A pedagogical history of compactness

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> Modern mathematics tends to obliterate history: each new school rewrites the foundations of its subject in its own language, which makes for fine logic but poor pedagogy.
>
> *R. Hartshorne*

1   Why study the history of compactness?

Compactness has come to be one of the most important and useful notions in advanced mathematics. It can also be seen as a kind of a gate-keeper topic: if an undergraduate mathematics student does not understand compactness, whatever it really means to understand, it is unlikely that he or she will be able to do higher level mathematics. However, for whatever reasons, when we teach compactness (and other topics of similar importance) we tend to do so without motivation, leaving students on their own to figure out, if they ever do, how various definitions and theorems relate to each other and why they take the specific forms they do.

This paper is an attempt to fill in some of the information that the standard textbook treatment of compactness leaves out. It is not a historical article, per se, but a synthesis of historical documents with an eye towards clarifying the main ideas related to compactness. In particular, the paper discusses the origins and development of both open-cover and sequential compactness, how and why open-cover compactness came to be favored, and some modern developments involving nets and filters.

A list of terms related to compactness is given in the Appendix. Since the terms have changed names at various points in history, the list can be useful for keeping the concepts straight. In the main text we will use a combination of

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historical and modern formulations of the main definitions, lemmas, and theorems (choosing the ones best for readability, with originals in footnotes.)

2 Possible motivations for compactness

Compactness grew out of one of the most productive periods of mathematical activity. In middle to late nineteenth century Europe, advanced mathematics began to take the form in which we know it today. In the background was Cantor's work establishing the beginning of a systematic study of set theory and point-set topology (though Cantor himself turned his interests to transfinite sets. The significance of Cantor's work for topology should be credited to Poincaré[1]). Also, many mathematicians—including Weierstrass, Hausdorff and Dedekind—were worried about the foundations of mathematics and began to make rigorous many of the ideas that had for centuries been taken for granted. While some of the nineteenth century work can be traced to mathematical concerns of the early Greeks, the level of rigor and abstraction reflects a revolution in mathematical thought.

It is in this context that we will discuss some specific problems that appear to have motivated the concept of compactness. In particular, we will discuss the influence of the study of properties of closed, bounded intervals of real numbers (which I will denote $[a, b]$), spaces of continuous functions, and solutions to differential equations.

2.1 Properties of $[a, b]$

In mid to late nineteenth century, mathematicians began to really understand and specify essential properties of the real line. There were essentially two characterizations that were developed during this time. One characterization, developed by Bolzano and Weierstrass among others, grew out of the study of functions defined on sequences of real numbers. The other characterization, which grew out of work by Heine, Borel, and Lebesgue, was based on topological features, such as the covering of sets by open neighborhoods. We will examine both of these characterizations in more detail.

The origin of sequential compactness is often traced to a theorem, proved rigorously by Weierstrass in 1877, which concerns the behavior of continuous functions defined on closed, bounded intervals of the real line.[2] The following statement of the theorem comes from Fréchet, who referred to this theorem as a result of Weierstrass.

Theorem 2.1 (Weierstrass) *Each function continuous in a limited [equivalent to modern-day "closed and bounded"] interval attains there at least once its*

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[1]Thanks to Jeremy Gray for this comment.

[2]This date refers to one of the earliest publications of the theorem, see [17]. However it is likely that Weierstrass actually proved it years before and disseminated it orally, via lectures, perhaps ten years earlier.
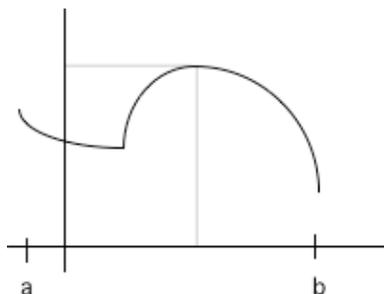
maximum.[3]

Figure 1: A continuous function on $[a, b]$.

Fréchet, who defined sequential compactness in his 1906 thesis, said that his definition came from his desire to generalize this theorem to abstract topological spaces [38, p. 244]. Weierstrass's theorem owes its essential ideas to Bolzano in 1817, who working in relative isolation, both politically and mathematically, in Bohemia, remarkably stated and proved the following:

Lemma 2.2 (Bolzano) *If a property M does not apply to all values of a variable quantity x, but to all those that are* smaller *than a certain u, there is always a quantity U which is the greatest of those of which it can be asserted that all smaller x possess the property M.*[4]

This lemma, today called the greatest lower bound property for real numbers, was somewhat of a breakthrough in the conceptualization of real numbers. The proof of this lemma provided the first real account of the limiting process, and was used to prove what we now call the Intermediate Value Theorem: if f is continuous on $[a, b]$ with $f(a) < 0$ and $f(b) > 0$, then for some x between a and b, $f(x)$ will be exactly 0.

The idea behind Bolzano's proof of the lemma was to use interval bisection, that is to narrow in on the least upper bound by throwing away points of the set that were below it. This iterative process was essentially the same process used in Weierstrass's proof of the maximum value theorem (see [29, p.953]). In particular, Bolzano's lemma allowed Weierstrass to prove that every bounded infinite set of real numbers has a limit point. It is this property that Fréchet used

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[3]From the original French: Weierstrass a en effete démontré que toute fonction continué dans un intervalle limité y atteint au moins une fois son maximum. [7, p. 848]

[4]From the original German: Wenn eine Eigenschaft M nicht allen Werthen einer veränderlichen Grösse x, wohl aber allen, die kleiner sind, als ein gewisser u, zukömmt: so gibt es allemahl eine Grösse U, welche die grösste derjenigen ist, von denen behauptet werden kann, dass alle kleineren x die Eigenschaft M besitzen. [4, p. 41]

when he generalized Weierstrass's theorem to abstract spaces. We now know this property as the *Bolzano-Weierstrass property*, or *limit-point compactness*.

