The Zero Lower Bound and Estimation Accuracy

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MOTIVATION

- Estimating linear DSGE models is common
  - Fast and easy to implement
  - Used by many central banks
- Recent ZLB period calls into question linear methods
  - Creates a kink in the monetary policy rule
  - Linear methods ignore the ZLB
  - Might lead to inaccurate estimates
  - Lower natural rate makes ZLB events more likely

Alternative Methods

1. Estimate fully nonlinear model (NL-PF)
   - Uses a projection method and particle filter (PF)
   - Most comprehensive treatment of the ZLB
   - Numerically very intensive
2. Estimate piecewise linear model (OB-IF)
   - Uses OccBin (OB) and an inversion filter (IF)
   - Almost as fast as linear methods
   - Captures the kink in the monetary policy rule
   - Ignores precautionary savings effects of the ZLB

Contribution

- Compare the accuracy of the two methods
- Generate datasets from a medium-scale nonlinear model
  - No ZLB events
  - A single 30Q ZLB event
- For each dataset, estimate a small-scale model
- Misspecification provides role for positive ME variances

Related Literature

- Estimation accuracy using artificial datasets
  - Fernandez-Villaverde and Rubio-Ramirez (2005): RBC model using linear and nonlinear methods
  - Hirose and Inoue (2016): New Keynesian model with a ZLB constraint using linear methods
  - Hirose and Sunakawa (2015): Nonlinear DGP with ZLB

- Estimates of global nonlinear models with actual data: (Gust et al., 2017; Iiboshi et al., 2018; Plante et al., 2018; Richter and Throckmorton, 2016)

- Effect of positive ME variances on estimation: (Canova et al., 2014; Cuba-Borda et al., 2017; Herbst and Schorfheide, 2017)

KEY FINDINGS

- NL-PF and OB-IF produce similar parameter estimates
- NL-PF predictions typically more accurate than OB-IF
  - Notional interest rate estimates
  - Expected ZLB duration
  - Probability of a $4+$ quarter ZLB event
  - Forecasts of the policy rate
- Increase in accuracy is often small due to weak precautionary savings effects and other nonlinearities

Data Generating Process

- Familiar medium-scale New Keynesian model
- One-period nominal bond
- Elastic labor supply and sticky wages
- Habit persistence and variable capital utilization
- Quadratic investment adjustment costs
- Monopolistically competitive intermediate firms
- Rotemberg quadratic price adjustment costs
- Occasionally binding ZLB constraint
- Risk premium, tech. growth, and interest rate shocks

Details

Estimation Methods

- Generate data by solving the nonlinear model ▸ Details
- Datasets: 50 for each ZLB duration, 120 quarters ▸ Details
- Estimated small-scale model is the DGP without: ▸ Details
  - Capital accumulation
  - Sticky wages
- Random walk Metropolis-Hastings algorithm:
  1. Mode Search ($5{,}000$ draws): initial covariance matrix
  2. Initial MH ($25{,}000$ draws): update covariance matrix
  3. Final MH ($50{,}000$ draws): calculate posterior mean
- Priors: Centered around truth ▸ Details
- Observables: Output growth, inflation rate, and nominal interest rate ▸ Details

Estimation Algorithms

- NL-PF: Fully nonlinear model with particle filter
  - Solve the model with the algorithm that generates the data
  - Filter uses $40{,}000$ particles and is adapted to incorporate information contained in the current observation Details
  - Likelihood evaluated on each of 16 cores, where the median determines whether to accept or reject the draw.
- OB-IF: Piecewise linear model with inversion filter
  - Solves the model with OccBin (Guerrieri & Iacoviello, 2015)
  - Filter solves for shocks where the observables equal the model predictions (Guerrieri & Iacoviello, 2017)
- Lin-KF: Unconstrained linear model with Kalman filter
  - Uses Sims's (2002) gensys algorithm

Speed Tests

	NL-PF (16 Cores)	OB-IF (1 Core)	Lin-KF (1 Core)
	No ZLB Events		
Seconds per draw	6.7 (6.1, 7.9)	0.035 (0.031, 0.040)	0.002 (0.002, 0.004)
Hours per dataset	148.8 (134.9, 176.5)	0.781 (0.689, 0.889)	0.052 (0.044, 0.089)
	30 Quarter ZLB Events		
Seconds per draw	8.4 (7.5, 9.5)	0.096 (0.051, 0.135)	0.002 (0.001, 0.003)
Hours per dataset	186.4 (167.6, 210.7)	2.137 (1.133, 3.000)	0.049 (0.022, 0.067)

Accuracy: Root Mean Squared Error

- True value for parameter j is $\tilde{\theta}_j$ and estimate is $\hat{\theta}_{j,h,k}$ given solution/estimation method h and artificial dataset k
- The normalized RMSE is

$$\mathrm{NRMSE}_h^j = \frac{1}{\tilde{\theta}_j}\sqrt{\frac{1}{N}\sum_{k=1}^{N}(\hat{\theta}_{j,h,k} - \tilde{\theta}_j)^2}$$

- N is the number of datasets. The RMSE is normalized by $\tilde{\theta}_j$ to remove differences in the scales of the parameters and measure the total error.

Parameter Estimates: No ZLB Events

Ptr	Truth	NL-PF-5%	OB-IF-0%	Lin-KF-5%
φ_p	100	151.1 (134.2, 165.8) [0.52]	142.6 (121.1, 157.3) [0.44]	151.4 (134.0, 165.7) [0.52]
h	0.8	0.66 (0.62, 0.70) [0.18]	0.64 (0.61, 0.67) [0.20]	0.66 (0.62, 0.69) [0.18]
ρ_s	0.8	0.76 (0.72, 0.80) [0.06]	0.76 (0.73, 0.81) [0.05]	0.76 (0.72, 0.80) [0.06]
ρ_i	0.8	0.79 (0.75, 0.82) [0.03]	0.76 (0.71, 0.79) [0.06]	0.79 (0.75, 0.82) [0.03]
σ_z	0.005	0.0032 (0.0023, 0.0039) [0.37]	0.0051 (0.0044, 0.0058) [0.09]	0.0032 (0.0023, 0.0039) [0.36]
σ_s	0.005	0.0052 (0.0040, 0.0066) [0.15]	0.0051 (0.0042, 0.0063) [0.13]	0.0053 (0.0040, 0.0067) [0.15]
σ_i	0.002	0.0017 (0.0014, 0.0020) [0.17]	0.0020 (0.0018, 0.0023) [0.08]	0.0017 (0.0015, 0.0020) [0.16]
ϕ_π	2.0	2.04 (1.88, 2.19) [0.06]	2.01 (1.84, 2.16) [0.06]	2.04 (1.88, 2.20) [0.06]
ϕ_y	0.5	0.35 (0.21, 0.54) [0.36]	0.32 (0.17, 0.48) [0.41]	0.35 (0.22, 0.54) [0.35]
Σ		[1.90]	[1.53]	[1.88]

