Chapter 8 Solutions to Selected Exercises

Notes:

- The questions are in a separate PDF on LongFormMath.com.
- For most problems there are many correct solutions, so the below are not the only correct ways to solve the problems.
- If you spot an error, please email it to me at LongFormMath@gmail.com. Thanks!

Solution to Question 1.
(a) This is a function. It is injective.
(b) This is not a function.
(c) This is a bijective function.
(d) This is a function that is neither injective nor surjective.

Solution to Question 2. A function is defined not only by its rule, but also its domain and codomain. While the domains and codomains were rarely explicitly mentioned in precalculus, the standard interpretation is for it to be the largest real domain possible. With this, f's domain is $\mathbb{R} \setminus \{-3\}$ and g's domain is $\mathbb{R}$. Since these have different domains, they are different functions.

Solution to Question 3. The range is the set of all positive integers which are divisible by both 2 and 3, and are not divisible by any other prime.

Solution to Question 4. We will prove that f is a bijection by proving it is injective and surjective.

Injective. Suppose $x, y \in \mathbb{R}$ and $f(x) = f(y)$. That is,

$$4x + 3 = 4y + 3.$$

Simplifying,

$$\begin{aligned} 4x &= 4y \\ x &= y. \end{aligned}$$

We have shown that if $f(x) = f(y)$, then $x = y$. Thus, f is injective.

Surjective. Suppose that $b \in \mathbb{R}$. We wish to find an x from the domain for which $f(x) = b$.
Let $x = \frac{b-3}{4}$. Notice that $x \in \mathbb{R}$ and

$$f(x) = f\left(\frac{b-3}{4}\right) = 4\frac{b-3}{4} + 3 = b - 3 + 3 = b.$$

That is, for any $b \in \mathbb{R}$ we found an $x \in \mathbb{R}$ for which $f(x) = b$. Thus, f is surjective.

Since f is both injective and surjective, f is bijective. □

Solution to Question 5.
Part (j): r is not injective. To see this, note that

$$r(-5) = -|-5 + 4| = -1$$

and

$$r(-3) = -|-3 + 4| = -1.$$

That is, we have found numbers $x, y \in (\infty, 0)$ for which $x \neq y$ and yet $r(x) = r(y)$, proving that r is not injective.

We will now show that r is surjective. Indeed, let $b \in (-\infty, 0)$ and let $x = b - 4$. Note that $b \in (-\infty, 0)$ and that

$$r(x) = r(b-4) = -|b-4+4| = -|b| = -(-b) = b.$$

Thus, given any $b \in (-\infty, 0)$, there exists some $b \in (-\infty, 0)$ for which $r(x) = b$, proving that r is surjective.

Part (k): s is injective. To see this, observe that if $s(x) = s(y)$, then $(x, x) = (y, y)$. For two ordered pairs to be equal, they must be equal in each coordinate—both give the same thing: $x = y$. We have shown that if $s(x) = s(y)$, then $x = y$, and hence s is injective.

We will now show that s is not surjective. To see this, note that $(1, 2) \in \mathbb{N} \times \mathbb{N}$, and if $s(x) = (1, 2)$, then $(x, x) = (1, 2)$, implying that $x = 1$ and $x = 2$, a contradiction. Thus, $(1, 2)$ is in the codomain but not the range, showing that s is not surjective. $\square$

Solution to Question 6. Assume A and B are finite sets and $|A| = |B|$ and $f : A \to B$. First we will prove that if f is injective, then f is surjective. We do so using the contrapositive. To that end, assume f is not surjective. Then there is some $b \in B$ which nothing maps to; that is,

$$f(x) \in B \setminus \{b\}$$

for all $x \in A$. Consider each element in A to be in object and each element in $B \setminus \{b\}$ to be a box. Place object a in box $f(a)$, for each $a \in A$. Notice that there are $|A|$ objects and $|B| - 1$ boxes. And since $|A| = |B|$ this means there are more objects than boxes and hence by the pigeonhole principle there must be some box with at least two objects in it.

Suppose x and y are two objects that ended up in the same box. Since x was placed in box $f(x)$ and y was placed in box $f(y)$, and the pigeonhole principle tells us that these two boxes are the same, this means that $f(x) = f(y)$, proving that f is not injective. Since f being not surjective implies it is not injective, by the contrapositive f being injective implies it is surjective.

Next we prove that if f is surjective, then it is injective. Again, we use the contrapositive. To that end, assume that f is not injective. Then there are $x, y \in A$ such that $f(x) = f(y)$ but yet $x \neq y$. Consider now the remaining $|A| - 2$ points from A, and the remaining $|B| - 1$ points from B. Assume for a contradiction that each of those points in B were hit by a point in A. Then, every $b \in B$ has some $a \in A$ for which $f(a) = B$.

