Solving, Simulating and Estimating Nonlinear DSGE Models

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Outline

TOOLBOX FUNCTIONS

- `script.m`: assigns options to `O` and runs the algorithm
- `parameters.m`: assigns model parameters to `P`
- `steadystate.m`: assigns steady-state values to `S(P)`
- `variables.m`: outputs a structure, `V`, containing indices of variables, forecast errors, and shocks and variable titles
- `grids.m`: assigns the discretized state space to `G(O, P)`
- `guess.m`: assigns the initial conjectures to `pf(P,S,G)`
- `linmodel.m`: outputs the linear transition matrix, `T`, the impact matrix, `M`, and a 2-element vector of flags, `eu`, indicating existence and uniqueness of the linear solution
- `eqm.m`: outputs a vector, `R`, containing the residuals to a subsystem of expectational equations that are constrained by all of the other equations in the equilibrium system

RBC MODEL: FIRMS

A firm chooses $\{n_t, k_t\}$ to maximize:

$$y_t - w_t n_t - r_t^k k_t$$

subject to

$$y_t = z_t k_t^\alpha n_t^{1-\alpha}$$

Optimality conditions:

$$w_t = (1-\alpha) y_t / n_t$$
$$r_t^k = \alpha y_t / k_t$$

RBC MODEL: HOUSEHOLDS

A household chooses $\{c_t, n_t, i_t, k_{t+1}\}$ to maximize:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{n_t^{1+\eta}}{1+\eta} \right)$$

subject to

$$c_t + i_t = w_t n_t + r_t^k k_t$$
$$k_{t+1} = (1-\delta) k_t + i_t$$

Optimality conditions:

$$w_t = \chi n_t^{\eta} c_t^{\sigma}$$
$$1 = \beta E_t [\underbrace{(c_t / c_{t+1})^{\sigma} (r_{t+1}^k + 1 - \delta)}_{\equiv \Phi(k_t, z_{t+1})}]$$

RBC MODEL: EQUILIBRIUM

Nonlinear system of equations:

$$w_t = \chi n_t^{\eta} c_t^{\sigma}$$
$$1 = \beta E_t[(c_t/c_{t+1})^{\sigma}(r_{t+1}^k + 1 - \delta)]$$
$$w_t = (1-\alpha) y_t / n_t$$
$$r_t^k = \alpha y_t / k_t$$
$$y_t = z_t k_t^{\alpha} n_t^{1-\alpha}$$
$$c_t + i_t = y_t$$
$$k_{t+1} = (1-\delta) k_t + i_t$$
$$z_t = (1-\rho)\bar{z} + \rho z_{t-1} + \sigma_z \varepsilon_t,\ \varepsilon_t \sim \mathbb{N}(0,1)$$

Variables(8): n, w, c, y, r^k, k, i, z

DISCRETIZED STATE SPACE

- State variables: k_t, z_t
- Number of grid points: N_k, N_z
- Grid boundaries: $[k_{\min}, k_{\max}]$ and $[z_{\min}, z_{\max}]$
- Create evenly spaced grids:

$$x_{grid} = \texttt{linspace}(x_{\min}, x_{\max}, N_x), \quad x \in \{k, z\}$$

- State space contains $N = N_k \times N_z$ independent nodes
- Create an array for each state variable, where every position is a unique permutation of the state space:

$$[k_{gr}, z_{gr}] = \texttt{ndgrid}(k_{grid}, z_{grid})$$

FUNCTIONAL APPROXIMATION

- True RE solution only exists in very special cases
- Find an approximating function that maps the state space to the optimal decision rule for consumption:

$$\underbrace{n(k,z)}_{\text{True RE Solution}} \approx \underbrace{\mathcal{P}_n(k,z)}_{\text{Approximating Function}}$$

- Basic elements of policy function iteration:
  1. Interpolation: Linear, Least squares
  2. Integration: Trapezoid, Gauss-Hermite, Rouwenhorst
  3. Iteration: Time, Fixed-point

ALGORITHM OUTLINE

1. Use the log-linear solution to obtain $\mathcal{P}_n^0$ and set $q = 1$

On node $i = 1, 2, \ldots$

2. Solve for the updated state (k', z'), given (k_i, z_i)
3. Use linear interpolation to compute $\mathcal{P}_n^q(k', z')$
4. Solve for the time $t + 1$ variables that enter expectations
5. Evaluate expectations using numerical integration:

$$E[\Phi(z')] = \beta E[c(k', z')^{-\sigma}(\alpha z' k'^{\alpha-1} n'^{1-\alpha} + 1 - \delta)]$$

6. Use a solver to find $\mathcal{P}_n^q(k_i, z_i)$ that satisfies the system

Then...

7. Calculate the distance between updates: $\mathcal{P}_n^q - \mathcal{P}_n^{q-1}$
8. If $|\text{dist}| < \text{tol}$, then stop. If not, then set $q = q + 1$ and repeat steps 2-7 using $\mathcal{P}_n^q = \mathcal{P}_n^{q-1}$ as the new initial conjecture

INITIAL CONJECTURE

- Use the linear solution from `gensys` as a guess for $\mathcal{P}_n$
- Map the model to the following form:

$$G_0\hat{Y}_t = G_1\hat{Y}_{t-1} + \Psi\varepsilon_t + \Pi\eta_t + C$$

  where $\hat{Y}$ is a vector of variables, ε is a vector of shocks, C is a vector of constants, and η is a vector of forecast errors
- Linear solution takes the form:

$$\hat{Y}' = T\hat{Y} + M\varepsilon$$

- Convert the state space to deviations from steady state
- Compute an initial conjecture for all nodes ($i = 1, \ldots, N$):

$$\hat{\mathcal{P}}_n = \underbrace{T(n_{idx}, [k_{idx}, z_{idx}])}_{1\times 2}\underbrace{[\text{vec}(\hat{k}_{gr}), \text{vec}(\hat{z}_{gr})]^T}_{2\times N}$$

- Convert $\hat{\mathcal{P}}_n$ to levels ($\mathcal{P}_n = \bar{n}(1 + \hat{\mathcal{P}}_n)$) and assign to `pf.n`

RBC MODEL: LINEAR SYSTEM

Log-Linear system of equations:

$$\hat{w}_t = \eta \hat{n}_t + \sigma \hat{c}_t$$
$$\sigma(\hat{c}_t - E_t \hat{c}_{t+1}) + (1 - \beta(1-\delta)) E_t \hat{r}^k_{t+1} = 0$$
$$\hat{w}_t = \hat{y}_t - \hat{n}_t$$
$$\hat{r}^k_t = \hat{y}_t - \hat{k}_t$$
$$\hat{y}_t = \hat{z}_t + \alpha \hat{k}_t + (1-\alpha)\hat{n}_t$$
$$\bar{c}\hat{c}_t + \bar{\imath}\hat{\imath}_t = \bar{y}\hat{y}_t$$
$$\hat{k}_t = (1-\delta)\hat{k}_{t-1} + \delta \hat{\imath}_{t-1}$$
$$\hat{z}_t = \rho \hat{z}_{t-1} + \sigma_z \varepsilon_t$$

Variables(8): $\hat{n}, \hat{w}, \hat{c}, \hat{y}, \hat{r}^k, \hat{k}, \hat{\imath}, \hat{z}$

PIECEWISE LINEAR INTERPOLATION

- Goal: Find the policy function value $\mathcal{P}_n(k', z')$
- Interpolate in the k direction:

$$\begin{aligned}\mathcal{P}_n(k', z_j) &= \mathcal{P}_n(k_i, z_j) + (k' - k_i)\frac{\mathcal{P}_n(k_{i+1}, z_j) - \mathcal{P}_n(k_i, z_j)}{k_{i+1} - k_i} \\ &= \underbrace{\frac{k_{i+1} - k'}{k_{i+1} - k_i}}_{\omega_{k_i}} \mathcal{P}_n(k_i, z_j) + \underbrace{\frac{k' - k_i}{k_{i+1} - k_i}}_{\omega_{k_{i+1}}} \mathcal{P}_n(k_{i+1}, z_j)\end{aligned}$$

- Then interpolate in the z direction:

$$\mathcal{P}_n(k', z') = \underbrace{\frac{z_{j+1} - z'}{z_{j+1} - z_j}}_{\omega_{z_j}} \mathcal{P}_n(k', z_j) + \underbrace{\frac{z' - z_j}{z_{j+1} - z_j}}_{\omega_{z_{j+1}}} \mathcal{P}_n(k', z_{j+1})$$

- Combine these two equations:

$$\mathcal{P}_n(k', z') = \sum_{a=0}^{1} \sum_{b=0}^{1} \omega_{k_{i+a}} \omega_{z_{j+b}} \mathcal{P}_n(k_{i+a}, z_{j+b})$$

