Higher Trigonometry, Hyperreal Numbers, and Euler's Analysis of Infinities

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In a textbook published in 1748, without the barest mention of the derivative, Euler derived the fundamental equations of a subject that was later to become known as *higher trigonometry*: he explained the series for the exponential and logarithmic functions,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$
$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots,$$

proved the Euler identity,

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

computed the series for the sine and cosine,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots,$$

proved the factorization formula for the sine,

$$\sin x = x\left(1 - \frac{x^2}{(1\pi)^2}\right)\left(1 - \frac{x^2}{(2\pi)^2}\right)\left(1 - \frac{x^2}{(3\pi)^2}\right)\cdots,$$

and deduced his celebrated formula,

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6},$$

among many other facts. The textbook is Euler's *Introductio in Analysin Infinitorum* (Introduction to the Analysis of Infinities). "All this follows from ordinary algebra," he claimed, and all this in a textbook for beginners!

> Often I have considered the fact that most of the difficulties which block the progress of students trying to learn analysis stem from this: that although they understand little of ordinary algebra, still they attempt this more subtle art. From this it follows not only that they remain on the fringes, but in addition they entertain strange ideas about the concept of the infinite, which they must try to use. Although analysis does not require an exhaustive knowledge of algebra,

> even of all the algebraic techniques so far discovered, still there are topics whose consideration prepares a student for a deeper understanding. However, in the ordinary treatise on the elements of algebra, these topics are either completely omitted or are treated carelessly. For this reason, I am certain that the material I have gathered in this book is quite sufficient to remedy that defect.... There are many questions which are answered in this work by means of ordinary algebra, although they are usually discussed with the aid of analysis. In this way the interrelationship between the two methods becomes clear. [**12**, p. *v*]

What is this "ordinary algebra" that Euler spoke of, and how did it allow him to deduce results that we now classify as requiring differential calculus? The answer lies here: although Euler did not use the notion of the derivative to deduce these results (and certainly not theorems like Taylor's Theorem, which depend on the derivative) his notion of ordinary algebra went beyond what most of our contemporaries would include. In particular, Euler explicitly included the arithmetic of infinite and infinitesimal quantities, and implicitly used a general principle for simplifying calculations involving infinitely many infinitesimals. Because of this, Euler is often portrayed in popular accounts and classroom lectures as a reckless symbol-manipulator, who worked in a number system fraught with nonsense and contradiction, but who through sheer intuitive brilliance somehow came to correct conclusions. The following passages, taken from popular books on the history of mathematics, are typical.

> It is perhaps only fair to point out that some of Euler's works represent outstanding examples of eighteenth-century formalism, or the manipulation, without proper attention to matters of convergence and mathematical existence, of formulas involving infinite processes. He was incautious in his use of infinite series, often applying to them laws valid only for finite sums. Regarding power series as polynomials of infinite degree, he heedlessly extended to them well-known properties of finite polynomials. Frequently, by such careless approaches, he luckily obtained truly profound results .... [**13**, p. 435]

> Today, we recognize that Euler was not so precise in his use of the infinite as he should have been. His belief that finitely generated patterns and formulas automatically extend to the infinite case was more a matter of faith than science, and subsequent mathematicians would provide scores of examples showing the folly of such hasty generalizations. [**7**, p. 222]

In contrast we take Euler's calculations involving infinite and infinitesimal numbers seriously, and find that Euler's *Introductio* is written with grace, wit, and care. There is the occasional misstep, but on the whole, Euler's use of the infinite and infinitesimal is consistent and clear. Furthermore, there is a modern context, replete with infinite and infinitesimal numbers, in which Euler's methods can be made intelligible, rigorous, and useful to modern readers.

What follows is our own version of Euler's mathematical tale, sensitively rehabilitated to contemporary tastes for rigor.

Exponentials and logarithms in Euler's *Introductio*

Euler began his introductory chapter on exponentials and logarithms [**12**, Chap. VI] by saying,

> Although the concept of a transcendental function depends on integral calculus, there are certain kinds of functions which are more obvious, which can be conveniently developed, and which open the door to further investigations.

He went on to explain the usual laws of exponents and logarithms, and illustrated the usefulness of tables of logarithms, much as one would in a precalculus course today, with examples from business and the life sciences.

> A certain man borrowed 400,000 florins at the usurious rate of five percent annual interest. Suppose that each year he repays 25,000 florins. The question is, how long will it be before the debt is repaid completely?...
>
> Since after the flood all men descended from a population of six, if we suppose that the population after two hundred years was 1,000,000, we would like to find the annual rate of growth.

To demonstrate the usefulness of tables of logarithms, Euler asked,

> If the progression 2, 4, 16, 256, ... is formed by letting each term be the square of the preceding term, find the value of the twenty-fifth term.

In the succeeding chapter, Euler developed the series for the exponential and logarithmic functions, and showed how to use series to compile tables of logarithms. What interests us here is the means by which Euler obtained those series. Euler began his discussion of the series for the exponential function as follows [**12**, Chap. VII]:

> Since $a^0 = 1$, when the exponent on a increases, the power itself increases, provided that a is greater than 1. It follows that if the exponent is infinitely small and positive, then the power also exceeds 1 by an infinitely small amount. Let ϵ be an infinitely small number, or, a fraction so small that, although not equal to 0, still $a^\epsilon = 1 + \psi$, where ψ is also an infinitely small number. From the preceding chapter we know that unless ψ were infinitely small, then neither would ϵ be infinitely small. It follows that $\psi = \epsilon$ or $\psi > \epsilon$ or $\psi < \epsilon$. Which of these is true depends on the value of a, which is not now known, so we let $\psi = \lambda\epsilon$. Then we have $a^\epsilon = 1 + \lambda\epsilon$ .... [**12**, §114]

(We have changed Euler's ω to ϵ and his j, in what follows, to K.) Euler then reasoned that if x is any finite, positive, noninfinitesimal number, and K is x/ϵ, then by a simple calculation using the Binomial Theorem (discussed in §71 of the *Introductio*), a series for a^x is given by

$$
\begin{aligned}
a^x = a^{K\epsilon} = (a^\epsilon)^K = (1+\lambda\epsilon)^K &= \left(1 + \frac{\lambda x}{K}\right)^K \\
&= 1 + \frac{1}{1}\lambda x + \frac{1(K-1)}{1\cdot 2K}\lambda^2 x^2 + \frac{1(K-1)(K-2)}{1\cdot 2K\cdot 3K}\lambda^3 x^3 + \cdots \\
&= 1 + \lambda x + \frac{K-1}{K}\cdot\frac{1}{1\cdot 2}\lambda^2 x^2 + \frac{K-1}{K}\frac{K-2}{K}\cdot\frac{1}{1\cdot 2\cdot 3}\lambda^3 x^3 + \cdots.
\end{aligned}
$$

Euler then reasoned that since x is noninfinitesimal and ϵ is infinitesimal, K will necessarily be infinite, and hence one may substitute 1 for the fractions $\frac{K-1}{K}$, $\frac{K-2}{K}$, $\frac{K-3}{K}$, and so on, to obtain

$$
a^x = 1 + \lambda x + \frac{1}{2!}\lambda^2 x^2 + \frac{1}{3!}\lambda^3 x^3 + \cdots.
$$

Finally, Euler examined the case in which the base a is taken to correspond to λ being equal to unity—the natural exponential function—and showed that in general λ is the natural logarithm of a.

This argument, also discussed by Edwards [**9**, pp. 272–274] and Dunham [**8**], among others, is a gem of eighteenth-century mathematical reasoning, but there are several issues that must be dealt with before something like it could honestly be given in a modern context.

- Euler freely uses the arithmetic of infinite and infinitesimal numbers. If such numbers are to be used in a modern context, the rules for dealing with them must be presented as clearly, concisely, and consistently as the rules for ordinary numbers.
- Even granted a sound treatment of infinite and infinitesimal numbers, the reasoning by which one is allowed to make infinitely many substitutions—the numbers $\frac{K-1}{K}$, $\frac{K-2}{K}$, $\frac{K-3}{K}$, and so on, each being replaced by 1—must be explained. In each substitution instance, an error is incurred; for example, the difference between 1 and $\frac{K-1}{K}$ is $\frac{1}{K}$. Individually these differences are infinitesimal, but (as Euler was well aware) it is possible for infinitely many infinitesimals to add up to a noninfinitesimal amount.
- The argument as given employs the Binomial Theorem for nonintegral exponents, a theorem that Euler chose not to prove in the *Introductio*, and something that we would hesitate to assume in a modern precalculus course.

In our rehabilitation of Euler's methods for modern use, we deal with these issues as follows.

- We work in a consistent axiomatic system that clearly specifies the properties of infinite and infinitesimal numbers.
- We provide a criterion, based on the intuitive notion of *determinacy*, for deciding whether neglecting infinitely many infinitesimals leads to a negligible difference in an infinite sum.
- In our construction of the series for the exponential function, we find that the Binomial Theorem for natural exponents, a theorem that is verified by mathematical induction in traditional precalculus courses, suffices. (Later, in connection with the series for the logarithm, we give an elementary proof of the Binomial Theorem for fractional exponents.)

Once these issues are dealt with, we will return to Euler's argument and show how it can be rigorously rehabilitated in this context. We will then go on to obtain the series for the sine, cosine, and logarithm.

The arithmetic of the infinite and infinitesimal

The first requirement of our rehabilitation of Euler's arguments is that his methods be formulated within a mathematical system in which the properties of infinite and infinitesimal numbers are explained at least as clearly as the properties of the real numbers. For this we turn to the system of *hyperreal* numbers, as described axiomatically in Keisler's textbook, *Calculus: An Infinitesimal Approach* [**23**].

In elementary courses, the real numbers are not defined explicitly; instead they are defined implicitly by their arithmetic properties, an approach that is essentially axiomatic. In more advanced courses one builds a model for the real numbers, typically using equivalence classes of Cauchy sequences of rational numbers. Similarly, the hyperreal numbers can either be introduced axiomatically or by building a model using equivalence classes of sequences of real numbers.

Keisler's textbook is intended for use in an introductory calculus course. He introduces the properties of the hyperreal numbers gradually, with appropriate examples and exercises, over the first forty pages of the book. The real numbers are described informally in the main body of the textbook, but presented more precisely in an appendix by citing the field axioms, the order axioms, the definition of the natural numbers, the root axiom (that principal n^{th} roots exist for positive numbers), and the completeness axiom. Further axioms describe the hyperreal numbers as a field containing infinite and infinitesimal numbers in addition to all the real numbers. (He discusses a set-theoretic construction of the hyperreals in his guide for teachers [**22**].) Keisler sets the stage for extending the real numbers by reminding students that successive extensions of the notion of number have been the milestones of their mathematical educations.

