Macroeconomic Responses to Uncertainty Shocks: The Perils of Recursive Orderings

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Introduction

- Structural VAR analysis is widely used in the literature
- Despite methodological advances, many applied researchers still use recursive identification schemes
- Well known that recursive VAR models will not recover the population responses when the true model is not recursive
- Yet many people believe recursive orderings may be used to learn about the quantitative importance of causal effects
- Literature on uncertainty shocks is a leading example

Popular Beliefs (1/3)

Altig et al. (2020, JPubE) concedes that

> *"drawing causal inferences from VARs is challenging—in part because policy, and policy uncertainty, can respond to current and anticipated future economic conditions"*

but argues that

> *"despite the challenges, [recursively identified] VARs are useful for...gauging whether uncertainty innovations foreshadow weaker macroeconomic performance conditional on standard macro and policy variables."*

This view is widely held in the uncertainty literature

Popular Beliefs (2/3)

Gao et al. (2022, JFE) claims that

> *"To bracket the effect of...volatility shocks on fundamentals, we consider two different schemes: in the first approach, we let implied...variance be the first among all the variables in the VAR, while in the second approach, it is ordered last. Thus,...volatility innovations from the VAR are treated as most exogenous to the system under the first ordering and as least exogenous under the second ordering. By comparing the impulse responses under these two specifications, we can assess the range of possible responses to...volatility shocks in the data."*

Popular Beliefs (3/3)

Similarly, Jurado et al. (2015, AER) say that

> *"as uncertainty is placed last in the VAR, the effects of uncertainty shocks on the other variables in the system are measured after we have removed all the variation in uncertainty that is attributable to shocks to the other endogenous variables in the system. That the effects of uncertainty shocks are still non-trivial is consistent with the view that uncertainty has important implications for economic activity."*

This bounding interpretation of alternative orderings of recursive VARs is also widely held in the uncertainty literature

CONTRIBUTION TO THE LITERATURE (1/3)

- A common practice in the literature has been to report impulse response estimates based on alternative orderings with uncertainty ordered first or last
- When the implied impulse responses are robust to alternative orderings, they are considered trustworthy
- When they are not, the estimates are used to bound the true response without addressing the identification problem
- The validity of these arguments has never been examined, even though many have expressed skepticism about the validity of recursively identified models of uncertainty

Contribution to the Literature (2/3)

- We consider the merit of these arguments using:
  - Analytically based structural VAR examples
  - Simulation studies using data from DSGE model with endogenously determined uncertainty
- We prove by counterexample that both arguments are incorrect, calling into question many empirical results
- We also examine the ability of recursive and several non-recursive identification procedures to identify uncertainty shocks when uncertainty is endogenous
  - Recursive VAR models lead to biased IRFs
  - Bias persists even if exogenous uncertainty shocks play a dominant role in driving aggregate uncertainty
  - Instrument variables (IV) estimator may overcome identification issues

Contribution to the Literature (3/3)

Understanding these issues is important because

1. The findings of earlier studies based on recursive identification continue to shape the academic discourse about the economic effects of uncertainty shocks

   (e.g., Bachmann et al., 2013; Baker et al., 2016; Basu and Bundick, 2017; Bekaert et al., 2013; Bloom, 2009; Caggiano et al., 2014; Altig et al., 2020; Fernandez-Villaverde et al., 2015; Jurado et al., 2015; Leduc and Liu, 2016)

2. Recursively identified VAR models remain a common empirical approach in the uncertainty literature

   (e.g., Ahir et al., 2022; Beraja and Wolf, 2021; Born and Pfeifer, 2021; Cacciatore and Ravenna, 2021; Caldara and Iacoviello, 2022; Smiech et al., 2021 Chen and Tillmann, 2021; Gao et al., 2022; Larsen, 2021; Londono et al., 2021)

3. These issues apply to other VAR applications

   (e.g., Goldman Sachs financial conditions index; Chin and Lin, 2023; Cesa-Bianchi and Miranda-Agrippino 2024)

