Integrals Don't Have Anything to Do with Discrete Math, Do They?

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To students just beginning their study of mathematics, the subject appears to come in two distinct flavours: continuous and discrete. The former is embodied by the calculus, into which many math majors delve extensively, while the latter has its own introductory course (often entitled Discrete Mathematics) whose overlap with calculus is slight. The distinction persists as we learn more mathematics, since most advanced undergraduate math courses have their focus on one side or the other of this apparent divide.

This article attempts to bridge the divide by describing one surprising connection between continuous and discrete mathematics. Its goal is to convince readers that the two worlds are not so very far apart. Though they may frequently feel like polar opposites, there are also times when they join to become one, like antipodal points in projective space. Therefore, any serious study of discrete math ought to include a healthy dose of the continuous, and vice versa.

Before we are done, various players from both worlds will make their appearance: rook polynomials, derangements, the gamma function, and the Gaussian density (just to name the headliners).

Teaser To whet the reader's appetite, we begin with a challenge.

PROBLEM 1. Give a *combinatorial proof* that

$$\int_0^\infty (t^3 - 6t^2 + 9t - 2)e^{-t}\,dt = 1; \tag{1}$$

i.e., count something that, on one hand, is easily seen to number the left side of (1) and on the other, the right.

For a delightful treatment of combinatorial proofs in general, see [**4**].

At first blush, Problem 1 may appear to be out of reach—a *combinatorial* proof of an integral identity—what in heavens should we count? The answer provides part of the fun of writing (and hopefully reading) this article.

Entities: continuous and discrete

After introducing our objects of study, we reveal some of their connections in the next few sections, and also present a solution to Problem 1. In an attempt to make the article self-contained, we include an appendix containing some basic facts and other curiosities about these objects.

Math. Mag. **84** (2011) 108–119. doi:10.4169/math.mag.84.2.108.

Integrals The integral on the left side of (1) belongs to a family of integrals enjoying discrete connections. The family's matriarch is Euler's *gamma function*, which can be defined, for $0 < x < \infty$, by

$$\Gamma(x) := \int_0^{\infty} t^{x-1} e^{-t}\, dt. \tag{2}$$

One can check that this improper integral converges for such x; see, e.g., [**2**, pp. 11–12]. (In fact, Γ need not be confined to the positive real numbers—it is possible to extend its definition so that Γ becomes a meromorphic function on the complex plane, with poles at the origin and each negative integer; see, e.g., [**1**, p. 199] or [**11**, p. 54]—but we'll restrict our attention to positive real x.)

Some close cousins of the gamma function are certain 'probability moments.' For integers $n \geq 0$, the *nth moment* (of a Gaussian random variable with mean 0 and variance 1, i.e., a standard normal random variable) is defined by

$$\mathcal{M}_n := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^n e^{-t^2/2}\, dt.$$

These integrals also converge (see, e.g., [**8**, p. 148]), and though probability language enters in their naming, we won't be making much use of this connection. Since we do need the fact that $\mathcal{M}_0 = 1$ (see Theorem 4), we present a standard proof of this identity in the Appendix (Lemma 6).

Graphs The right side of (1)—i.e., the number 1—counts the 'perfect matchings' in a certain graph. While we shall assume that the reader is familiar with graphs, we nevertheless introduce the few required elementary notions. Any standard graph theory text should suffice to close our expositional gaps; see, e.g., [**5**].

Recall that a *graph* $G = (V, E)$ consists of a finite set V (of *vertices*), together with a set E of unordered pairs $\{x, y\}$ (*edges*) with $x \neq y$ and both of $x, y \in V$. (Such graphs are called *simple graphs* in [**5**, p. 3].) A graph is *complete* if, for each pair x, y of distinct vertices, the edge $\{x, y\}$ appears in E. FIGURE 1 depicts the complete graphs with $1 \leq |V| \leq 6$ and introduces the standard notation K_n for the complete graph on $n \geq 1$ vertices.