While Bolzano and Weierstrass were trying to characterize properties of the real line in terms of sequences, other mathematicians, such as Borel and Lebesgue, were trying to characterize it in terms of open covers. Borel proved the following lemma in his 1894 thesis:

Lemma 2.3 (Borel) *If on a line one has an infinite number of subintervals, such that every point of the line is interior to at least one of the intervals then one can determine effectively a bounded number of intervals from among the given intervals that have the same property (every point of the line is interior to at least one of them.)*[5]

Here a line means a bounded interval. It turns out that Borel's approach was similar to the approach Heine used in 1872 to prove that a continuous function on a closed interval was uniformly continuous [10, p. 188]. This theorem was first proven by Dirichlet in his lectures of 1852, with a more explicit use of coverings and subcoverings than in Heine's theorem [23, p. 91]. However Dirichlet's notes were not published until 1904, which might explain why he does not get credit for the generalized version of the Borel lemma (now referred to as Borel Theorem). The reason that Heine's name is attached to the theorem is that Schönflies, a student of Weierstrass, noticed the connection between Heine's work and Borel's [37, p. 51]. The generalized theorem, which is now commonly called the Heine-Borel theorem,[6] with modern language and notation, is:

Theorem 2.4 (Heine-Borel) *A subset of $\mathbb{R}$ is compact iff it is closed and bounded.*

While Heine is credited with a theorem he did not prove, it appears that Cousin was largely overlooked for a lemma he did prove. In 1895, he generalized the Borel lemma to arbitrary covers. The following is often referred to as Cousin's Theorem, but it appears in the original as a lemma. The plane YOX below is just $\mathbb{R}^2$, and the region S would, in today's language, be described as closed and bounded.

Lemma 2.5 (Cousin) *In the plane YOX let S be a connected area bounded by a closed contour, simple or complex; one supposes that at each point of S or its perimeter there is a circle, of non-zero radius, having this point as its centre; it is then always possible to subdivide S into regions, finite in number and*

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[5]From the original French: Si l'on a sur une droite une infinitè d'intervalles partiels, tels que tout point de la droite soit intérieur à l'un au moins des intervalles, on peut déterminer effectivement un nombre limité d'intervalles choisis parmi les intervalles donnés et ayant la même propriété (tout point de la droite est intérieur à au moins l'un d'eux. [5, p. 51]

[6]For an accessible history of this theorem, along with discussion of a number of different original statements of this theorem, see [18].

sufficiently small for each one of them to be entirely inside a circle corresponding to a suitably chosen point in S or on its perimeter.[7]

In other words, if for every point of a closed, bounded region, there corresponds a finite neighborhood, then the region can be divided into a finite number of subregions such that each subregion is contained in a circle having its center in the subregion.[8] Cousin's theorem is generally attributed to Lebesgue, who was said to be aware of the result in 1898 and published his proof in 1904 [11][9]. The Lebesgue lemma is considered itself to be an important consequence of compactness.

While there is some debate over who was really responsible for the ideas and proofs, the idea that any closed, bounded subset of $\mathbb{R}$ has the open-cover property (sometimes called the Borel-Lebesgue property) was known when Fréchet first defined compactness formally.

2.2   Spaces of continuous functions

A second motivation for the notion of compactness was the study of abstract topological spaces such as spaces of continuous functions,[10] $C^0[a, b]$. In $C^0[a, b]$, points are functions (whereas in $[a, b]$ points are real numbers). The properties of $[a, b]$ alone might not have been seen as important to generalize if it weren't the case that these properties seemed to be important in more abstract spaces as well. However, it turned out that infinite dimensional spaces like $C^0[a, b]$ were not as well-behaved as finite dimensional spaces like $\mathbb{R}$. For instance, closed, bounded subsets of continuous functions on $\mathbb{R}$ do not necessarily have the Bolzano-Weierstrass or open-cover property. The work in this area was done by Ascoli and Arzelà in the last decades of the 1800's.

The following example illustrates that a closed, bounded subset of continuous functions on $\mathbb{R}$ is not, in our modern language, sequentially compact. Consider B, the set of continuous functions, f, defined on $[0, 1]$ with $\|f\| \leq 1$. (This is the closed unit ball in $C^0[a, b]$ and $\| \|$ is the sup norm.) We will show that there is a sequence in B that does not have a convergent subsequence. Let $f_n(x) = x^n$. This sequence lies in B, but we cannot find a subsequence that converges to a function in $C^0[a, b]$. Suppose to the contrary f is such a function. Then

$$f(x) = \lim_{k \to \infty} f_{n_k}(x)$$

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[7]From the original French: Soit, sur le plan YOX, une aire connexe S limitée par un contour fermé simple ou complexe; on suppose qu'à chaque point de S ou de son périmètre correspond un cercle, de rayon non nul, ayant ce point pour centre: il est alors toujours possible de subdiviser S en régions, en nombre fini et assez petites pour que chacune d'elles soit complètement intérieure au cercle correspondant à un point convenablement choisi dans S ou sur son périmètre. [6, p. 22]

[8]Note that the original definition was formulated without the term 'neighborhood,' or 'voisinage' in French. The idea of neighborhood was obviously around during Cousin's time, but was not used consistently. Formal definitions of the term can be found in [8] and [9]).