IDENTIFICATION PROBLEM

- Stylized model of the impact of uncertainty on real GDP growth, where the reduced-form shocks, u_t, are linked to the structural shocks, w_t, through the structural impact multiplier matrix, B_0^{-1} with elements $b_0^{ij} = \partial u_{i,t}/\partial w_{j,t}$
- Two potential recursive models of uncertainty shocks:

$$\begin{pmatrix} u_t^{\Delta gdp} \\ u_t^{\text{uncertainty}} \end{pmatrix} = \begin{bmatrix} b_0^{11} & 0 \\ b_0^{21} & b_0^{22} \end{bmatrix} \begin{pmatrix} w_t^{\text{other}} \\ w_t^{\text{uncertainty}} \end{pmatrix} \quad (1)$$

$$\begin{pmatrix} u_t^{\text{uncertainty}} \\ u_t^{\Delta gdp} \end{pmatrix} = \begin{bmatrix} b_0^{11} & 0 \\ b_0^{21} & b_0^{22} \end{bmatrix} \begin{pmatrix} w_t^{\text{uncertainty}} \\ w_t^{\text{other}} \end{pmatrix} \quad (2)$$

- No compelling reason to prefer one model over another
  - Changes in economic activity may affect uncertainty
  - Uncertainty shocks could affect GDP growth on impact
  - Neither model is consistent with both economic arguments

Alternative Orderings (1/2)

- If the impulse responses across the two orderings are similar, they are viewed as robust and hence trustworthy
- The two impulse responses will be identical only when the reduced-form error correlation is zero, but even then they may not look anything like the population responses
- Consider the following model ($\Sigma \equiv \mathrm{Var}(u_t) = B_0^{-1}(B_0^{-1})'$):

$$\begin{pmatrix} u_t^{\Delta gdp} \\ u_t^{\text{uncertainty}} \end{pmatrix} = \begin{bmatrix} 1 & 0.5 \\ -0.75 & 1 \end{bmatrix} \begin{pmatrix} w_t^{\text{other}} \\ w_t^{\text{uncertainty}} \end{pmatrix}$$

$$\Sigma = \begin{bmatrix} 1.25 & 0 \\ 0 & 2.8125 \end{bmatrix} \quad \mathrm{chol}(\Sigma) = \begin{bmatrix} 1.118 & \mathbf{0} \\ 0 & 1.6771 \end{bmatrix}$$

- Population response of GDP growth to uncertainty is 0.5
- Estimated response is 0 under both recursive orderings

Alternative Orderings (2/2)

- Given the robustness of the reduced-form estimates, an applied user would be tempted to conclude that the population response is zero, when in reality it is 0.5.
  - Can occur when off-diagonal of B_0^{-1} has opposite signs
  - Even if the responses are the same, can never know whether the population response has been identified
- If the off-diagonal values are the same sign, the reduced-form error correlation would typically be far from zero
  - Responses would not be invariant to alternative orderings
  - Trying to establish robustness makes little sense
- Led some researchers to provide an alternative justification for the use of the two recursive orderings

Bounding the Responses (1/2)

- Belief is that the two recursive orderings place an upper and lower bound on the effects of uncertainty shocks
- The following model shows this is **not** valid in general:

$$\begin{pmatrix} u_t^{\Delta gdp} \\ u_t^{\text{uncertainty}} \end{pmatrix} = \begin{bmatrix} 1 & 0.5 \\ -0.9 & 1 \end{bmatrix} \begin{pmatrix} w_t^{\text{other}} \\ w_t^{\text{uncertainty}} \end{pmatrix}$$

$$\Sigma = \begin{bmatrix} 1.25 & -0.4 \\ -0.4 & 1.81 \end{bmatrix} \quad \text{chol}(\Sigma) = \begin{bmatrix} 1.118 & \mathbf{0} \\ -0.3578 & 1.2969 \end{bmatrix}$$

- Population response of GDP growth to uncertainty is 0.5
- Estimated response of GDP growth to uncertainty is 0

Bounding the Responses (2/2)

- Reversing the order of the variables yields

$$\begin{pmatrix} u_t^{\text{uncertainty}} \\ u_t^{\Delta gdp} \end{pmatrix} = \begin{bmatrix} 1 & -0.9 \\ 0.5 & 1 \end{bmatrix} \begin{pmatrix} w_t^{\text{uncertainty}} \\ w_t^{\text{other}} \end{pmatrix}$$

$$\Sigma = \begin{bmatrix} 1.81 & -0.4 \\ -0.4 & 1.25 \end{bmatrix} \quad \text{chol}(\Sigma) = \begin{bmatrix} 1.3454 & 0 \\ -\mathbf{0.2973} & 1.0778 \end{bmatrix}$$