Parameter Estimates: 30Q ZLB Events

Ptr	Truth	NL-PF-5%	OB-IF-0%	Lin-KF-5%
φ_p	100	188.4 (174.7, 202.7) [0.89]	183.4 (169.2, 198.5) [0.84]	191.6 (175.3, 204.1) [0.92]
h	0.8	0.68 (0.64, 0.71) [0.16]	0.63 (0.60, 0.67) [0.21]	0.67 (0.63, 0.70) [0.17]
ρ_s	0.8	0.81 (0.78, 0.84) [0.03]	0.82 (0.79, 0.86) [0.04]	0.82 (0.78, 0.86) [0.04]
ρ_i	0.8	0.80 (0.75, 0.84) [0.03]	0.77 (0.73, 0.81) [0.05]	0.84 (0.80, 0.88) [0.06]
σ_z	0.005	0.0040 (0.0030, 0.0052) [0.23]	0.0059 (0.0050, 0.0069) [0.22]	0.0043 (0.0030, 0.0057) [0.20]
σ_s	0.005	0.0050 (0.0039, 0.0062) [0.13]	0.0046 (0.0036, 0.0056) [0.15]	0.0047 (0.0037, 0.0061) [0.15]
σ_i	0.002	0.0015 (0.0013, 0.0019) [0.24]	0.0020 (0.0019, 0.0024) [0.09]	0.0016 (0.0014, 0.0019) [0.20]
ϕ_π	2.0	2.13 (1.94, 2.31) [0.09]	1.96 (1.77, 2.14) [0.06]	1.73 (1.52, 1.91) [0.15]
ϕ_y	0.5	0.42 (0.27, 0.62) [0.28]	0.44 (0.27, 0.61) [0.25]	0.32 (0.17, 0.47) [0.40]
Σ		[2.08]	[1.91]	[2.28]

Lower Misspecification: No ZLB Events

Ptr	Truth	OB-IF-0%	OB-IF-0%-Sticky Wages	OB-IF-0%-DGP
φ_p	100	142.6 (121.1, 157.3) [0.44]	100.1 (76.9, 119.6) [0.13]	101.4 (80.1, 120.7) [0.12]
h	0.8	0.64 (0.61, 0.67) [0.20]	0.82 (0.78, 0.86) [0.04]	0.81 (0.75, 0.85) [0.04]
ρ_s	0.8	0.76 (0.73, 0.81) [0.05]	0.82 (0.76, 0.86) [0.04]	0.80 (0.76, 0.85) [0.03]
ρ_i	0.8	0.76 (0.71, 0.79) [0.06]	0.80 (0.77, 0.83) [0.02]	0.79 (0.75, 0.82) [0.03]
σ_z	0.005	0.0051 (0.0044, 0.0058) [0.09]	0.0038 (0.0031, 0.0044) [0.24]	0.0047 (0.0039, 0.0054) [0.11]
σ_s	0.005	0.0051 (0.0042, 0.0063) [0.13]	0.0085 (0.0056, 0.0134) [0.81]	0.0060 (0.0043, 0.0084) [0.30]
σ_i	0.002	0.0020 (0.0018, 0.0023) [0.08]	0.0020 (0.0018, 0.0022) [0.08]	0.0020 (0.0018, 0.0022) [0.08]
ϕ_π	2.0	2.01 (1.84, 2.16) [0.06]	1.91 (1.74, 2.04) [0.07]	1.92 (1.72, 2.08) [0.06]
ϕ_y	0.5	0.32 (0.17, 0.48) [0.41]	0.40 (0.24, 0.58) [0.28]	0.41 (0.24, 0.57) [0.26]
Σ		[1.53]	[1.71]	[1.03]

Lower Misspecification: 30Q ZLB Events

Ptr	Truth	OB-IF-0%	OB-IF-0%-Sticky Wages	OB-IF-0%-DGP
φ_p	100	183.4 (169.2, 198.5) [0.84]	129.8 (105.5, 152.3) [0.33]	128.4 (109.0, 148.1) [0.31]
h	0.8	0.63 (0.60, 0.67) [0.21]	0.80 (0.77, 0.85) [0.03]	0.77 (0.72, 0.84) [0.06]
ρ_s	0.8	0.82 (0.79, 0.86) [0.04]	0.84 (0.80, 0.88) [0.06]	0.82 (0.79, 0.86) [0.04]
ρ_i	0.8	0.77 (0.73, 0.81) [0.05]	0.80 (0.77, 0.84) [0.03]	0.79 (0.75, 0.83) [0.03]
σ_z	0.005	0.0059 (0.0050, 0.0069) [0.22]	0.0047 (0.0039, 0.0055) [0.12]	0.0055 (0.0047, 0.0066) [0.15]
σ_s	0.005	0.0046 (0.0036, 0.0056) [0.15]	0.0074 (0.0050, 0.0107) [0.60]	0.0051 (0.0039, 0.0068) [0.19]
σ_i	0.002	0.0020 (0.0019, 0.0024) [0.09]	0.0020 (0.0018, 0.0023) [0.08]	0.0020 (0.0018, 0.0024) [0.09]
ϕ_π	2.0	1.96 (1.77, 2.14) [0.06]	1.81 (1.63, 1.99) [0.11]	1.81 (1.62, 2.03) [0.11]
ϕ_y	0.5	0.44 (0.27, 0.61) [0.25]	0.50 (0.33, 0.73) [0.23]	0.50 (0.32, 0.74) [0.24]
Σ		[1.91]	[1.59]	[1.23]

ME Variance: No ZLB Events

Ptr	Truth	NL-PF-2%	NL-PF-5%	NL-PF-10%
φ_p	100	150.2 (133.5, 165.3) [0.51]	151.1 (134.2, 165.8) [0.52]	149.5 (132.6, 163.8) [0.50]
h	0.8	0.66 (0.62, 0.69) [0.18]	0.66 (0.62, 0.70) [0.18]	0.66 (0.61, 0.70) [0.17]
ρ_s	0.8	0.76 (0.71, 0.79) [0.06]	0.76 (0.72, 0.80) [0.06]	0.76 (0.72, 0.79) [0.06]
ρ_i	0.8	0.77 (0.73, 0.80) [0.05]	0.79 (0.75, 0.82) [0.03]	0.80 (0.77, 0.84) [0.03]
σ_z	0.005	0.0038 (0.0031, 0.0043) [0.25]	0.0032 (0.0023, 0.0039) [0.37]	0.0027 (0.0020, 0.0035) [0.46]
σ_s	0.005	0.0052 (0.0039, 0.0065) [0.15]	0.0052 (0.0040, 0.0066) [0.15]	0.0051 (0.0041, 0.0065) [0.14]
σ_i	0.002	0.0019 (0.0017, 0.0021) [0.10]	0.0017 (0.0014, 0.0020) [0.17]	0.0015 (0.0012, 0.0018) [0.25]
ϕ_π	2.0	2.01 (1.84, 2.16) [0.06]	2.04 (1.88, 2.19) [0.06]	2.06 (1.89, 2.21) [0.07]
ϕ_y	0.5	0.31 (0.18, 0.48) [0.42]	0.35 (0.21, 0.54) [0.36]	0.41 (0.26, 0.59) [0.27]
Σ		1.79	1.90	1.95

