Chapter 6 Solutions to Selected Exercises

Notes:

- The questions are in a separate PDF on LongFormMath.com.
- For most problems there are many correct solutions, so the below are not the only correct ways to solve the problems.
- If you spot an error, please email it to me at LongFormMath@gmail.com. Thanks!

Solution to Question 1. Answers vary.

Solution to Question 2. Answers vary.

Solution to Question 3. Part (a). The contrapositive is: If n is not even, then $2n^2 - 5n + 3$ is not odd. (Note: you can not change "not even" to "odd" for part (a), because it was not assumed that $n \in \mathbb{Z}$.)

Part (b). We will use the contrapositive. Suppose $n \in \mathbb{Z}$ and n is not even, implying that n is odd. By Definition 2.2, $n = 2k + 1$ for some $k \in \mathbb{Z}$. Then,

$$\begin{aligned} 2n^2 - 5n + 3 &= 2(2k+1)^2 - 5(2k+1) + 3 \\ &= 2(4k^2 + 4k + 1) - 10k - 5 + 3 \\ &= 8k^2 - 2k \\ &= 2(4k^2 - k). \end{aligned}$$

And since $k \in \mathbb{Z}$, also $(4k^2 - k) \in \mathbb{Z}$. Thus, by Definition 2.2, $2n^2 - 5n + 3$ is even, which also means that it is not odd.

We have shown that if n is not even, then $n^2 - 4n + 7$ is not odd. Therefore, by the contrapositive, if $2n^2 - 5n + 3$ is odd, then n is even. □

Solution to Question 4. We will use the contrapositive. Suppose $n \in \mathbb{Z}$ and n is not odd, implying that n is even. By Definition 2.2, $n = 2k$ for some $k \in \mathbb{Z}$. Then,

$$\begin{aligned} n^2 + 2n + 3 &= (2k)^2 + 2(2k) + 3 \\ &= 4k^2 + 4k + 2 + 1 \\ &= 2(2k^2 + 2k + 1) + 1. \end{aligned}$$

And since $k \in \mathbb{Z}$, also $(2k^2 + 2k + 1) \in \mathbb{Z}$. Thus, by Definition 2.2, $n^2 + 2n + 3$ is odd, which also means that it is not even.

We have shown that if n is not odd, then $n^2 + 2n + 3$ is not even. Therefore, by the contrapositive, if $n^2 + 2n + 3$ is even, then n is odd. □

Solution to Question 5. We will use the contrapositive. Suppose $n \in \mathbb{Z}$ and $3 \nmid n$. By the division algorithm (Theorem 2.11), either $n = 3q + 1$ for some $q \in \mathbb{Z}$, or $n = 3q + 2$ for some $q \in \mathbb{Z}$. We will analyze these two cases separately.

<u>Case 1:</u> If $n = 3q + 1$ for some integer q, then

$$n^2 - 1 = (3q+1)^2 - 1 = 9q^2 + 6q + 1 - 1 = 3(3q^2 + 2q).$$

Since $q \in \mathbb{Z}$, also $(3q^2 + 2q) \in \mathbb{Z}$. And so by Definition 2.8, since $n^2 - 1 = 3k$ where $k = 3q^2 + 2q$ is an integer, we have $3 \mid (n^2 - 1)$.

Case 2: If $n = 3q + 2$ for some integer q, then

$$n^2 - 1 = (3q + 2)^2 - 1 = 9q^2 + 12q + 4 - 1 = 3(3q^2 + 4q + 1).$$

Since $q \in \mathbb{Z}$, also $(3q^2 + 4q + 1) \in \mathbb{Z}$. And so by Definition 2.8, since $n^2 - 1 = 3k$ where $k = 3q^2 + 4q + 1$ is an integer, we have $3 \mid (n^2 - 1)$.

In either case, if $3 \nmid n$, then $3 \mid (n^2 - 1)$. Therefore, by the contrapositive, if $3 \nmid (n^2 - 1)$, then $3 \mid n$. □

Solution to Question 6. We will use the contrapositive. Suppose that $x \in \mathbb{R}$ and $x \leq 0$. We aim to show that $x^3 + x \leq 0$. To do so, note that since $x^2 > 0$, we can multiply both sides of $x \leq 0$ by x^2 to get

$$x^3 \leq 0.$$

Then, since $x^3 \leq 0$ and $x \leq 0$, their sum

$$x^3 + x \leq 0.$$

We have shown that if $x \leq 0$, then $x^3 + x \leq 0$. Thus, by the contrapositive, if $x^3 + x > 0$, then $x > 0$. □

Solution to Question 7. We will use contrapositive. Suppose that it is not true that $n \notin \{1, 2, 6\}$. That is, $n \in \{1, 2, 6\}$. This produces three cases:

Case 1: if $n = 1$ then $F_1 = 1$, which is perfect cube.

Case 2: if $n = 2$ then $F_2 = 1$, which is perfect cube.

Case 3: if $n = 6$ then $F_6 = 8$, which is perfect cube.

We proved that if $n \in 1, 2, 6$, then it is F_n is a perfect cube. Hence, by contrapositive, if F_n is not a perfect cube, then $n \notin \{1, 2, 6\}$. □

Solution to Question 8. Suppose $n \in \mathbb{Z}$. First we will show that n being even implies $(n+1)^2 - 1$ is even by a direct proof. Since n is even, by Definition 2.2 we have $n = 2k$ for some $k \in \mathbb{Z}$. Thus,

$$(n + 1)^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 2(2k^2 + 2k).$$

And since $k \in \mathbb{Z}$, also $(2k^2 + 2k) \in \mathbb{Z}$, implying that $(n + 1)^2 - 1$ is even.

Next we show that $(n + 1)^2 - 1$ being even implies that n is even by using the contrapositive. To that end, assume that n is not even; being an integer, this means that n is odd. Thus, by Definition 2.2 we have $n = 2k + 1$ for some $k \in \mathbb{Z}$, and so

$$(n + 1)^2 - 1 = (2k + 2)^2 - 1 = 4k^2 + 8k + 4 - 1 = 4k^2 + 8k + 2 + 1 = 2(2k^2 + 2k + 1) + 1.$$

And since $k \in \mathbb{Z}$, also $(2k^2 + 2k + 1) \in \mathbb{Z}$, showing that $(n + 1)^2 - 1$ is odd and hence is not even. We have shown that n being not even implies that $n^2 + 1$ is not even. Therefore, by the contrapositive, if $(n + 1)^2 - 1$ is even, then n is even.

We have demonstrated that if n is even, then $(n + 1)^2 - 1$ is even, and also that if $(n + 1)^2 - 1$ is even, then n is even. Combined, this completes the proof. □

Solution to Question 9. Claim: No math textbook has ever included a meme.

Counterexample: *Proofs: A Long-Form Mathematics Textbook*

Solution to Question 10.

(d) Let $x = 0$ and $y = 4$. Then $|x + y| = 4 = |x - y|$.

(i) Let $n = 31$. Then

$$2n^2 - 4n + 31 = 2(31)^2 - 4(31) + 31 = 31(62 - 4 + 1) = 31 \cdot 59,$$

which is not prime.

(o) If $A = \{1, 2, 3\}$, $B = \{2\}$ and $C = \{3\}$, then

$$A \setminus (B \cap C) = \{1, 2, 3\} \neq \{1\} = (A \setminus B) \cap (A \setminus C).$$