> In grade school and high school mathematics, the real number system is constructed gradually in several stages. Beginning with the positive integers, the systems of integers, rational numbers, and finally real numbers are built up....
>
> What is needed [for an understanding of the calculus] is a sharp distinction between numbers which are small enough to be neglected and numbers which aren't. Actually, no real number except zero is small enough to be neglected. To get around this difficulty, we take the bold step of introducing a new kind of number, which is infinitely small and yet not equal to zero....
>
> The real line is a subset of the hyperreal line; that is, each real number belongs to the set of hyperreal numbers. Surrounding each real number r, we introduce a collection of hyperreal numbers infinitely close to r. The hyperreal numbers infinitely close to zero are called infinitesimals. The reciprocals of nonzero infinitesimals are infinite hyperreal numbers. The collection of all hyperreal numbers satisfies the same algebraic laws as the real numbers....
>
> We have no way of knowing what a line in physical space is really like. It might be like the hyperreal line, the real line, or neither. However, in applications of the calculus it is helpful to imagine a line in physical space as a hyperreal line. The hyperreal line is, like the real line, a useful mathematical model for a line in physical space. [**23**, pp. 1, 24, 25, 27]

In the picture of the hyperreal line (FIGURE 1), observe that $-\epsilon$, 0, and $1/2H$ are infinitesimal; $\pi+\epsilon$ is a finite, noninfinitesimal number that is infinitely close to π; H

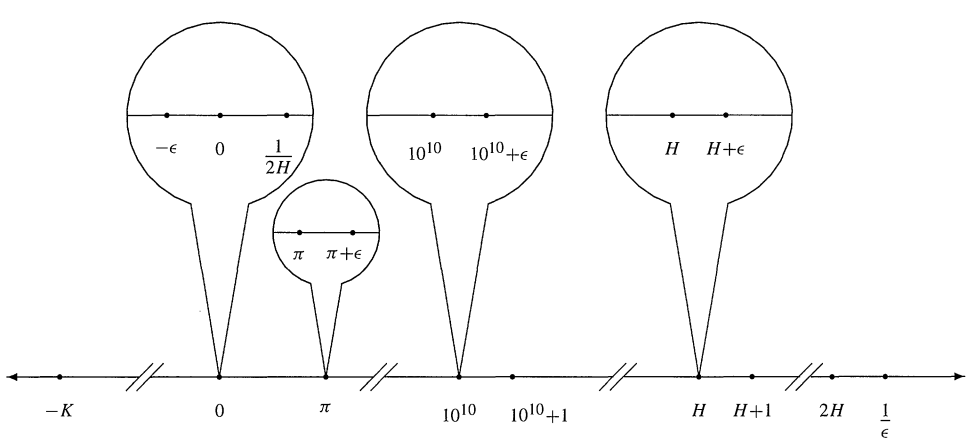

Figure 1 The hyperreal line

is infinite, but infinitely close to $H + \epsilon$; H is a finite, noninfinitesimal distance from $H + 1$, and infinitely far from $2H$.

Key computational properties of the hyperreal numbers are given in the following table.

RULES FOR INFINITE, FINITE, AND INFINITESIMAL NUMBERS. *Assume that ϵ, δ are infinitesimals; b, c are hyperreal numbers that are finite but not infinitesimal; H, K are infinite hyperreal numbers; and n is a finite natural number.*

- Real numbers. *The only infinitesimal real number is* 0. *Every real number is finite.*
- Negatives. $-\epsilon$ *is infinitesimal;* $-b$ *is finite but not infinitesimal;* $-H$ *is infinite.*
- Reciprocals. *If* $\epsilon \neq 0$, *then* $1/\epsilon$ *is infinite;* $1/b$ *is finite but not infinitesimal;* $1/H$ *is infinitesimal. Note that* $1/0$ *remains undefined.*
- Sums. $\epsilon + \delta$ *is infinitesimal;* $b + \epsilon$ *is finite but not infinitesimal;* $b + c$ *is finite (possibly infinitesimal);* $H + \epsilon$ *and* $H + b$ *are infinite.*
- Products. $\delta \cdot \epsilon$ *and* $b \cdot \epsilon$ *are infinitesimal;* $b \cdot c$ *is finite but not infinitesimal;* $H \cdot b$ *and* $H \cdot K$ *are infinite.*
- Quotients. ϵ/b, ϵ/H, *and* b/H *are infinitesimal;* b/c *is finite but not infinitesimal;* b/ϵ, H/ϵ, *and* H/b *are infinite, provided that* $\epsilon \neq 0$.
- Powers. ϵ^n *is infinitesimal;* b^n *is finite but not infinitesimal;* H^n *is infinite.*
- Roots. *If* $\epsilon > 0$ *then* $\sqrt[n]{\epsilon}$ *is infinitesimal; if* $b > 0$ *then* $\sqrt[n]{b}$ *is finite but not infinitesimal; if* $H > 0$ *then* $\sqrt[n]{H}$ *is infinite.*

Notice that there are no general rules for deciding whether the combinations ϵ/δ, H/K, $H\epsilon$, *and* $H + K$, *are infinitesimal, finite, or infinite.*

DEFINITION. We write $x \simeq y$ to mean that $x - y$ is infinitesimal. If $x \simeq y$, we say that x is *infinitely close* to y.

Keisler's entire course is based on three fundamental principles relating the real and hyperreal numbers: the Extension Principle, the Transfer Principle, and the Standard Part Principle. The *Extension Principle* posits the existence of nonzero infinitesimals in the hyperreal field, and for each real function f, a function $^*f$ extending f to the hyperreal numbers. The function $^*f$ is called the *hyperreal extension* of f. (A function is a set of ordered pairs such that no two pairs have the same first element and different second elements. If f and g are functions, then by "g extends f" or "g is an extension of f" we mean that f is a subset of g. A *real function of one variable* is a function in which the domain and range are sets of real numbers. A *real function of n variables* is a function in which the domain is a set of n-tuples of real numbers and the range is a set of real numbers.) The *Transfer Principle* says that every *real statement* that holds for a particular real function holds for its hyperreal extension as well. Equations and inequalities are examples of real statements.

> Here are seven examples that illustrate what we mean by a *real statement*.... (1) Closure law for addition: for any x and y, the sum $x + y$ is defined. (2) Commutative law for addition: $x + y = y + x$. (3) A rule for order: If $0 < x < y$ then $0 < 1/y < 1/x$. (4) Division by zero is never allowed: $x/0$ is undefined. (5) An algebraic identity: $(x - y)^2 = x^2 - 2xy + y^2$. (6) A trigonometric iden-

tity: $\sin^2 x + \cos^2 x = 1$. (7) A rule for logarithms: If $x > 0$ and $y > 0$ then $\log_{10}(xy) = \log_{10} x + \log_{10} y$. [**23**, pp. 28–29]

(Keisler later gives a precise characterization of the real statements [**23**, p. 907].) A consequence of the Transfer Principle is that one does not ordinarily need to distinguish between $^*f$ and f, since any real statement true of one of these functions will be true of the other: for simplicity we use the same function symbol f for both $^*f$ and f. Finally, the *Standard Part Principle* says that every finite hyperreal number is infinitely close to exactly one real number; this principle is useful for translating results about finite hyperreal quantities into equivalent statements about real quantities.

In our development, which emphasizes discrete mathematics, the natural numbers play a larger role than they do in most presentations of the calculus. Key properties of the natural numbers are that they contain 0 and 1, are closed under $+$ and $\cdot$, and that they satisfy the Natural Induction Principle (also known as the Principle of Mathematical Induction). For example, the binomial formula,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \cdots + b^n,$$

for all real a, b and all natural n, and the geometric sum formula,

$$\frac{1-a^{n+1}}{1-a} = 1 + a + a^2 + \cdots + a^n,$$

for all a except 1 and all natural n, are often proved by induction. We will only require the Natural Induction Principle for equations and inequalities. In the following, a *real sequence* is a real function in which the domain is the set of natural numbers.

NATURAL INDUCTION PRINCIPLE. *Let $\phi(n)$ be an equation or inequality of real sequences; that is, let $\phi(n)$ be of the form $a_n = b_n$, $a_n \neq b_n$, $a_n < b_n$, or $a_n \leq b_n$, where a and b are real sequences. If $\phi(0)$ holds and if for all natural m, we have that $\phi(m+1)$ holds whenever $\phi(m)$ holds, then $\phi(n)$ holds for all natural n.*

Another important tool is the Principle of Definition by Recursion, which says that one may define a real sequence by specifying its value at 0, and specifying for each natural n its value at $n+1$ as determined by its value at n. (See [**2**] for an elementary discussion of recursion schemes and their solutions.) For example, the *factorial-power* function,

$$x^{\underline{n}} = \underbrace{x(x-1)(x-2)\cdots(x-n+1)}_{n \text{ factors}},$$

is defined for all natural n by the equations,

$$x^{\underline{0}} = 1, \quad x^{\underline{n+1}} = x^{\underline{n}} \cdot (x-n).$$

A *real series* is a real sequence of partial sums $a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots,$ where a is a real sequence; we use the notation $a_0 + a_1 + a_2 + \cdots$ to denote this series. Real series are defined more formally by recursion. For example, the sum of the first n square numbers is defined for all natural n by the equations

$$s_0 = 0, \quad s_{n+1} = s_n + (n+1)^2.$$

The *integers* are defined to be the natural numbers together with their negatives. An important function from the real numbers to the integers is the *greatest-integer* function, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Following Keisler's presentation, we define the *hyperintegers* to be the range of $^*\lfloor\ \rfloor$. The *hypernatural* numbers are defined to be the nonnegative hyperintegers. They extend the natural numbers to include infinite elements that satisfy the same real statements (such as the recursive definitions of addition and multiplication) as the ordinary, finite natural numbers.

There are several ways of defining *hypersequences* having hypernatural indices and hyperreal values. The simplest way is to start with any real sequence s and take its hyperreal extension, $^*s$. By the Transfer Principle, $^*s_n = s_n$ for all finite n, but $^*s_N$ has new, hyperreal values for infinite N.

Example: $0, 1, 4, 9, 16, \ldots, N^2, \ldots$ (N hypernatural). If t is the sequence $0, 1, 4, 9, 16, \ldots, n^2, \ldots$ (n natural), then $^*t$ is $0, 1, 4, 9, 16, \ldots, N^2, \ldots$ (N hypernatural). In terms of sets, t is $\{(n, n^2) : n \text{ natural}\}$ and $^*t$ is $\{(N, N^2) : N \text{ hypernatural}\}$.

Example: $0, 1, 5, 14, 30, \ldots, (0^2 + 1^2 + 2^2 + 3^2 + \ldots + N^2), \ldots$ (N hypernatural). If s is the real series (sequence of partial sums) defined on the natural numbers by $s_0 = 0$, $s_{n+1} = s_n + (n + 1)^2$, then by the Extension Principle $^*s$ is defined on the hypernatural numbers, and by the Transfer Principle $^*s$ satisfies the same real statements as s—in particular, the same recursion equations. Thus s_N, also written $\sum_{n=0}^{N} n^2$ or even $0^2 + 1^2 + 2^2 + 3^2 \cdots + N^2$, makes sense for infinite as well as finite N.

More generally, one may start with a real function of one or more variables, take its hyperreal extension, and then substitute hyperreal values for some of its arguments.

Example:

$$1,\ \lambda,\ \frac{K-1}{K} \cdot \frac{\lambda^2}{1 \cdot 2},\ \frac{K-1}{K} \cdot \frac{K-2}{K} \cdot \frac{\lambda^3}{1 \cdot 2 \cdot 3},\ \ldots.$$

These are the coefficients of the binomial expansion of $(1 + (\lambda x/K))^K$, which Euler used in his construction of the series for a^x. These terms are given by

$$\beta_n = \frac{K^{\underline{n}}}{K^n} \frac{\lambda^n}{n!},$$

where λ and K are fixed hyperreal numbers and n ranges over the hypernatural numbers. The hyperreal function β arises from the three-argument real function b defined by

$$b(k, l, n) = \frac{k^{\underline{n}}}{k^n} \frac{l^n}{n!}$$

by fixing K and λ and setting $\beta_n = {}^*b(K, \lambda, n)$ for all hypernatural n.