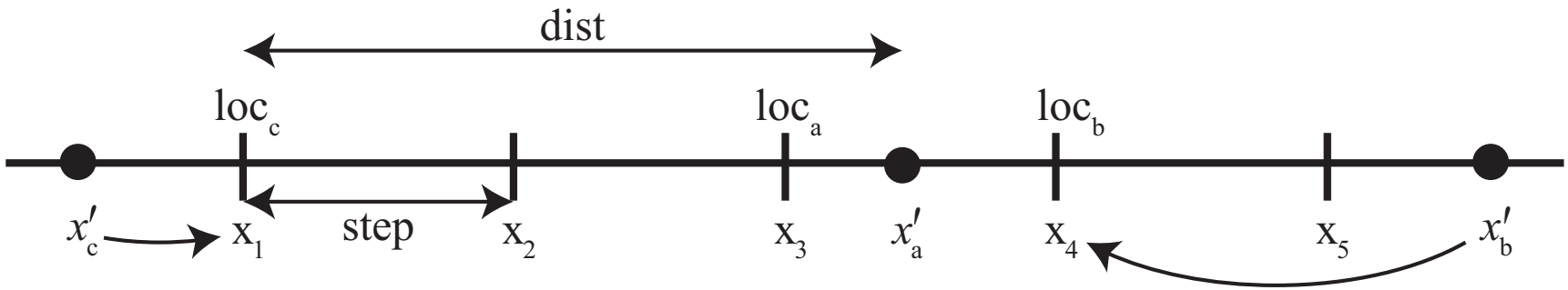
PIECEWISE LINEAR INTERPOLATION

- We have policy function values on the nearest nodes:

$$[\mathcal{P}_n(k_i, z_j), \mathcal{P}_n(k_i, z_{j+1}), \mathcal{P}_n(k_{i+1}, z_j), \mathcal{P}_n(k_{i+1}, z_{j+1})]$$

  once we determine the grid indices, i, j
- Locate the grid point to left of x', $x \in \{k, z\}$

$$\text{step} = x_2 - x_1, \qquad \text{dist} = x' - x_1$$
$$\text{loc} = \min(N_x - 1, \max(1, \text{floor}(\text{dist}/\text{step}) + 1))$$



PIECEWISE LINEAR INTERPOLATION

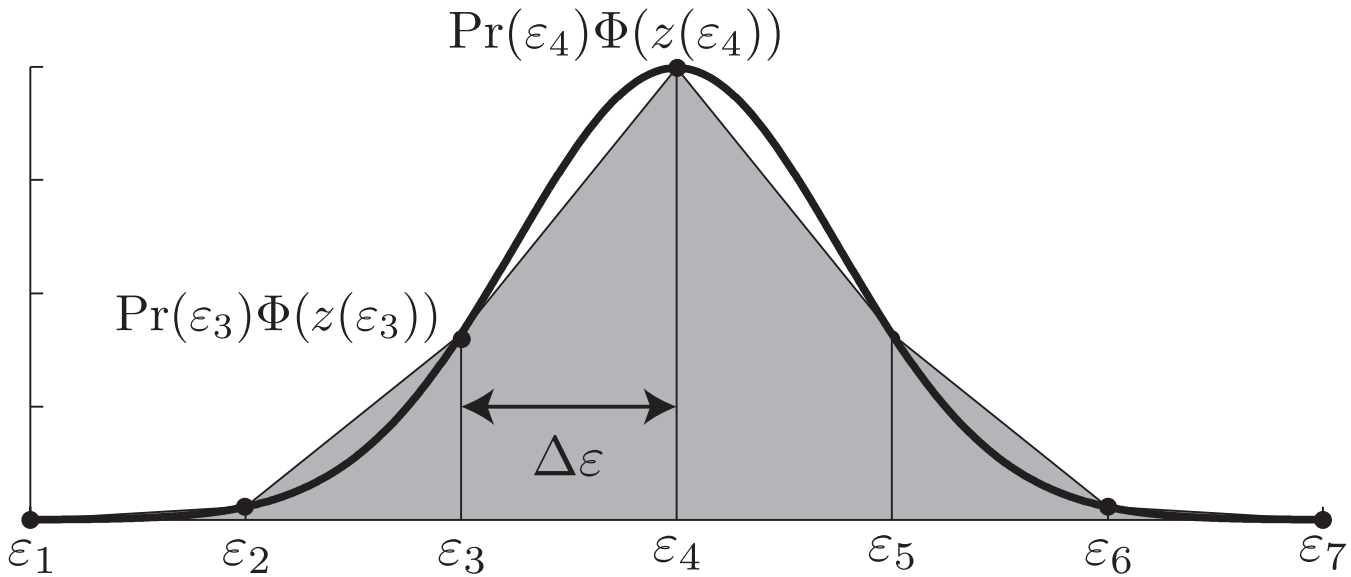
- Use a nested loop to calculate the interpolated value:

```
nestedsum = 0;   %initialize
for a = 0:1      %loop for k
  for b = 0:1    %loop for z
    nestedsum = nestedsum + ...
      wk(1+a)*wz(1+b)*pf.c(kloc+a,zloc+b);
  end
end
```

- Must calculate the interpolated value for each realization of the stochastic variable(s), each of which requires calculating a different set of locations and weights
- Number of loops equals the number of states

INTEGRATION: TRAPEZOID RULE

$$
\begin{aligned}
E[\Phi(z)] \approx & \frac{\Pr(\varepsilon_1)\Phi(z(\varepsilon_1)) + \Pr(\varepsilon_2)\Phi(z(\varepsilon_2))}{2}\Delta\varepsilon \\
& + \frac{\Pr(\varepsilon_2)\Phi(z(\varepsilon_2)) + \Pr(\varepsilon_3)\Phi(z(\varepsilon_3))}{2}\Delta\varepsilon + \cdots \\
& + \frac{\Pr(\varepsilon_{m-1})\Phi(z(\varepsilon_{m-1})) + \Pr(\varepsilon_m)\Phi(z(\varepsilon_m))}{2}\Delta\varepsilon \\
& = \frac{\Delta\varepsilon}{2}\left[2\sum_{i=1}^{m}\Pr(\varepsilon_i)\Phi(z(\varepsilon_i)) - \Pr(\varepsilon_1)\Phi(z(\varepsilon_1)) - \Pr(\varepsilon_m)\Phi(z(\varepsilon_m))\right]
\end{aligned}
$$



INTEGRATION: GAUSS-HERMITE

- Expectation given a shock, $\varepsilon \sim N(\mu, \sigma^2)$:

$$E[\Phi(z(\varepsilon))] = (2\pi\sigma^2)^{-1/2} \int_{-\infty}^{\infty} \Phi(z(\varepsilon)) e^{-(\varepsilon-\mu)^2/(2\sigma^2)} d\varepsilon$$

- Apply change of variables, $\upsilon = (\varepsilon - \mu)/(\sqrt{2}\sigma)$:

$$E[\Phi(z(\upsilon))] = \pi^{-1/2} \int_{-\infty}^{\infty} \Phi(z(\sqrt{2}\sigma\upsilon + \mu)) e^{-\upsilon^2} d\upsilon$$

$$\approx \pi^{-1/2} \sum_{i=1}^{n} \omega_i \Phi(z(\sqrt{2}\sigma\upsilon_i + \mu))$$

- υ_i are the roots of $H_n(x)$ and ω_i are weights given by

$$\omega_i = 2^{n+1} n! \sqrt{\pi} [H_{n+1}(\upsilon_i)]^{-2}$$

H_n is the physicist's Hermite polynomial of order n

INTEGRATION: ROUWENHORST

- Used to approximate an exogenous $AR(1)$ process
- Kopecky and Suen (2010) show the Rouwenhorst method outperforms other approximations of an $AR(1)$ process
- Approximation is an n-state Markov switching process
- Method determines the bounds of the exogenous state variables, the nodes, and the transition probabilities
- Let $z \sim AR(1)$ with persistence ρ, mean μ_z, and variance

$$\sigma_z^2 = \sigma_\varepsilon^2/(1-\rho^2).$$

- The bounds and transition matrix are chosen to match conditional and unconditional mean and variance of z

INTEGRATION: ROUWENHORST

- The n states for the discretized process z are evenly spaced on $[\mu_z - \sigma_z\sqrt{n-1}, \mu_z + \sigma_z\sqrt{n-1}]$
- The transition matrix is computed recursively:
  - For $n = 2$, let $q = (\rho + 1)/2$:

$$P_2 = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} q & 1-q \\ 1-q & q \end{bmatrix}$$

