Introduction to Linear Algebra

Chapter 3:

Matrices & Systems of Linear Equations

Foundation Year - 2023/2024
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Outline

1. Systems of Linear Equations

2. Solving Systems With Guassian Elimination

3. Solving Systems with Inverses

4. Solving 2×2 Systems with Cramers's Rule

5. Application: Linear Independence, Linear Dependence

6. Subspaces

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Problem 1: With one unknown value

If all the cows in a pasture have 124 legs, how many cows are in the pasture?

We can model this real-life situation using an equation with one unknown value, represented by one variable, to say x.

1. Identify the unknown and define your variable:
   - the unknown value is the number of cows,
   - set x to be the number of cows.
2. Analyse the problem and write your equation accordingly:
   - every cow has four legs,
   - $4\times$ *the number of cows* $= 124$ legs
   - $\boxed{4\,x = 124}$

Problem 1: With one unknown value

If all the cows in a pasture have 124 legs, how many cows are in the pasture?

$$4\,x = 124 \quad \Longleftrightarrow \quad x = 124/4 = 31$$

There are 31 cows in the pasture ☺

Problem 2: With two unknown values

If all the cows and chickens in a pasture have 35 heads and 110 legs, how many cows are in the pasture?

We can model this real-life situation using a **system of linear equations**.

- A system of equations is a set of two or more linear equations with the same set of unknown values, which are represented by the same variables such that all equations in the system are considered simultaneously.
- They are also called **simultaneous equations**
- To find the unique solution to a system of linear equations, we must find a numerical value for each variable in the system that will satisfy all equations in the system at the same time.
- Some linear systems may not have a solution and others may have an infinite number of solutions.
- A system of equations is called inconsistent if it has no solutions. It is called consistent otherwise.
- In order for a linear system to have a unique solution, there must be at least as many equations as there are variables. Even so, this does not guarantee a unique solution.

Definition

- A solution of a system of equations with n unknowns, is a list of numbers x, y, z, ... that make all of the equations true simultaneously.
- The solution set of a system of equations is the collection of all solutions.
- Solving the system means finding all solutions with formulas involving some number of parameters.

For example: The solution of the system

$$\begin{cases} y - \frac{1}{2}x = 2 \\ y + x = -1 \end{cases}$$

is the ordered pair $(-2, 1)$.

- We can check this by substituting the values in this solution for the variables in the equations in the system and get true statements:

For example: We check that the ordered pair $(-2, 1)$ is the solution of the system:

$$\begin{cases} 1 - \frac{1}{2}(-2) = 2 \\ 1 + (-2) = -1 \end{cases}$$

$$\begin{cases} 1 + 1 = 2 \\ 1 - 2 = -1 \end{cases}$$

$$\begin{cases} 2 = 2 & \textit{TRUE!} \\ -1 = -1 & \textit{TRUE!} \end{cases}$$

Problem 2: With two unknown values

If all the cows and chickens in a pasture have 35 heads and 110 legs, how many cows are in the pasture?

We can model this real-life situation using a **system of linear equations**.

- To begin writing appropriate equations to any system, we must first define our variables:

"What unknown values will I need to find for this system?"

Problem 2: With two unknown values

If all the cows and chickens in a pasture have 35 heads and 110 legs, how many cows are in the pasture?

We can model this real-life situation using a **system of linear equations**.

- To begin writing appropriate equations to any system, we must first define our variables:

"The number of cows and the number of chickens "

Problem 2: With two unknown values

If all the cows and chickens in a pasture have 35 heads and 110 legs, how many cows are in the pasture?

We can model this real-life situation using a **system of linear equations**.

- Set x = number of cows and y = number of chickens

Problem 2: With two unknown values

If all the cows and chickens in a pasture have 35 heads and 110 legs, how many cows are in the pasture?

We can model this real-life situation using a **system of linear equations**.

- Set x = number of cows and y = number of chickens

1. Heads equation: $x + y = 35$
2. Legs equation: $4x + 2y = 110$

Problem 2: With two unknown values

If all the cows and chickens in a pasture have 35 heads and 110 legs, how many cows are in the pasture?

We can model this real-life situation using a **system of linear equations**.

- Set x = number of cows and y = number of chickens

1. Heads equation: $x + y = 35$
2. Legs equation: $4x + 2y = 110$

- And here we go!

$$\begin{cases} x + y = 35 \\ 4x + 2y = 110 \end{cases}$$

Problem 2: With two unknown values

If all the cows and chickens in a pasture have 35 heads and 110 legs, how many cows are in the pasture?

$$\begin{cases} x + y = 35 \\ 4\,x + 2\,y = 110 \end{cases}$$

"How do we solve it?"

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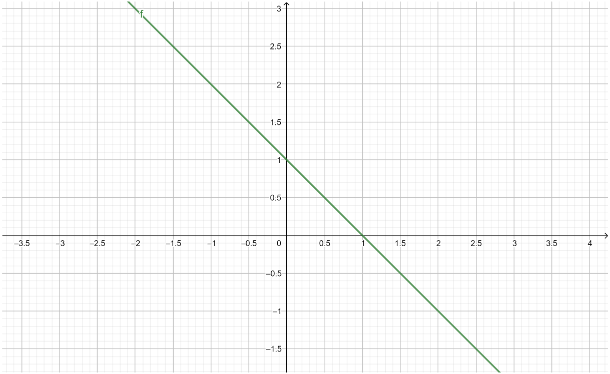
- ☺ Before discussing how to solve a system of linear equations below, it is helpful to see some pictures of what these solution sets look like geometrically!

- **One Equation in Two Variables**: Consider the linear equation $x + y = 1$. We can rewrite this as $y = -1x + 1$, which defines a line in the plane: the slope is -1, and the x-intercept is 1.

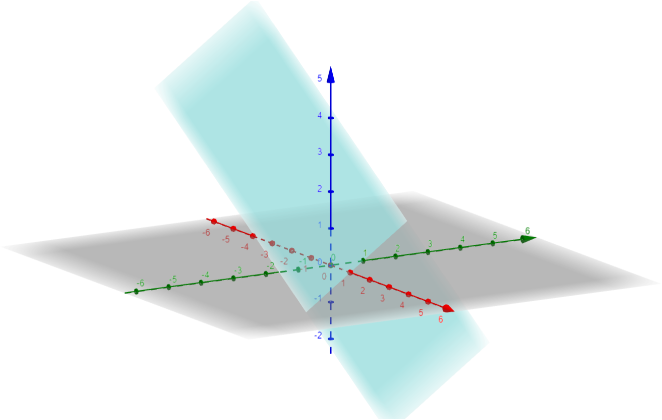


Definition

- For our purposes, a line is a ray that is straight and infinite in both directions.
- Slope is a measure of the steepness of a line.
- It can be seen as

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{\Delta y}{\Delta x}$$

⋆ **One Equation in Three Variables**: Consider the linear equation $x + y + z = 1$ (*click on the equation and visualise it on GeoGebra*). This is the **implicit equation** for a plane in space.



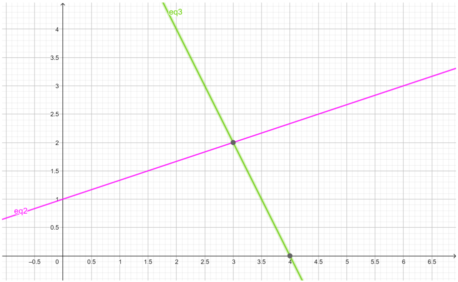
Definition

A **plane** is a flat sheet that is infinite in all directions.