DEFINITION. A *hypersequence* is any function defined on the hypernatural numbers by composing the hyperreal extensions of real functions of one or more variables and allowing hyperreal arguments. A *hyperseries* is the hypersequence of partial sums of a hypersequence. We often use the term *series* to refer to either a real series or a hyperseries.

By the Transfer Principle one can extend the binomial formula and the geometric sum formula to hyperreal terms and hypernatural exponents.

BINOMIAL THEOREM. *For all hyperreal a, b, and all hypernatural n,*

$$(a+b)^n = \sum_{k=0}^{n} \frac{n^{\underline{k}}}{k!} a^{n-k} b^k = a^n + \frac{n^{\underline{1}}}{1!} a^{n-1} b^1 + \frac{n^{\underline{2}}}{2!} a^{n-2} b^2 + \frac{n^{\underline{3}}}{3!} a^{n-3} b^3 + \cdots + b^n.$$

GEOMETRIC SUM THEOREM. *For all hyperreal a except unity and all hypernatural n,*

$$\frac{1-a^{n+1}}{1-a} = \sum_{k=0}^{n} a^k = 1 + a + a^2 + \cdots + a^n.$$

Hypersequences are examples of the *internal* sequences of Robinson's theory; see [**21**, pp. 94ff]. Because of the special role of hypersequences in our exposition, we will find it convenient to assume one further principle.

HYPERNATURAL INDUCTION PRINCIPLE. *Let $\phi(n)$ be an equation or inequality of hypersequences; that is, let $\phi(n)$ be of the form $a_n = b_n$, $a_n \neq b_n$, $a_n < b_n$, or $a_n \leq b_n$, where a and b are hypersequences. If $\phi(0)$ holds and if for all hypernatural m, we have that $\phi(m+1)$ holds whenever $\phi(m)$ holds, then $\phi(n)$ holds for all hypernatural n.*

In more advanced treatments, the Hypernatural Induction Principle can be seen to follow from the Natural Induction Principle and a version of the Transfer Principle that takes into account statements involving quantifiers, in addition to the (quantifier-free) real statements.

Sullivan, in her article in the *American Mathematical Monthly* [**44**], provides evidence that elementary calculus can be effectively taught to high school and college students using Keisler's system of hyperreal numbers. A recent reform-calculus book that uses infinitesimal methods is Stroyan's *Calculus using Mathematica* [**42**]. Interested readers might also consult Luxemburg's article in the *Monthly* [**32**], Lightstone's articles in the *Monthly* [**30**] and this MAGAZINE [**31**], Davis and Hersh's "Nonstandard analysis" in *Scientific American* [**6**], Simpson's article from the *Mathematical Intelligencer* [**41**], and Henle and Kleinberg's slender volume, *Infinitesimal Calculus* [**18**]. For more advanced treatments, see [**40**], [**43**], or [**21**]. Keisler's article [**24**] contains a brief history of infinitesimals. For a nonstandard connection between Euler's mathematics and modern functional analysis, see [**45**].

Determinate series

Much of the *Introductio* concerns the expansion of well-known functions into series:

> Since both rational functions and irrational functions of x are not of the form of polynomials $A + Bx + Cx^2 + Dx^3 + \cdots$, where the number of terms is finite, we are accustomed to seek expressions of this type with an infinite number of terms which give the value of the rational or irrational function. Even the nature of a transcendental function seems to be better understood when it is expressed in this form, even though it is an infinite expression. Since the nature of polynomial functions is very well understood, if other functions can be expressed by different powers of x in such a way that they are put in the form $A + Bx + Cx^2 + Dx^3 + \cdots$, then they seem to be in the best form for the mind to grasp their nature, even though the number of terms is infinite. [**12**, §59]

Implicit in this statement is the assumption that "infinite polynomials" share well-known properties of finite polynomials. In our rehabilitation of Euler's methods, the *polynomials with an infinite number of terms* become *polynomials of infinite hypernatural degree*: $a_0 + a_1x + a_2x^2 + \cdots + a_Nx^N$, where N is an infinite hypernatural number. By the Transfer Principle, such hyperreal polynomials satisfy the same real statements as real polynomials of finite degree, and in particular can be algebraically manipulated according to the usual rules. Hyperreal polynomials cannot provide *exact* expressions for nonpolynomial real functions, but the extension to the hyperreals does present the opportunity for approximating a real function to within *infinitesimal* error—and for most practical purposes this is close enough. In our rehabilitation of Euler's methods, the goal of expressing a real function as a polynomial with an infinite number of terms becomes: for a real function f, to find a hypersequence a and an infinite hypernatural N such that for all real x (or for all real x in some range),

$$f(x) \simeq a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N.$$

It would be computationally inconvenient if the value of $a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N$ were to depend perceptibly on the *particular* infinite value of N. Therefore we give special consideration to series that are *determinate* in the sense that once one has taken the summation to an infinite number of terms, the contribution made by adding still more terms is infinitesimal.

Euler did not discuss the notion of determinacy in the *Introductio* or anywhere else—with one exception. In a paper on harmonic series presented in 1734, Euler stated a principle which may be read as follows: "A series that has a finite sum when continued infinitely will receive insignificant growth even if it is continued further; in fact, that which is added after infinitely many terms will be infinitely small." (*Series, quae infinitum continuata summum habet finitam, etiamsi ea duplo longius continuetur, nullum accipiet augmentum, sed id, quod post infinitum adiicitur cogitatione, re vera erit infinite parvum.* [**11**, §2]) He used this principle to show that a harmonic series

$$\frac{c}{a} + \frac{c}{a+b} + \frac{c}{a+2b} + \frac{c}{a+3b} + \cdots$$

does not have a finite sum, but that series such as

$$\frac{c}{a} + \frac{c}{a+b} + \frac{c}{a+4b} + \frac{c}{a+9b} + \cdots$$

and in general series whose n^{th} term is $c/(a + n^{\alpha} b)$, $\alpha > 1$, do have finite sums. (In all of these cases, the assumption that a, b, and c are positive is implicit.) Because of its essential use of infinite and infinitesimal numbers, we find it worthwhile to recount Euler's argument that the harmonic series is not determinate [**11**, §3]:

> Let the series c/a, $c/(a+b)$, $c/(a+2b)$, etc., be continued infinitely to the infinitesimal term $c/(a+(i-1)b)$, where i denotes an infinite number, the index of this term. Now if this series is continued from the next term $c/(a+ib)$ through the ni^{th} term $c/(a+(ni-1)b)$, the number of these added terms is $(n-1)i$. The sum of these terms is less than
>
> $$\frac{(n-1)ic}{a+ib},$$
>
> and greater than
>
> $$\frac{(n-1)ic}{a+(ni-1)b}.$$
>
> Since i is infinitely large, a is negligible in each denominator; thus the sum is greater than
>
> $$\frac{(n-1)c}{nb},$$

and less than

$$\frac{(n-1)c}{b}.$$

Note the salient features of this argument. The number i is explicitly taken to be *infinite*, and a sum of i terms is taken, *terminating* with $c/(a+(i-1)b)$. After summing these infinitely many terms there is a *next term*, $c/(a+ib)$. A tail sum is taken of the next $(n-1)i$ terms, and a lower bound is obtained for this tail sum using *ordinary algebra*, which can then be simplified because *a/i is infinitesimal*:

$$\frac{(n-1)ic}{a+(ni-1)b} = \frac{(n-1)c}{a/i+(n-1/i)b} \simeq \frac{(n-1)c}{nb}.$$

For example, if we take $n=2$ (continuing the sum twice as far), we have a tail sum that is greater than or infinitely close to $c/2b$, and hence not infinitesimal. Therefore the series is not determinate.

What does the notion of determinacy have to do with the *Introductio*? Euler's techniques for expanding functions into series depend at various points on the negligibility of infinitely many infinitesimals in an infinite sum. There are easy examples to the contrary (arising, for example, in the computation of areas as infinite sums of infinitesimal rectangles) so to be rigorous, one must have a criterion for deciding when one can neglect infinitesimals in an infinite sum. The notion of determinacy provides such a criterion.

DEFINITION OF DETERMINACY. *A hypersequence $s_0, s_1, s_2, \ldots$ is said to be* determinate *iff $s_M \simeq s_N$ for all infinite M and N. If $a_0, a_1, a_2, \ldots$ is a hypersequence, then a series $a_0+a_1+a_2+\cdots$ is said to be determinate iff the hypersequence of partial sums defined by $s_n = a_0+a_1+a_2+\cdots+a_n$ is determinate.*

The following theorem says that one can neglect infinitely many infinitesimals in an infinite sum provided the relevant series are both determinate.

SUMMATION COMPARISON THEOREM. *If the series $a_0+a_1+a_2+\cdots$ and $b_0+b_1+b_2+\cdots$ are determinate, and if for each natural n, $a_n \simeq b_n$, then for all hypernatural n, $a_0+a_1+\cdots+a_n \simeq b_0+b_1+\cdots+b_n$.*

We will postpone the proof of this theorem to a later section. As an example of a determinate series we verify that a geometric series $1+x+x^2+\cdots$ is determinate for certain values of x. Let x be a hyperreal number such that $|x|<1$ and $|x| \not\simeq 1$, and let J and K be infinite hypernatural numbers with $K>J$. Then by the Geometric Sum Theorem, $x^J+\cdots+x^K = (x^J - x^{K+1})/(1-x)$. This is infinitesimal because both x^J and x^{K+1} are infinitesimal and $1/(1-x)$ is finite.

The following theorem contains two general tests for determinacy. The proof is left to the reader.

COMPARISON TEST FOR DETERMINACY. ***(i)** Let $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ be sequences of positive terms. If $b_0+b_1+b_2+\cdots$ is determinate and if there is a finite k such that $a_n \le b_n$ for $n \ge k$, then $a_0+a_1+a_2+\cdots$ is determinate as well. **(ii)** If $|c_0|+|c_1|+|c_2|+\cdots$ is determinate, then $c_0+c_1+c_2+\cdots$ is determinate as well.*

The requirement that *once one has added an infinite number of terms, the contribution made by adding still more terms must be infinitesimal* bears a striking resemblance to the Cauchy condition for convergence of real series, which says that *once one has added a sufficiently large finite number of terms, the contribution made by adding still* *more terms must always be less than some previously specified amount.* This resemblance has been discussed by Eneström [**10**] and Pringsheim [**39**], and more recently by Laugwitz [**29**, 205–208]. Our presentation was inspired by Laugwitz's discussion. Did Euler anticipate the "Cauchy" criterion for convergence? The answer is far from being free of controversy (see McKinzie [**35**]) and moreover, even if he did, the discovery seems to have made no difference to the historical development of the calculus. Eneström, compiler of the definitive catalog of Euler's published works, lamented:

> I have looked in vain for a reference to the Eulerian convergence condition in the accessible mathematical writings of the 18th century. The discovery appears therefore to remain completely unheeded, and the mathematicians who attack the convergence question at the start of the 19th century were surely not influenced by Euler. [**10**]

That much said, we still have no qualms about using our own definition of determinacy, simply and clearly stated in Eulerian language, in our rehabilitation of Euler's methods: as we shall see it is precisely what we need to make Euler's arguments rigorous.

The exponential series

Having outlined an axiomatic system that specifies the properties of infinite and infinitesimal numbers, and having provided a criterion for the negligibility of infinitesimals in an infinite sum, we are now ready to present our rehabilitation of Euler's derivation of the series for the exponential function.