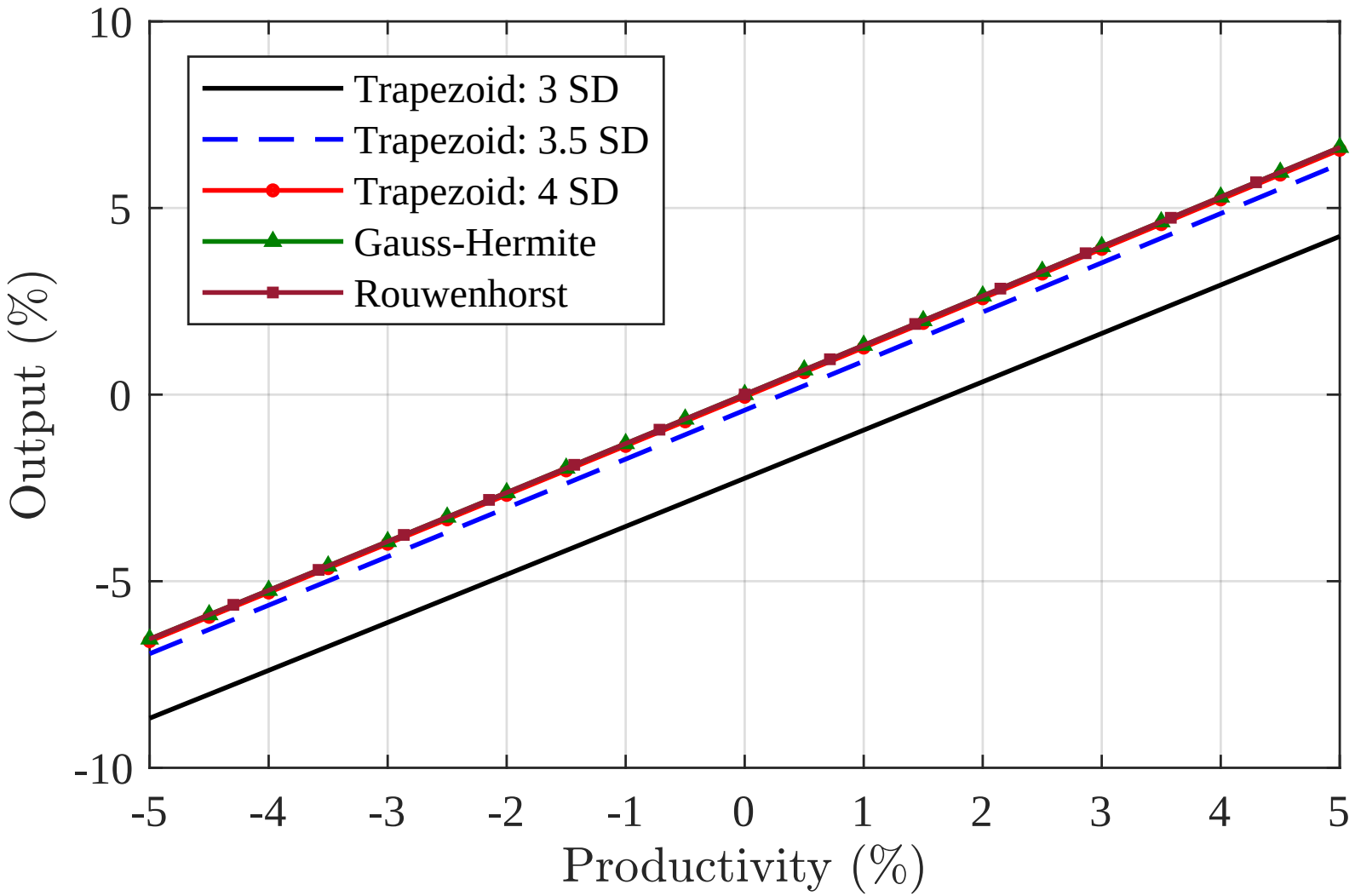
  - For $n > 2$:

$$P_n = \underbrace{q \begin{bmatrix} P_{n-1} & \mathbf{0}_{n-1\times 1} \\ \mathbf{0}_{1\times n-1} & 0 \end{bmatrix}}_{\text{Prob of staying in } 1{:}n-1} + \underbrace{(1-q) \begin{bmatrix} \mathbf{0}_{n-1\times 1} & P_{n-1} \\ 0 & \mathbf{0}_{1\times n-1} \end{bmatrix}}_{\text{Prob. of going to } 2{:}n|1{:}n-1}$$

$$+ \underbrace{(1-q) \begin{bmatrix} \mathbf{0}_{1\times n-1} & 0 \\ P_{n-1} & \mathbf{0}_{n-1\times 1} \end{bmatrix}}_{\text{Prob. of going to } 1{:}n-1|2{:}n} + \underbrace{q \begin{bmatrix} 0 & \mathbf{0}_{1\times n-1} \\ \mathbf{0}_{n-1\times 1} & P_{n-1} \end{bmatrix}}_{\text{Prob of staying in } 2{:}n}$$

Divide rows 2 through $n-1$ by 2 so they sum to 1

POLICY FUNCTION COMPARISON



ADVANTAGES OF ROUWENHORST

- Improves accuracy:
  - Matches 5 statistics of an $AR(1)$ process: autocorrelation and the conditional and unconditional mean and SD
  - No interpolation or extrapolation of the policy function at future realizations of the exogenous state variables
- Reduces computation time:
  - Requires fewer nodes relative to the other methods
  - Unnecessary to locate and obtain weights for the exogenous state variables if using linear interpolation
  - Reduces the dimension of the nested loop in the linear interpolation step by the number of exogenous states

GAUSS-HERMITE: INTERPOLATION

Suppose there are 4 states: 2 endogenous and 2 exogenous:

```
% Calling function
for i2 = 1,ne2
 for i1 = 1,ne1
   pfMat(i1,i2) = interp(grids,updated state,pfs)
 end
end

% Interpolation function
for m4 = 0,1
 for m3 = 0,1
  for m2 = 0,1
   for m1 = 0,1
     wtemp = w1(m1+1)*w2(m2+1)*w3(m3+1)*w4(m4+1)
     sum = sum + wtemp*pf(loc1+m1,loc2+m2,loc3+m3,loc4+m4)
   end
  end
 end
end
```

ROUWENHORST: INTERPOLATION

Suppose there are 4 states: 2 endogenous and 2 exogenous:

```
% Calling function
pfMat = interp(grids,updated endogenous state,pfs)

% Interpolation function
for m2 = 0,1
 for m1 = 0,1
   sum = sum + w1(m1+1)*w2(m2+1)*pf(:,:,loc1+m1,loc2+m2)
 end
end
```

No longer need to interpolate on the exogenous dimensions, which significantly reduces the number of computations

ROUWENHORST: INTEGRATION

- Conditional expectation of an n-state Markov process:

$$E\left[\Phi(z')|s=i\right] = \sum_{j=1}^{n} p_{ij}\Phi(z_j)$$

- Let $\nu \sim AR(1)$ in addition to z. Computing expectations across multiple exogenous processes generalizes to:

$$E\left[\Phi(z',\nu')|s_z=i_z, s_\nu=i_\nu\right] = \sum_{j_\nu=1}^{n_\nu}\sum_{j_z=1}^{n_z} p_{i_\nu j_\nu} p_{i_z j_z}\,\Phi(z_{j_z},\nu_{j_\nu})$$

where $i_z, j_z \in \{1,2,\ldots,n_z\}$ and $i_\nu, j_\nu \in \{1,2,\ldots,n_\nu\}$

IMPLEMENTATION: TWO SHOCKS

- Script: Compute shock realizations:

```
zpMat = G.z_nodes(:,ones(O.epsnu_pts,1));
nupMat = permute(G.nu_nodes(:,ones(O.epsz_pts,1)),[2,1]);
```

- Script: Compute transition matrix:

```
z_weightVec = G.z_weight(G.z_gr(inode) == G.z_grid,:)';
nu_weightVec = G.nu_weight(G.nu_gr(inode) == G.nu_grid,:)';
z_weightMat = z_weightVec(:,ones(O.epznu_pts,1));
nu_weightMat = permute(nu_weightVec(:,ones(O.epsz_pts,1),[2,1]));
weightMat = z_weightMat.*nu_weightMat
argzero = csolve('eqm',start,[],crit,itmax,state,...
                 structures,zpMat,nupMat,weightMat);
```

- Eqm: Compute all realizations and integrate:

```
Econs = sum(weightMat.*PhiMat,'all');
```

EXTENSION 1: REAL FRICTIONS

Introduce investment adjustment costs, variable capital utilization, and habit formation to the baseline model

$$x_{t+1} = \beta((c_t - hc_{t-1})/(c_{t+1} - hc_t))^\sigma$$

$$w_t = \chi(c_t - hc_{t-1})^\sigma n_t^\eta$$

$$q_t = E_t[x_{t+1}(v_{t+1}r_{t+1}^k + q_{t+1}(1 - \delta_{t+1}))]$$

$$1 = q_t\left(1 - \frac{\nu}{2}\left(\frac{i_t}{i_{t-1}} - 1\right)^2 - \nu\frac{i_t}{i_{t-1}}\left(\frac{i_t}{i_{t-1}} - 1\right)\right) + \nu E_t\left[x_{t+1}q_{t+1}\left(\frac{i_{t+1}}{i_t}\right)^2\left(\frac{i_{t+1}}{i_t} - 1\right)\right]$$

$$r_t^k = q_t(\delta_1 + \delta_2(v_t - 1))$$

$$r_t^k = \alpha y_t/(v_t k_{t-1})$$

$$w_t = (1 - \alpha)y_t/n_t$$

$$c_t + i_t = y_t$$

$$k_t = (1 - \delta_t)k_{t-1} + i_t(1 - \nu(i_t/i_{t-1} - 1)^2/2)$$

$$\delta_t = \delta_0 + \delta_1(v_t - 1) + \delta_2(v_t - 1)^2/2$$

$$y_t = z_t(v_t k_{t-1})^\alpha n_t^{1-\alpha}$$

State variables: $\{c, i, k, z\}$; Policy functions: $\{n, v\}$ (not unique)

