases, if $x \in (A \cup B) \cap (A \cup C)$, then $x \in A \cup (B \cap C)$. This proves that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$, as desired.

We have shown that

$$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$$

and

$$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C),$$

and hence

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

$\square$

Solution to Question 10. First we prove A exists. Let $A = \emptyset$ and observe that $A \in \mathcal{P}(C)$. Next note that for every $B \in \mathcal{P}(C)$ we have $A \cup B = B$. Thus, this choice of A does indeed satisfy the desired condition.

Next we will prove that A is unique. Let A and A' be the two sets that satisfy the hypothesis, and note what happens when we apply the "$A \cup B = B$" property. Since A has this property, then letting $B = A'$ implies that $A \cup A' = A'$.

On the other hand, since A' has this property, then letting $B = A$ implies that $A' \cup A = A$.

Thus we have shown that

$$A' = A \cup A' = A' \cup A = A,$$

and so $A = A'$. This shows that there cannot be two different sets with this property. Any two sets with this property must be the same set. $\square$

Another way to see it is to note that if A were a set that had this property and $A \neq \emptyset$, then A must have at least one element, call it a. Thus, by applying the property to $B = \emptyset$, we get $A \cup B = B$, which is $A \cup \emptyset = \emptyset$. But this can't possibly hold since A has at least one element and so $A \cup B$ must also have at least one element—it can't be $\emptyset$.