Chapter 7 Solutions to Selected Exercises

Notes:

- The questions are in a separate PDF on LongFormMath.com.
- For most problems there are many correct solutions, so the below are not the only correct ways to solve the problems.
- If you spot an error, please email it to me at LongFormMath@gmail.com. Thanks!

Solution to Question 1. Assume that $a, b \in \mathbb{R}$ and $a \in \mathbb{Q}$ and $ab \in \mathbb{R} \setminus \mathbb{Q}$. Assume for a contradiction that $b \in \mathbb{Q}$. Since $a, b \in \mathbb{Q}$, by Definition 3.3 there exist $m, n, s, t \in \mathbb{Z}$ for which $n, t \neq 0$ and

$$a = \frac{m}{n} \qquad \text{and} \qquad b = \frac{s}{t}.$$

Then,

$$ab = \frac{ms}{nt},$$

and since $m, n, s, t \in \mathbb{Z}$, so are ms and nt; and since $n, t \neq 0$, also $nt \neq 0$. Thus, again by Definition 3.3, $ab \in \mathbb{Q}$. This gives the contradiction and completes the proof. □

Solution to Question 2. By Theorem 7.6, $\sqrt{2}$ is irrational. We will now show that $-\sqrt{2}$ is irrational. Indeed, if it weren't, then $-\sqrt{2} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ and $b \neq 0$. But then $\sqrt{2} = \frac{-a}{b}$ shows that $\sqrt{2}$ would then be rational, giving the contradiction.

Thus, both $-\sqrt{2}$ and $\sqrt{2}$ are irrational, and yet their sum is rational:

$$-\sqrt{2} + \sqrt{2} = 0.$$

□

Solution to Question 3. Assume for a contradiction that there do exist integers m and n for which $21m + 35n = 1$. Since $m, n \in \mathbb{Z}$, also $(3m + 5n) \in \mathbb{Z}$. Dividing both sides by 7 gives

$$3m + 5n = \frac{1}{7}.$$

This is a contradiction, since we had said that $3m + 7n$ is an integer, and $\frac{1}{5}$ is not an integer. □

Solution to Question 4. Assume for a contradiction that A and B are sets inside a universal set U, and $A^c \cap (B \cap A) \neq \emptyset$. To not be the empty set means that you contain an element, so there must be some $x \in A^c \cap (B \cap A)$. By the definition of the intersection, this means $x \in A^c$ and $x \in (B \cap A)$. Again by the definition of the intersection, $x \in B$ and $x \in A$.

We have shown that $x \in A^c$, which means that $x \notin A$; but this contradicts our deduction that $x \in A$, and gives the contradiction. □

Solution to Question 5. Assume for a contradiction that $\sqrt{5}$ is rational. Then, there must be some nonzero integers p and q where

$$\sqrt{5} = \frac{p}{q}.$$

Moreover, we may assume that this fraction is written in *lowest terms*, meaning that p and q have no common divisors. Then,

$$\sqrt{5}q = p.$$

And by squaring both sides,

$$5q^2 = p^2.$$

Since $q^2 \in \mathbb{Z}$, by the definition of divisibility this implies that $5 \mid p^2$, and hence $5 \mid p$ by Lemma 2.17 part (iii). By a second application of the definition of divisibility, this means that $p = 5k$ for some nonzero integer k. Plugging this in,

$$\begin{aligned} 5q^2 &= p^2 \\ 5q^2 &= (5k)^2 \\ 5q^2 &= 25k^2 \\ q^2 &= 5k^2. \end{aligned}$$

Therefore, $5 \mid q^2$, and hence $5 \mid q$, again by Lemma 2.17 part (iii). But this is a contradiction: We had assumed that p and q had no common factors, and yet we proved that 5 divides each. Therefore $\sqrt{5}$ cannot be rational, meaning it is irrational. □

Solution to Question 6. Assume for a contradiction that $\sqrt{10}$ is rational. Then, there must be some nonzero integers p and q where

$$\sqrt{10} = \frac{p}{q}.$$

Moreover, we may assume that this fraction is written in *lowest terms*, meaning that p and q have no common divisors. Then,

$$\sqrt{10}q = p.$$

And by squaring both sides,

$$\begin{aligned} 10q^2 &= p^2 \\ 5(2q^2) &= p^2. \end{aligned}$$

Since $2q^2 \in \mathbb{Z}$, by the definition of divisibility this implies that $5 \mid p^2$, and hence $5 \mid p$ by Lemma 2.17 part (iii). By a second application of the definition of divisibility, this means that $p = 5k$ for some nonzero integer k. Plugging this in,

$$\begin{aligned} 10q^2 &= p^2 \\ 10q^2 &= (5k)^2 \\ 10q^2 &= 25k^2 \\ 2q^2 &= 5k^2. \end{aligned}$$