ME VARIANCE: 30Q ZLB EVENTS

Ptr	Truth	NL-PF-2%	NL-PF-5%	NL-PF-10%
φ_p	100	192.0 (176.5, 207.1) [0.93]	188.4 (174.7, 202.7) [0.89]	182.7 (168.6, 197.3) [0.83]
h	0.8	0.67 (0.64, 0.71) [0.17]	0.68 (0.64, 0.71) [0.16]	0.68 (0.65, 0.72) [0.15]
ρ_s	0.8	0.81 (0.78, 0.84) [0.03]	0.81 (0.78, 0.84) [0.03]	0.81 (0.79, 0.85) [0.03]
ρ_i	0.8	0.79 (0.75, 0.83) [0.03]	0.80 (0.75, 0.84) [0.03]	0.81 (0.76, 0.85) [0.03]
σ_z	0.005	0.0043 (0.0035, 0.0052) [0.18]	0.0040 (0.0030, 0.0052) [0.23]	0.0038 (0.0025, 0.0050) [0.28]
σ_s	0.005	0.0051 (0.0040, 0.0061) [0.13]	0.0050 (0.0039, 0.0062) [0.13]	0.0049 (0.0037, 0.0061) [0.14]
σ_i	0.002	0.0018 (0.0016, 0.0021) [0.14]	0.0015 (0.0013, 0.0019) [0.24]	0.0013 (0.0011, 0.0017) [0.34]
ϕ_π	2.0	2.14 (1.96, 2.31) [0.09]	2.13 (1.94, 2.31) [0.09]	2.12 (1.92, 2.28) [0.08]
ϕ_y	0.5	0.39 (0.24, 0.60) [0.32]	0.42 (0.27, 0.62) [0.28]	0.46 (0.30, 0.66) [0.24]
Σ		2.01	2.08	2.13

Small Scale DGP: No ZLB Events

Ptr	Truth	NL-PF-5%	OB-IF-0%	Lin-KF-5%
φ_p	100	96.8 (81.6, 109.9) [0.09]	94.3 (81.8, 108.3) [0.11]	103.7 (92.6, 118.4) [0.09]
h	0.8	0.79 (0.76, 0.82) [0.02]	0.79 (0.75, 0.82) [0.02]	0.80 (0.76, 0.83) [0.02]
ρ_s	0.8	0.80 (0.76, 0.83) [0.03]	0.81 (0.76, 0.85) [0.04]	0.82 (0.77, 0.86) [0.05]
ρ_i	0.8	0.82 (0.79, 0.84) [0.03]	0.79 (0.77, 0.82) [0.02]	0.82 (0.79, 0.84) [0.03]
σ_z	0.005	0.0037 (0.0029, 0.0046) [0.27]	0.0051 (0.0044, 0.0056) [0.08]	0.0038 (0.0029, 0.0046) [0.26]
σ_s	0.005	0.0047 (0.0035, 0.0058) [0.19]	0.0049 (0.0039, 0.0060) [0.16]	0.0047 (0.0034, 0.0059) [0.21]
σ_i	0.002	0.0016 (0.0013, 0.0020) [0.20]	0.0020 (0.0017, 0.0022) [0.07]	0.0016 (0.0013, 0.0019) [0.20]
ϕ_π	2.0	2.00 (1.81, 2.21) [0.06]	1.95 (1.74, 2.14) [0.06]	1.97 (1.76, 2.18) [0.07]
ϕ_y	0.5	0.45 (0.29, 0.61) [0.22]	0.46 (0.30, 0.63) [0.21]	0.46 (0.31, 0.63) [0.22]
Σ		[1.12]	[0.78]	[1.14]

Small Scale DGP: 30Q ZLB Events

Ptr	Truth	NL-PF-5%	OB-IF-0%	Lin-KF-5%
φ_p	100	109.8 (89.5, 130.3) [0.15]	110.6 (95.3, 125.1) [0.15]	128.5 (111.2, 145.3) [0.30]
h	0.8	0.79 (0.77, 0.82) [0.02]	0.79 (0.77, 0.82) [0.02]	0.79 (0.76, 0.82) [0.03]
ρ_s	0.8	0.83 (0.78, 0.86) [0.04]	0.84 (0.80, 0.87) [0.06]	0.87 (0.83, 0.91) [0.10]
ρ_i	0.8	0.82 (0.78, 0.85) [0.03]	0.79 (0.74, 0.82) [0.03]	0.86 (0.83, 0.88) [0.08]
σ_z	0.005	0.0035 (0.0025, 0.0045) [0.33]	0.0052 (0.0043, 0.0061) [0.11]	0.0034 (0.0026, 0.0044) [0.33]
σ_s	0.005	0.0043 (0.0032, 0.0058) [0.22]	0.0046 (0.0034, 0.0057) [0.17]	0.0036 (0.0027, 0.0046) [0.32]
σ_i	0.002	0.0014 (0.0010, 0.0018) [0.31]	0.0019 (0.0016, 0.0022) [0.10]	0.0015 (0.0012, 0.0017) [0.27]
ϕ_π	2.0	2.01 (1.82, 2.20) [0.06]	1.80 (1.58, 2.06) [0.12]	1.62 (1.42, 1.86) [0.20]
ϕ_y	0.5	0.48 (0.28, 0.61) [0.18]	0.52 (0.32, 0.73) [0.23]	0.50 (0.34, 0.66) [0.19]
Σ		[1.35]	[0.99]	[1.82]

Notional Interest Rate Accuracy

- Nominal interest rate

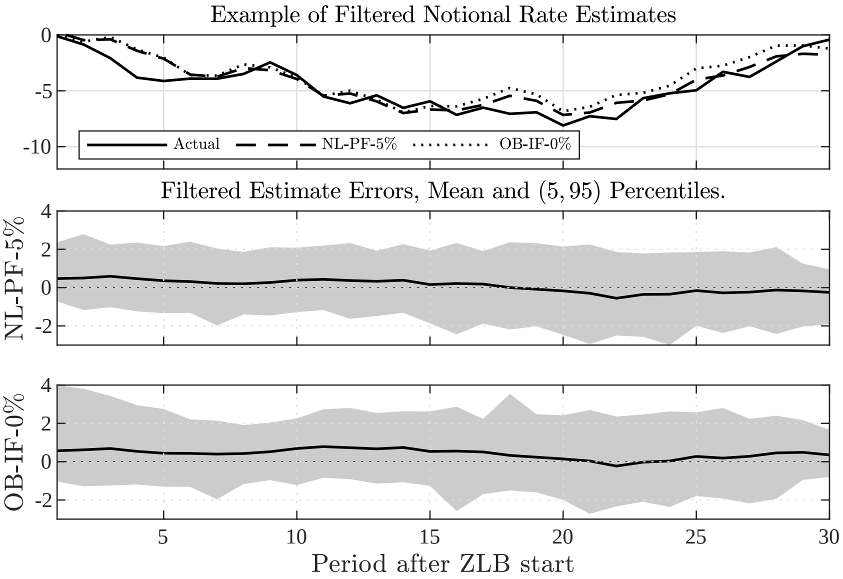
$$i_t = \max\{1, i_t^n\}$$

- Notional interest rate

$$i_t^n = (i_{t-1}^n)^{\rho_i} (\bar{\imath}(\pi_t/\bar{\pi})^{\phi_\pi} (y_t^{gdp}/(y_{t-1}^{gdp}\bar{z}))^{\phi_y})^{1-\rho_i} \exp(\sigma_i \varepsilon_{i,t})$$