⋆ **Two Equations in Two Variables**: Now consider the system of two linear equations

$$\begin{cases} x - 3y & = & -3 \\ 2x + y & = & 8 \end{cases}$$

Each equation individually defines a line in the plane:



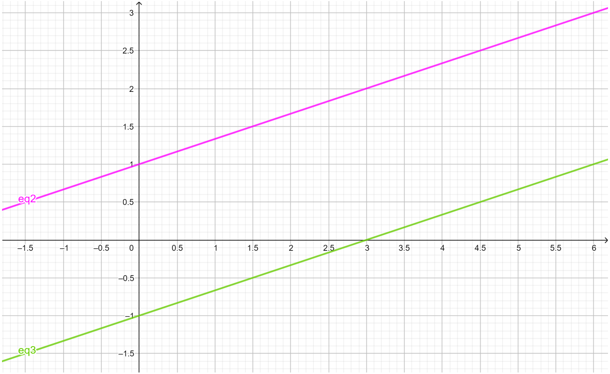
- We recall that a solution to the system of both equations is a pair of numbers (x, y) that makes both equations true at once.
- In other words, it is a point that lies on both lines simultaneously.
- We can see in the picture above that there is only one point where the lines intersect: therefore, this system has exactly one solution. (This solution is $(3, 2)$, as we can verify.)

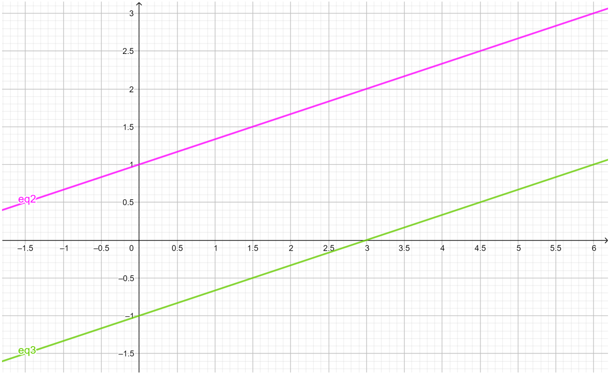
This is said to be an Independent System

- Usually, two lines in the plane will intersect in one point, but of course this is not always the case. Consider now the system of equations

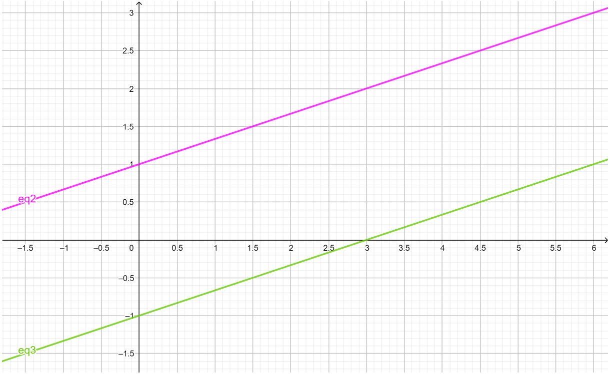
$$\begin{cases} x - 3y = -3 \\ x - 3y = 3 \end{cases}$$

These define parallel lines in the plane.





- The fact that the lines do not intersect means that the system of equations has no solution. Of course, this is easy to see algebraically: if $x - 3y = -3$, then it is cannot also be the case that $x - 3y = 3$.

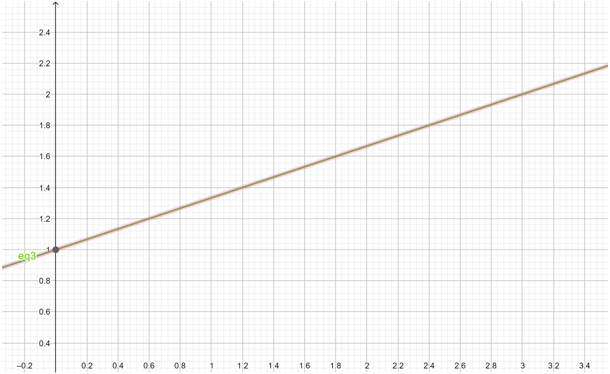


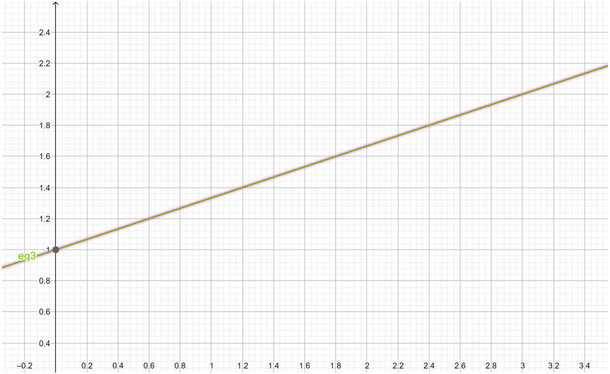
This is said to be an Inconsistent System

- There is one more possibility. Consider the system of equations

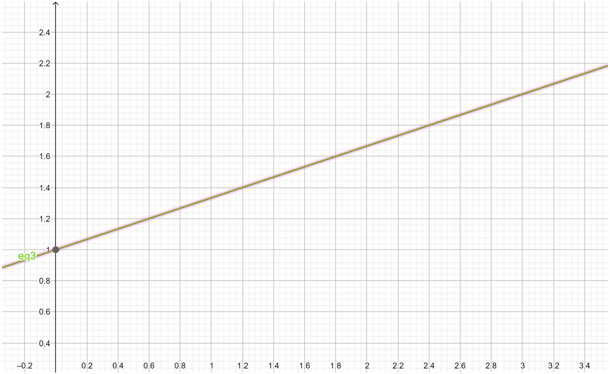
$$\begin{cases} x - 3y & = & -3 \\ 2x - 6y & = & -6 \end{cases}$$

The second equation is a multiple of the first, so these equations define the same line in the plane.





- In this case, there are infinitely many solutions of the system of equations.

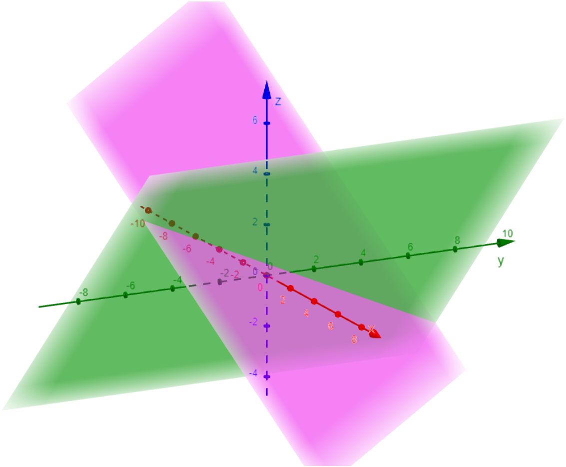


This is said to be a Dependent System

⋆ **Two Equations in Three Variables**: Consider the system of two linear equations

$$\begin{cases} x + y + z = 1 \\ x - z = 0. \end{cases}$$

- Each equation individually defines a plane in space.
- The solutions of the system of both equations are the points that lie on both planes.
- We can see in the picture in the next slide (as well as in GeoGebra) that the planes intersect in a line. In particular, this system has infinitely many solutions.


z
y
x
10
8
6
4
2
0
-2
-4
-6
-8
-10

The Elimination Method

- We will solve systems of linear equations algebraically using the elimination method.
- In other words, we will combine the equations in various ways to try to eliminate as many variables as possible from each equation.
- Take for instance our example of Problem 2 and let's find the solution.