[9]Cited in [27, p. 29].

[10]We could just as well take C^0 on any set Ω.

which would imply that

$$f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases}.$$

Since f is a discontinuous function, it is not in $C^0[a,b]$. Hence the sequence $f_n(x)$ has no convergent subsequence.

The problem in this example comes from how the functions converge. If convergence means pointwise convergence, we do not get behavior analogous to that of, say, sequences in closed unit balls of $\mathbb{R}$. In order to avoid this problem, Ascoli introduced the notion of equicontinuity [2, p. 566].[11] Equicontinuity requires functions to converge to a limit all at once instead of pointwise. A set E is *equicontinuous* iff for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|s - t| < \delta$ and $f \in E$ imply $|f(s) - f(t)| < \epsilon$.

The Arzelà-Ascoli theorem, in modern language, then states:

Theorem 2.6 (Arzelà-Ascoli) *Any bounded equicontinuous sequence of functions in $C^0[a,b]$ has a uniformly convergent subsequence.*[12]

Using modern terminology we can state a consequence of this theorem, analogous to the Heine-Borel theorem:

Theorem 2.7 *A subset of $C^0[a,b]$ is compact iff it is closed, bounded and equicontinuous.*

Ascoli proved the sufficiency of this condition in 1884 [2, p.567] and Arzelà the necessity in 1889 [1, p. 345] (with a clearer proof presented in 1894 [3, p. 226]). This generalization of Bolzano-Weierstrass's theorem (although not stated in terms of compactness) was apparently well known after 1880. Moreover, Hilbert seems to have discovered this "compactness" property independently and published it in 1900 [21, p. 82]. It is unclear whether Arzelà and Ascoli themselves were aware of how their work was connected with compactness, but Fréchet's work was influenced by theirs [38, p. 255].

2.3 Solutions to differential equations

A third motivation for the notion of compactness came from the desire to find solutions to differential equations. Peano, a contemporary of Arzelà and Ascoli as well as a fellow Italian, realized that the Arzelà-Ascoli theorem might be

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[11]See also [19, p. 27].

[12]From the original Italian: La condizione necessaria e sufficiente affinché una successione data di funzioni $u_1(x), u_2(x), ...u_n(x), ...$ abbia una funzione limite, nel senso detto sopra, é che, preso un numero positivo σ piccolo a piacere, si possa sempre determinare un numero intero corrispondente m_σ tale che per ogni x, nell'intervallo $a...b$, si abbia qualunque sia p intero positivo, $| u_{m_\sigma(x)} - u_{m_\sigma + p}(x) | < \sigma$. Translated to English: The necessary and sufficient conditions that a given sequence of functions $u_1(x), u_2(x), ...u_n(x), ...$ defined on an interval $a...b$ may converge to a limiting function is that, given an arbitrarily small positive number σ there can always be determined a corresponding integer m_σ such that for all values of x in the interval $a...b$, and for all positive integers p, $| u_{m_\sigma(x)} - u_{m_\sigma + p}(x) | < \sigma$ [3, p. 226].

useful for demonstrating the existence of such solutions. He searched for solutions by making a sequence of approximations. He then used what we now call compactness to show that there will be a subsequence that converges to a limit (which will be the solution to the differential equation).

To this end, Peano proved the following theorem in 1890:

Theorem 2.8 (Peano) *Suppose we are given a system of differential equations in normal form:*

$$\frac{dx_1}{dt} = \varphi_1(t, x_1, \ldots, x_n),$$
$$\ldots\ldots$$
$$\frac{dx_n}{dt} = \varphi_n(t, x_1, \ldots, x_n)$$

where the functions $\varphi_1, \ldots, \varphi_n$ are continuous in a neighbourhood of $(b, a_1, \ldots, a_n)$. [Then there exists] an interval (b, b'), and n functions of t from this interval, $x_1, \ldots, x_n$, satisfying our system of equations and evaluating to $a_1, \ldots, a_n$ respectively at $t = b$.[13]

While it is not clear if Fréchet was aware of this application, it does seem to be the case that applications for the notion of compactness were known before the term was formally defined.

3 Developing the definition

We will trace below the development of the two central notions of compactness discussed above, those stemming from sequences and open covers of real numbers. Again, it is useful to know something about the climate of the mathematics community at the time of these historical developments. We will focus on the contributions of only a few central people, but there was actually a large community of mathematicians who were developing ideas that are now the foundations for analysis and topology. Many of these mathematicians were in very close contact with each other, so it is difficult to tease apart their contributions. Among them, in France, were Hadamard, Lebesgue, and Fréchet; in Russia, Alexandroff[14] and Urysohn; in Germany, Hausdorff, Hilbert, Schönflies, and Cantor; in Hungary, F. Riesz; in the Netherlands, Brouwer; and in the U.S., Chittenden, Hedrick, and Moore.