- $i_t = i_t^n$ if $i_t^n \geq 1$

Notional Interest Rate Accuracy



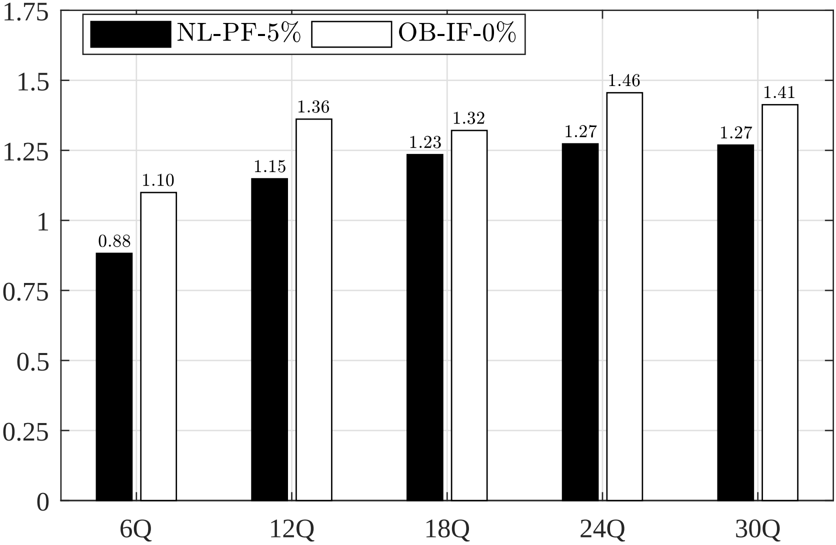
Accuracy: Root Mean Squared Error

- True value for the notional rate is $\tilde{i}^n_j$ and estimate is $\hat{i}^n_{j,h,k}$ given solution/estimation method h and artificial dataset k
- The RMSE is

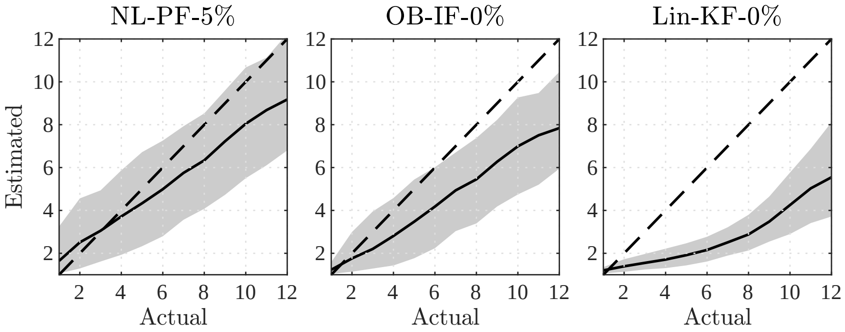
$$\mathrm{RMSE}^{in}_h = \sqrt{\frac{1}{N}\frac{1}{\tau}\sum_{k=1}^{N}\sum_{j=t}^{t+\tau-1}(\hat{i}^n_{j,h,k} - \tilde{i}^n_j)^2}$$