Exponentiation is defined for 0 and other natural n by

$$a^0 = 1$$
$$a^n = \underbrace{a \cdots a}_{n \text{ factors}},$$

or more formally using recursion, then extended to positive rational exponents $\frac{m}{n}$ using the root axiom:

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m \quad (a > 0);$$

then extended to negative rational numbers by taking reciprocals,

$$a^{-\frac{m}{n}} = 1/a^{\frac{m}{n}} \quad (a > 0).$$

From these definitions and the basic field and order axioms for the real numbers, one can show that for all positive a greater than 1 and all rational p and q, the following familiar rules hold for rational exponentiation: $a^p a^q = a^{p+q}$, $(a^p)^q = a^{pq}$, and $a^p < a^q$ if and only if $p < q$.

Extending the definition of a^x further, from rational to real x, and verifying that the rules just given also hold for the extension, is more involved. Instead, we *assume* that a^x is a real function (defined for all real x) and, using the properties mentioned in the previous paragraph as motivation, assume the following axioms for the exponential function.

AXIOMS. *For all positive real a greater than* 1 *and all real x and y, the following rules hold:* $a^0 = 1$; $a^{-x} = 1/a^x$; $a^x a^y = a^{x+y}$; $(a^x)^y = a^{xy}$; *and* $a^x < a^y$ *iff* $x < y$.

By the Transfer Principle, the axioms stated above hold for hyperreal numbers as well. In addition to these rules, we also require the following proposition.

PROPOSITION. *Assuming that a is finite and greater than* 1, *we have the following results. If $\epsilon > 0$ and $\epsilon \simeq 0$ then $a^\epsilon > 1$ and $a^\epsilon \simeq 1$. If x and y are finite, then $a^x \simeq a^y$ iff $x \simeq y$.*

Proof. We prove this in three steps. (i) Let N be an infinite hypernatural number. We want to conclude that $a^{1/N}$ exceeds 1 by an infinitesimal amount. By the axioms for exponentiation, $a^{1/N} > a^0 = 1$, so we may write $a^{1/N} = 1 + u$, where u is positive. By the Binomial Theorem, $a = (a^{1/N})^N = (1+u)^N = 1 + Nu +$ (positive terms), from which we conclude that $0 < u < (a-1)/N \simeq 0$, and hence $a^{1/N} > 1$ and $a^{1/N} \simeq 1$. (ii) Now let ϵ be positive, and take $N = \lfloor 1/\epsilon \rfloor$, the greatest hypernatural number less than or equal to $1/\epsilon$, so that $1/(N+1) < \epsilon \leq 1/N$. Then by the axioms for exponentiation, $a^{1/(N+1)} < a^\epsilon \leq a^{1/N}$, from which it follows that $\epsilon \simeq 0$ iff N is infinite iff $a^\epsilon \simeq 1$. Furthermore, $a^{-\epsilon}$, which is $1/a^\epsilon$, is infinitely close if 1 iff a^ϵ is as well. (iii) Assuming that both x and y are finite, we conclude that

$$a^x \simeq a^y \quad \text{iff} \quad \frac{a^x}{a^y} \simeq 1 \quad \text{iff} \quad a^{x-y} \simeq 1 \quad \text{iff} \quad x - y \simeq 0 \quad \text{iff} \quad x \simeq y.$$

It is important in the first step that a^x and a^y are neither infinite nor infinitesimal. ∎

Our goal for this section is to show that there is a finite λ such that for all finite x and infinite N,

$$a^x \simeq a + (\lambda x) + \frac{1}{2!}(\lambda x)^2 + \frac{1}{3!}(\lambda x)^3 + \cdots + \frac{1}{N!}(\lambda x)^N.$$

Let x be finite, and for the moment, positive. We choose an infinite hypernatural number K, which we hold fixed for the rest of this section, and choose a fraction J/K that is infinitely close to x. This can be done by taking $J = \lfloor Kx \rfloor$, so that $0 < x - J/K < 1/K$, and hence that $x \simeq J/K$.

By the proposition, we know that $a^x \simeq a^{J/K}$, so let us now work with $a^{J/K}$. We write $a^{J/K}$ as $(a^{1/K})^J$, and consider $a^{1/K}$. By the proposition, $a^{1/K}$ exceeds 1 by an infinitesimal amount. We do not know whether that amount is greater or less than $1/K$, so (following Euler) we introduce a positive scaling factor, λ, depending on K:

$$a^{1/K} = 1 + \lambda\frac{1}{K}.$$

It is easy to see that λ must be finite: by the Binomial Theorem, $a = (a^{1/K})^K = (1 + \lambda/K)^K = 1 + \lambda +$ (positive terms), so that $0 < \lambda < a$.

We may now expand $a^{J/K}$, written as $(1 + \lambda\frac{1}{K})^J$, as follows:

$$\begin{aligned} a^x \simeq a^{J/K} = (a^{1/K})^J &= \left(1 + \lambda\frac{1}{K}\right)^J \\ &= 1 + J\left(\lambda\frac{1}{K}\right) + \frac{J^{\underline{2}}}{2!}\left(\lambda\frac{1}{K}\right)^2 + \cdots + \frac{J^{\underline{J}}}{J!}\left(\lambda\frac{1}{K}\right)^J \qquad (1) \\ &= 1 + \left(\lambda\frac{J}{K}\right) + \frac{1}{2!}\frac{J^{\underline{2}}}{J^2}\left(\lambda\frac{J}{K}\right)^2 + \cdots + \frac{1}{J!}\frac{J^{\underline{J}}}{J^J}\left(\lambda\frac{J}{K}\right)^J \\ &\simeq 1 + (\lambda x) + \frac{1}{2!}(\lambda x)^2 + \cdots + \frac{1}{J!}(\lambda x)^J \qquad (2) \end{aligned}$$

$$\simeq 1 + (\lambda x) + \frac{1}{2!}(\lambda x)^2 + \cdots + \frac{1}{N!}(\lambda x)^N \tag{3}$$

Line (1) follows from the Binomial Theorem, and lines (2) and (3) will be justified by the Summation Comparison Theorem once we have shown that the series in question are determinate and that their respective terms are infinitely close.

LEMMA. *The series* $1 + y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \cdots$ *is determinate for all finite* y.

Proof. Fix $y > 0$ and let $n_0 = \lfloor y \rfloor$. Then for $n > n_0$,

$$\frac{y^n}{n!} = \frac{y^{n_0}}{n_0!}\frac{y^{n-n_0}}{(n_0+1)(n_0+2)\cdots n} \le \frac{y^{n_0}}{n_0!}\left(\frac{y}{n_0+1}\right)^{n-n_0} = b\,c^{n-n_0},$$

where $b = y^{n_0}/n_0!$ and $c = y/(n_0+1)$, so that $|c| < 1$, $c \not\simeq 1$, and b is finite. As we saw earlier, the series $1 + c + c^2 + c^3 + \cdots$ is determinate for $0 < c < 1$, so the result follows from the Comparison Test for Determinacy. ∎

Setting y to λx in the lemma shows that the series in (2) is determinate, and since for positive x,

$$\frac{1}{k!}\frac{J^{\underline{k}}}{J^k}\left(\lambda\frac{J}{K}\right)^k \le \frac{1}{k!}(\lambda x)^k,$$

the Comparison Test for Determinacy implies that the series in (3) is determinate. We next note that since J is infinite, we have $J^{\underline{k}}/J^k \simeq 1$ and $(J/K)^k \simeq x^k$ for all finite k, and hence

$$\frac{J^{\underline{k}}}{J^k}\left(\frac{J}{K}\right)^k\frac{\lambda^k}{k!} \simeq \frac{\lambda^k}{k!}x^k$$

for all finite k as well. Using the Summation Comparison Theorem, and similar reasoning for negative exponents, we obtain the desired theorem.

THEOREM. *If* a *is finite and greater than* 1, *then there is a finite* λ *such that*

$$a^x \simeq 1 + (\lambda x) + \frac{1}{2!}(\lambda x)^2 + \frac{1}{3!}(\lambda x)^3 + \cdots + \frac{1}{N!}(\lambda x)^N$$

for all finite x *and infinite* N.

The natural exponential series

Suppose one wanted to compute 10^x using the previous theorem. What value of λ would one use? From our original equation, $a^{1/K} = 1 + \lambda\frac{1}{K}$, one can deduce that $\lambda = K(a^{1/K} - 1)$, but this formula is difficult to evaluate, in that it requires the extraction of a large-order root of a. Later in this article, we will use a series to compute λ, but in the mean time one can ask, why not take λ to be some value convenient for computation, and use the value of a corresponding to that value of λ? Euler noted that "[s]ince we are free to choose the base a . . . , we now choose a in such a way that $\lambda = 1$." [**12**, §§122–123] That is, we choose $a = (1 + 1/K)^K$, so that the corresponding series for a^x is

$$1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots.$$

But does this help? Now we need to know this special value of a. Noting this difficulty, Euler wrote,

> [If] we now choose a such that $\lambda = 1 \ldots$ then the series
>
> $$1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots$$
>
> is equal to a. If the terms are represented as decimal fractions and summed, we obtain the value for $a = 2.71828182845904523536028\ldots$. For the sake of brevity, for this number ... we will use the symbol e .... [**12**, §122]

The function e^x is called the *natural exponential function*. According to Cajori [**3**, §400], Euler first used the letter e to represent the natural exponential base in a manuscript of 1727 or 1728, published posthumously in 1862. The notation first found its way into print in Euler's *Mechanica sive motus scientia analytice exposita* of 1736. Its use in such influential works as the *Mechanica* and the *Introductio* established the notation as standard. See also Coolidge [**4**] and Maor [**34**].

Let us verify that $(1 + 1/K)^K$ is determinate in the sense that for different infinite values of K the corresponding values are all infinitely close.

PROPOSITION. *For all infinite M, N, and P,*

$$\left(1 + \frac{1}{M}\right)^M \simeq 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{P!} \simeq \left(1 + \frac{1}{N}\right)^N.$$

Proof. Expanding the binomial power and repeatedly applying the Summation Comparison Theorem, we find that for all infinite M and P,

$$\begin{aligned}\left(1 + \frac{1}{M}\right)^M &= 1 + M\left(\frac{1}{M}\right) + \frac{M^{\underline{2}}}{2!}\left(\frac{1}{M}\right)^2 + \cdots + \frac{M^{\underline{M}}}{M!}\left(\frac{1}{M}\right)^M \\ &= 1 + 1 + \frac{M^{\underline{2}}}{M^2}\frac{1}{2!} + \cdots + \frac{M^{\underline{M}}}{M^M}\frac{1}{M!} \\ &\simeq 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{M!} \simeq 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{P!}.\end{aligned}$$

Since this computation holds for infinite N as well as M, the result follows. ∎

At this stage one might be tempted to define e to be any one of the values $(1 + 1/K)^K$, for K infinite, and, by now-familiar computations obtain the relation,

$$e^x \simeq 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{N!}x^N \tag{4}$$

for all finite x and infinite N. But this does not seem to adequately pin down the value of e; one would prefer to have e stand for some specific real number rather than having it be an arbitrary choice from an anonymous class of hyperreal numbers, albeit all infinitely close. Our remedy is to use the Standard Part Principle, which says that every finite hyperreal number has exactly one real number that is infinitely close to it:

DEFINITION. *e is the unique real number that is infinitely close to $(1+1/K)^K$, where K is infinite.*

Finally, we must verify that this value of e, which actually differs (infinitesimally) from $(1+1/K)^K$ for each K, still satisfies (4). This is a consequence of the following proposition, about different exponential functions whose bases are infinitely close.