EXTENSION 2: EPSTEIN-ZIN PREFERENCES

Introduce Epstein-Zin preferences into the baseline model

$$\chi w_t(1-n_t) = (1-\chi)c_t$$

$$1 = E_t[x_{t+1}(r^k_{t+1} + 1 - \delta)]$$

$$x_t = \beta(u_t/u_{t-1})^{1-1/\psi}(c_{t-1}/c_t)(J_t/a_{t-1})^{1/\psi-\gamma}$$

$$u_t = c_t^{\chi}(1-n_t)^{1-\chi}$$

$$a_t = \begin{cases} \exp(E_t[\ln J_{t+1}]) & \text{if } \gamma = 1 \\ (E_t[J_{t+1}^{1-\gamma}])^{1/(1-\gamma)} & \text{if } \gamma \neq 1 \end{cases}$$

$$J_t = \begin{cases} u_t^{1-\beta} a_t^{\beta} & \text{if } \psi = 1 \\ \left((1-\beta)u_t^{1-1/\psi} + \beta a_t^{1-1/\psi}\right)^{\frac{1}{1-1/\psi}} & \text{if } \psi \neq 1 \end{cases}$$

State variables: $\{k, z\}$ (unchanged); Policy functions: $\{n, J\}$

EXTENSION 3: STOCHASTIC VOLATILITY

- Productivity is $AR(1)$ with stochastic volatility

$$\ln z_t = \rho_z \ln z_{t-1} + \sigma_{z,t-1}\varepsilon_{z,t}$$
$$\ln \sigma_{z,t} = (1-\rho_{sv}) \ln \bar{\sigma}_a + \rho_{sv} \ln \sigma_{z,t-1} + \sigma_{sv}\varepsilon_{sv,t}$$

- Cannot discretize $\ln z_t$ with Rouwenhorst
- Instead, discretize...
  - $\varepsilon_{z,t}$ with Rouwenhorst
  - $\ln \sigma_{z,t}$ with Rouwenhorst
  - $\ln z_t$ with evenly-spaced nodes after choosing bounds

SETTING UP THE GRID

```
% Rouwenhorst for productivity volatility shock
[G.epssigz_nodes,G.epssigz_weight] = ...
  rouwenhorst(log(P.sigz),P.rhosv,P.sigsv,O.epssigz_pts);
% Rouwenhorst for productivity shock
[G.epsz_nodes,G.epsz_weight] = ...
  rouwenhorst(0,0,1,O.epsz_pts);

% Grid points for productivity shock volatility
G.lnsigz_grid = G.epssigz_nodes';
% Grid points for productivity level
G.lnz_grid = linspace(O.lnzmin,O.lnzmax,O.lnz_pts);
% Grid points for capital
G.k_grid = linspace(O.kmin,O.kmax,O.k_pts);

% Grid arrays
[G.lnsigz_gr,G.lnz_gr,G.k_gr] = ...
  ndgrid(G.lnsigz_grid,G.lnz_grid,G.k_grid);
```

INTERPOLATION WITH SV

- Shock realizations

```
lnsigzpMat = repmat(G.epssigz_nodes,[1,O.epsz_pts]);
epszMat = permute(repmat(G.epsz_nodes,...
        [1,O.epssigz_pts]),[2,1]);
```

- Inside time-iteration loop, but before the call to `csolve`:

```
lnzpMat = (1-P.rhoz)*log(P.zbar)+P.rhoz*G.lnz_gr(i)
            + exp(G.lnsigz_gr(i))*epszMat;
```

- Interpolation in `eqm.m`:

```
npMat = zeros(O.epssigz_pts,O.epsz_pts);
kpMat = kp*ones(O.epssigz_pts,O.epsz_pts);
npMat(:) = Fallterp31(...
        G.lnsigz_grid,G.lnz_grid,G.k_grid,...
        lnsigzpMat(:),lnzpMat(:),kpMat(:),...
        pf.n);
```

ENDOGENOUS UNCERTAINTY

- In a nonlinear model, the expected volatility of forecast errors (i.e., uncertainty) endogenously varies over time
- Definition follows "Measuring Uncertainty" (AER, 2015)

$$\mathcal{U}_t = \sqrt{E_t[(\ln y_{t+h} - E_t[\ln y_{t+h}])^2]}$$

  where h is the forecast horizon

- After solving the model, create a policy function for $\mathcal{U}_t$
  - Loop across the nodes due to the expectation
  - Can simulate like any other policy function
  - Similar procedure can be used for asset prices

EXTENSION 4: MONETARY POLICY AND ZLB

- Phillips curve:

$$\varphi\left(\frac{\pi_t}{\bar{\pi}}-1\right)\frac{\pi_t}{\bar{\pi}}=1-\theta+\theta\frac{w_t}{z_t}+\varphi E_t\left[q_{t,t+1}\left(\frac{\pi_{t+1}}{\bar{\pi}}-1\right)\frac{\pi_{t+1}}{\bar{\pi}}\frac{y_{t+1}}{y_t}\right]$$

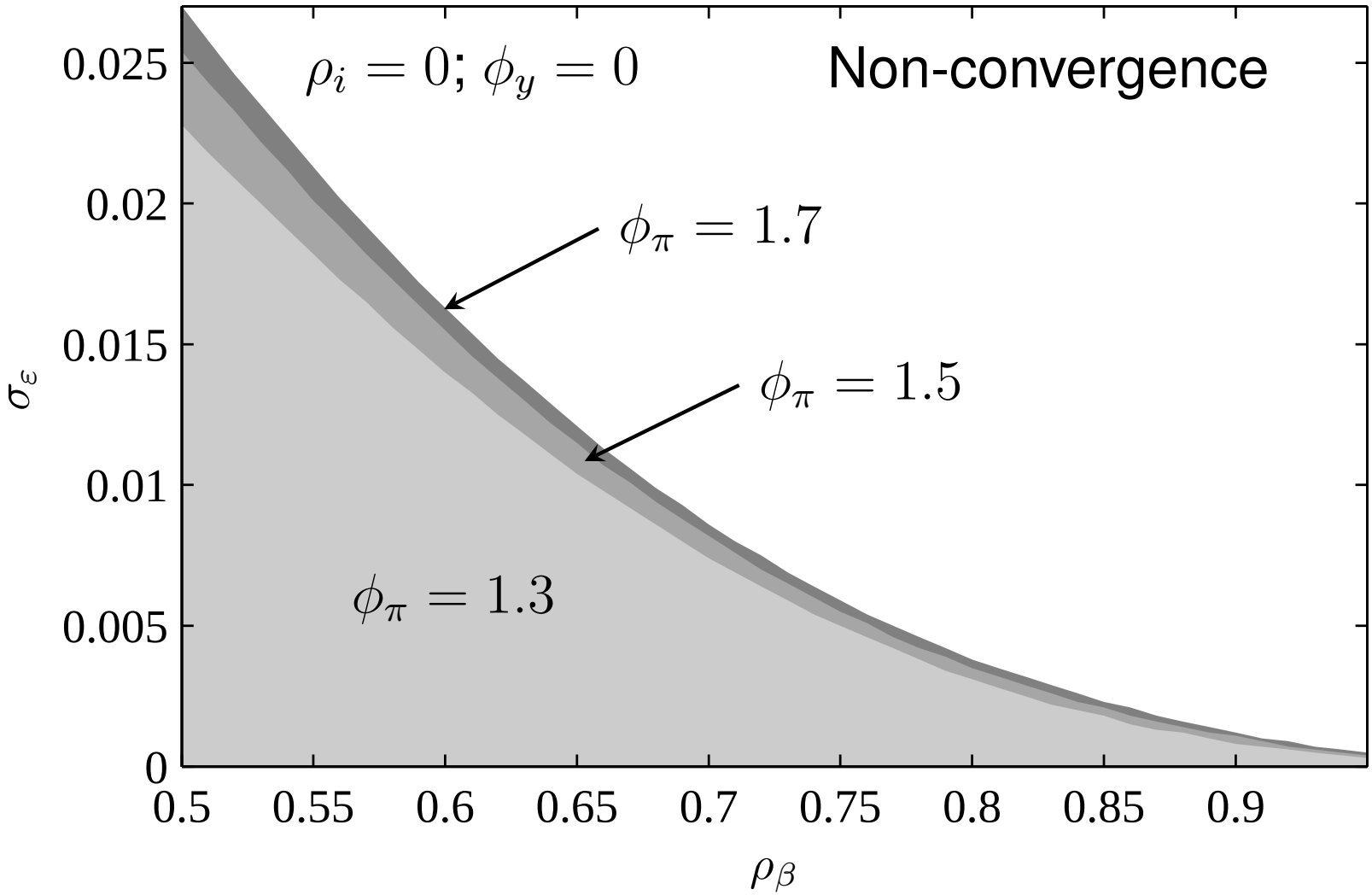
- Monetary policy rule:

$$i_t=\max\{\underline{i},i_t^*\}$$

$$i_t^*=(i_{t-1}^*)^{\rho_i}\left(\bar{\imath}(\pi_t/\bar{\pi})^{\phi_\pi}(y_t/y_{t-1})^{\phi_y}\right)^{1-\rho_i}\exp(\sigma_\nu\nu_t)$$