- t is the first period and τ is the duration of the ZLB event

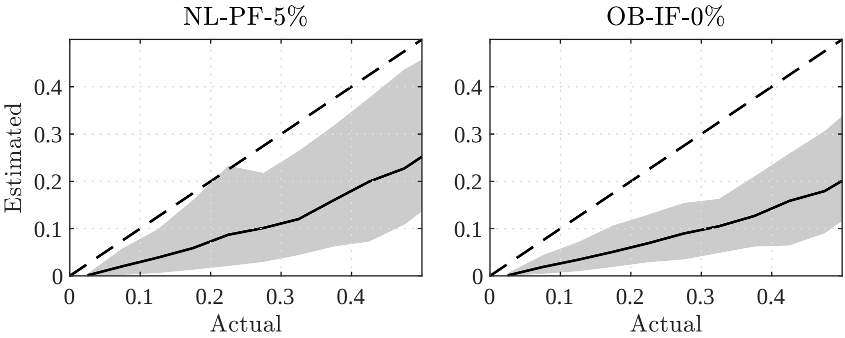
Notional Interest Rate Accuracy



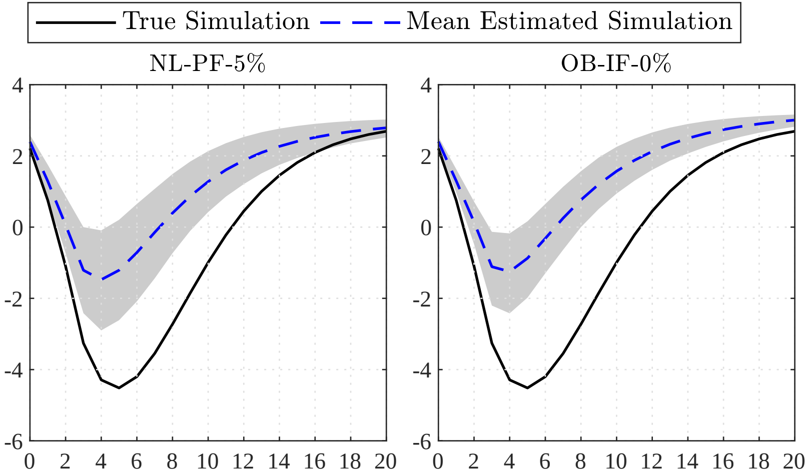
Expected ZLB Durations



4+ Quarter ZLB Event Probability

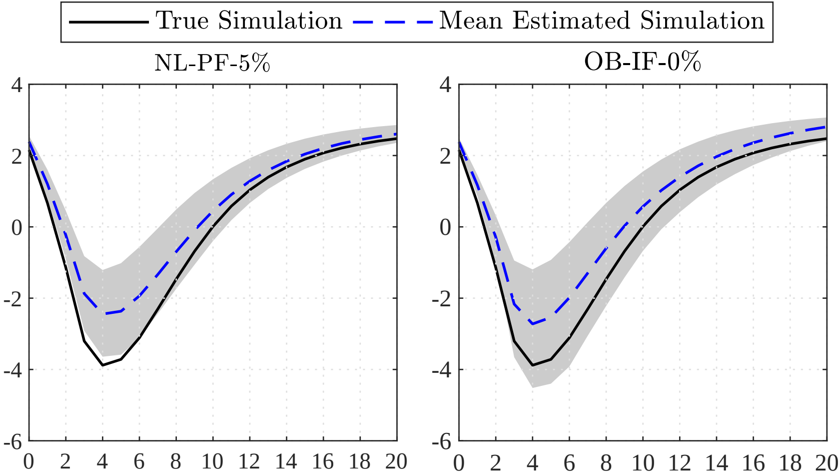


Notional Interest Rate Response



Output Growth
Inflation

Small Scale DGP: Notional Rate



Output Growth Inflation

Forecasts: Cont. Rank Prob. Score

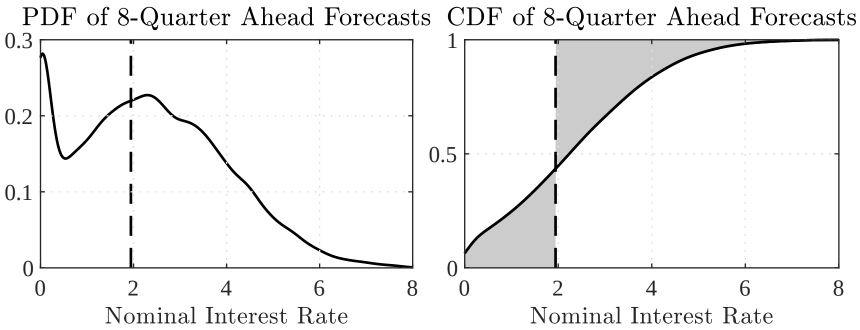
- $\mathrm{CRPS}^j_{m,k,t,\tau}$ for variable j given model/method m, dataset k, time t, and horizon τ

$$\int_{-\infty}^{\tilde{j}_{t+\tau}} [F_{m,k,t}(j_{t+\tau})]^2 dj_{t+\tau} + \int_{\tilde{j}_{t+\tau}}^{\infty} [1 - F_{m,k,t}(j_{t+\tau})]^2 dj_{t+\tau}$$

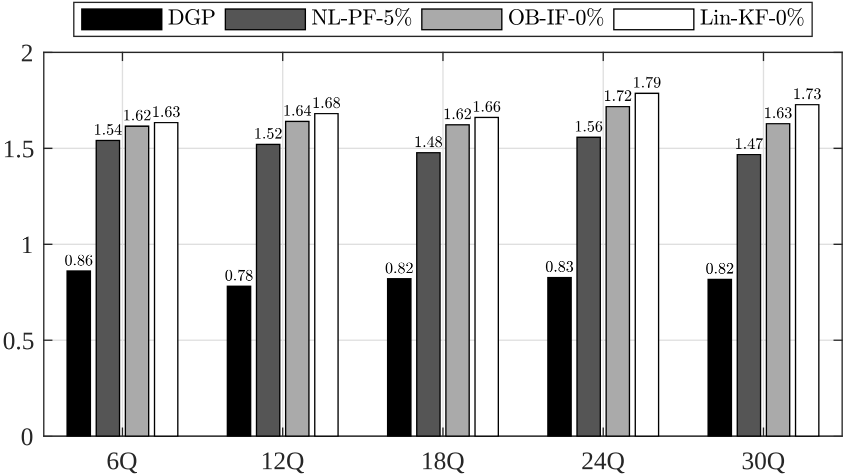
- $F_{m,k,t}(j_{t+\tau})$ is the cumulative distribution function (CDF) of the τ-quarter ahead forecast, and $\tilde{j}_{t+\tau}$ is the true realization
- CRPS penalizes probabilities assigned to outcomes that are not realized
- CRPS has the same units as the forecasted variables, which are percentages
- If forecast is deterministic, CRPS is mean absolute error

Forecast Accuracy Example

- Initialized at filtered state one quarter before ZLB binds
- Forecast horizon is 8-quarters ahead



Mean CRPS Interest Rate Forecasts



▸ Output Growth ▸ Inflation

Conclusion

- Two promising methods for dealing with ZLB:
  - Estimate the fully nonlinear model with a particle filter
  - Estimate the piecewise linear model with an inversion filter
- NL-PF is typically more accurate than OB-IF but the differences are often small
- Much larger gains in accuracy from estimating a richer, less misspecified piecewise linear model
- Important to examine whether findings are generalizable
- Nonlinear model is considerably more versatile

Detrended Equilibrium System

Medium-Scale Model 1

$$z_t = \bar{z} + \sigma_z \varepsilon_{z,t},\ \varepsilon_z \sim \mathbb{N}(0,1)$$

$$u_t = \bar{r}^k(\exp(\sigma_\upsilon(\upsilon_t - 1)) - 1)/\sigma_\upsilon$$

$$s_t = (1-\rho_s)\bar{s} + \rho_s s_{t-1} + \sigma_s \varepsilon_{s,t},\ \varepsilon_s \sim \mathbb{N}(0,1)$$

$$r^k_t = \bar{r}^k \exp(\sigma_\upsilon(\upsilon_t - 1))$$

$$i_t = \max\{1, i^n_t\}$$

$$i^n_t = (i^n_{t-1})^{\rho_i}(\bar{\imath}(\pi_t/\bar{\pi})^{\phi_\pi}(y^{gdp}_t/(y^{gdp}_{t-1}\bar{z}))^{\phi_y})^{1-\rho_i}\exp(\sigma_i \varepsilon_{i,t}),\ \varepsilon_i \sim \mathbb{N}(0,1)$$

$$\tilde{y}_t = (\upsilon_t \tilde{k}_{t-1}/z_t)^\alpha n_t^{1-\alpha}$$

$$r^k_t = \alpha mc_t z_t \tilde{y}_t/(\upsilon_t \tilde{k}_{t-1})$$

$$\tilde{w}_t = (1-\alpha) mc_t \tilde{y}_t / n_t$$

$$w^g_t = \pi_t z_t \tilde{w}_t/(\bar{\pi}\bar{z}\tilde{w}_{t-1})$$

Medium-Scale Model 2

$$\tilde{y}_t^{gdp} = [1 - \varphi_p(\pi_t/\bar{\pi} - 1)^2/2 - \varphi_w(w_t^g - 1)^2/2]\tilde{y}_t - u_t\tilde{k}_{t-1}/z_t$$

$$y_t^g = z_t\tilde{y}_t^{gdp}/(\bar{z}\tilde{y}_{t-1}^{gdp})$$

$$\tilde{\lambda}_t = \tilde{c}_t - h\tilde{c}_{t-1}/z_t$$

$$\tilde{w}_t^f = \chi n_t^\eta \tilde{\lambda}_t$$

$$\tilde{c}_t + \tilde{x}_t = \tilde{y}_t$$

$$x_t^g = z_t\tilde{x}_t/(\bar{z}\tilde{x}_{t-1})$$

$$\tilde{k}_t = (1-\delta)(\tilde{k}_{t-1}/z_t) + \tilde{x}_t(1 - \nu(x_t^g - 1)^2/2)$$