- Estimated response of GDP growth to uncertainty is -0.3
- Conventional wisdom suggests the population response is between -0.3 and 0, but the population response is 0.5
- Failure can occur with positively correlated, uncorrelated or negatively correlated reduced-form errors

Sufficiency Conditions (1/3)

- Consider the following model,

$$\begin{pmatrix} u_t^{\text{uncertainty}} \\ u_t^{\Delta gdp} \end{pmatrix} = \begin{bmatrix} b_0^{11} & b_0^{12} \\ b_0^{21} & b_0^{22} \end{bmatrix} \begin{pmatrix} w_t^{\text{uncertainty}} \\ w_t^{\text{other}} \end{pmatrix}$$

- The variance-covariance matrix is given by

$$\Sigma = B_0^{-1}(B_0^{-1})' = \begin{bmatrix} (b_0^{11})^2 + (b_0^{12})^2 & b_0^{11}b_0^{21} + b_0^{12}b_0^{22} \\ b_0^{11}b_0^{21} + b_0^{12}b_0^{22} & (b_0^{21})^2 + (b_0^{22})^2 \end{bmatrix}.$$

- Lower triangular Cholesky decomposition

$$\Sigma = \underbrace{\begin{bmatrix} c_1 & 0 \\ c_2 & c_3 \end{bmatrix}}_{\text{chol}(\Sigma)} \underbrace{\begin{bmatrix} c_1 & c_2 \\ 0 & c_3 \end{bmatrix}}_{\text{chol}(\Sigma)'} = \begin{bmatrix} c_1^2 & c_1c_2 \\ c_1c_2 & c_2^2 + c_3^2 \end{bmatrix}.$$

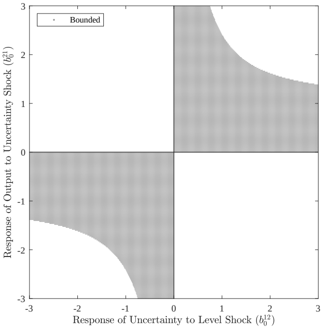
Sufficiency Conditions (2/3)

- Bounds on the population response of GDP growth to an uncertainty shock are $[0, c_2]$, where

$$c_2 = \frac{b_0^{11} b_0^{21} + b_0^{12} b_0^{22}}{\sqrt{(b_0^{11})^2 + (b_0^{12})^2}}.$$

- The value of c_2 depends non-linearly on response of:
  - Uncertainty to own shock (b_0^{11}) and growth shock (b_0^{12})
  - Growth to uncertainty shock (b_0^{21}) and own shock (b_0^{22})
- If $b_0^{21} > 0$, then $c_2 > b_0^{21}$ is sufficient for the population response to be bounded. Similarly, if $b_0^{21} < 0$, the sufficient condition is that $c_2 < b_0^{21}$.

Sufficiency Conditions (3/3)



- Example: $b_0^{11} = b_0^{22} = 1$ then $c_2 = (b_0^{21} + b_0^{12})/\sqrt{1 + (b_0^{12})^2}$
- Values of b_0^{11} and b_0^{22} influence bounding region
- Conditions much more complicated for larger VARs

Figure

Practicality of Bounding Conditions

- Bounding condition cannot be checked before implementing approach
- Population parameter values are unknown in practice
- Estimating those parameters requires identifying assumptions

Summary of Analytical Results

1. Examples show the issues with ad hoc recursive orderings, without any estimation or model misspecification error
2. No reason to expect recursive estimates to recover the population response when it is not recursive
3. No reason for recursive estimates based on alternative orderings to bound the population response in general
4. True whether the reduced-form errors are positively correlated, uncorrelated, or negatively correlated or whether alternative orderings produce the same responses
5. Sufficiency conditions can be derived but are of limited use as population parameters unknown in practice

Simulation Results

- Ad hoc recursive VAR orderings fail, in particular, when output and uncertainty are simultaneously determined
- Best illustrated using a fully specified economic model
- DSGE-based simulation results useful for four reasons:
  1. Formally demonstrate uncertainty is endogenous under reasonable model specifications
  2. Allow for dynamics, which may affect asymptotic bias
  3. Can vary importance of exogenous uncertainty shocks in driving aggregate uncertainty
  4. Examine merits of alternative non-recursive identification approaches

DSGE Model of Uncertainty (1/3)