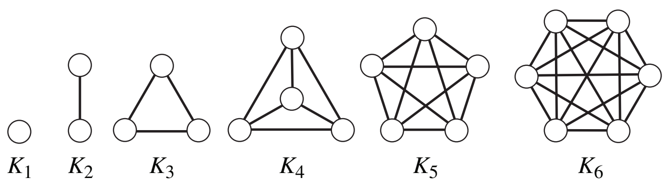
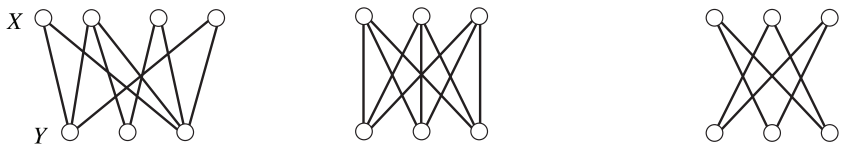


Figure 1 Complete graphs on up to six vertices

The second graph family of primary interest in this article is the collection of *bipartite graphs* G, i.e., those for which the vertex set admits a partition $V = X \oplus Y$ into nonempty sets X, Y such that each edge of G is of the form $\{x, y\}$, with $x \in X$ and $y \in Y$. One often forms a mental picture of a bipartite graph by imagining two rows of dots—a row for X and a row for Y—together with a collection of line segments xy joining an $x \in X$ to a $y \in Y$ whenever $\{x, y\} \in E$. In the next definition, we fix two positive integers n, m. The bipartite graph $(X \oplus Y, E)$ for which $|X| = n$, $|Y| = m$, and E consists of all nm possible edges between X and Y is called a *complete bipartite graph* and denoted by $K_{n,m}$. The bipartite graphs arising in this article are the complete bipartite ones for which $n = m$ (for $n \geq 1$) and their *spanning subgraphs*, i.e., those bipartite graphs $(X \oplus Y, E)$ with $|X| = |Y| = n$. It's worth noting that a subgraph H of G is a spanning subgraph exactly when they share a common vertex set; the edge set of H may form any subset of the edge set of G, including the empty set. FIGURE 2 depicts a few small bipartite graphs.



(a) a bipartite graph (b) the complete bipartite graph $K_{3,3}$ (c) a spanning subgraph of $K_{3,3}$

Figure 2 Bipartite and complete bipartite graphs

A first brush between continuous and discrete For the gamma function (2), it is easy to check that $\Gamma(1) = 1$, and integration by parts yields the recurrence

$$\Gamma(x+1) = x\Gamma(x), \tag{3}$$

valid for positive real numbers x. It follows by mathematical induction that each nonnegative integer n satisfies $\Gamma(n+1) = n!$; i.e., the gamma function generalizes the factorial function to the real numbers.

Given this generalization, a natural question to ponder might be: What values does Γ take on at half-integers? The reader might enjoy showing that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{4}$$

and then using (3) to prove that

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{4^n n!}$$

whenever n is a nonnegative integer. (Corollary 7 in the Appendix provides a key step in this exercise.) The ease in determining Γ at half-integers belies the dearth of known exact values; for example, no simple expression is known for $\Gamma(1/3)$ or $\Gamma(1/4)$—see [**11**, p. 55], or, for a more recent and specific discussion, [**15**].