Consider each of these $|B| - 1$ points to be objects and each of the $|A| - 2$ remaining points in A to be boxes, and for each $b \in B$, pick an $a \in A$ which maps to b and place b in that Box a. Since $|A| = |B|$, there are more objects than boxes. Thus, by the pigeonhole principle, there must be some Box a which has more than one point in it; meaning, there are two distinct points $b_1, b_2 \in B$ for which $f(a) = b_1$ and $f(a) = b_2$. This contradicts the definition of a function, which insists that each point in the domain is sent to only one element of the codomain. This proves that $f : A \to B$ is not surjective. Since f being not injective implies f is not surjective, by the contrapositive f being surjective implies that f is injective. $\square$

Second Solution. Assume A and B are finite sets and $|A| = |B|$ and $f : A \to B$. First we will prove that if f is injective, then f is surjective. We do so using the contrapositive. To that end, assume f is not surjective. Then there is some $b \in B$ which nothing maps to; that is,

$$f(x) \in B \setminus \{b\}$$

for all $x \in A$. Thus, f's rule acts as a function from A to $B \setminus \{b\}$, where $|A| = n$ and $|B \setminus \{b\}| = n - 1$, by the func-y pigeonhole principle f is not injective. Since f being not surjective implies it is not injective, by the contrapositive f being injective implies it is surjective.

Next we prove that if f is surjective, then it is injective. Again, we use the contrapositive. To that end, assume that f is not injective. Then there are $x, y \in A$ such that $f(x) = f(y)$ but yet $x \neq y$. Consider now the remaining $n - 2$ points from A. Consider the portion of f which maps $A \setminus \{x, y\}$ to $B \setminus \{f(x)\}$, which is a function from a set of size $n-2$ to a set of size $n-1$. By the func-y pigeonhole principle this function is not surjective. This proves that $f : A \to B$ is not surjective. Since f being not injective implies f is not surjective, by the contrapositive f being surjective implies that f is injective. $\square$

Solution to Question 7.

(c) Not invertible. Not surjective because no element maps to -3.

(d) Invertible; $h^{-1} : [-1, 1] \to [0, \pi]$ where $h^{-1}(x) = \arccos(x)$

Solution to Question 8.

$$(f \circ g)(x) = f(g(x)) = f(2x+3) = (2x+3)^2 - 2(2x+3) + 1.$$

$$(g \circ f)(x) = g(f(x)) = g(x^2 - 2x + 1) = 2(x^2 - 2x + 1) + 3.$$

$\square$

Solution to Question 9. There are many examples. Here's one:

$$\begin{aligned} f : \mathbb{R} \to \mathbb{R} \qquad & \text{where} \qquad f(x) = x^2 \\ g : \mathbb{R}^+ \to \mathbb{R} \qquad & \text{where} \qquad g(x) = x^2 \\ f \circ g : \mathbb{R}^+ \to \mathbb{R} \qquad & \text{where} \qquad (f \circ g)(x) = x^4 \end{aligned}$$

Here's another one:

$$\begin{aligned} f : \mathbb{R} \to \mathbb{R} \qquad & \text{where} \qquad f(x) = x^2 \\ g : \mathbb{R} \to \mathbb{R} \qquad & \text{where} \qquad g(x) = e^x \\ f \circ g : \mathbb{R} \to \mathbb{R} \qquad & \text{where} \qquad (f \circ g)(x) = e^{2x} \end{aligned}$$

$\square$

Solution to Question 10.

Conjecture 1 is true. The expression $(f+g) \circ h$ refers to the function obtained by plugging $h(x)$ into the function $f(x) + g(x)$. Doing this gives $f(h(x)) + g(h(x))$, which is also $(f \circ h) + (g \circ h)$.

Conjecture 2 is false. For example, consider the functions $f(x) = x$ and $g(x) = 1$ and $h(x) = x^2$, each with a domain and codomain of $\mathbb{R}$. For these functions,

$$\begin{aligned} h \circ (f+g) &= h(f(x) + g(x)) \\ &= h(x+1) \\ &= (x+1)^2 \\ &= x^2 + 2x + 1 \\ &\neq x^2 + 1 \\ &= h(x) + h(1) \\ &= h(f(x)) + h(g(x)) \\ &= (h \circ f) + (h \circ g). \end{aligned}$$

Since there is a counterexample to Conjecture 2, this shows that the conjecture is false.