- Consider a textbook real business cycle model
  - Disaster risk (Gourio, 2012; Shen, 2015)
  - Recursive preferences (Epstein-Zin, 1989)
- The representative household solves

$$J_t = \max_{c_t, n_t, b_{t+1}} \left[(1-\beta) u_t^{1-1/\psi} + \beta (E_t[J_{t+1}^{1-\gamma}])^{\frac{1-1/\psi}{1-\gamma}} \right]^{\frac{1}{1-1/\psi}}$$

subject to

$$u_t = c_t^{\eta}(1-n_t)^{1-\eta}$$
$$c_t + b_{t+1}/r_t = w_t n_t + b_t + d_t$$

DSGE Model of Uncertainty (2/3)

- The representative firm solves

$$V_t = \max_{n_t, i_t, k_{t+1}} d_t + E_t[x_{t+1} V_{t+1}]$$

subject to

$$d_t = y_t - w_t n_t - i_t$$
$$y_t = a_t k_t^{\alpha} n_t^{1-\alpha}$$
$$k_{t+1} = \Theta_{t+1}((1-\delta)k_t + i_t)$$

- Θ_t is a capital quality (or disaster) shock determined by

$$\Theta_t = \mathbb{I}(e_t \geq e^*) + \theta \mathbb{I}(e_t < e^*)$$

If $e_t \geq e^*$, there is no capital quality loss ($\Theta_t = 1$)
If $e_t < e^*$, a disaster causes capital quality loss ($\Theta_t = \theta < 1$)

DSGE Model of Uncertainty (3/3)

- Stochastic processes evolve according to

$$\begin{aligned}
\ln a_t &= \rho_a \ln a_{t-1} + \sigma_{a,t-1}\varepsilon_{a,t} \\
\ln e_t &= \rho_e \ln e_{t-1} + \sigma_{e,t-1}\varepsilon_{e,t} \\
\ln \sigma_{a,t} &= (1-\rho_{av}) \ln \bar{\sigma}_a + \rho_{av} \ln \sigma_{a,t-1} + \sigma_{av}\varepsilon_{av,t} \\
\ln \sigma_{e,t} &= (1-\rho_{ev}) \ln \bar{\sigma}_e + \rho_{ev} \ln \sigma_{e,t-1} + \sigma_{ev}\varepsilon_{ev,t}
\end{aligned}$$

- Aggregate uncertainty:

$$\mathcal{U}_t = \sqrt{E_t[(\ln(y_{t+1}/y_t) - E_t[\ln(y_{t+1}/y_t)])^2]}$$

Endogenously fluctuates over time due to the nonlinear propagation of the level shocks $(\varepsilon_a, \varepsilon_e)$ and exogenously fluctuates over time due to the volatility shocks $(\varepsilon_{av}, \varepsilon_{ev})$

DSGE Model Parameterization

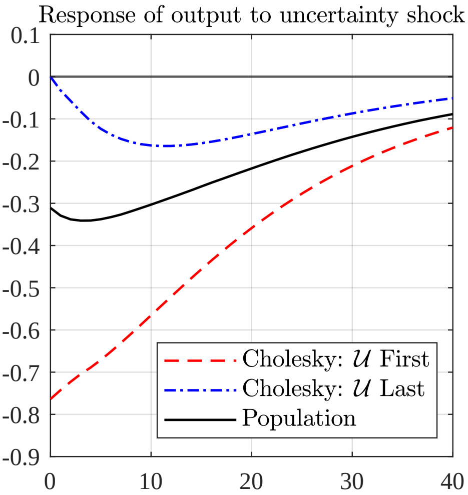
Parameters	Value	Parameters	Value
Discount Factor (β)	0.995	Risk Aversion (γ)	80
Cost Share of Capital (α)	0.333	Elasticity of Substitution (ψ)	1
Capital Depreciation Rate (δ)	0.025	Frisch Elasticity of Labor Supply (η^{λ})	2
Size of Disaster (θ)	0.95	Disaster Risk Threshold (e^*)	0.97
Disaster Risk AC (ρ_e)	0.90	Disaster Risk Shock SD ($\bar{\sigma}_e$)	0.0065
Technology AC (ρ_a)	0.90	Technology Shock SD ($\bar{\sigma}_a$)	0.007
Disaster Risk Vol. Shock AC (ρ_{ev})	0.90	Disaster Risk Vol. Shock SD (σ_{ev})	0.175
Technology Vol. Shock AC (ρ_{av})	0.90	Technology Vol. Shock SD (σ_{av})	0.0275