Therefore, $5 \mid 2q^2$, and hence, by Lemma 2.17 part (iii), $5 \mid 2$ or $5 \mid q^2$. Since $5 \nmid 2$, we have $5 \mid q^2$. And so, by one more application of Lemma 2.17 part (iii), $5 \mid q$. But this is a contradiction: We had assumed that p and q had no common factors, and yet we proved that 5 divides each. Therefore $\sqrt{10}$ cannot be rational, meaning it is irrational. □

Solution to Question 7. Assume for a contradiction that there are only finitely many composite numbers. Then there must be a largest composite number, which we call M. Since every positive integer is either prime or composite, every integer larger than M must be prime. Now consider $2M$. This is a composite number larger than M, and hence gives the contradiction. □

Solution to Question 8. Suppose that $a, b, c \in \mathbb{Z}$ and $a^2 + b^2 = c^2$. Assume for a contradiction that a and b are both not even—meaning they are odd. By Definition 2.2, there exist integers x and y such that

$a = 2x + 1$ and $b = 2y + 1$. Thus,

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= (2x+1)^2 + (2y+1)^2 \\ &= 4x^2 + 4x + 1 + 4y^2 + 4y + 1 \\ &= 4\left(x^2 + x + y^2 + y\right) + 2. \end{aligned}$$

This shows that c^2 is a sum of even numbers and hence is even. And since c^2 is even, c must also be even. Thus there exists some $z \in \mathbb{Z}$ such that $c = 2z$. Observe

$$\begin{aligned} c^2 &= 4\left(x^2 + x + y^2 + y\right) + 2 \\ (2z)^2 &= 4\left(x^2 + x + y^2 + y\right) + 2 \\ 4z^2 &= 4\left(x^2 + x + y^2 + y\right) + 2 \\ z^2 &= \left(x^2 + x + y^2 + y\right) + \frac{1}{2} \\ z^2 - \left(x^2 + x + y^2 + y\right) &= \frac{1}{2}. \end{aligned}$$

This, however, means that the difference of two integers is equal to a non-integer, which is a contradiction. □

Solution to Question 9. Direct Proof: Consider $(x-y)^2$. Note that since this is squared, it must be non-negative. Then,

$$\begin{aligned} 0 &\le (x-y)^2 \\ 0 &\le x^2 - 2xy + y^2 \\ 2xy &\le x^2 + y^2 \\ 2xy + 2xy &\le x^2 + 2xy + y^2 \\ 4xy &\le (x+y)^2. \end{aligned}$$

And since both sides are non-negative, we can take the square root of both sides to get

$$\begin{aligned} \sqrt{4xy} &\le x + y \\ 2\sqrt{xy} &\le x + y. \end{aligned}$$

□

Proof by contradiction: Assume for a contradiction that $x + y < 2\sqrt{xy}$. Then by doing some arithmetic,

$$x + y - 2\sqrt{xy} < 0.$$

By factoring,

$$(\sqrt{x} - \sqrt{y})^2 < 0.$$

This is a contradiction, though, since squaring a real number will not produce a negative result. □

Solution to Question 10. Assume for a contradiction that there was a way to complete that square to form a magic square. Since one diagonal is already complete, we see that the magic number must be

$$1 + 4 + 8 + 10 = 23.$$

With this information, we can slowly start deducing what some squares must be. First,

1	2	3	17
	4	5	6
7		8	
	9		10

Second,

1	2	3	17
	4	5	6
7		8	-10
	9		10

Third,

1	2	3	17
8	4	5	6
7		8	-10
	9		10

And with this, by looking at the columns, we can complete the square:

1	2	3	17
8	4	5	6
7	8	8	-10
7	9	7	10

This produces a contradiction, though. Since every one of our placements was forced, if this were to be a magic square, we have argued what the magic square must look like. However, note that the last two rows do not have the magic sum of 23. Thus, no magic square is possible. □