Chapter 4 Solutions to Selected Exercises

Notes:

- The questions are in a separate PDF on LongFormMath.com.
- For most problems there are many correct solutions, so the below are not the only correct ways to solve the problems.
- If you spot an error, please email it to me at LongFormMath@gmail.com. Thanks!

Solution to Question 1. We aim to prove that $1 + 3 + 5 + ... + (2n-1) = n^2$. We proceed by induction.

Base Case. The base case is when $n = 1$, and

$$1 = 1^2,$$

as desired.

Inductive Hypothesis. Assume the result holds for some $k \in \mathbb{N}$. Since the k^{th} odd natural number is $(2k-1)$, this means that

$$1 + 3 + 5 + \cdots + (2k-1) = k^2.$$

Induction Step. We aim to prove the result for $k+1$. The $(k+1)^{\text{st}}$ odd number is $(2k+1)$. Adding $(2k+1)$ to both sides of the induction hypothesis,

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k-1) + (2k+1) &= k^2 + (2k+1) \\ &= (k+1)^2. \end{aligned}$$

Conclusion. Therefore, by induction, the result holds for every $n \in \mathbb{N}$. □

Solution to Question 2. We proceed by induction.

Base Case. The base case is $n = 1$, and $5 \mid (6^1 - 1)$ is true as it simply asserts $5 \mid 5$.

Inductive Hypothesis. Assume that for some $k \in \mathbb{N}$, we have that $5 \mid (6^k - 1)$.

Induction Step. Note that by the definition of divisibility, $5 \mid (6^k - 1)$ means that $6^k - 1 = 5m$ for some integer m. Multiplying both sides by 6 gives

$$6^{k+1} - 6 = 5(6m).$$

Adding 5 to each side gives $6^{k+1} - 1 = 5(6m) + 5 = 5(6m + 1)$. Since m is an integer, $6m + 1$ is also an integer, and so

$$6^{k+1} - 1 = 5n$$

where $n = 6m + 1$ is an integer. Thus, by the definition of divisibility,

$$5 \mid (6^{k+1} - 1),$$

as desired.

Conclusion. Therefore, by induction, $5 \mid (6^n - 1)$ holds for all $n \in \mathbb{N}$.

Solution to Question 3.

Part (e). We proceed by induction.

Base Case. $n = 1$, then $1^3 = 1 = 1^2$, as desired.

Inductive Hypothesis. Assume that for some $k \in \mathbb{N}$, we have that

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = (1 + 2 + 3 + \cdots + k)^2.$$

Induction Step. By Proposition 4.2, recall that $1 + 2 + 3 + ... + k = \frac{k(k+1)}{2}$. Then,

$$\begin{aligned}
1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 &= (1 + 2 + 3 + \cdots + k)^2 + (k+1)^3 \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2 + 4(k+1))}{4} \\
&= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2 \\
&= (1 + 2 + 3 + \cdots + k + k + 1)^2,
\end{aligned}$$

as desired, where the final equality made use of Proposition 4.2 again (with a $k+1$ in place of a k).

Conclusion. Therefore, by induction, the property holds for all $n \in \mathbb{N}$. □

Part (f). We proceed by induction.

Base Case. $n = 1$, then $1 \cdot 1! = 1 = (1+1)! - 1$, as desired.

Inductive Hypothesis. Assume that for some $k \in \mathbb{N}$, we have that

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + k \cdot k! = (k+1)! - 1$$

Induction Step. By applying the induction hypothesis,

$$\begin{aligned}
1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + k \cdot k! + (k+1) \cdot (k+1)! &= (k+1)! - 1 + (k+1) \cdot (k+1)! \\
&= (k+1)!(1 + k + 1) - 1 \\
&= (k+1)!(k+2) - 1 \\
&= (k+2)! - 1,
\end{aligned}$$

as desired.

Conclusion. Therefore, by induction, the result holds for all $n \in \mathbb{N}$. □

Solution to Question 4. We proceed by induction.
Base Case. The base case is when $n = 1$, and

$$1 + 2^1 = 3 \leq 3 = 3^1,$$

as desired.

Inductive Hypothesis. Assume that

$$1 + 2^k \leq 3^k$$

for some $k \in \mathbb{N}$.

Induction Step. We aim to prove the result for $k + 1$. By using the inductive hypothesis in the third line and the fact that $2(3^k) \leq 3(3^k)$ in the fourth line (note that $3^k > 0$), we see that

$$\begin{aligned} 1 + 2^{k+1} &\leq 2 + 2^{k+1} \\ &= 2(1 + 2^k) \\ &\leq 2(3^k) \\ &\leq 3(3^k) \\ &= 3^{k+1}. \end{aligned}$$

Conclusion. Therefore, by induction, the result holds for every $n \in \mathbb{N}$. $\square$

Solution to Question 5. We proceed by induction.
Base Case. The base case is when $n = 0$, and

$$\tilde{F}_0 = 2^{2^0} + 1 = 3 = \left(2^{2^1} + 1\right) - 2 = \tilde{F}_2 - 2,$$

as desired.