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Medium-Scale Model 3

$$1 = \beta E_t[(\tilde{\lambda}_t/\tilde{\lambda}_{t+1})(s_t i_t/(z_{t+1}\pi_{t+1}))]$$

$$q_t = \beta E_t[(\tilde{\lambda}_t/\tilde{\lambda}_{t+1})(r^k_{t+1}\upsilon_{t+1} - u_{t+1} + (1-\delta)q_{t+1})/z_{t+1}]$$

$$\begin{aligned} 1 = q_t[1 - \nu(x^g_t - 1)^2/2 - \nu(x^g_t - 1)x^g_t] + \ldots \\ \beta\nu\bar{z}E_t[q_{t+1}(\tilde{\lambda}_t/\tilde{\lambda}_{t+1})(x^g_{t+1})^2(x^g_{t+1} - 1)/z_{t+1}] \end{aligned}$$

$$\begin{aligned} \varphi_p(\pi_t/\bar{\pi} - 1)(\pi_t/\bar{\pi}) = 1 - \theta_p + \theta_p mc_t + \ldots \\ \beta\varphi_p E_t[(\tilde{\lambda}_t/\tilde{\lambda}_{t+1})(\pi_{t+1}/\bar{\pi} - 1)(\pi_{t+1}/\bar{\pi})(\tilde{y}_{t+1}/\tilde{y}_t)] \end{aligned}$$

$$\begin{aligned} \varphi_w(w^g_t - 1)w^g_t = [(1-\theta_w)\tilde{w}_t + \theta_w\tilde{w}^f_t]n_t/\tilde{y}_t + \ldots \\ \beta\varphi_w E_t[(\tilde{\lambda}_t/\tilde{\lambda}_{t+1})(w^g_{t+1} - 1)w^g_{t+1}(\tilde{y}_{t+1}/\tilde{y}_t)] \end{aligned}$$

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Small-Scale Model 1

$$z_t = \bar{z} + \sigma_z \varepsilon_{z,t}, \ \varepsilon_z \sim \mathbb{N}(0,1)$$

$$s_t = (1-\rho_s)\bar{s} + \rho_s s_{t-1} + \sigma_s \varepsilon_{s,t}, \ \varepsilon_s \sim \mathbb{N}(0,1)$$

$$i_t = \max\{1, i_t^n\}$$

$$i_t^n = (i_{t-1}^n)^{\rho_i} (\bar{\imath}(\pi_t/\bar{\pi})^{\phi_\pi} (y_t^{gdp}/(y_{t-1}^{gdp}\bar{z}))^{\phi_y})^{1-\rho_i} \exp(\sigma_i \varepsilon_{i,t}), \ \varepsilon_i \sim \mathbb{N}(0,1)$$

$$y_t^g = z_t \tilde{y}_t^{gdp} / (\bar{z}\tilde{y}_{t-1}^{gdp})$$

$$\tilde{\lambda}_t = \tilde{c}_t - h\tilde{c}_{t-1}/z_t$$

$$\tilde{y}_t = n_t$$

$$\tilde{y}_t^{gdp} = [1 - \varphi_p(\pi_t/\bar{\pi} - 1)^2/2]\tilde{y}_t$$

$$\tilde{c}_t = \tilde{y}_t^{gdp}$$

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Small-Scale Model 2

$$\tilde{w}_t = \chi n_t^\eta \tilde{\lambda}_t$$

$$1 = \beta E_t[(\tilde{\lambda}_t/\tilde{\lambda}_{t+1})(s_t i_t/(z_{t+1}\pi_{t+1}))]$$

$$\tilde{w}_t = mc_t \tilde{y}_t / n_t$$

$$\varphi_p(\pi_t/\bar{\pi} - 1)(\pi_t/\bar{\pi}) = 1 - \theta_p + \theta_p mc_t + \dots$$
$$\beta\varphi_p E_t[(\tilde{\lambda}_t/\tilde{\lambda}_{t+1})(\pi_{t+1}/\bar{\pi} - 1)(\pi_{t+1}/\bar{\pi})(\tilde{y}_{t+1}/\tilde{y}_t)]$$

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Additional Material

Adapted Particle Filter

1. Initialize the filter by drawing from the ergodic distribution.
2. For all particles $p \in \{1, \ldots, N_p\}$ apply the following steps:
   2.1 Draw $\mathbf{e}_{t,p} \sim \mathbb{N}(\bar{\mathbf{e}}_t, I)$, where $\bar{\mathbf{e}}_t$ maximizes $p(\xi_t|\mathbf{z}_t)p(\mathbf{z}_t|\mathbf{z}_{t-1})$.
   2.2 Obtain $\mathbf{z}_{t,p}$ and the vector of variables, $\mathbf{w}_{t,p}$, given $\mathbf{z}_{t-1,p}$
   2.3 Calculate, $\xi_{t,p} = \hat{\mathbf{x}}_{t,p}^{model} - \hat{\mathbf{x}}_t^{data}$. The weight on particle p is

$$\omega_{t,p} = \frac{p(\xi_t|\mathbf{z}_{t,p})p(\mathbf{z}_{t,p}|\mathbf{z}_{t-1,p})}{g(\mathbf{z}_{t,p}|\mathbf{z}_{t-1,p}, \hat{\mathbf{x}}_t^{data})} \propto \frac{\exp(-\xi_{t,p}' H^{-1} \xi_{t,p}/2) \exp(-\mathbf{e}_{t,p}' \mathbf{e}_{t,p}/2)}{\exp(-(\mathbf{e}_{t,p} - \bar{\mathbf{e}}_t)'(\mathbf{e}_{t,p} - \bar{\mathbf{e}}_t)/2)}$$

   The model's likelihood at t is $\ell_t^{model} = \sum_{p=1}^{N_p} \omega_{t,p}/N_p$.

   2.4 Normalize the weights, $W_{t,p} = \omega_{t,p}/\sum_{p=1}^{N_p} \omega_{t,p}$. Then use systematic resampling with replacement from the particles.
3. Apply step 2 for $t \in \{1, \ldots, T\}$. $\log \ell^{model} = \sum_{t=1}^{T} \log \ell_t^{model}$.