$$\begin{cases} x + y = 35 & \text{eq. 1} \\ 4\,x + 2\,y = 110 & \text{eq. 2} \end{cases}$$

1. **Eliminate** one variable, i.e., using eq. 1 and eq. 2, find a new equation depending only on 1 variable (to say y).
2. **Solve** the obtained equation (for y).
3. **Substitute** the obtained value for its corresponding variable in either eq. 1 or eq. 2 and then solve for the other variable (x).

$$\begin{cases} \boxed{x} + y = 35 & \text{eq. 1} \\ \boxed{4\,x} + 2\,y = 110 & \text{eq. 2} \end{cases}$$

- Apart from Swap (where we can swap two equations, since they all happen simultaneously!), there are two valid operations we can perform on our system of equations (and by valid we mean operations that don't change the meaning of the system, i.e., result in an equivalent system):

$$\begin{cases} \boxed{x} + y = 35 & \text{eq. 1} \\ \boxed{4\,x} + 2\,y = 110 & \text{eq. 2} \end{cases}$$

- Apart from Swap (where we can swap two equations, since they all happen simultaneously!), there are two valid operations we can perform on our system of equations:
    1. Scaling: we can multiply both sides of an equation by a nonzero number.

$$\begin{cases} \boxed{-4\,x} + (-4)y = (-4)35 & -4 \times \text{eq. 1} \\ \boxed{4\,x} + 2\,y = 110 & \text{eq. 2} \end{cases}$$

$$\begin{cases} \boxed{x} + y = 35 & \text{eq. 1} \\ \boxed{4\,x} + 2\,y = 110 & \text{eq. 2} \end{cases}$$

- Apart from Swap (where we can swap two equations, since they all happen simultaneously!), there are two valid operations we can perform on our system of equations:
  1. Scaling: we can multiply both sides of an equation by a nonzero number.
  2. Replacement: we can add a multiple of one equation to another, replacing the second equation with the result.

$$\begin{cases} \boxed{-4\,x} + (-4)y = (-4)35 & -4 \times \text{eq. 1} \\ \boxed{4\,x} + 2\,y = 110 & \text{eq. 2} \end{cases}$$

$$\begin{cases} x + y = 35 & \text{eq. 1} \\ \boxed{4\,x} + 2\,y + \boxed{(-4)\,x} + (-4)y = 110 + (-4)35 & \text{eq. 2} + (-4) \times \text{eq. 1} \end{cases}$$

$$\begin{cases} \boxed{x} + y = 35 & \text{eq. 1} \\ \boxed{4\,x} + 2\,y = 110 & \text{eq. 2} \end{cases}$$

$$\begin{cases} x + y = 35 & \text{eq. 1} \\ \boxed{4\,x} + 2\,y + \boxed{(-4)\,x} + (-4)y = 110 + (-4)35 & \text{eq. 2} + (-4) \times \text{eq. 1} \end{cases}$$

$$\begin{cases} x + y = 35 \\ \boxed{2\,y - 4y = 110 - 4 \times 35} \end{cases}$$

$$\begin{cases} x + y = 35 & \text{eq. 1} \\ 4\,x + 2\,y = 110 & \text{eq. 2} \end{cases}$$

- Solve the obtained equation (for y):

$$\begin{cases} x + y = 35 \\ \boxed{2\,y - 4y = 110 - 4 \times 35} \end{cases} \iff \begin{cases} x + y = 35 \\ \boxed{-2\,y = -30} \end{cases}$$

$$\begin{cases} x + y = 35 & \text{eq. 1} \\ 4x + 2y = 110 & \text{eq. 2} \end{cases}$$

- Solve the obtained equation (for y):

$$\begin{cases} x + y = 35 \\ -2y = -30 \end{cases} \iff \begin{cases} x + y = 35 \\ \boxed{y = 15} \end{cases}$$

$$\begin{cases} x + y = 35 & \text{eq. 1} \\ 4x + 2y = 110 & \text{eq. 2} \end{cases}$$

- Substitute the obtained value for its corresponding variable in either eq. 1 or eq. 2 and then solve for the other variable.

$$\begin{cases} x + y = 35 \\ \boxed{y = 15} \end{cases} \iff \begin{cases} x + 15 = 35 \\ \boxed{y = 15} \end{cases}$$

$$\begin{cases} x + y = 35 & \text{eq. 1} \\ 4\,x + 2\,y = 110 & \text{eq. 2} \end{cases}$$

$$\begin{cases} \boxed{x = 20} \\ \boxed{y = 15} \end{cases}$$

Problem 2: With two unknown values

If all the cows and chickens in a pasture have 35 heads and 110 legs, how many cows are in the pasture?

$$\begin{cases} x + y = 35 \\ 4\,x + 2\,y = 110 \end{cases} \iff \begin{cases} \boxed{x = 20} \\ \boxed{y = 15} \end{cases}$$

"There are 20 cows in the pasture"

- Solving equations by elimination requires writing the variables x, y, and the equals sign $=$ over and over again, merely as placeholders: all that is changing in the equations is the coefficient numbers.
- For instance, while solving Problem 2 system, we notice that what really matter in the operations done ($-4\times$ equ. 1 and equ. 2 $-4\times$ equ. 1) (during the elimination) are the numbers attached to the variables (the coefficients).
- We can make our life easier and represent the same system in an equivalent way, but without any letters (without the variables x and y)
- In this case we obtain the following **rectangular table of numbers**, that we call the **Augmented Matrix**.

$$\begin{cases} 1x + 1y = 35 \\ 4\,x + 2\,y = 110 \end{cases} \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 35 \\ 4 & 2 & 110 \end{array}\right)$$

- The word "augmented" refers to the vertical line, which we draw to remind ourselves where the equals sign belongs.
- In this notation, our three valid ways of manipulating our equations become **row operations**:
- Let R_1 and R_2 be respectively the first and second row of our matrix. By following the same operations done in the elimination step previously, that is replacing equ.2 by equ.2 $-4\times$ equ.1, we can, in an equivalent way, replace R_2 by $R_2 - 4 \times R_1$ in the matrix (that is a **row operation**) to obtain

$$\begin{cases} 1x + 1y = 35 \\ 4\,x + 2\,y = 110 \end{cases} \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 35 \\ 4 & 2 & 110 \end{array}\right)$$

$$\xrightarrow{\mathrm{R}_2 \leftarrow \mathrm{R}_2 - 4\times\mathrm{R}_1} \left(\begin{array}{cc|c} 1 & 1 & 35 \\ 0 & -2 & -30 \end{array}\right)$$

$$\begin{cases} 1x + 1y = 35 \\ 4\,x + 2\,y = 110 \end{cases} \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 35 \\ 4 & 2 & 110 \end{array}\right)$$

$$\xrightarrow{R_2 \leftarrow R_2 - 4 \times R_1} \left(\begin{array}{cc|c} 1 & 1 & 35 \\ 0 & -2 & -30 \end{array}\right)$$

- The number 0 represents the variable x being eliminated in a new equation of 1 variable y: $-2\,y = -4$.
- In this case, we can immediately see again the new equivalent system

$$\begin{cases} 1x + 1y = 35 \\ 4\,x + 2\,y = 110 \end{cases} \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 35 \\ 4 & 2 & 110 \end{array}\right)$$

$$\begin{cases} 1x + 1y = 35 \\ 0\,x + -2\,y = -30 \end{cases} \longleftarrow \left(\begin{array}{cc|c} 1 & 1 & 35 \\ 0 & -2 & -30 \end{array}\right)$$

$$\begin{cases} 1x + 1y = 35 \\ 4\,x + 2\,y = 110 \end{cases} \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 35 \\ 4 & 2 & 110 \end{array}\right)$$

$$\xrightarrow{R_2 \leftarrow R_2 - 4 \times R_1} \left(\begin{array}{cc|c} 1 & 1 & 35 \\ 0 & -2 & -30 \end{array}\right)$$