PROPOSITION. *If a and b are finite and greater than 1, and $a \simeq b$, then $a^x \simeq b^x$ for all finite x.*

Proof. If $a \simeq b$ and $a, b > 1$, then we may write $b = a(1+\epsilon)$ where $\epsilon \simeq 0$. Then $b^x = (a(1+\epsilon))^x = a^x(1+\epsilon)^x$. We need only verify that $(1+\epsilon)^x \simeq 1$. Let $n = \lfloor x \rfloor$. Then $n \leq x < n+1$, and by the order axiom for exponentiation, $(1+\epsilon)^n \leq (1+\epsilon)^x < (1+\epsilon)^{n+1}$. Since $\epsilon \simeq 0$ and n is finite, by the Binomial Theorem we have $1 \leq (1+\epsilon)^n = 1 + \epsilon \cdot$ (a sum of n finite terms) $\simeq 1$, and $(1+\epsilon)^{n+1} = (1+\epsilon)^n(1+\epsilon) \simeq 1$, and hence $(1+\epsilon)^x \simeq 1$, and finally $b^x \simeq a^x$. ∎

Since $e \simeq (1+1/K)^K$, by the previous proposition we conclude that

THEOREM. *For all finite x and infinite N,*

$$e^x \simeq 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{N!}x^N. \tag{5}$$

By the way, this theorem helps explain the relationship between λ and a. For if λ happens to satisfy the equation $e^\lambda = a$, then

$$a^x = (e^\lambda)^x = e^{\lambda x} \simeq 1 + (\lambda x) + \frac{1}{2!}(\lambda x)^2 + \frac{1}{3!}(\lambda x)^3 + \cdots + \frac{1}{N!}(\lambda x)^N,$$

for all finite x and infinite N. This shows that the λ we chose earlier can be taken to be the exponent to which e must be raised in order to yield a: in other words, the natural logarithm of a. We will return to the natural logarithm later in this article.

The Euler identities and the series for sine and cosine

Although series for the exponential function, logarithm, and trigonometric functions were known to Newton and others prior to 1670 (see [**25**, p. 436ff]), Euler's *Introductio in Analysin Infinitorum* of 1748 provided a systematic account of these formulas as deduced from basic principles. According to Boyer,

> [The *Introductio*] contains the earliest algorithmic treatment of logarithms as exponents and of the trigonometric functions as numerical ratios. It was the first textbook to list systematically the multiple-angle formulas, calling attention to the periodicities of the functions; and it included the first general analytic treatment of these as infinite products, as well as their expansion into infinite series. The well-known "Euler identities," relating the trigonometric functions to imaginary exponentials, are also found here. [**1**, pp. 224–225]

In this section, we will use the multiple-angle formulas to deduce a form of the Euler identities, and use these identities to derive the series for sine and cosine.

The Euler identities are well known to us in the form

$$\cos x = \tfrac{1}{2}[e^{ix} + e^{-ix}], \qquad \sin x = \tfrac{1}{2i}[e^{ix} - e^{-ix}],$$

or equivalently, $e^{ix} = \cos x + i \sin x$, but to appreciate these formulas one must understand what is meant by e^{ix}. This is difficult because the axioms for exponentiation discussed so far are silent on the subject of imaginary exponents. It is tempting to take our relation $e^x \simeq (1 + x/N)^N$ for finite x and infinite N, postulate that it holds for imaginary exponents as well,

$$e^{ix} \simeq \left(1 + \frac{ix}{N}\right)^N, \tag{6}$$

and then perform algebraic operations on the right-hand side. We will give a definition of e^{ix} very close to this one toward the end of the article, but in the mean time we avoid the difficulty of having to pin down the meaning of e^{ix} by providing a form of the Euler identities that does not require the relation in (6), nor even mention of e^{ix}, but instead uses algebraic terms of the form $(1 + ix/N)^N$. We show that for all finite x and infinite N,

$$\cos x \simeq \frac{1}{2}\left[\left(1 + \frac{ix}{N}\right)^N + \left(1 - \frac{ix}{N}\right)^N\right], \tag{7}$$

$$\sin x \simeq \frac{1}{2i}\left[\left(1 + \frac{ix}{N}\right)^N - \left(1 - \frac{ix}{N}\right)^N\right], \tag{8}$$

and then use these relations directly.

This maneuver frees us from having to define e^{ix}, but what about $(1 + ix/N)^N$? We still have to explain how complex numbers fit into the hyperreal framework. In elementary courses, the complex numbers are defined by starting with the real numbers and adjoining a new number i, together with the axiom $i^2 = -1$ and a "transfer principle" that says that the usual rules of algebra apply to the extended system of *complex* numbers. This method suits our purposes exactly, except that now we adjoin i to the hyperreals rather than the reals, and call the resulting numbers the *hypercomplex* numbers. Every hypercomplex number can be written as $a + bi$ where a and b are hyperreal. For two hypercomplex numbers c and d, we write $c \simeq d$ to mean that the modulus of their difference is infinitesimal (or equivalently, that their respective real and imaginary parts are infinitely close). We say that a hypercomplex number c is infinitesimal if its modulus is infinitesimal, and finite if its modulus is finite. We note that the Binomial Theorem holds for hypercomplex binomials by the Transfer Principle, and that the Summation Comparison Theorem holds for series of hypercomplex terms by the same argument (given in a later section) as for series of hyperreal terms.

We begin by proving two standard formulas.

PROPOSITION. *For all real x and natural n,*

$$\cos x = \frac{1}{2}\left[\left(\cos\frac{x}{n} + i\sin\frac{x}{n}\right)^n + \left(\cos\frac{x}{n} - i\sin\frac{x}{n}\right)^n\right], \tag{9}$$

$$\sin x = \frac{1}{2i}\left[\left(\cos\frac{x}{n} + i\sin\frac{x}{n}\right)^n - \left(\cos\frac{x}{n} - i\sin\frac{x}{n}\right)^n\right]. \tag{10}$$

Proof. Using the familiar formulas for the sine and cosine of a sum of angles, one can show by induction that for all n and θ, $\cos n\theta = \frac{1}{2}[(\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n]$ and $\sin n\theta = \frac{1}{2i}[(\cos\theta + i\sin\theta)^n - (\cos\theta - i\sin\theta)^n]$. Substituting x/n for θ yields the result. ■

We will obtain (7) and (8) from this proposition using small-angle approximations for the sine and cosine. If θ is an infinitesimal angle (that is, an angle subtending an

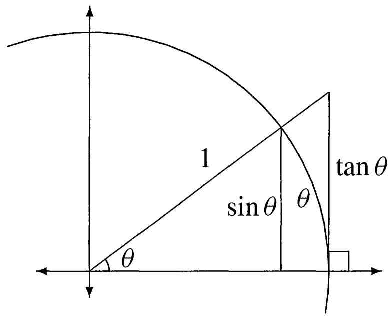

Figure 2 $0 < \sin\theta < \theta < \tan\theta$ in the first quadrant

infinitesimal arc) then it is obvious from the inequality in FIGURE 2 that $\sin\theta \simeq \theta$ and hence that $\cos\theta = (1-\sin^2\theta)^{1/2} \simeq 1$. But the presence of the exponents in (9) and (10) prevents us from using these results to take n infinite and substitute 1 for $\cos(x/n)$ and x/n for $\sin(x/n)$ to get (7) and (8). For these substitutions to be valid we need a sharper result that involves the notion of *relative infinitesimal*. For $\epsilon \neq 0$, we will say that *a is infinitesimal with respect to ϵ*, and write $a \simeq 0 \pmod{\epsilon}$, to mean that $a/\epsilon \simeq 0$. Similarly, $a \simeq b \pmod{\epsilon}$ means that $a - b \simeq 0 \pmod{\epsilon}$.

PROPOSITION. *If $0 < \theta < \pi/2$, then*

$$\theta - \frac{1}{2}\theta^3 < \sin\theta < \theta \quad \textit{and} \quad 1 - \frac{1}{2}\theta^2 < \cos\theta < 1.$$

If θ is a nonzero infinitesimal, then

$$\sin\theta \simeq \theta \pmod{\theta} \quad \textit{and} \quad \cos\theta \simeq 1 \pmod{\theta}.$$

Proof. Assume that $0 < \theta < \pi/2$; the case of negative θ will be an easy consequence. From geometry (see FIGURE 2) we have $\sin\theta < \theta < \tan\theta$. From $\theta < \tan\theta$ we deduce $\theta\cos\theta < \sin\theta$. By the double-angle formula, $\cos\theta = 1 - 2\sin^2(\frac{\theta}{2})$, we get $\theta(1 - 2\sin^2(\frac{\theta}{2})) < \sin\theta$. Since $\sin(\frac{\theta}{2}) < \frac{\theta}{2}$ in this range, we conclude (remarkably) that $\theta - \frac{1}{2}\theta^3 = \theta(1 - 2(\frac{\theta}{2})^2) < \theta(1 - 2\sin^2(\frac{\theta}{2})) < \sin\theta < \theta$, which implies $-\frac{1}{2}\theta^2 < (\sin\theta - \theta)/\theta < 0$. Then $\theta \simeq 0$ implies $(\sin\theta - \theta)/\theta \simeq 0$, and hence $\sin\theta \simeq \theta \pmod{\theta}$. For the cosine approximation, the formula $\cos\theta = 1 - 2\sin^2(\frac{\theta}{2})$ implies that $1 - \frac{1}{2}\theta^2 < 1 - 2\sin^2(\frac{\theta}{2}) = \cos\theta < 1$, and thus for $\theta \simeq 0$, we have $\cos\theta \simeq 1 \pmod{\theta}$. ∎

With this proposition we can now prove the theorem.

THEOREM. *For all finite x and infinite N,*

$$\cos x \simeq \frac{1}{2}\left[\left(1 + \frac{ix}{N}\right)^N + \left(1 - \frac{ix}{N}\right)^N\right],$$

$$\sin x \simeq \frac{1}{2i}\left[\left(1 + \frac{ix}{N}\right)^N - \left(1 - \frac{ix}{N}\right)^N\right].$$

Proof. Let x be finite and N infinite. By the Transfer Principle applied to (9) and (10) we get

$$\cos x = \frac{1}{2}\left[\left(\cos\frac{x}{N} + i\sin\frac{x}{N}\right)^N + \left(\cos\frac{x}{N} - i\sin\frac{x}{N}\right)^N\right], \tag{11}$$

$$\sin x = \frac{1}{2i}\left[\left(\cos\frac{x}{N} + i\sin\frac{x}{N}\right)^N - \left(\cos\frac{x}{N} - i\sin\frac{x}{N}\right)^N\right], \tag{12}$$

where $x/N \simeq 0$. Since $\cos\frac{x}{N} \simeq 1 \pmod{\frac{x}{N}}$ and $\sin\frac{x}{N} \simeq \frac{x}{N} \pmod{\frac{x}{N}}$, we may write $\cos\frac{x}{N} = 1 + \frac{x}{N}\epsilon$ and $\sin\frac{x}{N} = \frac{x}{N} + \frac{x}{N}\delta$, where ϵ and δ are infinitesimals depending on x and N (take $\epsilon = (\cos\frac{x}{N} - 1)/\frac{x}{N}$ and $\delta = (\sin\frac{x}{N} - \frac{x}{N})/\frac{x}{N}$; these are infinitesimal by the previous proposition). Then

$$\cos\frac{x}{N} \pm i\sin\frac{x}{N} = 1 \pm \frac{ix + (\epsilon + i\delta)x}{N}. \tag{13}$$

By the Binomial Theorem and the Summation Comparison Theorem, one can easily show that $(1 \pm c/N)^N \simeq (1 \pm d/N)^N$ whenever c and d are finite, hypercomplex numbers that are infinitely close. Thus (13) implies that

$$\left(\cos\frac{x}{N} \pm i\sin\frac{x}{N}\right)^N = \left(1 \pm \frac{ix + (\epsilon + i\delta)x}{N}\right)^N \simeq \left(1 \pm \frac{ix}{N}\right)^N,$$

which by (11) and (12) yields the result. ∎

The familiar series for sine and cosine can now be obtained by applying the Binomial Theorem to "multiply out" the N^{th} powers in (7) and (8), and then applying the Summation Comparison Theorem. This proves the following theorem.