Inductive Hypothesis. Assume that

$$\tilde{F}_0 \cdot \tilde{F}_1 \cdot \tilde{F}_2 \cdot \tilde{F}_3 \cdots \tilde{F}_k = \tilde{F}_{k+1} - 2$$

for some $k \in \mathbb{N}$.

Induction Step. We aim to prove the result for $k + 1$. By using the definition of the Fermat numbers,

$$\begin{aligned} \tilde{F}_0 \cdot \tilde{F}_1 \cdot \tilde{F}_2 \cdot \tilde{F}_3 \cdots \tilde{F}_k \cdot \tilde{F}_{k+1} &= \left(\tilde{F}_{k+1} - 2\right) \cdot \tilde{F}_{k+1} \\ &= \left(2^{2^{k+1}} + 1 - 2\right) \cdot \left(2^{2^{k+1}} + 1\right) \\ &= \left(2^{2^{k+1}} - 1\right) \cdot \left(2^{2^{k+1}} + 1\right) \\ &= \left(2^{2^{k+1}}\right)^2 - 1 \\ &= 2^{2^{k+1} + 2^{k+1}} - 1 \\ &= 2^{2^{k+2}} - 1 \\ &= \tilde{F}_{k+2} - 1. \end{aligned}$$

Conclusion. Therefore, by induction, the result holds for every $n \in \mathbb{N}$. $\square$

Solution to Question 6. The error is in the induction step. In the induction step is a picture that the step relies on. It relies on the fact that the "first k people" and the "last k people" have some overlap. This is a mistake in the $k = 1$ case. If $k = 1$, meaning that $k + 1 = 2$ people are being considered in the induction step, then the "first k people" and the "last k people" is just a single person each. So if you line up two people, and the first person and the second person each have a name, the argument fails to show that they must have the same name. So the induction stops at the very first step.

Solution to Question 7. We proceed by induction.
Base Case. The base case is when $n = 0$, and if $|A| = 0$, then $A = \emptyset$. And $\mathcal{P}(\emptyset) = \{\emptyset\}$, since there is only one subset of the empty set—the empty set itself. Thus, $|\mathcal{P}(A)| = 1 = 2^0$, as desired.

We will also show the $n = 1$ case explicitly. If $|A| = 1$ then A has a single element, say a. Then $A = \{a\}$. And $\mathcal{P}(A) = \mathcal{P}(\{a\}) = \{\emptyset, \{a\}\}$. So $|\mathcal{P}(A)| = 2 = 2^1$, as desired.

Inductive Hypothesis. Assume that $|A| = k$ for some $k \in \mathbb{N}$, and $|\mathcal{P}(A)| = 2^k$.

Induction Step. Since A has at least one element, pick any such element and call it a. We aim to prove that $|\mathcal{P}(A)| = 2^{k+1}$. That is, we wish to prove that there are 2^{k+1} subsets of A.

To count these subsets, observe that every subset of A either includes a or does not include a. We will count those two separately.

Those which do not include a: To count the subsets of A which do not include a, what we are counting is $|\mathcal{P}(A \setminus \{a\})|$. Since $|A| = k+1$, note that $|A \setminus \{a\}| = k$ and so by the inductive hypothesis $|\mathcal{P}(A \setminus \{a\})| = 2^k$.

Those which do include a: To count the subsets of A which do include a, notice that each of these subsets, if a is removed, becomes a subset of $A \setminus \{a\}$. Conversely, the set of subsets of $A \setminus \{a\}$ can be turned into the set of subsets of A which contain a by simply adding a to all of the subsets of $A \setminus \{a\}$. This shows that there must be 2^k subsets of A which do not include a.

Since both of these cases has 2^k subsets, the total number of subsets of A is

$$2^k + 2^k = 2^{k+1}.$$

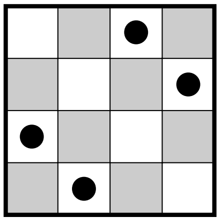
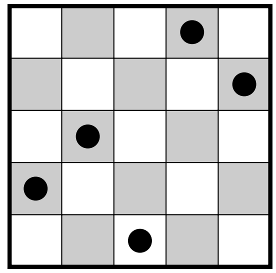
And so, $|\mathcal{P}(A)| = 2^{k+1}$, as desired.