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$$\begin{cases} x + y = 35 \\ -2\,y = -30 \end{cases} \longleftarrow \left(\begin{array}{cc|c} 1 & 1 & 35 \\ 0 & -2 & -30 \end{array}\right)$$

$$\begin{cases} 1x + 1y = 35 \\ 4\,x + 2\,y = 110 \end{cases} \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 35 \\ 4 & 2 & 110 \end{array}\right)$$

$$\begin{cases} x + y = 35 \\ -2\,y = -30 \end{cases} \longleftarrow \left(\begin{array}{cc|c} 1 & 1 & 35 \\ 0 & -2 & -30 \end{array}\right)$$

$$\begin{cases} \boxed{x = 20} \\ \boxed{y = 15} \end{cases}$$

- A system of equations can be represented by an **augmented matrix**.
- In an augmented matrix, each row represents one equation in the system and each column represents a variable coefficient (on the left side of the bar) or the constant terms (on the right side of the bar).
- Augmented matrices are a shorthand way of writing systems of equations. The organization of the numbers into the matrix makes it unnecessary to write various symbols like x, y, and $=$, yet all of the information is still there!

For example:

$$\begin{cases} -4x + 5y = -1 \\ 35\,x + 7\,y = 123 \end{cases} \longrightarrow \left(\begin{array}{cc|c} -4 & 5 & -1 \\ 35 & 7 & 123 \end{array}\right)$$

Example

Write the following system of equations as an augmented matrix.

$$\begin{cases} 3x - 2y &= 4 \\ x + 5z &= 3 \\ -4x - y + 3z &= 0 \end{cases}$$

Example

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$$\begin{cases} 3x - 2y &= 4 \\ x + 5z &= 3 \\ -4x - y + 3z &= 0 \end{cases}$$

Solution:

- First, in order to make things easier, we can rewrite the system in a way where we can see each of the coefficients clearly.
- P.S.: If a variable term is not written in an equation, it means that the coefficient is 0.

EXAMPLE

Write the following system of equations as an augmented matrix.

$$\begin{cases} 3x + (-2)y + 0z = 4 \\ 1x + 0y + 5z = 3 \\ -4x + (-1)y + 3z = 0 \end{cases}$$

Solution:

- First, in order to make things easier, we can rewrite the system in a way where we can see each of the coefficients clearly.
- P.S.: If a variable term is not written in an equation, it means that the coefficient is 0.

EXAMPLE

Write the following system of equations as an augmented matrix.

$$\begin{cases} 3x & + & (-2)y & + & 0z & = & 4 \\ 1x & + & 0y & + & 5z & = & 3 \\ -4x & + & (-1)y & + & 3z & = & 0 \end{cases}$$

Solution:

$$\left(\begin{array}{ccc|c} 3 & -2 & 0 & 4 \\ 1 & 0 & 5 & 3 \\ -4 & -1 & 3 & 0 \end{array}\right)$$

REMARK

In general, before converting a system into an augmented matrix, be sure that the variables appear in the same order in each equation, and that the constant terms are isolated on one side.

There are three elementary matrix row operations:

1. Switch any two rows: for example

$$\begin{pmatrix} 1 & 3 & 5 \\ 7 & 7 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 7 & 7 & 0 \\ 1 & 3 & 5 \end{pmatrix}$$

2. Multiply a row by a nonzero constant: for example

$$\begin{pmatrix} 1 & 3 & 5 \\ 7 & 7 & 0 \end{pmatrix} \xrightarrow{R_1 \leftarrow 2 \times R_1} \begin{pmatrix} 2 \times 1 & 2 \times 3 & 2 \times 5 \\ 7 & 7 & 0 \end{pmatrix}$$

3. Add a multiple of one row to another: for example

$$\begin{pmatrix} 1 & 3 & 5 \\ 7 & 7 & 0 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + 3 \times R_1} \begin{pmatrix} 1 & 3 & 5 \\ 7 + 3 \times 1 & 7 + 3 \times 3 & 0 + 3 \times 5 \end{pmatrix}$$

Systems of equations and matrix row operations:

- Recall that in an augmented matrix, each row represents one equation in the system and each column represents a variable or the constant terms.
- For example, the system on the left corresponds to the augmented matrix on the right.

$$\text{System:} \begin{cases} 3x + y = 5 \\ x + 2y = 6 \end{cases} \qquad \text{Matrix:} \left(\begin{array}{cc|c} 3 & 1 & 5 \\ 1 & 2 & 6 \end{array}\right)$$

When working with augmented matrices, we can perform any of the **elementary matrix row operations** to create a new augmented matrix that refers to an equivalent system of equations.

DEFINITION

Two matrices are called **row equivalent** if one can be obtained from the other by doing some number of matrix row operations.

We say that a matrix is in **row-echelon form** if it meets the following two requirements:

1. For each row, the first (leftmost) nonzero entry - called a **leading coefficient** or **pivot** - is to the right of the one above it.
2. Any non-zero rows are always above rows with all zeros.

Example of row-echelon form matrices:

$$\begin{pmatrix} 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & \boxed{5} & 0 \\ 0 & 0 & 0 & \boxed{-5} \end{pmatrix}, \quad \begin{pmatrix} \boxed{1} & 1 & -1 & 0 \\ 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \boxed{1} & 0 \\ 0 & \boxed{3} \end{pmatrix}$$

Example of matrices that are not in row echelon form:

$$\begin{pmatrix} 1 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} \boxed{1} & 0 & 2 & 2 \\ 0 & 0 & \boxed{1} & 3 \\ 0 & \boxed{1} & 0 & 4 \end{pmatrix}$$

- Any matrix can be transformed to row echelon form using one or more of the row operations
  1. Interchange one row with another.
  2. Multiply one row by a non-zero constant.
  3. Replace one row with: one row, plus a constant, times another row.
- In addition, it isn't enough just to know the rules, you have to be able to look at the matrix and make a logical decision about which rule you're going to use and when.

$$\begin{pmatrix} 0 & \boxed{1} & 3 & 4 \\ 0 & 0 & \boxed{2} & 3 \\ 0 & 0 & \boxed{4} & 1 \end{pmatrix} \longrightarrow \text{Not echelon form}$$

$$\begin{pmatrix} 0 & \boxed{1} & 3 & 4 \\ 0 & 0 & \boxed{2} & 3 \\ 0 & 0 & \boxed{4} & 1 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 - 2.R_2} \begin{pmatrix} 0 & \boxed{1} & 3 & 4 \\ 0 & 0 & \boxed{2} & 3 \\ 0 & 0 & 0 & \boxed{-5} \end{pmatrix} \longrightarrow \text{Echelon form}$$

Reduced row echelon form

Reduced row echelon form is a type of matrix used to solve systems of linear equations. A matrix is in *reduced row echelon form* if it is in row echelon form, and in addition:

- Each pivot is equal to 1.
- Each pivot is the only nonzero entry in its column.

Example

The following matrix is in reduced echelon form:

$$\begin{pmatrix} \boxed{1} & 0 & 2 & 0 & 4 \\ 0 & \boxed{1} & -3 & 0 & 7 \\ 0 & 0 & 0 & \boxed{1} & 12 \end{pmatrix}$$

Theorem

Every matrix is row equivalent to one and only one matrix in reduced row echelon form.