Particle Adaption

1. Given $\mathbf{z}_{t-1}$ and a guess for $\bar{\mathrm{e}}_t$, obtain $\mathbf{z}_t$ and $\mathbf{w}_{t,p}$.
2. Calculate $\xi_t = \hat{\mathbf{x}}_t^{model} - \hat{\mathbf{x}}_t^{data}$, which is multivariate normal:

$$p(\xi_t|\mathbf{z}_t) = (2\pi)^{-3/2}|H|^{-1/2}\exp(-\xi_t' H^{-1}\xi_t/2)$$
$$p(\mathbf{z}_t|\mathbf{z}_{t-1}) = (2\pi)^{-3/2}\exp(-\bar{\mathbf{e}}_t'\bar{\mathbf{e}}_t/2)$$

$H \equiv \mathrm{diag}(\sigma^2_{me,\hat{y}}, \sigma^2_{me,\pi}, \sigma^2_{me,i})$ is the ME covariance matrix.

3. Solve for the optimal $\bar{\mathrm{e}}_t$ to maximize

$$p(\xi_t|\mathbf{z}_t)p(\mathbf{z}_t|\mathbf{z}_{t-1}) \propto \exp(-\xi_t' H^{-1}\xi_t/2)\exp(-\bar{\mathbf{e}}_t'\bar{\mathbf{e}}_t/2)$$

We converted MATLAB's `fminsearch` routine to Fortran.

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Nonlinear Solution Method

- Use linear solution as an initial conjecture: $\tilde{c}^A(\mathbf{z}_t)$, $\pi^A(\mathbf{z}_t)$
- For all nodes $d \in D$, implement the following steps:
  1. Solve for $\{\tilde{w}_t, \tilde{y}_t, i_t^n, i_t, \tilde{\lambda}_t\}$ given $\tilde{c}_{i-1}^A(\mathbf{z}_t^d)$ and $\pi_{i-1}^A(\mathbf{z}_t^d)$
  2. Use piecewise linear interpolation to solve for updated values of consumption and inflation, $\{\tilde{c}_{t+1}^m, \pi_{t+1}^m\}_{m=1}^M$, given each realization of the updated state vector, $\mathbf{z}_{t+1}$
  3. Given $\{\tilde{c}_{t+1}^m, \pi_{t+1}^m\}_{m=1}^M$, solve for future output, $\{\tilde{y}_{t+1}^m\}_{m=1}^M$, which enters expectations. Then numerically integrate.
  4. Use Chris Sims' `csolve` to determine the values of the policy functions that best satisfy the equilibrium system
- On iteration i, $\text{maxdist}_i \equiv \max\{|\tilde{c}_i^A - \tilde{c}_{i-1}^A|, |\pi_i^A - \pi_{i-1}^A|\}$. Continue iterating until $\text{maxdist}_i < 10^{-6}$ for all d

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Prior Distributions

Parameter		Dist.	Mean	SD
Rotemberg Price Adjustment Cost	φ	Norm	100.0	25.00
Inflation Gap Response	ϕ_π	Norm	2.000	0.250
Output Gap Response	ϕ_y	Norm	0.500	0.250
Habit Persistence	h	Beta	0.800	0.100
Risk Premium Shock Persistence	ρ_s	Beta	0.800	0.100
Notional Rate Persistence	ρ_i	Beta	0.800	0.100
Growth Rate Shock SD	σ_z	IGam	0.005	0.005
Risk Premium Shock SD	σ_s	IGam	0.005	0.005
Notional Rate Shock SD	σ_i	IGam	0.002	0.002

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State and Observation Equations

- Linear model

$$\hat{\mathbf{s}}_t = T(\vartheta)\hat{\mathbf{s}}_{t-1} + M(\vartheta)\varepsilon_t$$
$$\hat{\mathbf{x}}_t = H\hat{\mathbf{s}}_t + \xi_t$$

- Nonlinear Model

$$\mathbf{s}_t = \Psi(\vartheta, \mathbf{s}_{t-1}, \varepsilon_t)$$
$$\mathbf{x}_t = H\mathbf{s}_t + \xi_t$$

$\mathbf{x}_t = [y_t^g, \pi_t, i_t]$ (observables), $\varepsilon_t = [\varepsilon_{z,t}, \varepsilon_{s,t}, \varepsilon_{i,t}]$ (shocks),
$\xi \sim \mathbb{N}(0, R)$ (measurement errors), ϑ (parameters),
$\mathbf{s}_t = [\tilde{c}, n, \tilde{y}, \tilde{y}^{gdp}, y^g, \tilde{w}, \pi, i, i^n, mc, \tilde{\lambda}, z, s]$ (states)

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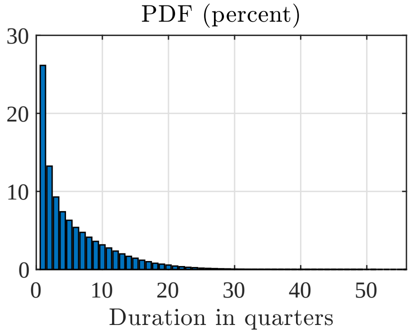
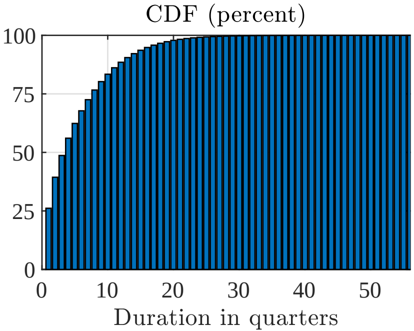
Dataset Statistics

	6Q	12Q	18Q	24Q	30Q
CDF of ZLB Durs	0.678	0.885	0.966	0.992	0.998
Sims to 50 Datasets	150,300	154,950	256,950	391,950	1,030,300





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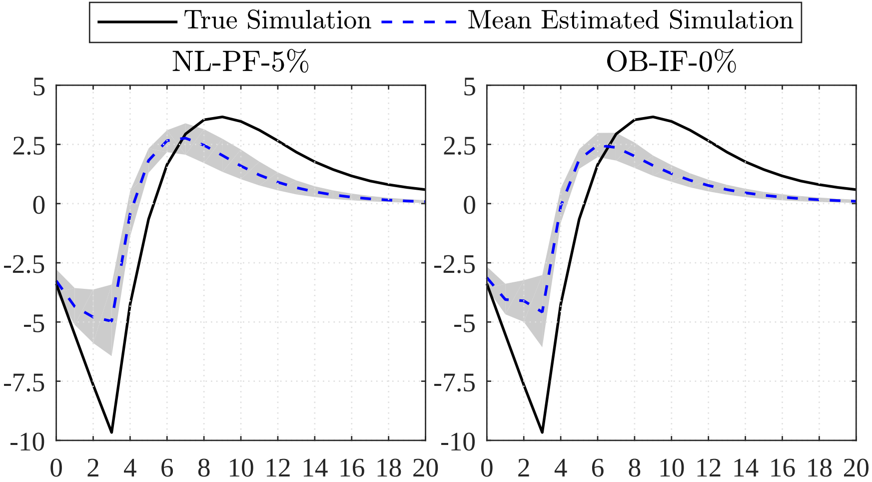
Output Growth Response