THEOREM. *For all finite x and infinite H,*

$$\cos x \simeq 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \pm \frac{x^{2H}}{(2H)!},$$

$$\sin x \simeq x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \pm \frac{x^{2H+1}}{(2H+1)!}.$$

The binomial series

Many times already we have used the formula,

$$(1+x)^m = 1 + mx + \frac{m^{\underline{2}}}{2!}x^2 + \frac{m^{\underline{3}}}{3!}x^3 + \cdots + \frac{m^{\underline{m}}}{m!}x^m, \quad \text{natural } m,$$

a result that was known centuries prior to Euler (though not in this notation), and which can be verified using induction. In 1665 Newton discovered a generalization of the coefficients of the binomial expansion using a complicated interpolation between the rows and columns of a tabular form of Pascal's triangle, and conjectured that these generalized coefficients could be used to obtain a binomial series for negative and fractional exponents. Newton tested the conjecture on many examples, including squaring the series for $(1+x^2)^{1/2}$,

$$(1+x^2)^{1/2}(1+x^2)^{1/2}$$
$$= \left(1 + \frac{1}{2}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^4 + \cdots\right)\left(1 + \frac{1}{2}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^4 + \cdots\right)$$

to obtain

$$1 + x^2 + 0x^4 + 0x^6 + 0x^8 + 0x^{10} + \cdots,$$

but he never published a deductive proof of the general formula. (Edwards [**9**, pp. 178–187] gives a detailed account of the discovery of the Binomial Theorem; see also [**25**, p. 438].) Euler states Newton's "universal theorem" [**12**, §71],

$$(P+Q)^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}P^{\frac{m-n}{n}}Q + \frac{m(m-n)}{n\cdot 2n}P^{\frac{m-2n}{n}}Q^2 + \frac{m(m-n)(m-2n)}{n\cdot 2n\cdot 3n}P^{\frac{m-3n}{n}}Q^3 + \cdots$$

but omits the proof "since it can be done so much more easily with the aid of some principles of differential calculus" [**12**, §76]. Surprisingly, we will show that the proof of the Binomial Theorem for fractional exponents—which we write as

$$(1+x)^{\frac{m}{n}} \simeq 1 + \frac{m}{n}x + \left(\frac{m}{n}\right)^{\underline{2}}\frac{x^2}{2!} + \left(\frac{m}{n}\right)^{\underline{3}}\frac{x^3}{3!} + \cdots + \left(\frac{m}{n}\right)^{\underline{H}}\frac{x^H}{H!},$$

where H is infinite, $|x| < 1$, and $x \not\simeq 1$—is well within the scope of this article, and forms a natural part of our rehabilitation of Euler's methods. The proof uses the Binomial Theorem for Factorial Powers, which can be verified by induction. (See [**16**] for other uses of factorial powers.)

BINOMIAL THEOREM FOR FACTORIAL POWERS. *For all real a, b and all natural n,*

$$(a+b)^{\underline{n}} = \sum_{k=0}^{n} \frac{n^{\underline{k}}}{k!} a^{\underline{n-k}} b^{\underline{k}}.$$

BINOMIAL THEOREM (FRACTIONAL EXPONENTS). *If $|x| < 1$, and m and n are finite and positive, and H is infinite, then*

$$(1+x)^{m/n} \simeq 1 + \frac{m}{n}x + \left(\frac{m}{n}\right)^{\underline{2}}\frac{x^2}{2!} + \cdots + \left(\frac{m}{n}\right)^{\underline{H}}\frac{x^H}{H!}.$$

Proof. Fix an infinite hypernatural H and a hyperreal x such that $|x| < 1$ and $x \not\simeq 1$. We introduce the notation $(1+x)^{\boxed{a}}$ for the sum,

$$(1+x)^{\boxed{a}} \stackrel{\text{def}}{=} 1 + ax + a^{\underline{2}}\frac{x^2}{2!} + \cdots + a^{\underline{H}}\frac{x^H}{H!}.$$

(The dependence on H is not explicit in our notation.) Generalizing Newton's calculation for $(1+x^2)^{1/2}$, we will show that

$$\left((1+x)^{\boxed{m/n}}\right)^n \simeq (1+x)^m$$

and hence that

$$(1+x)^{\boxed{m/n}} \simeq (1+x)^{m/n}, \tag{14}$$

which is the statement of our theorem in our new notation.

The key formula in the proof of (14) is that for finite positive a and b,

$$(1+x)^{\boxed{a+b}} \simeq (1+x)^{\boxed{a}}(1+x)^{\boxed{b}}. \tag{15}$$

This is easy to see for integral m, n:

$$(1+x)^{\boxed{m+n}} = (1+x)^{m+n} = (1+x)^m(1+x)^n = (1+x)^{\boxed{m}}(1+x)^{\boxed{n}}.$$

For the general case we make use of the Binomial Theorem for Factorial Powers, after first multiplying out, gathering like terms, and neglecting the tail. Using the lemma (following this theorem), we write

$$\begin{aligned}&(1+x)^{\boxed{a}}(1+x)^{\boxed{b}}\\ &= \left(1+ax+a^{\underline{2}}\frac{x^2}{2!}+\cdots+a^{\underline{H}}\frac{x^H}{H!}\right)\left(1+bx+b^{\underline{2}}\frac{x^2}{2!}+\cdots+b^{\underline{H}}\frac{x^H}{H!}\right)\\ &= \sum_{k=0}^{2H} c_k x^k, \quad \text{where } c_k = \sum_{i+j=k}\frac{a^{\underline{i}}}{i!}\frac{b^{\underline{j}}}{j!}.\end{aligned}$$

Observe that for all *finite* k,

$$c_k = \sum_{i+j=k}\frac{a^{\underline{i}}}{i!}\frac{b^{\underline{j}}}{j!} = \sum_{j=0}^{k}\frac{a^{\underline{k-j}}}{(k-j)!}\frac{b^{\underline{j}}}{j!} = \frac{1}{k!}\sum_{j=0}^{k}\frac{k^{\underline{j}}}{j!}a^{\underline{k-j}}b^{\underline{j}} = \frac{1}{k!}(a+b)^{\underline{k}}.$$

For the first step in this chain of equalities it is essential that k be finite; the last step follows from the Binomial Theorem for Factorial Powers. In the lemma (following) we will verify that both $\sum_{k=0}^{2H} c_k x^k$ and $\sum_{k=0}^{H}(a+b)^{\underline{k}}x^k/k!$ are determinate; if for the moment we assume this as fact, then by the Summation Comparison Theorem, we may conclude that

$$(1+x)^{\boxed{a}}(1+x)^{\boxed{b}} = \sum_{k=0}^{2H} c_k x^k \simeq \sum_{k=0}^{H}\frac{(a+b)^{\underline{k}}}{k!}x^k = (1+x)^{\boxed{a+b}},$$

which shows (15). If m and n are finite, then by applying (15) a total of n times, we get

$$(1+x)^m = (1+x)^{\boxed{m}} = (1+x)^{\underbrace{\boxed{\frac{m}{n}+\cdots+\frac{m}{n}}}_{n \text{ terms}}} \simeq \left((1+x)^{\boxed{m/n}}\right)^n,$$

and hence that $(1+x)^{m/n} \simeq (1+x)^{\boxed{m/n}}$, as required. ∎

The Binomial Theorem can be extended to negative rational exponents by a similar argument, and, by the Sequential Theorem (see the next section) to the case where m and n are infinite, so long as m/n is finite. From there it is but a very small step to the theorem for real exponents. This is left as an exercise for the reader.

The previous theorem requires the following lemma.

LEMMA. ***(i)*** *The series* $1+|a|+|a^{\underline{2}}x^2/2!|+|a^{\underline{3}}x^3/3!|+\cdots$ *is determinate for finite positive* a, $|x|<1$, $x \not\simeq 1$. ***(ii)*** *If* $a_0, a_1, a_2, \ldots$ *and* $b_0, b_1, b_2, \ldots$ *are hypersequences then for all hypernatural* H,

$$\left(\sum_{i=0}^{H} a_i\right)\left(\sum_{j=0}^{H} b_j\right) = \sum_{k=0}^{2H} c_k \quad \textit{where} \quad c_k = \sum_{i+j=k} a_i b_j.$$

Moreover, if $|a_0| + |a_1| + |a_2| + \cdots$ *and* $|b_0| + |b_1| + |b_2| + \cdots$ *are determinate and have finite partial sums, then* $|c_0| + |c_1| + |c_2| + \cdots$ *is determinate and has finite partial sums.*

Proof. **(i)** We ask the reader to verify that for integral $k > a > 0$, we have $|a^{\underline{k}}| < k!$, and hence that $|a^{\underline{k}}x^k/k!| \leq |x^k|$. This shows that for $k > a > 0$ the absolute values of the terms in $(1+x)^{\boxed{a}}$ are bounded by a determinate geometric series. **(ii)** Note that the product $(\sum_{i=0}^{H} a_i)(\sum_{j=0}^{H} b_j)$ when multiplied out, is the sum of all terms $a_i b_j$ for i and j between 0 and H. These terms can be arranged in a table.

	0	1	2	$\cdots$	H
0	a_0b_0	a_0b_1	$\boldsymbol{a_0b_2}$	$\cdots$	a_0b_H
1	a_1b_0	$\boldsymbol{a_1b_1}$	a_1b_2	$\cdots$	a_1b_H
2	$\boldsymbol{a_2b_0}$	a_2b_1	a_2b_2	$\cdots$	a_2b_H
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
H	a_Hb_0	a_Hb_1	a_Hb_2	$\cdots$	a_Hb_H

$$\boldsymbol{c_2 = a_2b_0 + a_1b_1 + a_0b_2}$$

For each k, $c_k = \sum_{i+j=k} a_i b_j$ is the sum of all terms on the northeasterly diagonal at k, making $\sum_{k=0}^{2H} c_k$ the sum of all of the diagonals, and hence the sum of all terms in the table. To see that $|c_0| + |c_1| + |c_2| + \cdots$ is determinate, observe that for N infinite,

$$\sum_{k=N}^{N+M} |c_k| \leq \sum_{k=N}^{N+M} \sum_{i+j=k} |a_i b_j| \leq \sum_{i=\frac{N}{2}}^{H} \sum_{j=0}^{H} |a_i b_j| + \sum_{j=\frac{N}{2}}^{H} \sum_{i=0}^{H} |a_i b_j|$$

$$= \left(\sum_{i=\frac{N}{2}}^{H} |a_i|\right)\left(\sum_{j=0}^{H} |b_j|\right) + \left(\sum_{i=0}^{H} |a_i|\right)\left(\sum_{j=\frac{N}{2}}^{H} |b_j|\right) \simeq 0.$$

The second inequality is obtained by noting that if $i + j \geq N$, then either $i \geq N/2$ or $j \geq N/2$. The final step ($\simeq 0$) follows because the series are determinate and have finite partial sums. ∎

Proof of the Summation Comparison Theorem

In contrast with the other theorems in this article, which concern concrete functions and equations, the Summation Comparison Theorem is a result about all functions and equations of a general class. It should not be surprising then that the proof is more abstract and relies on more basic definitions and principles than the proofs of the other theorems. We will show how the Summation Comparison Theorem follows from the Least Counterexample Principle, an equivalent of the Hypernatural Induction Principle, by way of the Sequential Theorem. The Sequential Theorem is an important result due to Robinson [**40**, Theorem 3.3.20]. In a course of study, the proof could be delayed.