- The uniqueness statement is interesting-it means that, no matter how you row reduce, you always get the same matrix in reduced row echelon form.
- ⋆ This assumes, of course, that you only do the three legal row operations, and you don't make any arithmetic errors.

Gaussian Elimination

- **Gaussian Elimination** is a set of well-defined instructions to solve a system of linear equations.
- It consists of a sequence of **elementary row operations** performed on the corresponding **augmented matrix**, in order to get in a **row-echelon form**.
- Then the system of linear equations corresponding to the row-echelon form is said to be **triangular**.

Example

- Supposing it exists and is unique, find the solution to the following system of linear equations:

$$\begin{cases} x + 2y + 3z & = & 2 \\ 2x + y + 4z & = & 5 \\ x + 3y + 2z & = & 1 \end{cases}$$

Example

- Supposing it exists and is unique, find the solution to the following system of linear equations:

$$\begin{cases} x + 2y + 3z & = & 2 \\ 2x + y + 4z & = & 5 \\ x + 3y + 2z & = & 1 \end{cases}$$

- Solution:

The table below is the **Gaussian Elimination** process applied simultaneously to the system of equations and its associated augmented matrix.

- The Guassian Eimination procedure may be summarized as follows:
  - eliminate x from all equations below R_1, and then eliminate y from all equations below R_2. This will put the system into **triangular form**,
  - then, using back-substitution, each unknown can be solved for.

System of equations	Row operations	Augmented matrix
$\begin{cases} x+2y+3z = 2 \\ 2x+y+4z = 5 \\ x+3y+2z = 1 \end{cases}$		$\left(\begin{array}{ccc\|c} 1 & 2 & 3 & 2 \\ 2 & 1 & 4 & 5 \\ 1 & 3 & 2 & 1 \end{array}\right)$
$\begin{cases} x+2y+3z = 2 \\ 0x-3y-2z = 1 \\ 0x+y-z = -1 \end{cases}$	$R_2 \leftarrow R_2 - 2R_1$ $R_3 \leftarrow R_3 - R_1$	$\left(\begin{array}{ccc\|c} 1 & 2 & 3 & 2 \\ 0 & -3 & -2 & 1 \\ 0 & 1 & -1 & -1 \end{array}\right)$
$\begin{cases} x+2y+3z = 2 \\ 0x-3y-2z = 1 \\ 0x+0y-\frac{5}{3}z = -\frac{2}{3} \end{cases}$	$R_3 \leftarrow R_3 + \frac{1}{3}R_2$	$\left(\begin{array}{ccc\|c} 1 & 2 & 3 & 2 \\ 0 & -3 & -2 & 1 \\ 0 & 0 & -\frac{5}{3} & -\frac{2}{3} \end{array}\right)$
The matrix now in echelon form and the system is triangular		

System of equations	Row operations	Augmented matrix
$\begin{cases} x+2y+3z = 2 \\ 2x+y+4z = 5 \\ x+3y+2z = 1 \end{cases}$		$\left(\begin{array}{ccc\|c} 1 & 2 & 3 & 2 \\ 2 & 1 & 4 & 5 \\ 1 & 3 & 2 & 1 \end{array}\right)$
$\begin{cases} x+2y+3z = 2 \\ 0x-3y-2z = 1 \\ 0x+y-z = -1 \end{cases}$	$R_2 \leftarrow R_2 - 2R_1$ $R_3 \leftarrow R_3 - R_1$	$\left(\begin{array}{ccc\|c} 1 & 2 & 3 & 2 \\ 0 & -3 & -2 & 1 \\ 0 & 1 & -1 & -1 \end{array}\right)$
$\begin{cases} x+2y+3z = 2 \\ 0x-3y-2z = 1 \\ 0x+0y-\frac{5}{3}z = -\frac{2}{3} \end{cases}$	$R_3 \leftarrow R_3 + \frac{1}{3}R_2$	$\left(\begin{array}{ccc\|c} 1 & 2 & 3 & 2 \\ 0 & -3 & -2 & 1 \\ 0 & 0 & -\frac{5}{3} & -\frac{2}{3} \end{array}\right)$
The matrix now in echelon form and the system is triangular		

1. From R_3: $-\frac{5}{3}z = -\frac{2}{3} \iff \boxed{z = \frac{2}{5}}$.
2. From R_2: $-3y - 2z = 1 \overset{z=\frac{2}{5}}{\iff} -3y = \frac{9}{5} \iff \boxed{y = -\frac{3}{5}}$.
3. From R_1: $x + 2y + 3z = 2 \overset{z=\frac{2}{5},\ y=-\frac{3}{5}}{\iff} x - \frac{6}{5} + \frac{6}{5} = 2 \iff \boxed{x = 2}$.

One may proceed further in order to get a reduced row echelon form.

System of equations	Row operations	Augmented matrix
$\begin{cases} x+2y+3z = 2 \\ 2x+y+4z = 5 \\ x+3y+2z = 1 \end{cases}$		$\left(\begin{array}{ccc\|c} 1 & 2 & 3 & 2 \\ 2 & 1 & 4 & 5 \\ 1 & 3 & 2 & 1 \end{array}\right)$
$\begin{cases} x+2y+3z = 2 \\ 0x-3y-2z = 1 \\ 0x+y-z = -1 \end{cases}$	$R_2 \leftarrow R_2 - 2R_1$ $R_3 \leftarrow R_3 - R_1$	$\left(\begin{array}{ccc\|c} 1 & 2 & 3 & 2 \\ 0 & -3 & -2 & 1 \\ 0 & 1 & -1 & -1 \end{array}\right)$
$\begin{cases} x+2y+3z = 2 \\ 0x-3y-2z = 1 \\ 0x+0y-\frac{5}{3}z = -\frac{2}{3} \end{cases}$	$R_3 \leftarrow R_3 + \frac{1}{3}R_2$	$\left(\begin{array}{ccc\|c} 1 & 2 & 3 & 2 \\ 0 & -3 & -2 & 1 \\ 0 & 0 & -\frac{5}{3} & -\frac{2}{3} \end{array}\right)$
The matrix now in echelon form and the system is triangular		
$\begin{cases} x+2y+3z = 2 \\ 0x+y+\frac{2}{3}z = -\frac{1}{3} \\ 0x+0y+z = \frac{2}{5} \end{cases}$	$R_3 \leftarrow -\frac{3}{5}R_3$ $R_2 \leftarrow -\frac{1}{3}R_2$	$\left(\begin{array}{ccc\|c} 1 & 2 & 3 & 2 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{2}{5} \end{array}\right)$
$\begin{cases} x+2y+0z = \frac{4}{5} \\ 0x+y+0z = -\frac{3}{5} \\ 0x+0y+z = 0 \end{cases}$	$R_2 \leftarrow R_2 - \frac{2}{3}R_3$ $R_1 \leftarrow R_1 - 3R_3$	$\left(\begin{array}{ccc\|c} 1 & 2 & 0 & \frac{4}{5} \\ 0 & 1 & 0 & -\frac{3}{5} \\ 0 & 0 & 1 & \frac{2}{5} \end{array}\right)$
$\begin{cases} x+0y+0z = 2 \\ 0x+y+0z = -\frac{3}{5} \\ 0x+0y+z = \frac{2}{5} \end{cases}$	$R_1 \leftarrow R_1 - 2.R_2$	$\left(\begin{array}{ccc\|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -\frac{3}{5} \\ 0 & 0 & 1 & \frac{2}{5} \end{array}\right)$

Get a 1 here

$$\begin{pmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{pmatrix}$$

Clear down

$$\begin{pmatrix} 1 & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{pmatrix}$$