- Consider two cases with different uncertainty shocks:
  1. Disaster risk volatility shocks
  2. TFP volatility shocks
- Both models match persistence and standard deviation of observed TFP; standard deviation of detrended output

Monte Carlo Procedure

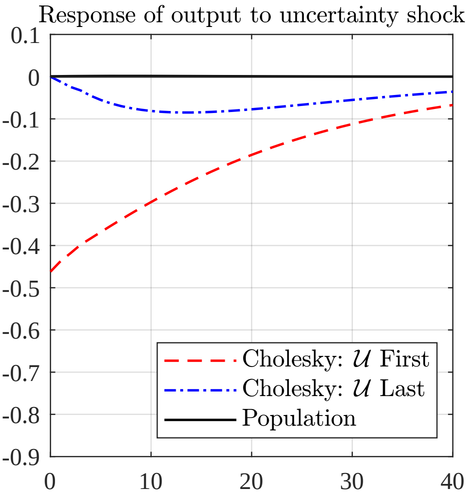
- Generate time series for log output, output uncertainty and investment of length $T = 1{,}000{,}000$
- Construct VAR(4) response estimates from alternative recursive orderings
- Close approximation to the asymptotic limit of the VAR

Model 1 $(\varepsilon_e, \varepsilon_a, \varepsilon_{ev})$



- Both orderings fail to recover the true response
- Bounding of the population response does occur in this case

Model 2 $(\varepsilon_e, \varepsilon_a, \varepsilon_{av})$



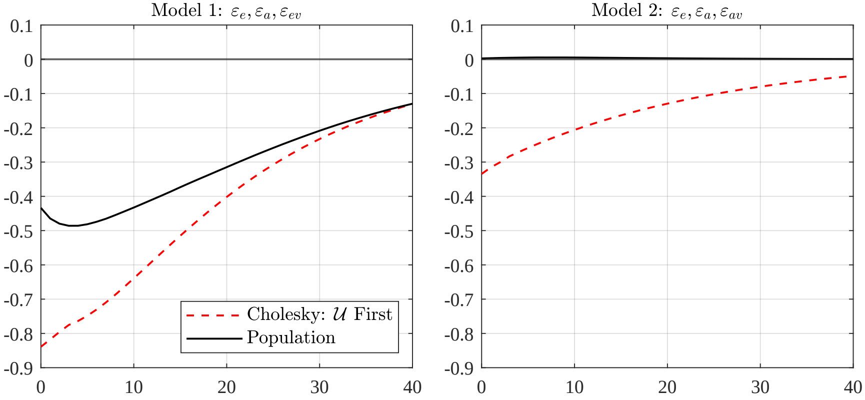
- Both orderings again fail to recover true response
- Bounding fails: Both responses indicate a decline in output but population response is positive

Quasi-recursive Specification

	Model 1 $(\varepsilon_e, \varepsilon_a, \varepsilon_{ev})$		Model 2 $(\varepsilon_e, \varepsilon_a, \varepsilon_{av})$	
Parameters	Baseline	Quasi-Recursive	Baseline	Quasi-Recursive
Disaster Risk Shock SD ($\bar{\sigma}_e$)	0.0065	0.0045	0.0065	0.0065
Disaster Risk Vol. Shock AC (ρ_{ev})	0.90	0.90	–	–
Disaster Risk Vol. Shock SD (σ_{ev})	0.175	0.365	–	–
Technology Vol. Shock AC (ρ_{av})	–	–	0.90	0.90
Technology Vol. Shock SD (σ_{av})	–	–	0.0275	0.08

- Increase the importance of the exogenous uncertainty shocks
- 90% of uncertainty variation due to uncertainty shock
- Investigate bias in recursively identified VAR model with uncertainty ordered first

Bias Persists



Alternatives to Recursive Identification

- Recursively identified VARs failed to recover the true uncertainty shock
- We consider three non-recursive alternatives:
  1. Max share identification
  2. Generalized max share (Carriero and Volpicella, 2024)
  3. Instrumental variables (Carriero et al., 2015)
- Re-do the simulation for each alternative and report RMSE for estimated output response