- Adds π_t as a policy function, y_{t-1} and i_{t-1}^* as endogenous state variables, and ν_t as an exogenous state variable
- Convergence region imposes a tradeoff between the expected frequency and average duration of ZLB events (See "The Zero Lower Bound: Frequency, Duration, and Numerical Convergence," BEJM, 2015)

CONVERGENCE (SHADED) REGIONS



EXTENSION 5: POLICY SWITCHING

- "Monetary and Fiscal Policy Switching" (Chung, Davig and Leeper, JMCB, 2007)
- Monetary and fiscal policy rules:

$$r_t = \bar{r}(\pi_t/\bar{\pi})^{\phi(s_t)} \exp(\varepsilon_{r,t})$$

$$\tau_t = \bar{\tau}(b_{t-1}/\bar{b})^{\gamma(s_t)} \exp(\varepsilon_{\tau,t})$$

$$\phi(s_t) = \begin{cases} \phi & \text{for } s_t = 1 \\ 0 & \text{for } s_t = 2 \end{cases} \qquad \gamma(s_t) = \begin{cases} \gamma & \text{for } s_t = 1 \\ 0 & \text{for } s_t = 2 \end{cases}$$

- Transition probabilities:

$$\begin{bmatrix} \Pr[s_t = 1 | s_{t-1} = 1] & \Pr[s_t = 2 | s_{t-1} = 1] \\ \Pr[s_t = 1 | s_{t-1} = 2] & \Pr[s_t = 2 | s_{t-1} = 2] \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

CONTINUOUS+DISCRETE SHOCKS

- Discretize the continuous shocks using Rouwenhorst
- Create a grid for the policy regime

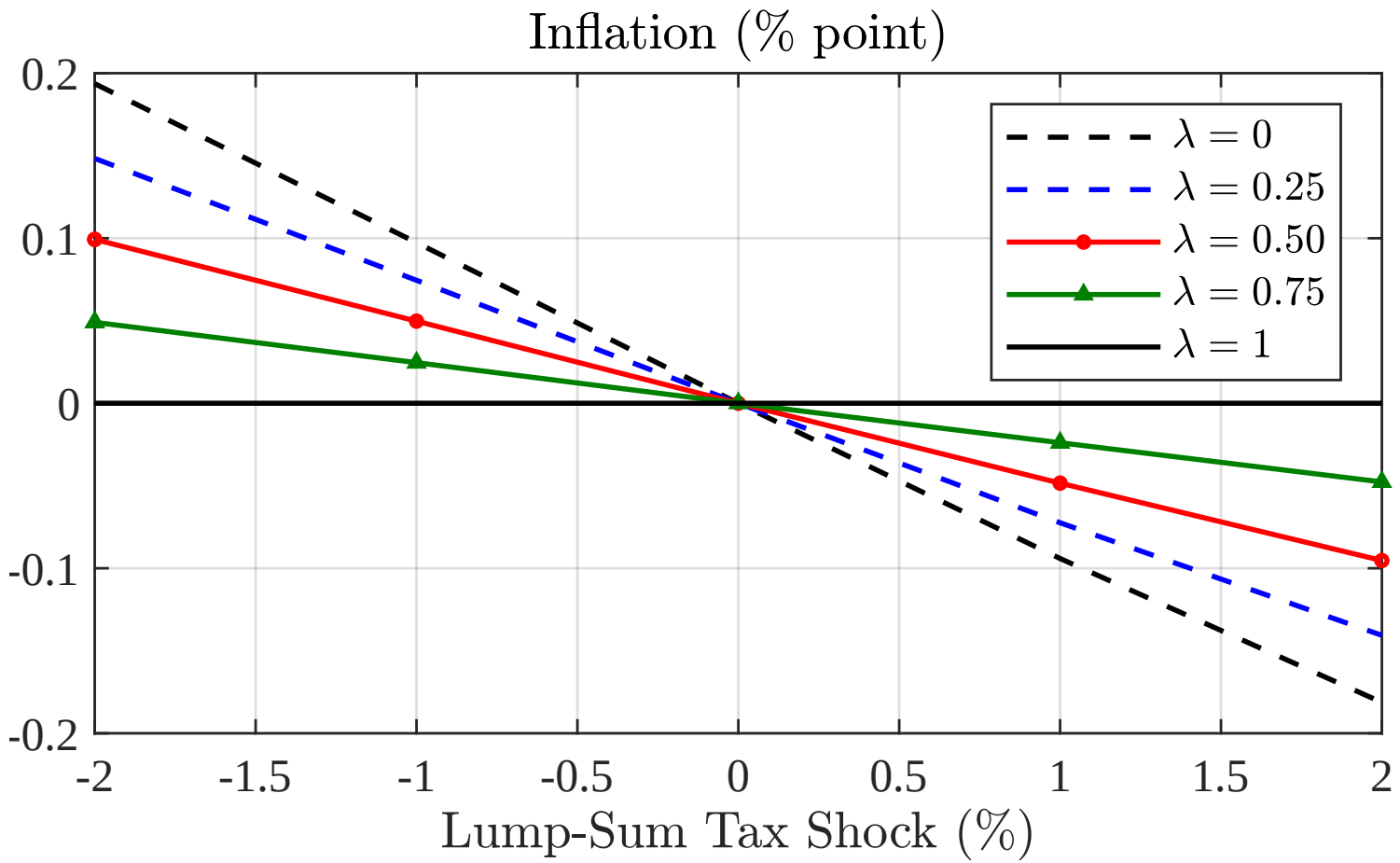
```
G.s_nodes = [0,1]';
G.s_weight = [P.p11 1-P.p11;
              1-P.p22 P.p22];
```

- Set the integration weights by creating arrays

```
% Integration weights
epsfp_weightVec = G.epsfp_weight(1,:)';
epsmp_weightVec = G.epsmp_weight(1,:)';
s_weightVec = G.s_weight(G.s_gr(i) == G.s_grid,:)';
epsfp_weightArr3 = ...
 epsfp_weightVec(:,ones(O.mp_pts,1),ones(O.s_pts,1));
epsmp_weightArr3 = permute(...
 epsmp_weightVec(:,ones(O.fp_pts,1),ones(O.s_pts,1)),[2,1,3]);
s_weightArr3 = permute(...
 s_weightVec(:,ones(O.epsfp_pts,1),ones(O.epsmp_pts,1)),[2,3,1]);
weightArr3 = epsfp_weightArr3.*epsmp_weightArr3.*s_weightArr3;
```

Expectational Effects: Lump-Sum Tax

Average duration of time spent in Regime 1 in the ergodic distribution: $\lambda = (1 - p_{22})/(2 - p_{11} - p_{22})$



EXTENSION 6: DMP AND NET FIRM ENTRY

- "Cyclical Net Entry and Exit" (EER, 2021)
- Active firm value function:

$$\begin{cases} J^F_{A,t} \in (0, \psi_n), & \text{if } \Delta Z = 0 \\ J^F_{A,t} = \psi_n, & \text{if } \Delta Z \geq 0 \\ J^F_{A,t} = 0, & \text{if } \Delta Z \leq 0 \end{cases}$$

- Introduce auxiliary variable $\mu_{A,t}$ to impose these conditions

$$J^F_{A,t} = \min\left\{\max\left\{0, \mu_{A,t}\right\}, \psi_n\right\}, \quad \Delta Z_t = \mu_{A,t} - J^F_{A,t}$$