Get a 1 here

$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \end{pmatrix}$$

Clear down

$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \end{pmatrix}$$

(maybe these are already zero)

$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & \star \\ 0 & 0 & 0 & \star \end{pmatrix}$$

Get a 1 here

$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & \star \\ 0 & 0 & 0 & \star \end{pmatrix}$$

Clear down

$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \star \end{pmatrix}$$

Matrix is in REF

$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Clear up

$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Clear up

$$\begin{pmatrix} 1 & \star & \star & 0 \\ 0 & 1 & \star & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Matrix is in RREF

$$\begin{pmatrix} 1 & 0 & \star & 0 \\ 0 & 1 & \star & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Example

Solve the following system of equations using row operations:

$$\begin{cases} x + y = 2 \\ 3x + 4y = 5 \\ 4x + 5y = 9 \end{cases}$$

Solution:

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 4 & 5 \\ 4 & 5 & 9 \end{array}\right) \xrightarrow[R_3:=R_3-4R_1]{R_2:=R_2-3R_1} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{R_3:=R_3-R_2} \left(\begin{array}{cc|c} \boxed{1} & 1 & 2 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & \boxed{2} \end{array}\right)$$

The last row implies that $0z = 2 \implies 0 = 2$, contradiction. Hence the system doesn't have any solution.

Understanding the pivot positions in a row echelon form matrix is crucial for determining solutions in systems and various geometric situations, as we will explore later. Hence, the upcoming terminology is essential.

Definition

- A pivot position of a matrix is an entry that is a pivot of a row echelon form of that matrix.
- A pivot column of a matrix is a column that contains a pivot position.

Example

Find the pivot positions and pivot columns of this matrix

$$A = \left(\begin{array}{ccc|c} 0 & -7 & -4 & 2 \\ 2 & 4 & 6 & 12 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

Solution: We can check that a row echelon form of the matrix is

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right)$$

The pivot positions of A are the entries that become pivots in a row echelon form; they are marked in red below:

$$\left(\begin{array}{ccc|c} \color{red}{0} & -7 & -4 & 2 \\ 2 & \color{red}{4} & 6 & 12 \\ 3 & 1 & \color{red}{-1} & -2 \end{array}\right)$$

The first, second, and third columns of matrix A are pivot columns.

☺ There are three possibilities for the reduced row echelon form of the augmented matrix of a linear system.

- ⋆ The last (augmented) column is a pivot column. In this case, the system is inconsistent. There are zero solutions, i.e., the solution set is empty. For example, the matrix

$$\left(\begin{array}{cc|c} \boxed{1} & 5 & 0 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & \boxed{1} \end{array}\right) \xrightarrow{\text{translates to}} \begin{cases} x + 5y = 0 \\ y = 2 \\ 0 = 1 \end{cases}$$

  comes from a linear system with no solutions.

- ⋆⋆ Every column except the last column is a pivot column. In this case, the system has a unique solution. For example, the matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right)$$

  tells us that the unique solution is $(x, y, z) = (a, b, c)$.

⋆⋆⋆ The last column is not a pivot column, and some other column is not a pivot column either. In this case, the system has infinitely many solutions, corresponding to the infinitely many possible values of the variable(s) corresponding to the non-pivot column(s) – we call them free variable(s)

- For example, the system corresponding to the matrix

$$\left(\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right) \quad \text{is} \quad \left\{ \begin{array}{rcr} x - 2y & = & 1 \\ z & = & -1 \end{array} \right.$$

which we can rewrite as $\left\{ \begin{array}{lcl} x & = & 2y + 1 \\ z & = & -1 \end{array} \right.$

- For any value of y, there is exactly one value of x and z that make the equations true. But we are free to choose any value of y.

- For this example in this last case, have found all the solutions which are of the form (x, y, z), where

$$\begin{cases} x &= 2y+1 \\ y &= y \\ z &= -1 \end{cases}, \quad \text{for } y \in \mathbb{R}$$

- This is called the **parametric form** for the solution to the linear system.
- The variable y is called a free variable.

- Given the parametric form for the solution to a linear system, we can obtain specific solutions by replacing the free variables with any specific real numbers.
- For instance, setting $y = 1$ in the last example gives the solution $(x, y, z) = (3, 1 - 1)$, and setting $y = 0$ gives the solution $(x, y, z) = (1, 0, -1)$.

Definition

- Consider a consistent system of equations in the variables $x_1, x_2, \ldots, x_n$.
- Let A be a row echelon form of the augmented matrix for this system.
- **We say that x_i is a free variable if its corresponding column in A is not a pivot column.**

Example

In the matrix

$$\left(\begin{array}{cccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array}\right)$$

the free variables are x_2 and x_4. (The augmented column is not free because it does not correspond to a variable.)

1. Systems of Linear Equations

2. Solving Systems With Guassian Elimination

3. Solving Systems with Inverses

4. Solving 2×2 Systems with Cramers's Rule

5. Application: Linear Independence, Linear Dependence

6. Subspaces

Four Equivalent Ways of Writing a Linear System

- Let's first understand the equivalence between a system of linear equations, an augmented matrix, a vector equation, and a matrix equation.

Vector Equations:

- An equation involving vectors with n components is the same as n equations involving only numbers.
- For example, let $\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$, and $\vec{w} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$, the equation

$$x\vec{u} + y\vec{v} = \vec{w} \iff x\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + y\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

simplifies to

$$\begin{pmatrix} x \\ 2x \\ 6x \end{pmatrix} + \begin{pmatrix} -y \\ -2y \\ -y \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix} \implies \begin{pmatrix} x - y \\ 2x - 2y \\ 6x - y \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

- $x\vec{u} + y\vec{v} = \vec{w} \iff \begin{pmatrix} x - y \\ 2x - 2y \\ 6x - y \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$
- For two vectors to be equal, all of their components must be equal, so this is just the system of linear equations

$$\begin{cases} x - y &= 8 \\ 2x - 2y &= 16 \\ 6x - y &= 3 \end{cases}$$

Definition

A vector equation is an equation involving a linear combination of vectors with possibly unknown coefficients.

Remark:

Asking whether or not a vector equation has a solution is the same as asking if a given vector is a linear combination of some other given vectors.

- For example the vector equation above is asking if the vector $\langle 8, 16, 3\rangle$ is a linear combination of the vectors $\langle 1, 2, 6\rangle$ and $\langle -1, -2, -1\rangle$.

Example

Is $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$?

Solution: $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ is a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$ if and only if

$$\begin{cases} x - y & = & 8 \\ 2x - 2y & = & 16 \\ 6x - y & = & 3 \end{cases}$$

is consistent.

Solution:

$$\left(\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array}\right) \xrightarrow[R_3:=R_3-6R_1]{R_2:=R_2-2R_1} \left(\begin{array}{cc|c} 1 & -1 & 8 \\ 0 & 0 & 0 \\ 0 & 5 & -45 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{cc|c} \boxed{1} & -1 & 8 \\ 0 & \boxed{5} & -45 \\ 0 & 0 & 0 \end{array}\right)$$

Then the system is consistent and hence $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ is a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$. Solving the system gives $x = -1$ and $y = -9$ and we can conclude that $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix} = -\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} - 9\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$

Now we have three equivalent ways of making the same statement:

1. A vector $\vec{b}$ is in the span of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$.
2. The vector equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_k\vec{v}_k = \vec{b}$$

   has a solution.
3. The linear system with augmented matrix

$$\left(\begin{array}{cccc|c} \vdots & \vdots & & \vdots & \vdots \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k & \vec{b} \\ \vdots & \vdots & & \vdots & \vdots \end{array}\right)$$

   is consistent.