Max Share Notation

1. **Reduced-form representation**

$$\mathbf{y}_t = \Phi(L)\mathbf{u}_t, \quad \Phi(L) = I_K + \Phi_1 L + \Phi_2 L^2 + \cdots$$

2. **Structural representation**

$$\mathbf{u}_t = B_0^{-1}\mathbf{w}_t, \quad B_0^{-1}(B_0^{-1})' = \Sigma = (PQ)(PQ)'$$

   - P is the lower triangular Cholesky decomposition of Σ
   - Q is an orthogonal matrix ($Q'Q = QQ' = I_K$)

3. **Identification**

$$\mathbf{y}_t = \Phi(L)B_0^{-1}\mathbf{w}_t = \Phi(L)PQ\mathbf{w}_t$$

   - Impact of shock j associated with jth column of Q
   - Objective is to pin down some or all columns of Q

Forecast Error Variance Share

- h-step ahead forecast error

$$\mathbf{y}_{t+h} - E_{t-1}\mathbf{y}_{t+h} = \sum_{\tau=0}^{h} \Phi_\tau PQ\mathbf{w}_{t+h-\tau}$$

- Share of the forecast error variance of variable i attributable to shock j at horizon h is given by

$$\Omega_{i,j}(h) = \frac{\sum_{\tau=0}^{h} \Phi_{i,\tau} P\gamma_j\gamma_{j'} P'\Phi'_{i,\tau}}{\sum_{\tau=0}^{h} \Phi_{i,\tau}\Sigma\Phi'_{i,\tau}}$$

  - γ_j is the jth column of Q
  - $\Phi_{i,\tau}$ is the ith row of the lag polynominal at lag τ

Standard Max share

- Identification involves pinning down values of relevant γ_j
- Order uncertainty first so that γ_1 is associated with identified "uncertainty" shock
- The value of γ_1 is determined by

$$\gamma_1 = \text{argmax } \Omega_{1,1}(h), \quad \Omega_{1,1}(h) \equiv \frac{\sum_{\tau=0}^{h} \Phi_{1,\tau} P \gamma_1 \gamma_1' P' \Phi_{1,\tau}'}{\sum_{\tau=0}^{h} \Phi_{1,\tau} \Sigma \Phi_{1,\tau}'}.$$

Generalized Max share

The objective function is given by

$$Q^*_{1:K} = \text{argmax} \sum_{i=1}^{K} \Omega_{i,i}(h_i), \quad \Omega_{i,i}(h_i) \equiv \frac{\sum_{\tau=0}^{h_i} \Phi_{i,\tau} P \gamma_i \gamma_i' P' \Phi_{i,\tau}'}{\sum_{\tau=0}^{h_i} \Phi_{i,\tau} \Sigma \Phi_{i,\tau}'},$$

subject to a series of inequality constraints,

$$\Omega_{i,i}(h_i) \geq \Omega_{j,i}(h_i), \quad j = 1, \ldots, K, \quad \forall i \neq j,$$

and the constraint that $QQ' = I$.

INSTRUMENTAL VARIABLES

- Instrument is the exogenous uncertainty in the DSGE model $(\ln \sigma_{e,t}, \ln \sigma_{a,t})$
- Estimate a block recursive VAR model with the instrument ordered first, uncertainty second, and output last.
  - IV estimator constructed from a Cholesky decomposition
  - Equivalent to using $\ln \sigma_{a,t}$ or $\ln \sigma_{e,t}$ as internal instrument
- Also consider case where instrument is contaminated with additive Gaussian measurement error
  - Replace instrument with $\ln \sigma^n_{i,t} \equiv \ln \sigma_{i,t} + \sigma_n \epsilon_{n,t}$ for $i \in \{a, e\}$, where $\epsilon_{n,t} \sim \mathbb{N}(0, 1)$.

RMSE FOR OUTPUT RESPONSE

	Model 1 $(\varepsilon_e, \varepsilon_a, \varepsilon_{ev})$		Model 2 $(\varepsilon_e, \varepsilon_a, \varepsilon_{av})$	
VAR Estimator	Baseline	Quasi-Recursive	Baseline	Quasi-Recursive
Cholesky: $\mathcal{U}$ First	7.1	5.0	8.7	6.2
Cholesky: $\mathcal{U}$ Last	4.6	–	2.5	–
Standard Max Share	7.1	5.0	8.7	6.2
Generalized Max Share	0.7	2.4	5.1	3.6
Internal Instrument	1.1	2.5	0.1	0.1

Notes: VAR(4) with $T = 1{,}000{,}000$. RMSE is a sum over 40 quarters.