What good, we might ask, is a continuous version of the factorial function? One answer is that a careful study of Γ can be used to establish Stirling's approximation for $n!$:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \tag{5}$$

published by James Stirling [**18**, p. 137] in 1730. (Here and below, the symbol $\sim$ means that the ratio of the left to the right side tends to 1 as $n \to \infty$.) See, e.g., [**14**] for an elementary proof of (5) starting from the definition (2) of Γ. A complex-analytic proof, based on the extension of Γ to $\mathbb{C} \setminus \{0, -1, -2, \dots\}$ to which we alluded earlier, appears in [**1**, pp. 201–204]. However it's reached, the estimate (5), involving two of the most famous mathematical constants and invoking only basic algebraic operations, is no doubt beautiful. Moreover, it is useful any time one wants to gain insight into the growth rate of functions involving factorials. For example, using (5), one easily shows that

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}},$$

and so learns something about the asymptotics of the *Catalan numbers* $\binom{2n}{n}/(n+1)$ (see, e.g., [**17**, pp. 219–229] for more on this pervasive sequence).

Our purpose is to refute the first part of this article's title, and as we move in that direction, we can't resist sharing a couple more fun facts about Γ that enhance the stature of Γ in the gallery of basic mathematical functions. First, as long as x is not an integer, we have

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)},$$

which generalizes (4). This 'complement formula' was first proved by Leonhard Euler; see, e.g., [**1**, pp. 198–199] or [**11**, p. 59] for modern proofs. Second, if

$$\zeta(x) := \sum_{k=1}^{\infty} \frac{1}{k^x}$$

denotes the Riemann zeta function, then whenever $\Gamma(x)$ is finite, we have

$$\zeta(x)\Gamma(x) = \int_0^{\infty} \frac{t^{x-1}}{e^t - 1}\, dt, \tag{6}$$

which bears a striking resemblance to (2); again, see [**1**, p. 214] or [**11**, pp. 59–60] for proofs. Because of ζ's central role in connecting number theory to complex analysis, the relation (6) opens deeper connections of Γ to number theory (beyond those stemming from the factorial function). Viewing number theory as falling within the discrete realm, we see in (6) a further refutation of this article's title.

Counting perfect matchings in $K_{n,n}$

A *matching* M in a bipartite graph $G = (X \oplus Y, E)$ is a subset $M \subseteq E$ such that the edges in M are pairwise disjoint. We think of M as 'matching up' some members of X with some members of Y. If every $x \in X$ appears in some $e \in M$, and likewise for Y, then we call M a *perfect matching*. It is a simple exercise to show that if G contains a perfect matching, then $|X| = |Y|$, so that G is a spanning subgraph of some $K_{n,n}$. FIGURE 3 highlights one matching within each of the graphs in FIGURE 2.

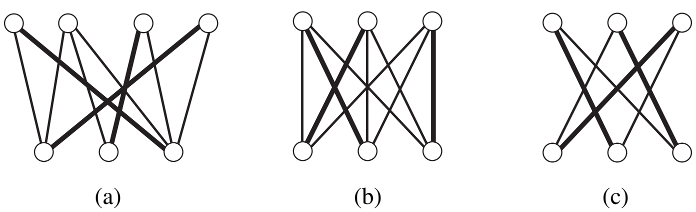


Figure 3 Matchings in the graphs of FIGURE 2 indicated by bold edges; those in (b) and (c) are perfect.

Given a bipartite graph G, we might be interested to know how many perfect matchings it contains; we use $\Xi(G)$ to denote this number.[1] Let's warm up by asking for the value of $\Xi(K_{n,n})$; a moment's reflection shows that for each integer $n \geq 1$, the answer is $n!$. (To see this, continue to denote the 'bipartition' by (X, Y), and notice that the perfect matchings of $K_{n,n}$ are in one-to-one correspondence with the bijections between X and Y.) Since $n! = \Gamma(n+1)$, we have proven our first result.

PROPOSITION 2. $\Xi(K_{n,n}) = \int_0^\infty t^n e^{-t}\, dt.$

If we replace $K_{n,n}$ by a different bipartite graph, how must we modify the formula in Proposition 2? It turns out that a so-called 'rook polynomial' should replace the polynomial t^n.

Rook polynomials Given a graph G and an integer r, we denote by $\mu_G(r)$ the number of matchings in G containing exactly r edges.