We will start with the work of Fréchet, who coined the term "compact" and gave definitions for what we now know as countable and sequential compactness. We will then briefly discuss contributions by Alexandroff and Urysohn

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[13]From the original French: Soit donné le système d'équations différentielles, ramené à la forme normale: $\frac{dx_1}{dt} = \varphi_1(t, x_1, ..., x_n), ... \frac{dx_n}{dt} = \varphi_n(t, x_1, ..., x_n)$, où les $\varphi_1, ..., \varphi_n$ sont des fonctions continues aux environs de $t = b, x_1 = a_1, ..., x_n = a_n$. [Alors il existe] un intervalle (b, b'), et, dans cet intervalle, n fonctions $x1 \ldots x_n$ de t, qui satisfont aux équations données, et qui, pour $t = b$, prennent les values $a_1 \ldots a_n$. [13, p.182]

[14]I will use the spelling Alexandroff, rather than the sometimes used Alexandrov, since that was the spelling he preferred.

who developed and stated what we now call open-cover compactness, or simply compactness. We will show why open-cover and sequential compactness are not equivalent in abstract topological spaces, providing motivation for a formulation of compactness in terms of nets and filters which is analogous to sequential compactness.

3.1 Fréchet: Countable and limit-point compactness

While Fréchet was influenced by many contemporaries and predecessors, it seems he deserves credit as the father of compactness. It was Fréchet who gave the concept a name, in a paper [7] leading to his 1906 doctoral thesis. Fréchet also defined metric spaces for the first time, though not using that term, and made inroads into functional analysis, thus providing a context for which the importance of compactness became clear.

Fréchet was a mathematician of big ideas. He preferred definitions that had an intuitive feel rather than analytic power. This preference can be seen in [7, p. 849] in which he defined a notion of compactness, introducing first a definition of what we now call countable compactness, using nested intersections, before introducing a characterisation using limit points.

In Fréchet's thesis, he considered three kinds of spaces, which he called L-class, V-class, and E-class. L-classes were the most general, in which a notion of sequential compactness was defined. E-classes, which we now call metric spaces, and V-classes,[15] a metric space with a weak version of the triangle inequality, were less general, but easier to work with. The goal was to define compactness for L-classes, but this turned out unsuccessful because sequential compactness did not have all the properties needed to generalize to abstract topological spaces (more on this in section 3.2). Fréchet focused instead on the V- and E-classes, in which notions of modern-day compactness and sequential or limit-point compactness were equivalent. The following definition was given for E-classes.

Definition 3.1 *A set E is called compact if, whenever E_n is a sequence of nonempty, closed subsets of E such that E_{n+1} is a subset of E_n for each n, there is at least one element that belongs to all of the E_n's.*[16]

The exact nature of Fréchet's intuition for this definition is unclear, but there might be two features of compact sets that he wanted to capture. The first is a sense of boundedness. The nested intersection property allows us to easily rule out sets that have tails running to ∞. For instance, we can show that $\mathbb{R}$ is not compact. Let $E_n = [n, \infty)$. Each E_n is closed because it contains the point n,

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[15]The letter V comes from the French *voisinage* meaning neighborhood. L stands for *limite*, or limit. E stands for *écart*, or gap, as in the non-zero distance between two points in a metric space.

[16]From the original French: Si on considère une suite d'ensembles $E_1, E_2, ..., E_n, ...$ formés d'éléments d'un même ensemble compact E, chacun fermé, contenu dans le précédent et possédent au moins un élément, il y a nécessairement un élément commun à tous ces ensembles. [8, p.7]

and clearly $E_{n+1} \subset E_n$. However, the infinite intersection of these intervals is empty, so $\mathbb{R}$ is not compact.
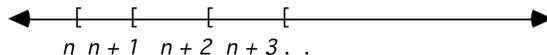

Figure 2: Nested tails.

A second feature the nested intersection definition allows us to quickly see is that sets that have "holes" are not compact. For instance, we can see that $X = [a, b]$ is compact and $Y = [a, b) \cup (b, c]$ is not.[17] In the latter case, consider $E_i = [a_i, b) \cup (b, c_i]$ where $a_{i+1} > a_i$ and $a_i \to b$, $c_{i+1} < c_i$, and $c_i \to b$. These sets are clearly nested and are closed in Y, but the infinite intersection of those intervals is empty. Hence Y is not compact.

Figure 3: Nested intersections.

While Fréchet preferred his intuitive definition involving nested intersections [35, p. 430], he realized the importance of also providing a more useful, if less intuitive, definition. Below is another definition from Fréchet, which uses the Bolzano-Weierstrass property. This definition applies to V- and E-classes where limit-point, countable, and sequential compactness are equivalent. Note that for Fréchet, a compact set need not be closed. So for his subsequent definitions and theorems, he often needed to require that a set be both compact and closed.

Definition 3.2 *We will say that a set is [relatively limit-point] compact if it contains only a finite number of points or if every one of its infinite subsets gives rise to at least one limit point.*[18]

Since Fréchet did not require that a compact set be closed, he defined the notion of an *extremal set*, which is closer to our modern day notion of compact.

Definition 3.3 *We shall call a set that is both closed and compact an extremal set; this nomenclature is justified later on. Within abstract set theory, extremal sets play a role akin to that of intervals in the theory of subsets of the real line.*

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[17]Of course one could construct a similar example *with* a hole that is compact such as $Z = [a, b] \cup [c, d]$, but the example illustrates what could have been the intuition behind the definition.