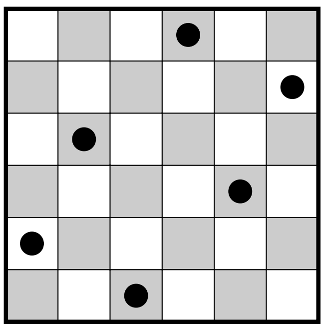
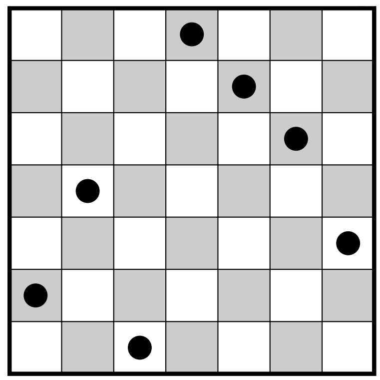
Conclusion. Therefore, by induction, the result holds for every $n \in \mathbb{N}$. $\square$

Solution to Question 8. A counterexample is $n = 11$. For this n, by using a calculator,

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{11} = 3.019\cdots > 3.$$

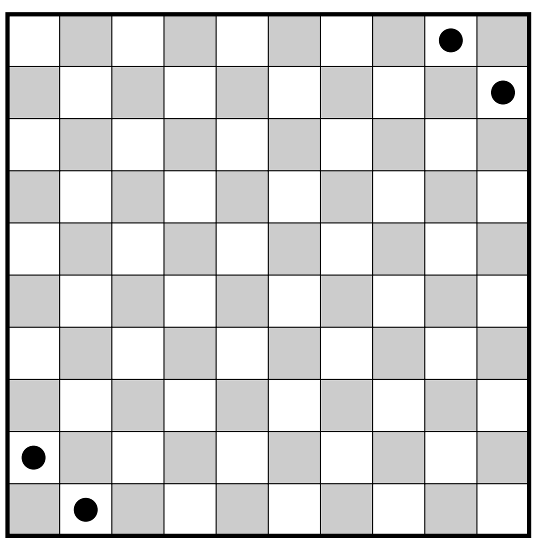
Solution to Question 9. We proceed by induction.
Base Case. Our proof will show that the kth stage implies the $(k+4)$th stage. Therefore, we will use four base cases.

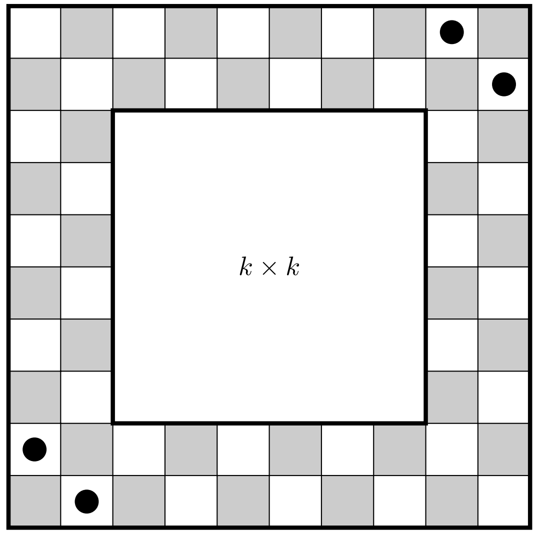



Inductive Hypothesis. Assume that there is a way to place k non-attacking rooks on a $k \times k$ chessboard, a way to place $k+1$ non-attacking rooks on a $(k+1) \times (k+1)$ chessboard, a way to place $k+2$ non-attacking rooks on a $(k+2) \times (k+2)$ chessboard, and a way to place $k+3$ non-attacking rooks on a $(k+3) \times (k+3)$ chessboard, for some natural number $k \geq 4$.

Induction Step. We will now show that the result holds for $k+4$. Assume you have a $(k+4) \times (k+4)$ chessboard. We can place four non-attacking rooks on the outer two rows and columns which are not on either diagonal.



Then, on the inside is a $k \times k$ board and so by the inductive hypothesis there is a way to place k non-attacking rooks on it so that non of those rooks are on either of the main diagonals (which will be the same as the diagonals on the $(k+4) \times (k+4)$ chessboard). Notice that those k rooks also do not attack any of the 4 rooks that were already placed. Therefore, these $k+4$ rooks are non-attacking and none of them are on either of the main diagonals.



Conclusion. Therefore, by induction, the result holds for every $n \geq 4$. $\square$

Solution to Question 10. We proceed by induction.

Base Case. The base case is when $n = 1$, and since $F_1 = 1$ and $F_3 = 2$, we do indeed have

$$F_1 = F_3 - 1,$$

as desired.

Inductive Hypothesis. Assume

$$F_1 + F_2 + F_3 + \cdots + F_k = F_{k+2} - 1,$$

for some $k \in \mathbb{N}$.

Induction Step. We aim to prove that

$$F_1 + F_2 + F_3 + \cdots + F_{k+1} = F_{k+3} - 1.$$

We show that this holds in the following. The second equality below uses the inductive hypothesis and the third equality makes use of the definition of the Fibonacci sequence, which implies that $F_{k+1} + F_{k+2} = F_{k+3}$.

$$\begin{aligned} F_1 + F_2 + F_3 + \cdots + F_{k+1} &= F_1 + F_2 + F_3 + \cdots + F_k + F_{k+1} \\ &= (F_{k+2} - 1) + F_{k+1} \\ &= (F_{k+1} + F_{k+2}) - 1 \\ &= F_{k+3} - 1, \end{aligned}$$

as desired.

Conclusion. Therefore, by induction, the result holds for every $n \in \mathbb{N}$. $\square$