EXAMPLE 1. (THE GRAPH $G = K_{3,3} - \{\{x_1, y_1\}, \{x_2, y_2\}, \{x_3, y_3\}\}$) This is the graph in FIGURE 2(c). Since the empty matching contains no edges, we have $\mu_G(0) = 1$; since each singleton edge forms a matching, we have $\mu_G(1) = 6$, and since G contains two perfect matchings, we have $\mu_G(3) = 2$. Fixing a vertex x, we see that there are three matchings of size two using either of the edges incident with x and three more two-edge matchings not meeting x; thus $\mu_G(2) = 9$.

Now suppose that G is a spanning subgraph of $K_{n,n}$. The *rook polynomial* of G is defined by

$$R_G(t) := \sum_{r=0}^{n} (-1)^r \mu_G(r) t^{n-r}.$$

See [**10**, p. 8] or [**16**, pp. 164–166] for the etymology of this term.

EXAMPLE 1. (CONTINUED) Based on our observations in the first part of this example, we see that

$$R_G(t) = t^3 - 6t^2 + 9t - 2;$$

we're getting a little ahead of ourselves, but this is the polynomial appearing in the integrand in Problem 1.

EXAMPLE 2. (EMPTY GRAPHS) If G is the empty graph on $2n$ vertices (i.e., $|V| = 2n$ and $E = \varnothing$), then

$$\mu_G(r) = \begin{cases} 0 & \text{if } r > 0 \\ 1 & \text{if } r = 0, \end{cases}$$

so that $R_G(t) = t^n$; keeping ahead of ourselves, notice that this polynomial appears in the integrand in Proposition 2.

EXAMPLE 3. (PERFECT MATCHINGS) If G consists of n pairwise disjoint edges (i.e., G is induced by a perfect matching), then one can easily see that $\mu_G(r) = \binom{n}{r}$ for $0 \leq r \leq n$. Thus, the binomial theorem shows that $R_G(t) = (t-1)^n$.

[1] We chose this notation because (whether we write it in English or Greek!) the letter XI (Ξ) resembles a perfect matching in a graph of order six, and, conveniently enough, six is a perfect number.

Continuing to let G denote a spanning subgraph of $K_{n,n}$, we now define its *bipartite complement* $\widetilde{G}$; this graph shares the vertex set of G and has for edges all the edges of $K_{n,n}$ that are not in G. We're ready to state a generalization of Proposition 2. To avoid possible confusion as to which graph is being complemented, we use next H instead of G to denote a generic graph.

THEOREM 3. (GODSIL [**9**, THEOREM 3.2]; JONI AND ROTA [**12**, COROLLARY 2.1]) *If H is a spanning subgraph of $K_{n,n}$, then*

$$\Xi(H) = \int_0^\infty R_{\widetilde{H}}(t)e^{-t}\,dt.$$

The proof of Theorem 3 is beyond our scope, but we'll present two applications in the following sections; [**7**] presents a recent proof. Theorem 3 generalizes Proposition 2 because the bipartite complement of $K_{n,n}$ is the empty graph on $2n$ vertices; see Example 2. Further generalizations of Theorem 3 are discussed in [**10**, pp. 9–10].

Solution to Problem 1. As noted in Example 1, the integral in Problem 1 is

$$\int_0^\infty (t^3 - 6t^2 + 9t - 2)e^{-t}\,dt = \int_0^\infty R_G(t)e^{-t}\,dt, \tag{7}$$

where, recall, G is the graph depicted in FIGURE 2(c) and defined at the start of Example 1. Thus, to bring Theorem 3 to bear, it will suffice to determine a spanning subgraph H of $K_{3,3}$ such that $\widetilde{H} = G$. The graph H in FIGURE 4 does the trick. Now ask: how many perfect matchings are contained in H? The answer is obviously $\Xi(H) = 1$ because H is induced by the edges of a perfect matching. On the other hand, Theorem 3 tells us that $\Xi(H)$ coincides with (7) because $\widetilde{H} = G$. ■