[18]From the original French: Nous dirons qu'un ensemble est *compact* lorsqu'il ne comprend qu'un nombre fini d'éléments ou lorsque toute infinité de ses éléments donne lieu à au moins un élément limite [8, p. 6].

We might not know exactly why Fréchet chose the word "compact", but we have some evidence of why he did not choose some other terms. When Fréchet first introduced the term, some mathematicians did not like his choice. For instance, Schönflies suggested that what Fréchet called compact be called something like "lückenlos" (without gaps, closer to the modern notion of completeness) or "abschliessbar" (closable) [38, p. 266]. Even to mature mathematicians the precise intuitions behind the term compactness was not yet clear.

Despite all of Fréchet's early concern with intuitive definitions and choice of terminology, it is surprising that at the end of his life, he could not remember why he chose the term:

> Doubtless I wanted to avoid a solid dense core with a single thread going off to infinity being called compact. This is a hypothesis because I have completely forgotten the reasons for my choice! [35, p. 440].[19]

So even in the lifetime of the mathematician who named the concept, the original intuition behind the concept was somewhat lost, and Fréchet's intuitive nested intersection definition was supplanted by less intuitive but more powerful notions of limit-point, sequential, and open-cover compactness.

3.2   Hausdorff: Compactness on metric spaces

One of the obstacles to defining compactness, as we know it today, was to define it in a way that would work for abstract topological spaces. This was a problem for Fréchet, and in the end he had to restrict his definition to V and E class, leaving open the question of defining compactness for L-classes, the abstract topological spaces. In 1914, Hausdorff made progress on this problem, introducing what we now call Hausdorff spaces, in which distinct points have distinct neighborhoods. In [9] he introduced the term metric spaces (E spaces to Fréchet), and he defined a set E to be compact if every infinite subset of E has a limit point in E, where limit point in this context means that every neighborhood of the point contains infinitely many elements of E. Hausdorff's notion of compactness, which we would call limit-point compactness and is equivalent to countable compactness for Hausdorff spaces, remained the standard notion of compactness throughout the rapid development of point-set topology in the 1920s.[20]

3.3   Alexandroff and Urysohn: Open-cover compactness

While Fréchet was the first to formally define compactness, his contemporaries in Russia, Alexandroff and Urysohn, appear to be the first to state it in its most general form (in the context of abstract topological spaces). It is perhaps for

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[19]From the original French: ... j'ai voulu sans doute éviter qu'on puisse appeler compact un noyau solide dense qui n'est agrémenté que d'un fil allant jusqu'à l'infini. C'est une supposition car j'ai complètement oublié les raisons de mon choix!

[20]Thanks to an anonymous reviewer for this information and formulation.

this reason that the two Russians are often credited with defining the notion (e.g. [31, p. 425S]). Alexandroff and Urysohn were actually in close contact with Fréchet [39, pp. 319-357].[21]

Also, though Alexandroff and Urysohn usually get credit for defining open-cover compactness, Fréchet was not unaware of the possibility of using neighborhoods to characterize compactness. In a correspondence in 1905, Hadamard, Fréchet's advisor, suggested that he think in terms of neighborhoods to generalize the properties of to abstract topological spaces.[22] The first definition that Fréchet gave, in terms of nested intersections, is the dual of (and hence logically equivalent to) countable open-cover compactness.

3.4 Open-cover vs. limit-point compactness

Though Fréchet may have been motivated originally to define compactness for abstract topological spaces, he in fact restricted himself to metric spaces. His approach of looking at sequences and limits was not as general as the approach of using open covers, which resulted in what we now take to be the correct definition of compactness. Here we look at examples which illustrate why sequential compactness and open-cover compactness are not equivalent. We will use the concept of the least upper bound property, namely that any non-empty set containing an upper bound necessarily has a least upper bound.

3.4.1 Sequentially compact does not imply compact

Consider $S_\Omega = \{\alpha \,|\, \alpha$ is an ordinal number and $\alpha < \Omega\}$ with the order topology, where Ω is the first uncountable ordinal number. (See diagram below. The first *infinite* ordinal, ω, is the first ordinal after "exhausting" the natural numbers. The first *uncountable* ordinal, Ω, is the ordinal after "exhausting" the countable ordinals.)
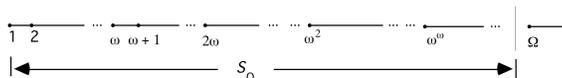

Figure 4: Representation of S_Ω.

We know that all closed subsets of compact sets are compact (and all compact sets are closed). So S_Ω is not compact since it is not closed in the compact set $S_\Omega \cup \{\Omega\}$.

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[21]Urysohn died, tragically, at the age of 26 in a swimming accident off the coast of France. Much of his work was published posthumously by Alexandroff, who kept up his correspondence with Fréchet after Urysohn's untimely death.

[22]See [30, p. 212]. The quotation comes from an undated letter from Hadamard to Fréchet and can be found in full in [38, p. 246]).

However, it turns out that S_Ω is limit-point compact. To see why this is true, we will use the fact that any countable subset of S_Ω has an upper bound in S_Ω. If we take any infinite subset of S_Ω, it has a countably infinite subset, which we will call X. Since X is countable, it has an upper bound, let's call it b, in S_Ω. But the interval $[1, b]$ is compact since S_Ω has the l.u.b. property. So there must be a point in $[1, b]$ which is a limit point (of both X and any set containing it). Thus, S_Ω is limit-point compact. Essentially the same argument shows that any sequence in S_Ω must have a convergent subsequence in S_Ω, so S_Ω is sequentially compact.