LEAST COUNTEREXAMPLE PRINCIPLE. *Let* $\phi(n)$ *be an equation or inequality of hypersequences; that is, let* $\phi(n)$ *be of the form* $a_n = b_n$, $a_n \neq b_n$, $a_n < b_n$, *or* $a_n \leq b_n$, *where a and b are hypersequences. Then either* $\phi(n)$ *holds for all hypernatural n, or else there is an m such that* $\phi(m)$ *fails but such that* $\phi(n)$ *holds for all hypernatural n less than m.*

For example, consider the inequality $\phi(n)$ given by $(1 - n/H) > 0$, for a fixed hypernatural H. This inequality is false for n equal to H, but it is true for all n less than H. Thus $n = H$ is a least counterexample for $\phi(n)$. On the other hand, the equation $a_n = 0$, where a_n is defined to be 0 for n finite and 1 for n infinite, does not have a least counterexample, even though there are counterexamples. It does not however disobey the Least Counterexample Principle, because the function a, so defined, is not a hypersequence (that is, a cannot be obtained by composition of natural extensions of real functions with hyperreal parameters).

SEQUENTIAL THEOREM. *Let $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ be hypersequences. If $a_n \simeq b_n$ for all natural n, then there is an infinite N such that $a_n \simeq b_n$ for all hypernatural n smaller than N.*

Proof. Since the relation $a_n \simeq b_n$ is not an equation or inequality, one cannot apply the Least Counterexample Principle directly. Instead, we note that if $a_n \simeq b_n$ for all finite n, then it is also true that $-\frac{1}{n} < a_n - b_n$ and $a_n - b_n < \frac{1}{n}$ for all finite $n > 1$. By applying the Least Counterexample Principle to these inequalities, we conclude that there is an infinite N such that $-\frac{1}{n} < a_n - b_n$ and $a_n - b_n < \frac{1}{n}$ for all n between 1 and N. By the original assumption that $a_n \simeq b_n$ for all finite n, and using the fact that $1/n$ is infinitesimal for infinite values of n, we conclude that $a_n \simeq b_n$ for all $n < N$. ∎

SUMMATION COMPARISON THEOREM. *If the series $a_0 + a_1 + a_2 + \cdots$ and $b_0 + b_1 + b_2 + \cdots$ are determinate, and if for each natural n, $a_n \simeq b_n$, then for all hypernatural n, $a_0 + a_1 + \cdots + a_n \simeq b_0 + b_1 + \cdots + b_n$.*

Proof. If $a_n \simeq b_n$ for all finite n then $a_0 + \cdots + a_n \simeq b_0 + \cdots + b_n$ for all finite n as well. By the Sequential Theorem, there is an infinite J such that for all n less than J, $a_0 + \cdots + a_n \simeq b_0 + \cdots + b_n$. Let N be greater than J. If the sums are determinate, then by definition, $a_J + \cdots + a_N$ and $b_J + \cdots + b_N$ are both infinitesimal, and hence *for all n*, $a_0 + \cdots + a_n \simeq b_0 + \cdots + b_n$. ∎

The logarithm and beyond

Immediately after defining the exponential function a^x and discussing the basic rules for exponentiation, Euler defined the logarithm for bases a greater than 1.

> Just as, given a number a, for any value of x, we can find a value of y [$= a^x$], so, in turn, given a positive value for y, we would like to give a value for x, such that $a^x = y$. This value of x, insofar as it is viewed as a function of y, is called the *logarithm* of y. ... It is customary to designate the logarithm of y by the symbol, $\log y$. [**12**, §102]

We usually write $\log_a y$, making the dependence on the base a explicit in the notation. From the definition that for $y > 0$, $\log_a y$ is the x such that $a^x = y$, and from the rules for exponentials given earlier, the following rules for logarithms (for $a > 1$ and $x, y > 0$) follow immediately: $\log_a 1 = 0$, $\log_a x^{-1} = -\log_a x$, $\log_a(xy) = \log_a x + \log_a y$, $\log_a x^y = y \log_a x$, and $\log_a x < \log_a y$ iff $x < y$.

Early in the *Introductio*, Euler explained how these properties of the logarithm, insofar as they reduce exponentiation to multiplication and multiplication to addition, make compiled tables of logarithms extremely useful for performing computations. One of his examples, employing a table of logarithms to the base 10, is as follows.

> If the population in a certain region increases annually by one-thirtieth and at one time there were 100,000 inhabitants, we would like to know the population

after 100 years. For the sake of brevity, we let the initial population be n, so that $n = 100{,}000$. After one year the new population will be $(1+\frac{1}{30})n = \frac{31}{30}n$. After two years it will equal $(\frac{31}{30})^2 n$. After three years it will equal $(\frac{31}{30})^3 n$. Finally after one-hundred years the population will be $(\frac{31}{30})^{100} n = (\frac{31}{30})^{100} 100{,}000$. The logarithm of this population is $100 \log\frac{31}{30} + \log 100{,}000$. But $\log\frac{31}{30} = \log 31 - \log 30 = 0.014240439$, so that $100\log\frac{31}{30} = 1.4240439$, which, when increased by $\log 100{,}000 = 5$, gives 6.424039, the logarithm of the desired population. The corresponding population is 2,654,874. So after one-hundred years the population will be more than twenty-six-and-a-half times as large. [**12**, §110]

The question remains: how might one compile such tables of logarithms? Euler gave an example to show how Briggs computed logarithms for his famous *Arithmetica Logarithmica* of 1624 using a calculation-intensive algorithm that required the manual extraction of many successive square roots, but noted that, "In the mean time, much shorter methods have been found by means of which logarithms can be computed more quickly." [**12**, §106] In the succeeding chapter, Euler explained how to calculate logarithms using series.

Let us find a series for the logarithm with base e. This logarithm can be written $\log_e$ but is more often written ln. To get started, we relax the requirement that for a given y we must find *the* x such that e^x is *exactly equal* to y, and instead try to find, for a given finite $y > 0$,

$$\text{an } x \text{ such that } e^x \simeq y. \tag{16}$$

Earlier we found that for finite x and infinite N, $e^x \simeq (1+x/N)^N$, so let us solve the equation $y = (1+x/N)^N$ and see whether that solution is of any use. A solution (taking the principal N^{th} root of y) is given by

$$x = N(y^{1/N} - 1).$$

Observe that this does satisfy (16),

$$e^x = e^{N(y^{1/N}-1)} \simeq \left(1 + \frac{N(y^{1/N}-1)}{N}\right)^N = (1 + y^{1/N} - 1)^N = (y^{1/N})^N = y,$$

so that indeed, $e^x \simeq y$. In our discussion of the logarithm, we also need to know that $N(y^{1/N} - 1) \simeq \ln y$.

THEOREM. *If* y *is finite and positive, then* $\ln y \simeq N(y^{1/N} - 1)$ *for all infinite* N.

Proof. Let y be finite and positive. By the preceding computation, $e^{N(y^{1/N}-1)} \simeq y = e^{\ln y}$. Then $N(y^{1/N}-1) \simeq \ln y$ follows from the proposition saying that $x \simeq y$ if and only if $e^x \simeq e^y$ for finite x and y. ∎

In [**12**, §119], Euler used the formula $\ln y = N(y^{1/N} - 1)$, for N infinite, to derive the series for the natural logarithm. He expanded the function $\log(1+y)$ using the Binomial Theorem for fractional exponents to get

$$\begin{aligned}
&\log(1+y)\\
&= N\left[(1+y)^{1/N} - 1\right]\\
&= N\left[\left[1 + \frac{1}{N}y + \frac{1}{N}\left(\frac{1}{N}-1\right)\frac{1}{2!}y^2 + \frac{1}{N}\left(\frac{1}{N}-1\right)\left(\frac{1}{N}-2\right)\frac{1}{3!}y^3 + \cdots\right] - 1\right]y\\
&= y + \left(\frac{1}{N}-1\right)\frac{1}{2!}y^2 + \left(\frac{1}{N}-1\right)\left(\frac{1}{N}-2\right)\frac{1}{3!}y^3 + \cdots.
\end{aligned}$$

Then, using the fact that N is infinite, Euler substituted 0 everywhere for $1/N$, and obtained the equation,

$$\log(1+y) \;=\; y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \cdots.$$

Such substitutions are somewhat more difficult to justify for this series than for our earlier examples, but it is nonetheless within reach of the methods we have discussed thus far.

THEOREM. *For all y with $|y| < 1$ but $|y| \not\simeq -1$, and all infinite H,*

$$\log(1+y) \simeq y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \cdots + (-1)^{H+1}\frac{1}{H}y^H.$$

Proof. Assume that $|y| < 1$, $|y| \not\simeq -1$, and let H be infinite. By the Binomial Theorem for fractional exponents, we conclude that for all *finite* n,

$$(1+y)^{1/n} \simeq 1 + \frac{1}{n}y + \left(\frac{1}{n}\right)^{\underline{2}}\frac{1}{2!}y^2 + \cdots + \left(\frac{1}{n}\right)^{\underline{H}}\frac{1}{H!}y^H. \tag{17}$$

For n infinite, both sides are infinitely close to 1, so (17) is actually true for all hypernatural n, infinite as well as finite. Thus it is tempting to follow Euler's lead and substitute an infinite N for n in (17), subtract 1 from both sides, then multiply by N. We cannot quite do this, because (17) has "$\simeq$" rather than "$=$", and because for N infinite it does not follow from $a \simeq b$ that $Na \simeq Nb$ (for a counterexample take $a = 0$, $b = 1/N$). Instead we can do this: (17) implies that for all *finite* n,

$$\begin{aligned}
&n\left[(1+y)^{1/n} - 1\right] \\
&\quad \simeq n\left[\left(1 + \frac{1}{n}y + \left(\frac{1}{n}\right)^{\underline{2}}\frac{y^2}{2!} + \left(\frac{1}{n}\right)^{\underline{3}}\frac{y^3}{3!} + \cdots + \left(\frac{1}{n}\right)^{\underline{H}}\frac{y^H}{H!}\right) - 1\right] \\
&\quad = y - \frac{\left(1-\frac{1}{n}\right)}{1}\cdot\frac{y^2}{2} + \frac{\left(1-\frac{1}{n}\right)}{1}\cdot\frac{\left(2-\frac{1}{n}\right)}{2}\cdot\frac{y^3}{3} \\
&\qquad - \cdots + (-1)^{H-1}\frac{\left(1-\frac{1}{n}\right)}{1}\cdot\frac{\left(2-\frac{1}{n}\right)}{2}\cdots\frac{\left(H-1-\frac{1}{n}\right)}{H-1}\cdot\frac{y^H}{H},
\end{aligned}$$

where the alternation in signs follows from the fact that for $k > 0$ and $n > 1$ we have $(1/n)^{\underline{k}} = (-1)^{k-1}\cdot\frac{1}{n}(1-\frac{1}{n})(2-\frac{1}{n})\cdots((k-1)-\frac{1}{n})$. Hence by the Sequential Theorem we conclude—not for all—but that for all *sufficiently small* infinite N,

$$\begin{aligned}
&N\left[(1+y)^{1/N} - 1\right] \\
&\quad \simeq y - \frac{\left(1-\frac{1}{N}\right)}{1}\cdot\frac{y^2}{2} + \frac{\left(1-\frac{1}{N}\right)}{1}\cdot\frac{\left(2-\frac{1}{N}\right)}{2}\cdot\frac{y^3}{3} \\
&\qquad - \cdots + (-1)^{H-1}\frac{\left(1-\frac{1}{N}\right)}{1}\cdot\frac{\left(2-\frac{1}{N}\right)}{2}\cdots\frac{\left(H-1-\frac{1}{N}\right))}{H-1}\cdot\frac{y^H}{H}.
\end{aligned}$$