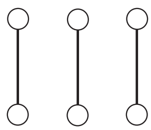


Figure 4 A graph H with $\widetilde{H} = G$ from FIGURE 2(c)

The fruit borne by the instantiation of Theorem 3 to the graphs in Examples 1 and 2 (respectively, a solution to Problem 1 and a proof of Proposition 2) might provide inspiration to consider this theorem in yet another instance, this time with $\widetilde{H}$ being the graph(s) in Example 3. This application of Theorem 3 takes us down an atypical path to a commonly studied class of combinatorial objects.

Derangements A *derangement* σ of a set S is a permutation of S with no fixed points; i.e., $\sigma : S \to S$ is a bijection such that $\sigma(x) \neq x$ for each $x \in S$. Counting the number of derangements of a finite set is a standard problem in introductory combinatorics and probability texts. We'll let $\mathcal{D}_n$ denote the set of derangements of $\{1, 2, \ldots, n\}$ and $d_n = |\mathcal{D}_n|$. We can easily determine these parameters for the smallest few values of n; TABLE 1 displays the results. We leave it as an exercise to show that $d_5 = 44$ and (for the punishment gluttons) $d_6 = 265$. But what is the pattern? Perhaps surprisingly, one way to obtain a general expression for d_n is to invoke Theorem 3.

Consider the bipartite graph G obtained from $K_{n,n}$ by removing the edges of a perfect matching; say, $G = K_{n,n} - \{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_n, y_n\}\}$. Notice that each perfect matching in G corresponds to exactly one derangement of $\{1, 2, \ldots, n\}$ and vice

TABLE 1: Derangement numbers and their corresponding derangements for $1 \le n \le 4$

n	d_n	$\mathcal{D}_n$
1	0	$\varnothing$
2	1	$\{21\}$
3	2	$\{231, 312\}$
4	9	$\{2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321\}$

versa. Thus, $d_n = \Xi(G)$. Since the bipartite complement of G is the graph considered in Example 3, Theorem 3 implies that

$$d_n = \int_0^\infty (t-1)^n e^{-t}\, dt. \tag{8}$$

If we separate the integral and change variables on the first subinterval, an evaluation of Γ presents itself:

$$\begin{aligned} d_n &= \int_1^\infty (t-1)^n e^{-t}\, dt + \int_0^1 (t-1)^n e^{-t}\, dt \\ &= \int_0^\infty x^n e^{-(x+1)}\, dx \; + \int_0^1 (t-1)^n e^{-t}\, dt \\ &= e^{-1}\Gamma(n+1) \qquad + E_n, \end{aligned} \tag{9}$$

where we now view the second integral as an error term E_n. It turns out that E_n doesn't contribute much to d_n; since $e^{-t} < 1$ on the interval (0, 1), we obtain

$$|E_n| \le \int_0^1 \left|(t-1)^n e^{-t}\right| dt < \int_0^1 (1-t)^n\, dt = \frac{1}{n+1}.$$

This shows that for each $n \ge 1$, the error satisfies $|E_n| < 1/2$, and it follows from (9) that d_n is the integer closest to $e^{-1}\Gamma(n+1)$, i.e., to $n!/e$.

Remarks The nonstandard derivation of d_n presented above is due to Godsil [**10**, pp. 8–9]. More typical approaches (e.g., [**6**, pp. 77–78] or [**13**, pp. 71, 109–110])—that apply either the principle of inclusion-exclusion or generating functions—lead to a perhaps more familiar expression

$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \tag{10}$$

for the derangement numbers. Starting from (8), this 'standard' expression (10) for d_n requires even less effort to derive than the former. We first apply the binomial theorem, obtaining

$$\begin{aligned} d_n &= \int_0^\infty \left(\sum_{k=0}^{n} \binom{n}{k} (-1)^k t^{n-k} \right) e^{-t}\, dt \\ &= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!\,(n-k)!} \int_0^\infty t^{n-k} e^{-t}\, dt, \end{aligned} \tag{11}$$

and then invoke the definition (2) of Γ to replace each integral by $(n-k)!$, after which (11) becomes (10). Alternately, via the MacLaurin series for $1/e$, (10) is easily seen to be equivalent to the 'integer closest to $n!/e$' description obtained via Godsil's derivation.