(**Equivalent** means that, for any given list of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k, \vec{b}$, either all three statements are true, or all three statements are false.)

Therefore, the following statements are also equivalent for vectors $\vec{b}, \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ in $\mathbb{R}^n$ (i.e. with n components)

1. Any vector $\vec{b}$ in $\mathbb{R}^n$ is in the span of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$.
2. The vector equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_k\vec{v}_k = \vec{b}$$

has a solution for any vector $\vec{b}$ in $\mathbb{R}^n$

3. The linear system with augmented matrix

$$\left(\begin{array}{cccc|c} \vdots & \vdots & & \vdots & \vdots \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k & \vec{b} \\ \vdots & \vdots & & \vdots & \vdots \end{array}\right)$$

is consistent for any vector $\vec{b}$ in $\mathbb{R}^n$

4. $\mathbb{R}^n = \text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ (or we can say that the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ span $\mathbb{R}^n$)

Therefore, we get the following proposition:

Proposition

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be vectors in R^n. Then $\mathbb{R}^n = \text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ if and only if the matrix

$$\begin{pmatrix} \vdots & \vdots & & \vdots \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \\ \vdots & \vdots & & \vdots \end{pmatrix}$$

has exactly n pivots.

- Thus, based on this proposition, we deduce that if $k < n$, then $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ can never span $\mathbb{R}^n$.

The Matrix Equation $AX = b$:

- Here A is a $m \times n$ matrix (m rows and n columns) and X, b are vectors (generally of different sizes).
- Of course, to have a defined product AX, X must be a column vector (that is matrix with one column) having n components. Then the product b is a column vector of m components.

Definition

A column vector is a matrix with one column and a row vector is a matrix with one row.

DEFINITION

- Let A be an $m \times n$ matrix with columns $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ and X a column vector in $\mathbb{R}^n$ (of n components in $\mathbb{R}$):

$$A = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

- The product of A with a vector X in $\mathbb{R}^n$ is the linear combination

$$AX = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n$$

This is a vector in $\mathbb{R}^m$.

Matrix Equation

Example

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1 \begin{pmatrix} 4 \\ 7 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ 8 \end{pmatrix} + 3 \begin{pmatrix} 6 \\ 9 \end{pmatrix}$$

Properties of Matrix-Vector Product

Let A be an $m \times n$ matrix, let $\vec{u}$, $\vec{v}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then:

- $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$
- $A(c\vec{u}) = cA\vec{u}$

Remark

- A matrix-vector product is a particular case of matrix-matrix product. So we either compute it as a linear combination of the right-matrix columns, or as we usually compute a matrix-matrix product.

DEFINITION

- A matrix equation is an equation of the form $AX = b$, where A is an $m \times n$ matrix, b is a vector in $\mathbb{R}^m$, and X is a vector whose components $x_1, x_2, \ldots, x_n$ are unknown.

Four Equivalent Ways of Writing a Linear System

1. As a system of equations:

$$\begin{cases} 2x_1 + 3x_2 - 2x_3 & = & 7 \\ x_1 - x_2 - 3x_3 & = & 5 \end{cases}$$

2. As an augmented matrix:

$$\left(\begin{array}{ccc|c} 2 & 3 & -2 & 7 \\ 1 & -1 & -3 & 5 \end{array}\right)$$

3. As a vector equation ($x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = b$):

$$x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

4. As a matrix equation ($AX = b$):

$$\begin{pmatrix} 2 & 3 & -2 \\ 1 & -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

Definition

- The inverse of a square matrix is another square matrix, which on multiplication with the given matrix gives the multiplicative identity.
- For a square matrix A, its inverse is A^{-1}, and $A \cdot A^{-1} = A^{-1} \cdot A = I$, where I is the identity matrix.
- The matrix whose determinant is non-zero, for which the inverse matrix can be calculated, is called an invertible matrix.

- When A is invertible, we can solve the matrix equation $AX = b$ in an elegant way:

$$\boxed{AX = b \iff X = A^{-1}b.}$$

Proposition

Let A and B be invertible $n \times n$ matrices.

1. A^{-1} is invertible, and its inverse is $(A^{-1})^{-1} = A$.
2. AB is invertible, and its inverse is $(AB)^{-1} = B^{-1}A^{-1}$ (note the order).

- The proof is left as an exercise.

General Way to Compute Matrix Inverses

- We have learned in the previous chapter how to compute the inverse of a 2×2 matrix after computing its determinant.
- The following theorem gives a procedure for computing A^{-1} in general.

Theorem

- *Let A be an $n \times n$ matrix, and let $(A \mid I_n)$ be the matrix obtained by augmenting A by the identity matrix.*
- *If the reduced row echelon form of $(A \mid I_n)$ has the form $(I_n \mid B)$, then A is invertible and $B = A^{-1}$.*
- *Otherwise, A is not invertible.*

- We will see the proof of this theorem next year second semester.

Example (An invertible matrix)

Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix}$$

Solution

We augment by the identity and row reduce:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3=R_3+3R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 3 & 1 \end{array}\right)$$

$$\xrightarrow[R_2=R_2-R_3]{R_1=R_1-2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -6 & -2 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 2 & 0 & 3 & 1 \end{array}\right)$$

$$\xrightarrow{R_3=R_3\div 2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -6 & -2 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 & 3/2 & 1/2 \end{array}\right).$$

Solution

By the theorem, the inverse matrix is

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -6 & -2 \\ 0 & -2 & -1 \\ 0 & 3/2 & 1/2 \end{pmatrix}.$$

We check:

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix}\begin{pmatrix} 1 & -6 & -2 \\ 0 & -2 & -1 \\ 0 & 3/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Example (A non-invertible matrix)

Is the following matrix invertible?

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -6 \end{pmatrix}.$$

Solution

We augment by the identity and row reduce:

$$\left(\begin{array}{rrr|rrr} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -6 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 = R_3 + 3R_2} \left(\begin{array}{rrr|rrr} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \end{array}\right).$$

At this point we can stop, because it is clear that the reduced row echelon form will not have I_3 in the non-augmented part: it will have a row of zeros. By the theorem, the matrix is not invertible.

- Let's learn to solve $AX = b$ by multiplying by A^{-1}.

Theorem

- *Let A be an invertible $n \times n$ matrix, and let b be a vector in $\mathbb{R}^n$.*
- *Then the matrix equation $AX = b$ has exactly one solution:*

$$X = A^{-1}b.$$

Proof

We calculate:

$$\begin{aligned} AX = b \quad &\Longrightarrow \quad A^{-1}(AX) = A^{-1}b \\ &\Longrightarrow \quad \left(A^{-1}A\right)X = A^{-1}b \\ &\Longrightarrow \quad I_nX = A^{-1}b \\ &\Longrightarrow \quad X = A^{-1}b. \end{aligned}$$

Here we used associativity of matrix multiplication, and the fact that $I_nX = X$ for any vector b.