When this last sum is compared with the sum

$$y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \cdots + (-1)^{H+1}\frac{1}{H}y^H,$$

it is clear that term by term the sums are infinitely close, so we need only verify that both sums are determinate. For $|y| < 1$, $|y| \not\simeq -1$, determinacy follows from the Comparison Test for Determinacy by comparison with a determinate geometric sum. By the Summation Comparison Theorem we finally conclude that

$$H\left[(1+y)^{1/H} - 1\right] \simeq y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \cdots + (-1)^{H+1}\frac{1}{H}y^H$$

for all infinite H. The result now follows from the previous theorem. ∎

Finally, Euler observed that though the series for the logarithm just given does not converge rapidly, and hence is not itself so effective for computing logarithms, it leads to other series that are quite effective. For example,

$$\log\left(\frac{1+x}{1-x}\right) = \log(1+x) - \log(1-x) = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \cdots.$$

Euler remarked,

> This last series is strongly convergent if we substitute an extremely small fraction for x. For instance, if $x = \frac{1}{5}$, then $\log\frac{6}{4} = \log\frac{3}{2} = \frac{2}{1\cdot 5} + \frac{2}{3\cdot 5^2} + \frac{2}{5\cdot 5^5} + \frac{2}{7\cdot 5^7} + \cdots$. If $x = \frac{1}{7}$, then $\log\frac{4}{3} = \frac{2}{1\cdot 7} + \frac{2}{3\cdot 7^2} + \frac{2}{5\cdot 7^5} + \frac{2}{7\cdot 7^7} + \cdots$, and if $x = \frac{1}{9}$, then $\log\frac{5}{4} = \frac{2}{1\cdot 9} + \frac{2}{3\cdot 9^2} + \frac{2}{5\cdot 9^5} + \frac{2}{7\cdot 9^7} + \cdots$. From the logarithms of these fractions, we can find the logarithms of integers. From the nature of logarithms we have $\log\frac{3}{2} + \log\frac{4}{3} = \log 2$, and $\log\frac{3}{2} + \log 2 = \log 3$, and $2\log 2 = \log 4$. Furthermore we have $\log\frac{5}{4} + \log 4 = \log 5$, $\log 2 + \log 3 = \log 6$, $3\log 2 = \log 8$, $2\log 3 = \log 9$, $\log 2 + \log 5 = \log 10$. [**12**, §123]

Using these series and relationships, Euler was able to show how to begin constructing a table of logarithms.

In the *Introductio*, Euler also exhibited series for other transcendental functions, including the tangent, cotangent, and arctangent, and went on to show how to use infinite products to compute the values of infinite sums. Using infinitesimal methods similar to those described here, Euler factored the sine function into an infinite product and used that factorization to deduce the celebrated formula $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}$. Both of these theorems can be rehabilitated, but the algebra turns out to be more taxing.

THEOREM. *For all finite x and infinite H,*

$$\sin x \simeq x\prod_{k=1}^{H}\left(1 - \frac{x^2}{(k\pi)^2}\right).$$

THEOREM. *For all infinite H,*

$$\sum_{k=1}^{H}\frac{1}{k^2} \simeq \frac{\pi^2}{6}.$$

A careful analysis of Euler's arguments for these two results is given in [**37**].

The connection between standard and nonstandard notions

Our theorem saying that $e^x \simeq \sum_{n=0}^{N} \frac{x^n}{n!}$, for all finite x and infinite N, is conceptually similar to the standard theorem that says

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

for all real x, but these theorems are not the same. The former refers to a hypernatural summation within a proper extension of the real numbers while the latter refers to something we have not yet discussed: the limit of a real sequence of partial sums. The notion of *limit* is usually taken to be the dividing line between algebra and analysis. In this section we give a brief sketch of how to cross that line. Let us first recall the standard definition of convergence for infinite series.

STANDARD DEFINITION OF CONVERGENCE OF SERIES. Let s be a real sequence and let r be real. We say that s *converges* to r if and only if for each positive real ϵ there is a natural n such that for all natural m greater than n, $|s_m - r| < \epsilon$. We write $\sum_{n=0}^{\infty} a_n = r$ to mean that a is a real sequence such that its sequence of partial sums $a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots$ converges to the real number r.

Except for our definition of the real number e given above, thus far we have not had to distinguish between the closely related notions of *real number* and *finite hyperreal number*. But we need this distinction if we are to convert our results about hyperreal numbers to results solely about real numbers. The real numbers are distinguished from other ordered fields by the Completeness Axiom. We will not prove this here but the Standard Part Principle is actually a consequence of the Completeness Axiom for the real numbers. (See Keisler [**23**, pp. 36–40, 908–909].)

STANDARD PART PRINCIPLE. *For every finite hyperreal b there is a unique real r such that $r \simeq b$. This real r is called the* standard part *of b, denoted ${}^{\circ}b$.*

Earlier we used the Standard Part Principle to define the real number e to be ${}^{\circ}(1 + 1/N)^N$, where N is infinite. This definition and (5) together with the assumption that e^x is a real function, imply that

$$e^x = {}^{\circ}\sum_{n=0}^{N} \frac{x^n}{N!} \tag{18}$$

for infinite N. Rather than assuming e^x to be defined for all real numbers (as we did above) one could instead take (18) to be the definition of the function e^x and derive the usual properties of exponentiation from this definition. We chose not to do that, but it is a reasonable alternative. However, for *complex* exponentiation the synthetic approach is all we have at our disposal, so we simply define e^{ix} by the identity

$$e^{ix} = {}^{\circ}\left(1 + \frac{ix}{N}\right)^N,$$

for real (and hyperreal) x, where ${}^{\circ}(a + bi) = ({}^{\circ}a) + ({}^{\circ}b)i$. From this definition one can deduce the Euler identities in their familiar form.

COROLLARY. *For all real x,*

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \textit{and} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

The connection between the infinite sum of a determinate hypersequence and the convergence of a real sequence of partial sums is given by the following theorem, which is a consequence of the Transfer Principle and the Least Counterexample Principle. The proof, although somewhat technical, is within the scope of Keisler's calculus book.

THEOREM. *Let β be a hypersequence such that $\beta_0 + \beta_1 + \beta_2 + \cdots$ is determinate, let b be a real sequence, and suppose that $b_n \simeq \beta_n$ for all natural n. Then the real sequence of partial sums of b is convergent in the standard sense, and for all infinite hypernatural N, $\sum_{n=0}^{\infty} b_n = {}^{\circ}\sum_{n=0}^{N} \beta_n$.*

(Note that the convergence of b is a consequence, not a hypothesis.) This theorem implies standard analogs of all the summations in this article.

COROLLARIES. *For all real x,*

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \qquad \log(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} \quad \textit{for } |x| < 1,$$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \qquad \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!},$$

$$(1+x)^{m/n} = \sum_{k=0}^{\infty} \left(\frac{m}{n}\right)^{\underline{k}} \frac{x^k}{k!} \quad \textit{for } |x| < 1.$$

Proof. For the first equation, let N be infinite. Then $e^x = {}^{\circ}\sum_{n=0}^{N} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. The others are similar. ∎

Lessons from Euler

The *Introductio* was expressly intended as a precalculus textbook, that is, a book for a course of study prior to differential and integral calculus. The point was not to give short and slick derivations from an extensive body of knowledge, but rather to educate beginners. Euler said,

> Although all of these nowadays are accomplished by means of differential calculus, nevertheless, I have here presented them using only ordinary algebra, in order that the transition from finite analysis to analysis of the infinite might be rendered easier.... At the same time I readily admit that these matters can be much more easily worked out by differential calculus. [**12**, pp. *ix–x*]

We might take a lesson from Euler's great textbook for our own courses. In the standard treatments, discrete mathematics is held disjoint from the calculus, and interesting and useful series are studied only after Taylor's Theorem is proved—usually at the end of the lectures on convergence of sequences and series, well after the derivative is thoroughly studied. In Euler's treatment, beginners get their hands on concrete examples of sequences and series even before the derivative is defined. As rehabilitated here, this approach might also give our own students practice with important topics from discrete mathematics—induction, recursion, finite summations, and axiomatics—in the course of proving elementary analogs of theorems of real analysis. But whether or not our rehabilitation of Euler's methods finds its way into the educational main stream, we hope that by focusing our attention on the intellectual beauty of the underlying mathematics, we have convinced the reader that Euler's insights and arguments, far from being reckless or nonsensical, are directly relevant to the understanding, appreciation, and application of elementary mathematics even in our day.

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Letter to the Editor

Dear Editor:

In my paper, "Avoiding your spouse at a bridge party," appearing in the February 2001 issue of this MAGAZINE, I calculated certain probabilities, associated with couples playing bridge, b_n, using the inclusion-exclusion principle. In an aside, I commented that the fact that the probabilities could be expressed as a sum with decreasing terms "is a consequence of our having formulated the expression for the b_n using the inclusion-exclusion principle." Professor Lajos Takacs pointed out to me in a letter that this claim is false. It is true that the terms in the sum are decreasing, but this fact is not a consequence of the inclusion-exclusion principle.

Recall the inclusion-exclusion principle, which can be proved by doing the following exercise from *Probability Theory and Examples, 2nd Ed.*, by Richard Durrett (p. 22, ex. 3.11):

Let $A_1, A_2, \ldots, A_n$ be events and $A = \bigcup_{i=1}^n A_i$. Prove that $1_A = 1 - \prod_{i=1}^n (1 - 1_{A_i})$. Expand out the right hand side, then take expected value to conclude

$$P\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n P(A_i) - \sum_{i<j} P\left(A_i \bigcap A_j\right)$$

$$+ \sum_{i<j<k} P\left(A_i \bigcap A_j \bigcap A_k\right) - \cdots + (-1)^{n-1} P\left(\bigcap_{i=1}^n A_i\right).$$

In the preceding, 1_A is the indicator function equal to 1 if $x \in A$, and 0 otherwise.

A trivial example of a case for which my statement is false is where we have n events $A_1, \ldots, A_n$ such that $A_1 = A_2 = \cdots = A_n$, and $P(A_i) = \alpha$; then

$$\sum_{i_1<i_2<\cdots<i_k} P\left(A_{i_1} \bigcap \cdots \bigcap A_{i_k}\right) = \binom{n}{k}\alpha.$$

In this case, then, the terms are just α times the binomial numbers $n, \binom{n}{2}, \ldots, \binom{n}{k}, \ldots, n, 1$ and this sequence is not decreasing. When $n = 3$, for example, the sequence is 3, 3, 1.

I regret the error.

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