Counting perfect matchings in K_n

Since matching enumeration is not confined to the realm of bipartite graphs, it is natural to seek analogues of Proposition 2 and Theorem 3 for determining $\Xi(K_n)$ and, more generally, $\Xi(G)$ for a spanning subgraph G of K_n. Here again, we will expose the speciousness of this article's title.

A *matching* M in a graph $G = (V, E)$ is defined as it is in a bipartite graph, and, as before, if each $v \in V$ is an end of some $e \in M$, then M is called *perfect*. FIGURE 5 displays all of the perfect matchings admitted by K_4 and some of those admitted by K_6. The bracketed numbers in FIGURE 5(b) indicate how many different perfect matchings result under the action of successive rotation by 60°; in this way, all $15 = 2 + 3 + 6 + 3 + 1$ perfect matchings of K_6 are obtained.

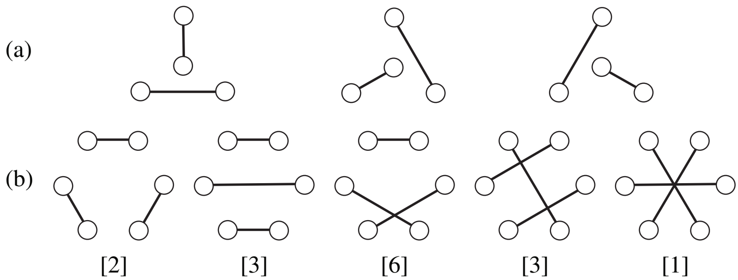


Figure 5 (a) All three perfect matchings in K_4; (b) five of fifteen perfect matchings in K_6

Following our earlier line of inquiry, we ask how many perfect matchings are contained in K_n. Since matchings pair off vertices, the question is interesting only when n is even; say $n = 2m$ for an integer $m \geq 1$. Let $V := V(K_{2m}) = \{1, 2, \ldots, 2m\}$. To determine a matching M, it is enough to decide, for each vertex $i \in V$, with which vertex i is paired under M. There are $(2m-1)$ choices for pairing with vertex 1. Having formed this pair, say $\{1, j\}$, it remains to decide how to pair the remaining $(2m-2)$ vertices. Selecting one of these, say k, there are $(2m-3)$ choices for pairing with vertex k, namely, any member of $V \setminus \{1, j, k\}$. Continuing in this fashion and applying the multiplication rule of counting, we find that

$$\Xi(K_n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (2m-1)(2m-3)\cdots 5\cdot 3\cdot 1 & \text{if } n = 2m \text{ for an integer } m \geq 1. \end{cases} \tag{12}$$

The last expression, reminiscent of a factorial, is sometimes called a *double factorial* which is defined, for a positive integer n, by $n!! := n(n-2)(n-4)\cdots(2 \text{ or } 1)$—see, e.g., [**20**]. This notation shortens (12) to

$$\Xi(K_n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (n-1)!! & \text{if } n \text{ is even.} \end{cases} \tag{13}$$

When n is even ($n = 2m$), we have

$$\Xi(K_{2m}) = (2m-1)!! = \frac{(2m)!}{2^m m!}, \tag{14}$$

which leads to an alternate way to count $\Xi(K_{2m})$: think of determining a matching by permuting the elements of V in a horizontal line (in $(2m)!$ ways) and then simply grouping the vertices into pairs from left to right. Of course, this over-counts $\Xi(K_{2m})$—by a factor of $m!$ since the resulting m matching edges are ordered, and by a factor of 2^m since each edge itself imposes one of two orders on its ends. After correcting for the over-counting, we arrive at (14) and thus have a second verification of (12).