3.4.2   Compact does not imply sequentially compact

Just as we can have a space that is compact but not sequentially compact, we can also have a space that is sequentially compact but not compact.[23] Consider the set of all functions from the interval $[0, 1]$ to itself with the topology of point-wise convergence. This can be thought of as the infinite product $[0, 1]^{[0,1]}$ with the product topology, which is compact by Tychonoff's theorem.[24] However, if $f_n(x)$ is the nth digit in the base-2 decimal expansion of x (using the expansion that terminates in 0's if x is a dyadic rational), the sequence f_n, which is a sequence in the set of all functions from $[0, 1]$ to itself, has no pointwise convergent subsequence. It does have convergent subnets,[25] a concept that will be defined in the next section, but not proper convergent sequences.
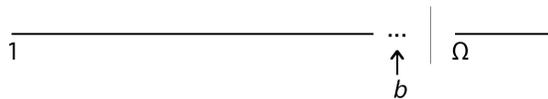

Figure 5: Illustration of b in S_Ω.

4   Nets and Filters

In the previous section we saw that the two important properties of compactness, those stemming from the Bolzano-Weierstrass property (sequential compactness) and the Borel-Lebesgue property (open-cover compactness), are not equivalent in abstract topological spaces. Open-cover compactness is more general and applicable, and for these reasons is considered to be compactness (and

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[23]Thanks to an anonymous reviewer for this example.

[24]The product of compact topological spaces is compact, or in German, Das Produkt von bikompakten Räumen ist wieder bikompakt, originally proved in [16, p. 772], though the theorem is sometimes credited to Čech [24].

[25]For example consider all functions which map $[0, 1]$ to the set $\{0, 1\}$. This will be a limit point for some subnets.

hence bears its name). However, it is possible to define open-cover compactness in a way that is analogous to sequential compactness, using modern notions of nets and filters, which we will develop here.[26] These two concepts are very different, on the surface, but they give rise to the same notion of convergence in abstract topological spaces.

4.1 Moore and Smith: Nets

The theory of nets was developed by E. H. Moore and his student H. L. Smith.[27] It is unclear whether Moore and Smith knew how nets could be used to define compactness. This connection is usually credited to Birkhoff [28, p. 64], who applied Moore-Smith theory to general topological spaces, but Moore and Smith did generalize some of Fréchet's compactness results in the same paper in which they defined nets [12, p. 118]. Our goal here is to express compactness in terms of nets, so we will use the S_Ω example to motivate and illustrate net compactness.

The problem in the S_Ω example is that while Ω is a limit point of S_Ω (any neighborhood of Ω contains points of S_Ω), no sequence in S_Ω converges to Ω [28, p. 76]. If we are limited to taking a countable number of elements in the sequence, we will never reach Ω. Nets provide one way of getting around this problem by allowing us to have something like uncountable sequences. In our discussion of both nets and filters, we will consider only topological spaces, on which the notion of neighborhood is defined.[28]

To see how nets are a generalization of sequences, it is useful to think of sequences as functions on the natural numbers.

Definition 4.1 *A* sequence, *(denoted $\{x_n\}_{n\in\mathbb{N}} = \{x_1, x_2, x_3, \ldots\}$) is a function which assigns to each element n of the natural numbers, $\mathbb{N}$, a functional value x_n in a set X.*

We would like to replace $\mathbb{N}$ with a set that can be uncountable but has an ordering similar to that of $\mathbb{N}$. In other words, we want to stipulate conditions for an ordering relation on a generic set that generalizes the way $>$ orders natural numbers. We will call this relation $\succ$ to suggest the connection to $>$, and we will say that this relation "directs" a given set.

Definition 4.2 *A non-empty set D, with the relation $\succ$ is called* directed *iff*
  (i) *if $d_1, d_2, d_3 \in D$ such that $d_1 \succ d_2$ and $d_2 \succ d_3$ then $d_1 \succ d_3$;*
  (ii) *if $d_1, d_2 \in D$, then there is a $d_3 \in D$ such that $d_3 \succ d_1$ and $d_3 \succ d_2$.*

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[26] In this section we are more interested in the concepts and not the historical development, so we will be less careful than in earlier sections about giving original definitions and theorems, though we provide references for anyone who wants to track down the original formulations.

[27] Little biographical information about Smith is available. He received his Ph.D. from University of Chicago under Moore and got a job at Louisiana State University, but apparently after his important work on nets and filters, he dropped into obscurity in 1922 [25, p. 563].

[28] This treatment follows [28, pp. 62-70] and [32, pp. 281-283, 286-289].

So the definition of a net is simply the definition of a sequence with $\mathbb{N}$ replaced by the notion of a directed set. From now on, D will stand for a directed set with the relation $\succ$ as defined above.

Definition 4.3 *A net (denoted $\{x_d\}_{d \in D}$ or simply $\{x_d\}$) is a function which assigns to each element d of a directed set D a functional value x_d in a set X.*

Once we know what a net is, we can state what it means for it to converge. Again we can derive the definition for net convergence and limit point by taking the definitions involving sequences and simply replacing $\mathbb{N}$ and $>$ with D and $\succ$.

