Online Appendix to: Macroeconomic Responses to Uncertainty Shocks: The Perils of Recursive Orderings*

Lutz Kilian[†] Michael D. Plante[‡] Alexander W. Richter[‡]

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[†]Corresponding author: Research Department, Federal Reserve Bank of Dallas, 2200 N Pearl Street, Dallas, TX 75201, and CEPR (lkilian2019@gmail.com).

[‡]Research Department, Federal Reserve Bank of Dallas, 2200 N Pearl Street, Dallas, TX 75201 (michael.plante@dal.frb.org; alex.richter@dal.frb.org).

A DSGE MODEL DERIVATIONS

Representative Household The representative household solves the Bellman equation

$$J(b_t) = \max_{c_t,n_t,b_{t+1}} \left[(1-\beta)u_t^{1-1/\psi} + \beta(E_t[J(b_{t+1})^{1-\gamma}])^{\frac{1-1/\psi}{1-\gamma}}\right]^{\frac{1}{1-1/\psi}} \tag{1}$$

subject to

$$u_t = c_t^{\eta}(1-n_t)^{1-\eta}, \tag{2}$$

$$c_t + b_{t+1}/r_t = w_t n_t + b_t + d_t. \tag{3}$$

The first order conditions are given by

$$\begin{aligned} c_t: \quad & \eta(1-\beta)J_t^{1/\psi}u_t^{1-1/\psi}/c_t = \lambda_t, \\ n_t: \quad & (1-\eta)(1-\beta)J_t^{1/\psi}u_t^{1-1/\psi}/(1-n_t) = \lambda_t w_t, \\ & \beta J_t^{1/\psi}(E_t[J_{t+1}^{1-\gamma}])^{\frac{\gamma-1/\psi}{1-\gamma}}E_t[J_{t+1}^{-\gamma}J_{b,t+1}] = \lambda_t/r_t. \end{aligned}$$

The envelope condition implies

$$J_{b,t} = \lambda_t.$$

Combining these results implies

$$w_t = (1-\eta)c_t/(\eta(1-n_t)), \tag{4}$$

$$1 = E_t[x_{t+1}r_t], \tag{5}$$

$$x_{t+1} = \beta(c_t/c_{t+1})(u_{t+1}/u_t)^{1-1/\psi}(J_{t+1}/z_t)^{1/\psi-\gamma}. \tag{6}$$

Representative Firm The representative firm solves the Bellman equation

$$V(k_t) = \max_{n_t,k_{t+1},i_t} d_t + E_t[x_{t+1}V(k_{t+1})]$$

subject to

$$d_t = y_t - w_t n_t - i_t,$$

$$y_t = a_t k_t^{\alpha} n_t^{1-\alpha}, \tag{7}$$

$$k_{t+1} = \Theta_{t+1}((1-\delta)k_t + i_t). \tag{8}$$

Substituting in all constraints yields

$$V(k_t) = \max_{n_t,i_t} a_t k_t^{\alpha} n_t^{1-\alpha} - w_t n_t - i_t + E_t[x_{t+1}V(\Theta_{t+1}((1-\delta)k_t + i_t))].$$

The first order conditions are given by

$$n_t: \quad w_t = (1-\alpha)a_t k_t^{\alpha} n_t^{-\alpha},$$
$$i_t: \quad 1 = E_t[x_{t+1}V_{k,t+1}\Theta_{t+1}].$$

The envelope condition implies

$$\begin{aligned} k_t: \quad V_{k,t} &= \alpha a_t k_t^{\alpha-1} n_t^{1-\alpha} + E_t[x_{t+1}V_{k,t+1}\Theta_{t+1}(1-\delta)], \\ &= \alpha a_t k_t^{\alpha-1} n_t^{1-\alpha} + 1 - \delta, \end{aligned}$$

after imposing $E_t[x_{t+1}V_{k,t+1}\Theta_{t+1}] = 1$. Combining this result with the first order condition for investment yields

$$1 = E_t[x_{t+1}\Theta_{t+1}(r_{t+1}^k + 1 - \delta)], \tag{9}$$
$$r_t^k = \alpha y_t / k_t. \tag{10}$$

Competitive Equilibrium The aggregate resource constraint is given by

$$c_t + i_t = y_t. \tag{11}$$

The equilibrium consists of infinite sequences of quantities $\{k_t, c_t, n_t, y_t, i_t, u_t, J_t, z_t\}_{t=0}^{\infty}$, prices $\{w_t, r_t^k\}_{t=0}^{\infty}$, and exogenous variables $\{\Theta_t, a_t, e_t, \sigma_{a,t}, \sigma_{e,t}\}_{t=0}^{\infty}$ that satisfy (1)-(11) and the exogenous processes, given an state of the economy $\{k_{-1}, a_{-1}, \sigma_{a,-1}, e_{-1}, \sigma_{e,-1}\}$ and the sequences of shocks $\{\varepsilon_{a,t}, \varepsilon_{av,t}, \varepsilon_{e,t}, \varepsilon_{ev,t}\}_{t=0}^{\infty}$.