As a final refutation of our title, we'll show that $\Xi(K_n)$ can also be expressed as an integral.

THEOREM 4. (GODSIL [**9**, THEOREM 1.2]; AZOR ET AL. [**3**, THEOREM 1])

$$\Xi(K_n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^n e^{-t^2/2}\, dt.$$

Proof. The right side of the identity is the moment $\mathcal{M}_n$. Since the integrand of each $\mathcal{M}_n$, for odd n, is an odd function, we have

$$\mathcal{M}_n = 0 \quad \text{whenever } n \text{ is odd.} \tag{15}$$

For even n, say $n = 2m$, we apply induction. Since $\mathcal{M}_0$ is the area under the curve for the probability density function of a standard normal random variable, we have

$$\mathcal{M}_0 = 1; \tag{16}$$

the proof of Lemma 6 below verifies this directly.

Fix an integer $m \geq 1$; starting with $\mathcal{M}_{2m-2}$ and integrating by parts yields the recurrence

$$\mathcal{M}_{2m} = (2m-1)\mathcal{M}_{2m-2} \quad \text{for } m \geq 1. \tag{17}$$

Now

$$\mathcal{M}_{2m} = (2m-1)!! \quad \text{for } m \geq 1 \tag{18}$$

follows easily from (16) and (17) by induction. Comparing (15) and (18) with (13) shows that Theorem 4 is proved. ∎

Just as Proposition 2 generalizes to Theorem 3, so too does Theorem 4 generalize. For a given (not necessarily bipartite) graph G (now with n vertices instead of the earlier $2n$ in the bipartite setting), the *matchings polynomial* is defined by $P_G(t) := \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r \mu_G(r) t^{n-2r}$. To determine $\Xi(G)$, we need to replace the factor t^n in the integrand of Theorem 4 by the matchings polynomial of the complementary graph $\overline{G}$ of G. We close this section by stating this analogue of Theorem 3 precisely.

THEOREM 5. (GODSIL [**9**, THEOREM 1.2]) *If G is a spanning subgraph of K_n, then*

$$\Xi(G) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} P_{\overline{G}}(t) e^{-t^2/2}\, dt.$$

A proof of Theorem 5 may be found in [**10**, p. 6].

Appendix

After establishing that the 0th moment $\mathcal{M}_0 = 1$ (which was needed in the proof of Theorem 4), we indicate how to obtain (4). Evaluating the integral in the definition of $\mathcal{M}_0$ is an enjoyable polar coordinates exercise.

LEMMA 6. $\int_{-\infty}^{\infty} e^{-u^2/2}\,du = \sqrt{2\pi}$.

Proof. Denoting the integral by $\mathfrak{J}$, we have

$$\begin{aligned}\mathfrak{J}^2 &= \left(\int_{-\infty}^{\infty} e^{-u^2/2}\,du\right)\left(\int_{-\infty}^{\infty} e^{-v^2/2}\,dv\right)\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(u^2+v^2)/2}\,du\,dv \qquad (19)\\ &= \int_{0}^{2\pi}\int_{0}^{\infty} r\,e^{-r^2/2}\,dr\,d\vartheta, \qquad (20)\end{aligned}$$

where we used Tonelli's Theorem to obtain (19) (see, e.g., [**21**, Theorem 6.10]) and a switch to polar coordinates to reach (20). Since the inner integral here is unity, the result follows. ■

Perhaps the simplicity of the preceding proof coloured the views of Lord Kelvin (1824–1907), as hinted in the following anecdote from [**19**, p. 1139]:

> Once when lecturing he used the word "mathematician," and then interrupting himself asked his class: "Do you know what a mathematician is?" Stepping to the blackboard he wrote upon it:—
>
> $$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$
>
> Then, putting his finger on what he had written, he turned to his class and said: "A mathematician is one to whom *that* is as obvious as that twice two makes four is to you. Liouville was a mathematician." Then he resumed his lecture.