Definition 4.4 *A net $\{x_d\}$ converges to $a \in X$ (denoted $\{x_d\} \to a$) iff for every neighborhood U of a, there is an index $d_0 \in D$ such that if $d \succ d_0$ then $x_d \in U$ (i.e. if the net is eventually in each neighborhood of a).*

Definition 4.5 *A point a is a limit point of $\{x_d\}$ if for every neighborhood U of a and every $d_0 \in D$ there is a $d \succ d_0$ such that $x_d \in U$.*

In order to state compactness in terms of nets, we also need the concept of subnet, the analog of subsequence.[29] Part of the definition of subsequence generalizes easily, but the other part requires us to think about subsequences in a slightly different way than we are accustomed. The first defining property of subsequence is that each element of the subsequence can be identified with an element of the sequence. This property is generalized in (i) below. The second defining property requires that the subsequence is ordered in a similar way as the sequence. Usually we require the indices of the subsequence, like the indices of the sequence, to be strictly increasing. In other words, for a subsequence $\{x_{n_k}\}$ of a sequence $\{x_n\}$, the n_k are positive integers such that $n_1 < n_2 < n_3 \cdots$. But the feature of this condition which turns out to be important is simply the fact that as $k \to \infty$, so do the n_k. This property is generalized in (ii) below.

Definition 4.6 *A subnet of a net $\{x_d\}_{d \in D}$ is a net $\{y_b\}_{b \in B}$ where B is a directed set and there is a function $\varphi : B \to D$ such that:*
(i) $y_b = x_{\varphi(b)}$ and
(ii) $\forall d \in D, \exists b_0 \in B$ such that if $b \succ b_0$ then $\varphi(b) \succ d$.

We are now ready to characterize compactness in terms of nets.

Theorem 4.7 *A topological space X is compact iff either ...*
- *Every net of points of X has a limit point in X, or*
- *Every net of points of X has a convergent subnet in X.*

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[29]Incidentally, Kelley, who first coined the term "net" had considered using the term "way" so the analog of subsequence would be "subway." McShane also proposed the term "stream" for net since he thought it was intuitive to think of the relation of the directed set as "being downstream from" [32, p. 282].

Notice that these definitions are precisely the same as limit point and sequential compactness for metric spaces with the term "net" substituted for "sequence."

Applying these definitions to the S_Ω example, we can show why S_Ω is not compact. If we take a net $\{x_d\}$ of elements of S_Ω, it is no longer the case that there will necessarily be a limit in S_Ω. In particular, let $D = S_\Omega$ and $x_d = d$. Then $\{x_d\}$ converges to Ω, which is not in S_Ω. Thus no subnet of $\{x_d\}$ will converge to a point in S_Ω.

4.2   Cartan (and Smith): Filters

Nets are not the only way of generalizing sequences. Another generalization of sequence is a filter, a notion suggested by Cartan in 1937.[30] While different from a net, both nets and filters give rise to the same notion of convergence on topological spaces. That is to say, on abstract topological spaces, they are essentially the same.[31] Nonetheless, some mathematicians find nets more intuitively appealing and useful, while others prefer filters.

The idea behind filters was foreshadowed by Riesz [14] in 1907 when he provided axioms for topology based on limit points instead of metrics, though his topological axioms are not equivalent to the standard ones we use today, and his work did not result in a fruitful line of research. Riesz defines a concept called an "ideal" which is essentially the same as what we now call an ultrafilter. Smith independently discovered filters as an attempt to explain what was lacking in the theory of nets that he and Moore proposed.

Again, we will define the notions we need to state compactness in terms of filters and then apply our compactness result to show S_Ω is not compact. As with nets, we can look at convergence of sequences to motivate the idea of convergence of filters. However, whereas with nets the focus was on the index set, with filters the focus is on neighborhoods.[32]

Definition 4.8 *Let X be a set. A set Φ of subsets of X is called a* filter *iff*
(i) $\emptyset \notin \Phi$
(ii) $A_1 \subset A_2 \subset X$ and $A_1 \in \Phi \Rightarrow A_2 \in \Phi$
(iii) $A_1, A_2 \in \Phi \Rightarrow A_1 \cap A_2 \in \Phi$

As with nets, we should define what it means for a filter to converge:

Definition 4.9 *A filter Φ converges to $a \in A$ (denoted $\Phi \to a$) iff each neighborhood of a is a member of Φ.*
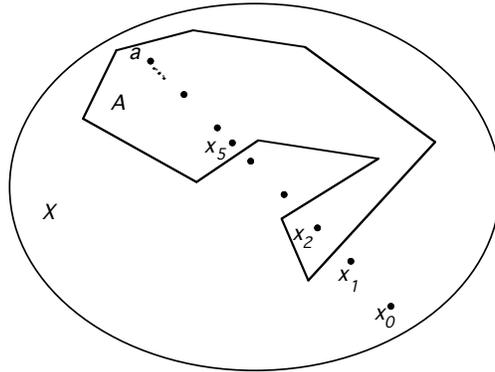
---

[30] See also [20, p. 8].

[31] To show equivalence on abstract topological spaces, there is for example an exercise in Kelley that establishes a dictionary mapping between them (i.e. given a net you can find a filter, and vice versa)[28, p. 83]. But there is a subtle distinction for a particular type of limit found in the advanced theory of integration [15, p. 371].

[32] This treatment follows [22].