Summary of Simulation Results

- Alternative recursive orderings can fail to bound the population response
- Recursively identified VAR models lead to biased IRFs when uncertainty is endogenous
- This issue persists even if exogenous uncertainty shocks are dominant driver of measured uncertainty
- Instrumental variables estimator provides significant RMSE reduction in all cases

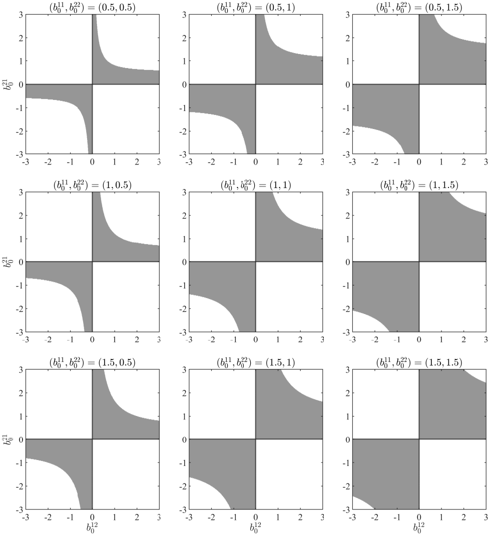
Discussion: Other Contexts

- Our results do not imply that recursive identifications can never be used
- Situations where their use may be justified:
    1. Exogenous instruments as internal instrument in VAR
    2. Predetermined variables
    3. Theoretically justified variables

Conclusion

- Verifying robustness of the response estimates under alternative recursive identification schemes is misleading
- Robustness does not ensure validity of the response estimates when the population model is non-recursive
- Also no guarantee that responses based on alternative recursive orderings bound the population response
- Recursive VAR models of uncertainty cannot be trusted

Additional Results



Back

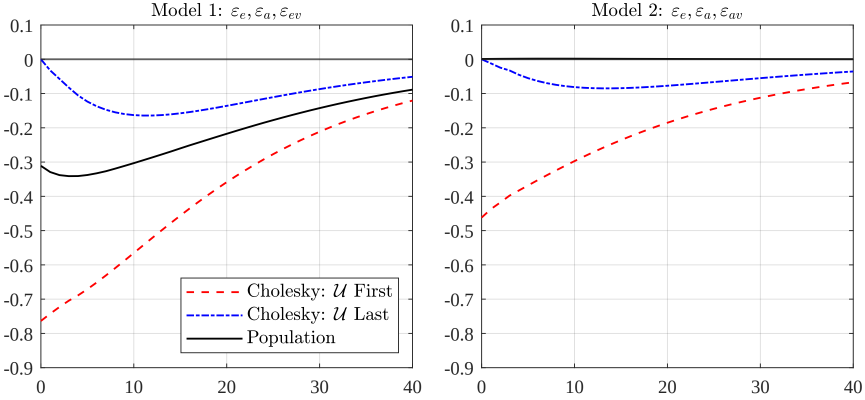
Sufficiency Conditions (2/3) (short)

- Bounds on the population response of GDP growth to an uncertainty shock are $[0, c_2]$, where

$$c_2 = \frac{b_0^{11} b_0^{21} + b_0^{12} b_0^{22}}{\sqrt{(b_0^{11})^2 + (b_0^{12})^2}}.$$

- The value of c_2 depends non-linearly on response of:
  - Uncertainty to own shock (b_0^{11}) and growth shock (b_0^{12})
  - Growth to uncertainty shock (b_0^{21}) and own shock (b_0^{22})
- Bounding conditions more complicated for larger VARs

Bounding is not Guaranteed



Alternatives Identification Methods

1. Max share identification
   - Uncertainty shock contributes maximum variance to uncertainty
2. Generalized max share (Carriero and Volpicella ,2024)
   - Identify multiple structural shocks at one time
   - Each shock explains more of the variance of target variable than other variables (e.g., uncertainty shock explains more of uncertainty than output)
3. Instrumental variables (Carriero et al., 2015)
   - Use exogenous uncertainty $(\ln \sigma_{e,t}, \ln \sigma_{a,t})$ as instrument
   - Block recursive VAR model with instrument ordered first