At any rate, now the relation (4) is almost immediate:

COROLLARY 7. $\Gamma(1/2) = \sqrt{\pi}$.

Proof. By definition, $\Gamma(1/2) = \int_0^\infty t^{-1/2}e^{-t}\,dt$. On putting $t = u^2/2$, we find that $\Gamma(1/2) = \sqrt{2}\int_0^\infty e^{-u^2/2}\,du$, or, since the last integrand is an even function, $\Gamma(1/2) = \sqrt{2}\int_{-\infty}^{\infty} e^{-u^2/2}\,du/2$. Now Lemma 6 gives the value of this integral to confirm the assertion. ■

Concluding remarks

Proposition 2 and Theorem 4 present just two examples of combinatorially interesting sequences that can be expressed in the form $\int_\Omega t^n\,d\nu$ for some measure ν and space Ω. This topic is considered in detail in [**10**, Chapter 9].

What is one to make of these connections between integrals and enumeration? We don't claim that integrals provide the preferred lens for viewing these counting problems. For example, nobody would make the case that the integral in Theorem 4 is the 'right way' to determine $\Xi(K_n)$ because the explicit formula (12) provides a direct route. However, perhaps surprisingly, integrals do offer *one* lens. And this connection between the continuous and the discrete reveals just one of the myriad ways in which mathematics intimately links to itself. These links can benefit the mathematical branches at either of their ends. The application to counting derangements illustrates how continuous methods can shed light on a discrete problem, while Problem 1 and its solution indicate how a discrete viewpoint might yield a fresh approach to an essentially continuous question. This symbiotic relationship between the different branches of mathematics should inspire students (and their teachers) not to overly specialize. As in life, it's better to keep one's mind as open as possible.

Acknowledgments The author drafted this article during a sabbatical at Université de Montréal. *Merci à mon hôte généreux, Geňa Hahn, et au Département d'informatique et de recherche opérationnelle. Un merci particulier à André Lauzon, qui a apprivoisé mon ordinateur.* Thanks also to Karel Stroethoff who pointed me to the Lord Kelvin connection and whose post-colloquium observation led to an improvement of the derangements discussion.

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Summary To students just beginning their study of mathematics, the discipline appears to come in two distinct flavours: continuous and discrete. This article attempts to bridge the apparent divide by describing a surprising connection between these ostensible opposites. Various inhabitants from both worlds make appearances: rook

polynomials, Euler's gamma function, derangements, and the Gaussian density. Uncloaking combinatorial proof of an integral identity serves as a thread tying these notions together.

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Letter to the Editor

The sequence discussed in G. Minton's Note, "Three approaches to a sequence problem," in the February issue [**4**] is known as Perrin's sequence and has a long history. (Perrin's sequence is defined by $x_1 = 0$, $x_2 = 2$, $x_3 = 3$, and $x_n = x_{n-2} + x_{n-3}$ for $n \geq 4$.) An important question is: Is an integer prime if and only if it satisfies the Perrin condition, n divides x_n? This question was raised by R. Perrin in 1899. A counterexample, now known as a *Perrin pseudoprime,* was not discovered until 1982: the smallest one is 271441. This is quite remarkable compared to, say, Fermat pseudoprimes with base 2, for which 341 is the smallest example. Recent work by J. Grantham [**3**] shows that there are infinitely many Perrin pseudoprimes. One can run the Perrin recurrence backward and verify that if p is prime then x_{-p} is divisible by p. When the Perrin condition is enhanced by this additional condition, then the first composite that satisfies both congruences, called a symmetric Perrin pseudoprime, is 27664033. For more information, see the references listed below.

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