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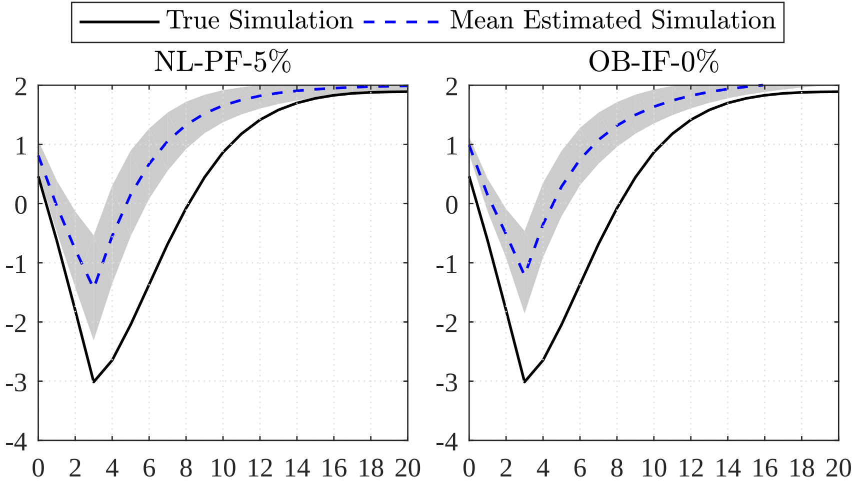
Inflation Rate Response



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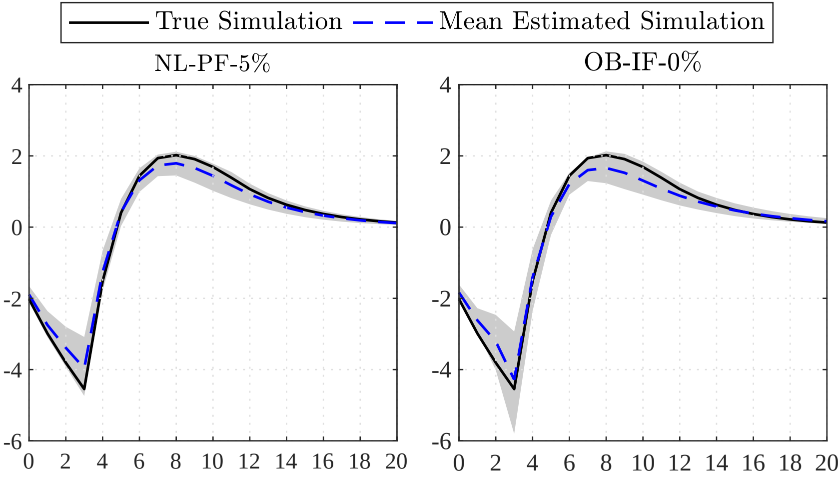
Small Scale DGP: Output Growth



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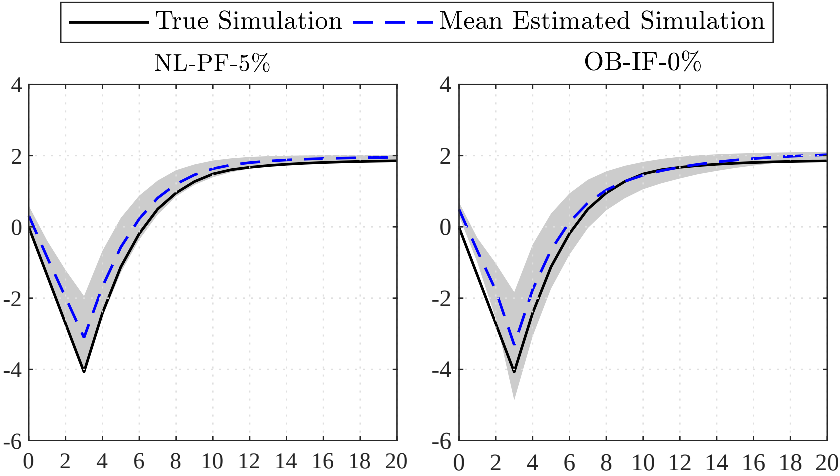
Small Scale DGP: Inflation Rate



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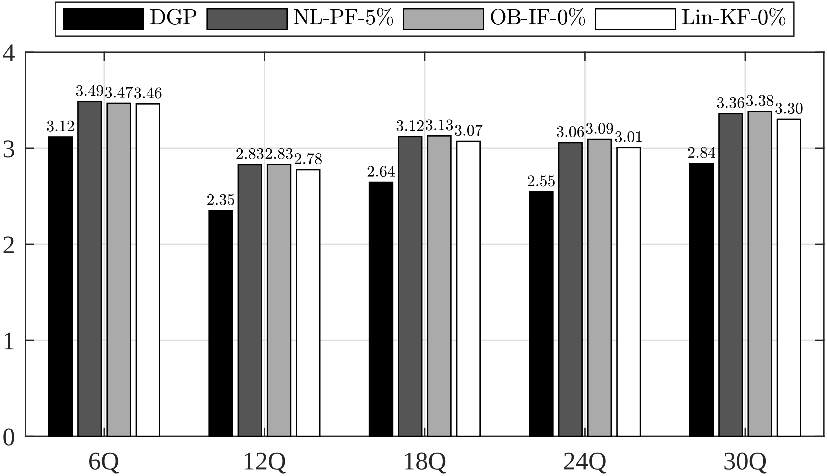
Output Growth Forecast Accuracy



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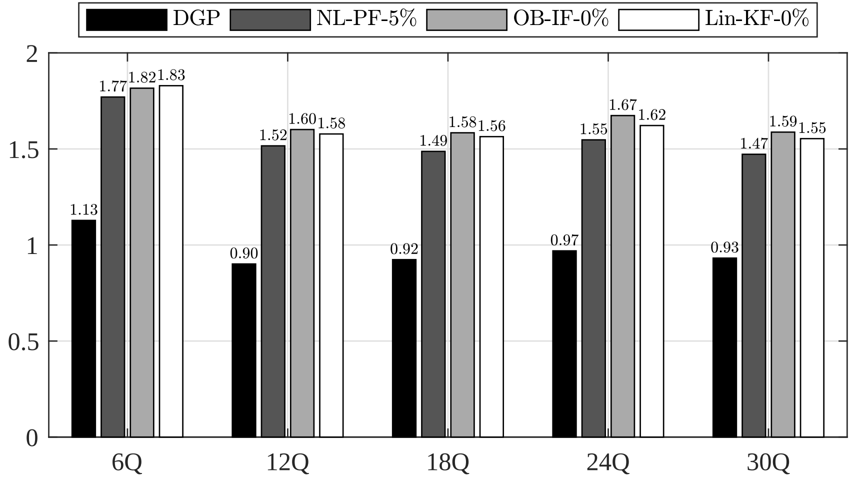
Inflation Rate Forecast Accuracy



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