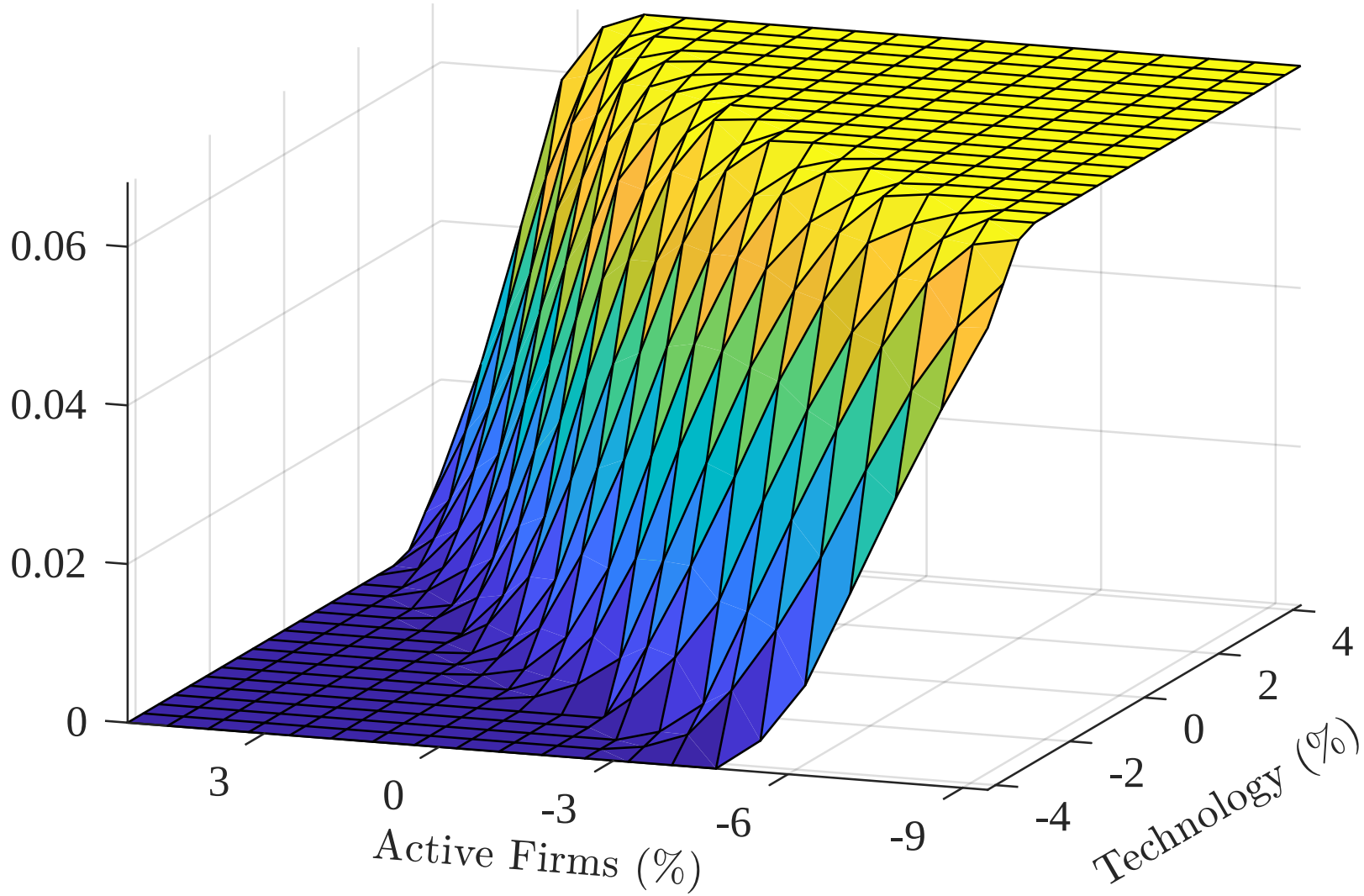
- Introduce second auxiliary variable $\mu_{V,t}$ to ensure $V_t \geq 0$

$$V_t = \max\{0, \mu_{V,t}\}^2, \quad \lambda_{V,t} = \max\{0, -\mu_{V,t}\}^2$$

$$\frac{\kappa - \lambda_{V,t}}{q_t} = w_t - w^n_t + E_t\left[x_{t+1}(1 - s_{t+1})\frac{\kappa - \lambda_{V,t+1}}{q_{t+1}}\right]$$

- Procedure follows Garcia and Zangwill (1981)

Value of Active Firms



GENERALIZED IMPULSE RESPONSES

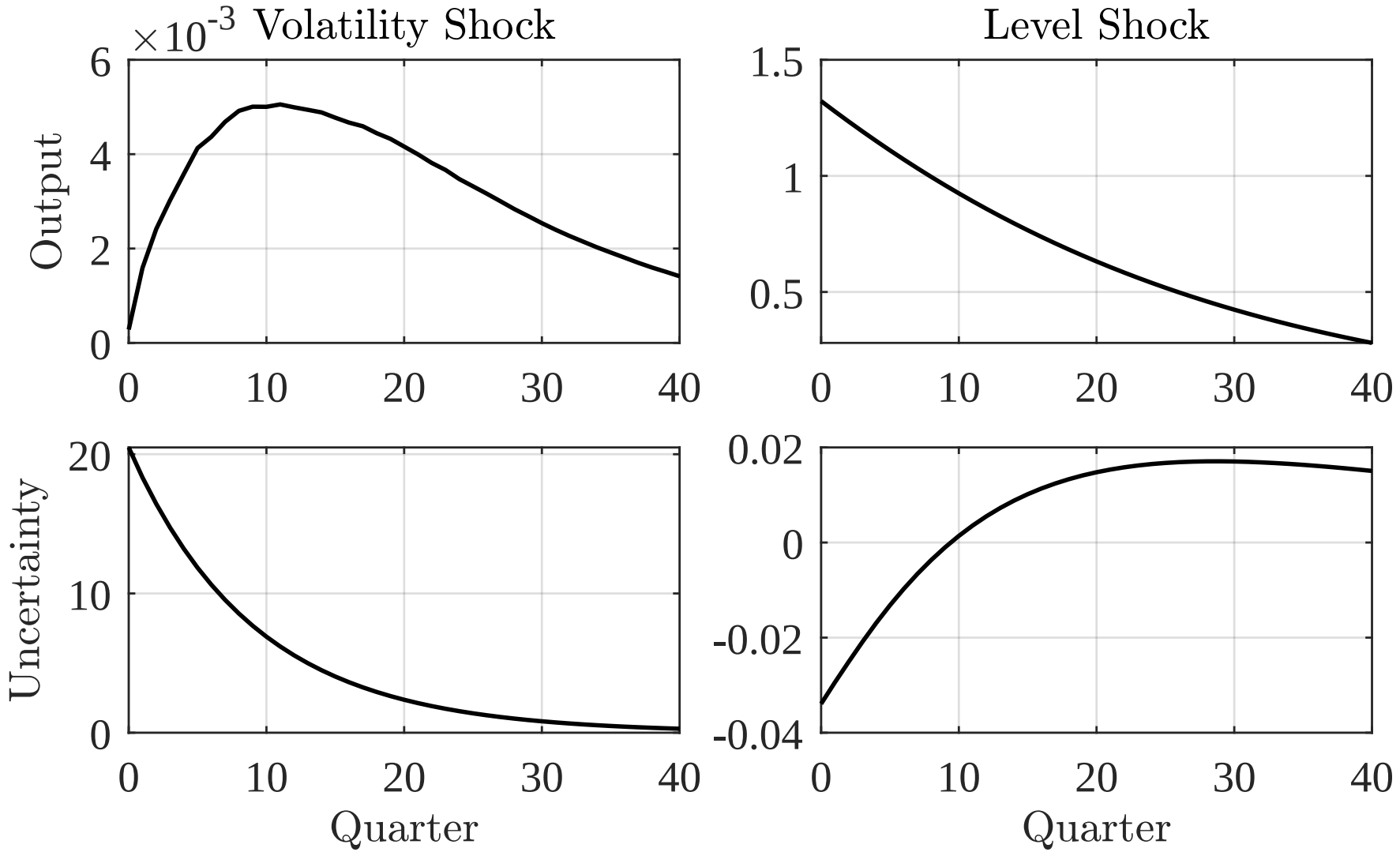
- Used to study model dynamics away from steady state
- The GIRF of x_{t+h} over horizon h is defined as:

$$\mathcal{G}(x_{t+h}|\varepsilon_{t+1} = \xi, \mathbf{z}_t) = E_t[x_{t+h}|\varepsilon_{t+1} = \xi, \mathbf{z}_t] - E_t[x_{t+h}|\mathbf{z}_t]$$

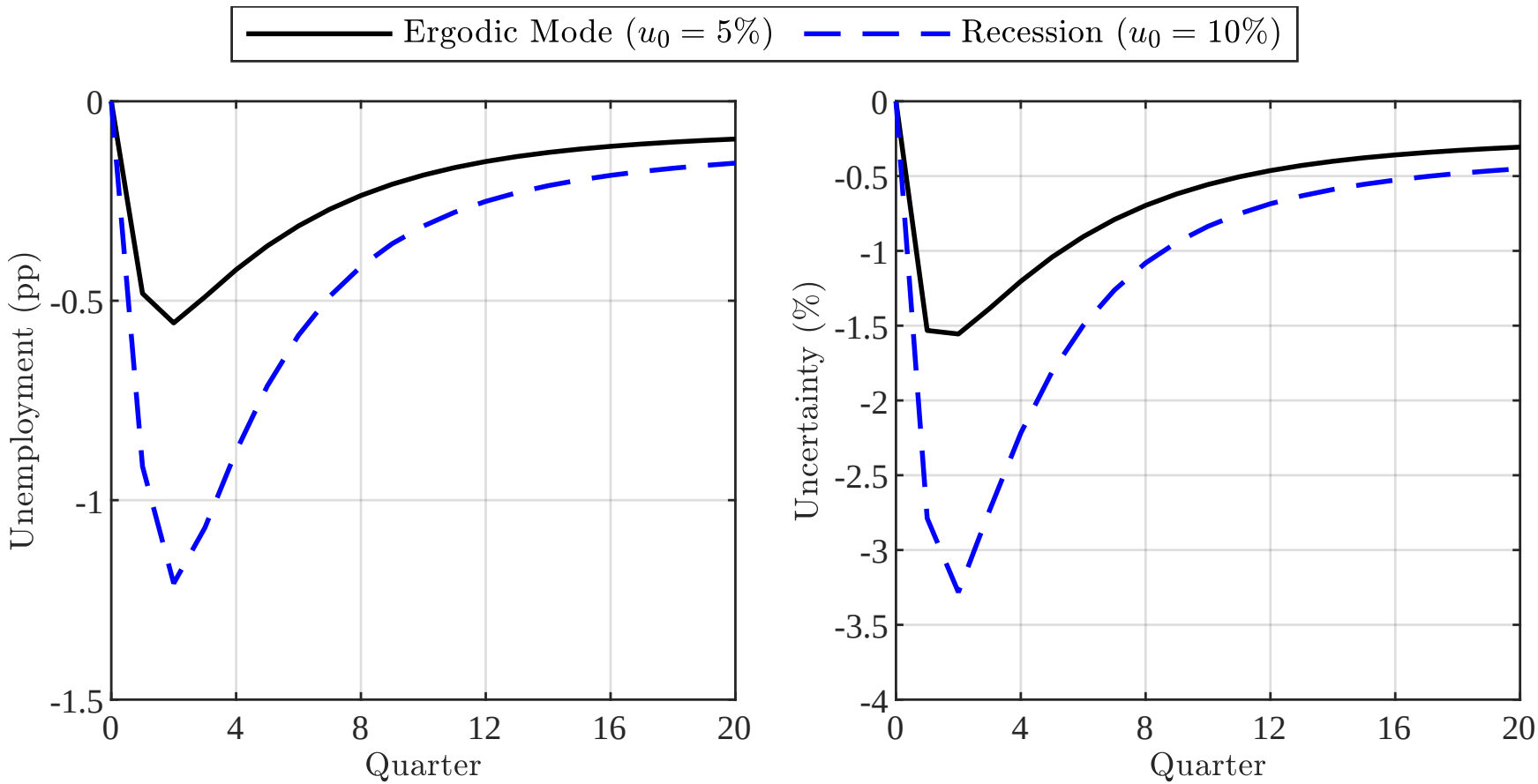
  where the conditional expectations are computed based on the mean path from repeated simulations of the model
- Initialize simulations at $\mathbf{z}_t$, which equals the average of periods where $z \in \mathbf{z}$ is within a certain distance of z^*

```
z_idx = exp(ergdraws(:,V.z)) > z0-0.005 & ...
        exp(ergdraws(:,V.z)) < z0+0.005;
initvec = mean(ergdraws(z_idx,:),1)';
```