B SOLUTION METHOD

The equilibrium system of the DSGE model is summarized by $E[g(\mathbf{x}_{t+1}, \mathbf{x}_t, \varepsilon_{t+1})|\mathbf{z}_t, \vartheta] = 0$, where g is a vector-valued function, $\mathbf{x}_t$ is the vector of model variables, ε_t is the vector of shocks, $\mathbf{z}_t$ is the vector of states, and ϑ is the vector of parameters. We discretize the level shocks and volatility processes using the Markov chain in Rouwenhorst (1995), which Kopecky and Suen (2010) show outperforms other methods for approximating autoregressive processes. In our model with a level shock to disaster risk and level and volatility shocks to technology, the bounds on a_t are set to $\pm 5\%$ of the deterministic steady state, while k_t ranges from -60% to 10% of the deterministic steady state to account for disasters. These bounds ensure that simulations contain at least 99% of the ergodic distribution. We specify 9 states for e_t, $\sigma_{a,t}$, and $\varepsilon_{a,t+1}$, and discretize a_t and k_t into 9 and 11 evenly-spaced points, respectively. The product of the points in each dimension, D, is the total number of nodes in the state space ($D = 8{,}019$). The realization of $\mathbf{z}_t$ on node d is denoted $\mathbf{z}_t(d)$. The Rouwenhorst method provides integration nodes, $[\varepsilon_{e,t+1}(m), \sigma_{a,t+1}(m), \varepsilon_{a,t+1}(m)]$, with weights, $\phi(m)$, for $m \in \{1, \ldots, M\}$, where $M = 729$ given that there are three shocks, each with 9 states. The setup in the 2-shock models is analogous.

The vector of policy functions and the realization on node d are denoted by $\mathbf{pf}_t$ and $\mathbf{pf}_t(d)$, where $\mathbf{pf}_t \equiv [n(\mathbf{z}_t), J(\mathbf{z}_t)]$. The following steps outline our policy function iteration algorithm:

1. Use Sims's (2002) `gensys` algorithm to solve the log-linear model. Then map the solution for the policy functions to the discretized state space. This provides an initial conjecture.
2. On iteration $j \in \{1, 2, \ldots\}$ and each node $d \in \{1, \ldots, D\}$, use Chris Sims' `csolve` to find the $\mathbf{pf}_t(d)$ that satisfies $E[g(\cdot)|\mathbf{z}_t(d), \vartheta] \approx 0$. Guess $\mathbf{pf}_t(d) = \mathbf{pf}_{j-1}(d)$. Then
   (a) Solve for all variables dated at time t, given $\mathbf{pf}_t(d)$ and $\mathbf{z}_t(d)$.
   (b) Linearly interpolate the policy functions, $\mathbf{pf}_{j-1}$, at the updated state variables, $\mathbf{z}_{t+1}(m)$, to obtain $\mathbf{pf}_{t+1}(m)$ on every integration node, $m \in \{1, \ldots, M\}$.
   (c) Given $\{\mathbf{pf}_{t+1}(m)\}_{m=1}^{M}$, solve for the other elements of $\mathbf{s}_{t+1}(m)$ and compute
   $$E[g(\mathbf{x}_{t+1}, \mathbf{x}_t(d), \varepsilon_{t+1})|\mathbf{z}_t(d), \vartheta] \approx \textstyle\sum_{m=1}^{M} \phi(m) g(\mathbf{x}_{t+1}(m), \mathbf{x}_t(d), \varepsilon_{t+1}(m)).$$
   When `csolve` has converged, set $\mathbf{pf}_j(d) = \mathbf{pf}_t(d)$.
3. Repeat step 2 until $\mathrm{maxdist}_j < 10^{-6}$, where $\mathrm{maxdist}_j \equiv \max\{|(\mathbf{pf}_j - \mathbf{pf}_{j-1})/\mathbf{pf}_{j-1}|\}$. When that criterion is satisfied, the algorithm has converged to an approximate solution.

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