There is a natural way to associate a filter with any sequence. If $x_1, x_2, x_3, \ldots$ is a sequence in X, we can associate with this sequence a filter Φ on X such that $\forall a \in X, \{x_n\} \to a$ iff $\Phi \to a$. In particular, let $\Phi = \{A \subset X \mid \exists k_A$ such that $\forall i \geq k_A, x_i \in A\}$. So the tails of the sequence are contained in neighborhoods which are members of the filter. The condition that each neighborhood of a is in the filter is then equivalent to the condition that the sequence is eventually in any neighborhood of a.

Example with $k_A = 5$

Figure 6: Member of filter with $k_A = 5$.

In order to define compactness in terms of filters, we need one more notion, that of an ultrafilter.

Definition 4.10 *A filter in X is an* ultrafilter *iff no filter in X properly contains it.*

The notion of ultrafilter is not exactly analogous to subsequence, but in the formulation of compactness, it serves the same purpose.

Theorem 4.11 *A topological space is* compact *iff every ultrafilter on X converges in X.*

Now we can return again to our example and get a sense in terms of filters for why S_Ω is not compact.[33] We want to show that there is an ultrafilter on S_Ω that does not converge. Consider all the neighborhoods of Ω in $S_\Omega \cup \Omega$. Let $\Phi = \{A \subset S_\Omega \mid \exists \alpha \in S_\Omega$ such that $\forall \beta \geq \alpha, \beta \in A\}$.

---

[33] The proof of this claim, in particular when we assert that there is an ultrafilter containing our filter, actually relies on the axiom of choice.

Figure 7: Member of filter on S_Ω.

This clearly satisfies the definition of a filter. Let Ψ be any ultrafilter containing Φ. We claim Ψ does not converge in S_Ω. Suppose it did. Say that $\Psi \to b$. Now pick some $\alpha > b, \alpha \in S_\Omega$. Then $A^+ = \{\beta : \beta \geq \alpha\} \in \Psi$ since $A^+ \in \Phi \subseteq \Psi$. We also have $A^- = \{\beta : \beta < \alpha\} \in \Psi$ since A^- is an open neighborhood of b (and we claim Ψ converges to b).
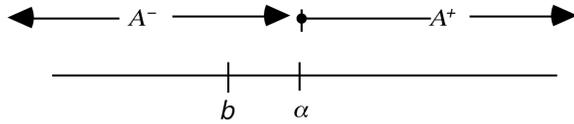

Figure 8: Illustration of A^- and A^+.

But $A^+ \cap A^- = \emptyset$, which violates the definition of a filter, so our assumption must be wrong. Thus, Ψ must not converge, and hence S_Ω is not compact.

4.3   Final Comments

Here ends the story, a sort of co-evolution of the two different, but related notions of sequential and open-cover compactness. Today when we use the term "compact" we mean open-cover compact, but this paper, and the terms listed in the Appendix shows that this was not always the case. The story of how open-cover compactness came to be seen as the *right* one is a story of developing mathematics without always knowing where it going, how important terms should be defined, and how widely they might be applied.

It might be worth noting, in closing, that even this paper, which attempts to characterize the evolution of compactness is only a sort of snapshot. In the time it took to write this paper, and revise it, textbooks have changed. For instance, the latest version of a standard general topology textbook [34] now includes a discussion of nets and filters, whereas the earlier version [33], available during the writing of the original version of this paper, did not.

The lesson to be drawn is simply that mathematics evolves and changes as concepts become clearer and are applied in more general situations. This may

be obvious to the mathematician who is involved in making these conceptual advances, but may be less clear to the student who sees a textbook as a collection of established facts. Textbooks are, as perhaps they should be, a distillation of what we currently know. They are also historical documents, in their own right, and being aware of that fact may help students mature as mathematicians.

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4.4 Appendix : Terminology

There are many notions related to (but not necessarily equivalent to) compactness. Table 1 contains a list of some of these notions.

[open-cover] *compact*: Every open cover has a finite subcover.
(also called the Borel-Lebesgue property)

sequentially compact: Every sequence has a convergent subsequence.

countably compact: Every countable open cover has a finite subcover.

limit-point compact: Every infinite subset of X has a limit point in X.
(also called Fréchet compact or the Bolzano-Weierstrass property)

relatively compact: The closure is compact.

quasi-compact: Compact to Bourbaki.

pseudo-compact: Each continuous real valued function on X is bounded.

finally compact: index of compactness is $\aleph_0$.
(also called Lindelöf compact)

Table 1: Flavors of compactness

Many of these concepts are related. For instance, compactness implies countable compactness implies limit point compactness. Sequential compactness implies countable compactness. And if we put further restrictions on our spaces we can get implications in the other direction. In T_1 spaces, limit-point compactness implies countable compactness. In first countable spaces, countable compactness implies sequential compactness. In second countable spaces, sequential compactness implies compactness. In particular, we know that in compact metric spaces, which turn out to be second countable, the first four notions of compactness in Table 1 are equivalent.

It took some time as compactness was applied to different types of spaces for relationships like these to be worked out. It also took time for names to stabilize. Table 2 lists different terms used for compactness-related ideas used by some of the most influential mathematicians in the historical development. In this paper, the modern names have been used.

Who	When	Their term	Modern term
Fréchet	1906	compact	relatively sequentially compact
		extremal	sequentially compact
Russian School (Alexandroff, etc.)	1920's	bicompact	compact
		compact	countably compact
Bourbaki	1930's	quasi-compact	compact
		compact	compact and Hausdorff

Table 2: Names of historical compactness-related terms