Chapter 9 Solutions to Selected Exercises

Notes:

- The questions are in a separate PDF on LongFormMath.com.
- For most problems there are many correct solutions, so the below are not the only correct ways to solve the problems.
- If you spot an error, please email it to me at LongFormMath@gmail.com. Thanks!

Solution to Question 1.
Part (a): $1 \sim 1$, $2 \sim 2$, $3 \sim 3$, $4 \sim 4$, $1 \sim 2$, $1 \sim 3$, $1 \sim 4$, $2 \sim 3$, $2 \sim 4$, $3 \sim 4$

Part (b): $1 \sim 1$, $2 \sim 2$, $3 \sim 3$, $1 \sim 3$, $3 \sim 1$, $2 \sim 4$, $4 \sim 2$

Solution to Question 2.
Part (a): There are two partitions: $\{\{1,2\}\}$ and $\{\{1\},\{2\}\}$.
Part(b): There are five partitions: $\{\{a,b,c\}\}$, $\{\{a,b\},\{c\}\}$, $\{\{a,c\},\{b\}\}$, $\{\{b,c\},\{a\}\}$ and $\{\{a\},\{b\},\{c\}\}$.

Solution to Question 3. First, $\sim$ is not guaranteed to be reflexive, because each person likely did not shake hands with themselves. Second, $\sim$ is symmetric, because if person a shakes hands with person b, then person b shakes hands with person a. Third, $\sim$ is not necessarily transitive, because if person a shakes hands with person b and if person b shakes hands with person c, then there is no guarantee that person a shakes hands with person c.

Solution to Question 4. The relation $\sim$ is reflexive, symmetric and transitive. It is therefore an equivalence relation with equivalence classes

$$\{w\}\ ,\ \{x,y\}\ ,\ \{z\}.$$

Solution to Question 5. Note that $\sim$ is reflexive because for any set a, we have $a \subseteq a$ (if $x \in a$, then $x \in a$). And $\sim$ transitive because for any sets a, b and c, if $a \subseteq b$ and $b \subseteq c$, then $a \subseteq c$ (if $x \in a$ and $a \subseteq b$, then $x \in b$; and if $x \in b$ and $b \subseteq c$, then $x \in c$; so, $x \in a$ implies $x \in c$).

But $\sim$ is not symmetric. For example, $\{1\} \subseteq \{1,2\}$ but $\{1,2\} \not\subseteq \{1\}$. Therefore $\sim$ is not an equivalence relation. $\square$

Solution to Question 6.
Reflexive: To see that $a \sim a$ for all $a \in \mathbb{Z}$, simply note that $2a + a = 3a \equiv 0 \pmod 3$. Therefore, $a \sim a$. This proves that $\sim$ is reflexive.

Symmetric: Assume that $a \sim b$ for some $a, b \in \mathbb{Z}$. This means that

$$2a + b \equiv 0 \pmod 3.$$

Next, since $3a \equiv 0 \pmod 3$ and $3b \equiv 0 \pmod 3$, note that

$$3a + 3b \equiv 0 \pmod 3.$$

Therefore,

$$(3a + 3b) - (2a + b) \equiv 0 - 0 \pmod 3.$$

Simplifying,

$$2b + a \equiv 0 \pmod 3.$$

This shows that $b \sim a$ and proves that $\sim$ is symmetric.

Transitive: Assume that $a \sim b$ for some $a, b \in \mathbb{Z}$. This means that

$$2a + b \equiv 0 \pmod 3 \qquad \text{and} \qquad 2b + c \equiv 0 \pmod 3.$$

Therefore

$$(2a + b) + (2b + c) \equiv 0 + 0 \pmod 3.$$

I.e.,

$$2a + 3b + c \equiv 0 \pmod 3.$$

And because $3b \equiv 0 \pmod 3$, this means that

$$2a + 0 + c \equiv 0 \pmod 3.$$

I.e.,

$$2a + c \equiv 0 \pmod 3.$$

This shows that $a \sim c$ and proves that $\sim$ is transitive and completes the proof that $\sim$ is an equivalence relation.

The equivalence classes of $\sim$ are

$$\{\dots, -6, -3, 0, 3, 6, \dots\}\ ,\ \{\dots, -5, -2, 1, 4, 7, \dots\}\ ,\ \{\dots, -4, -1, 2, 5, 8, \dots\}$$

Solution to Question 7. $\sim$ is not reflexive, since $a \neq a$ is clearly false for any $a \in \mathbb{Z}$.
$\sim$ is symmetric, since if $a \neq b$, then clearly also $b \neq a$.
$\sim$ is not transitive, since, for example, $1 \sim 2$ and $2 \sim 1$, but $1 \sim 1$.

Solution to Question 8.
Reflexive: To see that $\sim$ is reflexive, simply note that

$$ab = ba$$

implies that $(a, b) \sim (a, b)$ for any $a, b \in \mathbb{Z}$ (with $b \neq 0$).
Symmetric: Assume that $(a, b) \sim (c, d)$ for some $a, b, c, d \in \mathbb{Z}$ with $b, d \neq 0$. This means that

$$ad = bc,$$

and hence

$$cb = da,$$

which means that $(c, d) \sim (a, b)$. This shows that $\sim$ is symmetric.
Transitive: Assume that $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$ for some $a, b, c, d, e, f \in \mathbb{Z}$ with $b, d, f \neq 0$. This means that

$$ad = bc \qquad \text{and} \qquad cf = de.$$

Starting with the first equality and multiplying both sides by f, we get

$$adf = bcf.$$

Next, applying the equality $cf = de$ to the right hand side above,

$$adf = bde.$$

Since $d \neq 0$ we can cancel the d from both sides to get

$$af = be.$$

This shows that $(a, b) \sim (e, f)$ and proves that $\sim$ is transitive. And since $\sim$ is reflexive, symmetric and transitive, it is an equivalence relation.

Solution to Question 9. One answer: This is the relation $\sim$ on the set $\mathbb{Z}$ where $a \sim b$ if $a \equiv b \pmod 3$.

Solution to Question 10. Part (c): This is the relation $\sim$ on $\mathbb{R}$ where $a \sim b$ if $a \neq -b$.
Part (d): This is the relation $\sim$ on $\mathbb{Z}$ where $a \sim b$ if $a > b$.