RBC Model: Uncertainty



DMP MODEL: STATE-DEPENDENCE



VARIANCE DECOMPOSITION

Consider two total variance decompositions:

1. Let $\{\varepsilon_{-j}\}_{t+1}^{t+h}$ denote a realization of all shocks except j in periods $t+1$ to $t+h$, then law of total variance implies

$$V_t[y_{t+h}] = \underbrace{E_t[V_t[y_{t+h}|\{\varepsilon_{-j}\}_{t+1}^{t+h}]]}_{\text{Total Effect of } \varepsilon_j} + \underbrace{V_t[E_t[y_{t+h}|\{\varepsilon_{-j}\}_{t+1}^{t+h}]]}_{\text{Residual Variance of } \varepsilon_{-j}}$$

2. Conditioning on the jth shock, $\{\varepsilon_j\}_{t+1}^{t+h}$, implies

$$V_t[y_{t+h}] = \underbrace{E_t[V_t[y_{t+h}|\{\varepsilon_j\}_{t+1}^{t+h}]]}_{\text{Total Effect of } \varepsilon_{-j}} + \underbrace{V_t[E_t[y_{t+h}|\{\varepsilon_j\}_{t+1}^{t+h}]]}_{\text{Direct Effect of } \varepsilon_j}$$

Method parses out the direct and interaction effects of shock j

DECOMPOSITION IMPLEMENTATION

Suppose the model has a TFP level and volatility shock

```
for isim = 1:nsimsTVD
  % Simulate: Fixed level shocks
  shocks = cat(3,shocksTVD,repmat(shocksTVD(:,isim),[1,nsimsTVD]));
  ysim_Flev = simulation(pf,P,S,G,V,shocks,ergmean);
  % Simulate: Fixed volatility shocks
  shocks = cat(3,repmat(shocksTVD(:,isim),[1,nsimsTVD]),shocksTVD);
  ysim_Fvol = simulation(pf,P,S,G,V,shocks,ergmean);
  % Integrate: Across volatility shocks
  EysimVol(:,:,isim) = mean(ysim_Flev(2:end,:,:),3,'omitnan');
  VysimVol(:,:,isim) = var(ysim_Flev(2:end,:,:),[],3,'omitnan');
  % Integrate: Across level shocks
  EysimLev(:,:,isim) = mean(ysim_Fvol(2:end,:,:),3,'omitnan');
  VysimLev(:,:,isim) = var(ysim_Fvol(2:end,:,:),[],3,'omitnan');
end
% Integrate: Across level shock
TEVol = mean(VysimVol,3)';
DELev = var(EysimVol,[],3)';
% Integrate: Across volatility shock
TELev = mean(VysimLev,3)';
DEVol = var(EysimLev,[],3)';
```

ESTIMATION EXAMPLE 1: SMM

- Draw shocks, $\mathcal{E}_{T,N_s}$, simulate starting from a random draw from the ergodic distribution, and compute moments
- Run R simulations and compute the average moments
- Estimated parameter values, $\hat{\Theta}$, minimize:

$$J(\Theta, \mathcal{E}) = [\hat{\Psi}_T^D - \bar{\Psi}_{R,T}^M(\Theta, \mathcal{E})]' \hat{\Sigma}_T^D [\hat{\Psi}_T^D - \bar{\Psi}_{R,T}^M(\Theta, \mathcal{E})]$$

$\hat{\Psi}_T^D$: empirical targets based on GMM
$\hat{\Sigma}_T^D$: diagonal inverse of the covariance matrix

- Use `fminsearch` to find the optimal parameter values

```
[Phi,J] = fminsearch('objfunc',draw0,options,...
                     PsiD,OmegaD,O,P,V,...
                     shocks)
```

ESTIMATION EXAMPLE 2: PARTICLE FILTER

1. Initialize the filter by drawing from the ergodic distribution
2. For all particles $p \in \{1, \ldots, N_p\}$ apply the following steps:
   - 2.1 Draw $\mathbf{e}_{t,p} \sim \mathbb{N}(\bar{\mathbf{e}}_t, I)$, where $\bar{\mathbf{e}}_t$ maximizes $p(\xi_t|\mathbf{z}_t)p(\mathbf{z}_t|\mathbf{z}_{t-1})$
   - 2.2 Obtain $\mathbf{z}_{t,p}$ and the vector of variables, $\mathbf{w}_{t,p}$, given $\mathbf{z}_{t-1,p}$
   - 2.3 Calculate, $\xi_{t,p} = \hat{\mathbf{x}}_{t,p}^{model} - \hat{\mathbf{x}}_t^{data}$. The weight on particle p is

$$\omega_{t,p} = \frac{p(\xi_t|\mathbf{z}_{t,p})p(\mathbf{z}_{t,p}|\mathbf{z}_{t-1,p})}{g(\mathbf{z}_{t,p}|\mathbf{z}_{t-1,p}, \hat{\mathbf{x}}_t^{data})} \propto \frac{\exp(-\xi_{t,p}' H^{-1} \xi_{t,p}/2) \exp(-\mathbf{e}_{t,p}'\mathbf{e}_{t,p}/2)}{\exp(-(\mathbf{e}_{t,p} - \bar{\mathbf{e}}_t)'(\mathbf{e}_{t,p} - \bar{\mathbf{e}}_t)/2)}$$

     The model's likelihood at t is $\ell_t^{model} = \sum_{p=1}^{N_p} \omega_{t,p}/N_p$

   - 2.4 Normalize the weights, $W_{t,p} = \omega_{t,p}/\sum_{p=1}^{N_p} \omega_{t,p}$. Then use systematic resampling with replacement from the particles
3. Apply step 2 for $t \in \{1, \ldots, T\}$. $\log \ell^{model} = \sum_{t=1}^{T} \log \ell_t^{model}$

PARTICLE ADAPTATION

1. "Macroeconomic Dynamics Near the ZLB: A Tale of Two Countries" (ReStud, 2018)
2. Given $\mathbf{z}_{t-1}$ and a guess for $\bar{\mathbf{e}}_t$, obtain $\mathbf{z}_t$ and $\mathbf{w}_{t,p}$
3. Calculate $\hat{\mathbf{x}}_t^{model}$, such as $[\log(g_t \tilde{y}_t^{gdp} / \tilde{y}_{t-1}^{gdp}), \log(\pi_t), \log(i_t)]$
4. Calculate $\xi_t = \hat{\mathbf{x}}_t^{model} - \hat{\mathbf{x}}_t^{data}$, which is multivariate normal:

$$p(\xi_t|\mathbf{z}_t) = (2\pi)^{-3/2}|H|^{-1/2}\exp(-\xi_t' H^{-1}\xi_t/2)$$
$$p(\mathbf{z}_t|\mathbf{z}_{t-1}) = (2\pi)^{-3/2}\exp(-\bar{\mathbf{e}}_t'\bar{\mathbf{e}}_t/2)$$

   H is the diagonal of the ME covariance matrix
5. Use `fminsearch` to find $\bar{\mathbf{e}}_t$ that maximizes

$$p(\xi_t|\mathbf{z}_t)p(\mathbf{z}_t|\mathbf{z}_{t-1}) \propto \exp(-\xi_t' H^{-1}\xi_t/2)\exp(-\bar{\mathbf{e}}_t'\bar{\mathbf{e}}_t/2)$$

INTRODUCTION TO MEX

- Advantages of MATLAB:
  - Many built-in functions
  - Good documentation
  - Easy to debug code
  - Easy to store data in structures
  - Parallel processing easy to implement
- Main drawback: Slow at evaluating loops
- MEX (MATLAB Executable): Lets programmers write sections of their code using a compiled language (e.g., Fortran) and call it as a function in MATLAB
- Intermediate step toward full Fortran implementation
- Requires users must write a “Gateway” function that allows MATLAB to communicate with compiled code

WRITING A GATEWAY SUBROUTINE

```
1 #include "fintrf.h"
2 subroutine mexFunction(nlhs,plhs,nrhs,prhs)
  ! Declarations
3 implicit none
  ! mexFunc arguments
4 mwPointer plhs(*), prhs(*)
5 integer*4 nlhs, nrhs
```

- Line 1 defines pointer types in the MATLAB interface
- Function arguments:
  - prhs: Pointer to an array that holds the inputs
  - plhs: Pointer to an array that will hold the outputs
  - nrhs: number of right-hand inputs
  - nlhs: number of left-hand outputs
- Line 3 avoids Fortran's implicit type definitions

KEY MATLAB INTERFACE FUNCTIONS

- `mxGetPr`: Accesses the real data in an mxArray
- `mxGetScalar`: Grabs the value of the first real element of the mxArray (often one element)
- `mxGetM`/`mxGetN`: Determines the number of rows/columns in a specified mxArray
- `mxClassIDFromClassName`: Obtains an identifier for any MATLAB class (e.g., Double)
- `mxCreateNumericArray`: Creates an N-dimensional mxArray in which all data elements have the numeric data type specified by ClassID (7 dimensions max)

INPUTTING A POLICY FUNCTION

- Declare variable types and sizes (lines 1-3). If the dimension lengths are variable, use allocatable memory:

```
1 mwpointer n_pr
2 mwSize nk,nz
3 real*8, allocatable, dimension(:,:) :: n
4 nk = mxGetN(prhs(1)) !Capital grid
5 nz = mxGetN(prhs(2)) !Productivity grid
6 allocate(n(nk,nz))
7 n_pr = mxGetPr(prhs(3))
8 call mxCopyPtrToReal8(n_pr,n,nk*nz)
```

- Create pf (line 3), load the dimensions of `pf` from inputs (lines 4 and 5), and allocate the memory (line 6)
- Grab the address of the `pf` and store in `c_pr` (line 7), and then copy the information to Fortran variable `n` (line 8)

CREATE OUTPUT MATRIX

- Load output size: stochastic realizations (lines 1-2)

```
1 e = mxGetM(prhs(4))
2 allocate(o(e))
  !Create array for return argument
3 cid = mxClassIDFromClassName('double')
4 plhs(1) = mxCreateNumericArray(1,e,cid,0)
5 o_pr = mxGetPr(plhs(1))
  ! Call subroutine and load Fortran array
6 call interpfunction(inputs,o)
7 call mxCopyReal8ToPtr(o,o_pr,e)
```

- Create a $1 \times e$ output vector of type double (lines 3-4) and assign the address (line 5)
- Call the interpolation subroutine (line 6) and copy the data to the output address (line 7)

PARALLEL PROCESSING IN MATLAB

- Any calculations that are not dependent on the results of other calculations can be performed in parallel (e.g., solving for policy values at each node in the state space)
- MATLAB uses the Parallel Computing Toolbox (PCT)
- The PCT requires the following alterations to the code:
  - Replace `for` loops with `parfor` loops so MATLAB distributes each step in the loop across the processors
  - If a nested `for` loop is used to update policy functions across dimensions, reduce it to one loop by changing to a single index (as opposed to specifying coordinates)
  - Remove global variables and instead use parameter lists and variable arrays as direct inputs into functions

PARALLEL PROCESSING IN FORTRAN

OpenMP is a simple way to parallelize a `do`-loop in Fortran

```
!$omp parallel default(shared) private(g,s,arg)
!$omp do collapse(2)
do i2 = 1,Oz_pts
    do i1 = 1,Ok_pts
        g(1,1) = pf_n(i1,i2)
        s(1,1) = Gk_grid(i1)
        s(2,1) = Gz_grid(i2)
        call csolve(g,s,...,arg)
        pfn_up(i1,i2) = arg(1,1)
    end do
end do
!$omp end do
!$omp end parallel
```

MESSAGE PASSING INTERFACE (MPI)

- Initialize and finalize MPI only once:

```
mpi_init(ierr)
mpi_comm_rank(mpi_comm_world,myid,ierr)
mpi_comm_size(mpi_comm_world,nprocs,ierr)
...
mpi_finalize(rc)
```

- Processes do not communicate unless ordered:

```
mpi_bcast(var,n,type,0,mpi_comm_world,ierr)
```

- Need a way to merge results across cores:

```
mpi_allreduce(var_temp,var,n,type,&
                operation,mpi_comm_world,ierr)
```

- Cores may not finish at the same time:

```
mpi_barrier(mpi_comm_world,ierr)
```

PARALLEL PROCESSING WITH MPI

On a given iteration, apply the following:

```
do inode = myid + 1,Gnodes,nprocs
  g(1,1) = pf_n(inode)
  s(1,1) = Gk_gr(inode)
  s(2,1) = Gz_gr(inode)
  call csolve(g,s,...,arg)
  pfn_up(inode) = arg(1,1)
end do
! Impose temporal order
call mpi_barrier(MPI_COMM_WORLD,ierr)
! Combine argzero across processors
call mpi_allreduce(pfn_up_temp,pfn_up,Gnodes,&
                   mpi_double_precision, &
                   mpi_sum,mpi_comm_world,ierr)
```

SOFTWARE AND SCHEDULER COMMANDS

- Use SSH Client to connect
  - (Windows) PuTTY (`http://www.putty.org`)
  - (MacOS) Built-in terminal
- Use SFTP client to transfer files
  - (Windows) WinSCP (`http://winscp.net`)
  - (MacOS) FileZilla (`http://filezilla-project.org`)
- Scheduler commands for SLURM (`http://slurm.schedmd.com`):
  - `squeue`: displays all submitted jobs
  - `sinfo`: display cluster usage by queue type
  - `scancel`: cancels a running job
  - `sbatch runscript`: submits job to queue
  - `source`: read and execute the content of a file

EXAMPLE: RUN SCRIPT

- Maximum wall time is typically 48 hours
- Cluster may have partitions with different node types
- All program output is written to an `OUT` file

```
#!/bin/bash
#SBATCH --job-name=test
#SBATCH --out=test/out
#SBATCH --partition=normal-32g
#SBATCH --time=1:00:00
#SBATCH --nodes=1

prun ./test.out
```

EXAMPLE: JOB ARRAY

- Job arrays useful for code that only differs in the seed
- Add to the run script:

```
#SBATCH --array=[tasksids]
prun ./test.out >&
    test/out-'$SLURM_ARRAY_TASK_ID'
```

- Add to the main program:

```
call get_environment_variable
    ("SLURM_ARRAY_TASK_ID",taskidstr)
read(taskidstr,*) taskid
seed = taskid + 123456
call drandinitialize(3,1,seed,1,rs,lrs,info)
```

COMMAND LINE INPUTS

- Pass an input from the command line to the program
- On the run line, add the input(s) after the program name

```
prun ./results.out arg1 arg2
```

- In the main program, load the input(s) (as a string)

```
character(len=:), &
  allocatable :: arg1,arg2
call loadinput(1,arg1)
call loadinput(2,arg2)
```

where `loadinput` is a short subroutine

- If needed, convert to a number

```
real(dp) :: num1,num2
read(arg1,*) num1
read(arg2,*) num2
```

LOAD INPUT SUBROUTINE

```
subroutine loadinput(iarg,argout)

  ! Input
  integer :: iarg

  ! Output
  character(len=:), allocatable :: argout

  ! Internal
  integer :: arglen

  call get_command_argument(iarg,length=arglen)
  allocate(character(arglen) :: argout)
  call get_command_argument(iarg,value=argout)

end subroutine loadinput
```

EXAMPLE: BATCH RUN SCRIPT

```
#!/bin/bash
args='0.5 1.0 5.0'
for arg in $args
do
  out='arg'$arg'-out'
  echo $out
  jobname='arg'$arg'
  echo $jobname
  /bin/cp test.stub test.txt
  echo "#SBATCH --partition=normal-32g" >> test.txt
  echo "#SBATCH --time=1:00:00" >> test.txt
  echo "#SBATCH --nodes=1" >> test.txt
  echo "prun ./test.out ${arg} >& ${out}" >> test.txt
  sbatch --output=/dev/null
         --job-name=$jobname test.txt
  sleep 1
done
```

ADDITIONAL RESOURCES

- For more info on the solution method see
  - "Accuracy, Speed and Robustness of Policy Function Iteration" (*Computational Economics*, 2014)
- For more info on nonlinear estimation see
  - "The Zero Lower Bound and Endogenous Uncertainty" (*Economic Journal*, 2017)
  - "The Zero Lower Bound and Estimation Accuracy" (*Journal of Monetary Economics*, 2020)
- All of the code is available at: `http://alexrichterecon.com`
- Contact: `alex.richter@dal.frb.org`