Example (Solving a 2×2 system using inverses)

Solve the matrix equation

$$\begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix} X = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Solution

$$\det\begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix} = 1 \cdot 2 - (-1) \cdot 3 = 5 \neq 0$$

Thus, by the theorem, the only solution of our linear system is

$$X = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Example (Solving a 3×3 system using inverses)

Solve the system of equations

$$\begin{cases} 2x_1 + 3x_2 + 2x_3 = 1 \\ x_1 \qquad\quad + 3x_3 = 1 \\ 2x_1 + 2x_2 + 3x_3 = 1. \end{cases}$$

Solution

First we write our system as a matrix equation $AX = b$, where

$$A = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Next we find the inverse of A by augmenting and row reducing:

Solution

$$\left(\begin{array}{ccc|ccc} 2 & 3 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \longleftrightarrow R_2} \left(\begin{array}{ccc|ccc} \color{red}{1} & 0 & 3 & 0 & 1 & 0 \\ 2 & 3 & 2 & 1 & 0 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 = R_3 - 2R_1]{R_2 = R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ \color{red}{0} & 3 & -4 & 1 & -2 & 0 \\ \color{red}{0} & 2 & -3 & 0 & -2 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 - R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & \color{red}{1} & -1 & 1 & 0 & -1 \\ 0 & 2 & -3 & 0 & -2 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 - 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 \\ 0 & \color{red}{0} & -1 & -2 & -2 & 3 \end{array}\right).$$

Solution

$$\xrightarrow{R_3=R_3-2R_2}\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 \\ 0 & \color{red}{0} & -1 & -2 & -2 & 3 \end{array}\right)$$

$$\xrightarrow{R_3=-R_3}\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 \\ 0 & 0 & \color{red}{1} & 2 & 2 & -3 \end{array}\right)$$

$$\xrightarrow[R_2=R_3+R_3]{R_1=R_1-3R_3}\left(\begin{array}{ccc|ccc} 1 & 0 & \color{red}{0} & -6 & -5 & 9 \\ 0 & 1 & \color{red}{0} & 3 & 2 & -4 \\ 0 & 0 & 1 & 2 & 2 & -3 \end{array}\right).$$

By the theorem, the only solution of our linear system is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

The advantage of solving a linear system using inverses is that it becomes much faster to solve the matrix equation $AX = b$ for other, or even unknown, values of b. For instance, in the above example, the solution of the system of equations

$$\begin{cases} 2x_1 + 3x_2 + 2x_3 = b_1 \\ x_1 + 3x_3 = b_2 \\ 2x_1 + 2x_2 + 3x_3 = b_3, \end{cases}$$

where b_1, b_2, b_3 are unknowns, is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} =$$

$$\begin{pmatrix} -6b_1 - 5b_2 + 9b_3 \\ 3b_1 + 2b_2 - 4b_3 \\ 2b_1 + 2b_2 - 3b_3 \end{pmatrix}$$

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- Cramer's Rule is a method that uses determinants to solve systems of equations that have the same number of equations as variables.
- Consider a system of two linear equations in two variables.

$$\begin{cases} a_1x + b_1y & = & c_1 \\ a_2x + b_2y & = & c_2 \end{cases} \iff \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

- If A is invertible, then the solution of the system is given by:

$$x = \frac{\det \begin{pmatrix} c_1 & b_1 \\ c_2 & b_2 \end{pmatrix}}{\det(A)}, \quad y = \frac{\det \begin{pmatrix} a_1 & c_1 \\ a_2 & c_2 \end{pmatrix}}{\det(A)}$$

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☺ We study the matrix equation $AX = b$ in the particular case $b = 0$.

Homogeneous systems

Definition

- A system of linear equations of the form $AX = 0$ is called homogeneous.
- A system of linear equations of the form $AX = b$ for $b \neq 0$ is called inhomogeneous.

- A homogeneous system is just a system of linear equations where all constants on the right side of the equals sign are zero.

- ☺ A homogeneous system always has the solution $X = 0$. This is called the trivial solution.
- ☺ Any nonzero solution is called nontrivial.

Observation:

The equation $AX = 0$ has a nontrivial solution $\iff$ there is a free variable $\iff$ A has a column without a pivot position.
In this case, the solution set can be conveniently expressed as a span.

Example

What is the solution set of $AX = 0$, where

$$\begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$

SOLUTION

We form an augmented matrix and row reduce:

$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

The only solution is the trivial solution $X = 0$.

- **Remark:** When we row reduce the augmented matrix for a homogeneous system of linear equations, the last column will be zero throughout the row reduction process. So it is not really necessary to write augmented matrices when solving homogeneous systems.

Example

Compute the parametric vector form of the solution set of $AX = 0$, where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}$$

Solution: We row reduce (without augmenting, as suggested in the above observation):

$$\begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

This corresponds to the single equation $x_1 - x_2 + 2x_3 = 0$. We write the parametric form including the redundant equations $x_2 = x_2$ and $x_3 = x_3$:

$$\begin{cases} x_1 = x_2 - 2x_3 \\ x_2 = x_2 \\ x_3 = x_3. \end{cases}$$

Solution:

$$\begin{cases} x_1 = x_2 - 2x_3 \\ x_2 = x_2 \\ x_3 = x_3. \end{cases}$$

We turn these into a single vector equation:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

This is the parametric vector form of the solution set. Since x_2 and x_3 are allowed to be anything, this says that the solution set is the set of all linear combinations of $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$.

Solution:

$$\begin{cases} x_1 = x_2 - 2x_3 \\ x_2 = x_2 \\ x_3 = x_3. \end{cases}$$

We turn these into a single vector equation:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

This is the parametric vector form of the solution set.
In other words, the solution set is

$$\text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

We know how to draw the span of two noncollinear vectors in R^3 : it is a plane.

Application: Linear Independence, Linear Dependence

A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is linearly independent if and only if the vector equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_k\vec{v}_k = 0$$

has only the trivial solution, if and only if the matrix equation $AX = 0$ has only the trivial solution, where A is the matrix with columns $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$:

$$A = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \\ | & | & & | \end{pmatrix}$$

This is true if and only if A has a pivot position in every column.

Application: Linear Independence, Linear Dependence

- Recall that $AX = 0$ has a nontrivial solution if and only if A has a column without a pivot.
- Suppose that A has more columns than rows. Then A cannot have a pivot in every column (it has at most one pivot per row), so its columns are automatically linearly dependent.

A wide matrix (a matrix with more columns than rows) has linearly dependent columns.

EXAMPLE

The following vector families are linearly dependent:

- $\left\{\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}-1\\4\end{pmatrix},\begin{pmatrix}-2\\0\end{pmatrix}\right\}$.
- $\left\{\begin{pmatrix}1\\3\\4\end{pmatrix},\begin{pmatrix}-1\\4\\-7\end{pmatrix},\begin{pmatrix}-2\\0\\0\end{pmatrix},\begin{pmatrix}-2\\0\\7\end{pmatrix}\right\}$.

Application: Linear Independence, Linear Dependence

Example

Is the set

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$$

linearly independent?

Solution: Equivalently, we are asking if the homogeneous vector equation

$$x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + z \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has a nontrivial solution. We solve this by forming a matrix and row reducing:

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

This says $x = -2z$ and $y = -z$.

Application: Linear Independence, Linear Dependence

Solution: So there exist nontrivial solutions: for instance, taking $z = 1$ gives this equation of linear dependence:

$$-2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Example

Is the set

$$\left\{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}\right\}$$

linearly independent?

Solution: Equivalently, we are asking if the homogeneous vector equation

$$x\begin{pmatrix}1\\1\\-2\end{pmatrix}+y\begin{pmatrix}1\\-1\\2\end{pmatrix}+z\begin{pmatrix}3\\1\\4\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

has a nontrivial solution.
We solve this by forming a matrix and row reducing:

$$\begin{pmatrix}1&1&3\\1&-1&1\\-2&2&4\end{pmatrix}\xrightarrow{\text{row reduce}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

This says $x=y=z=0$, i.e., the only solution is the trivial solution. We conclude that the set is linearly independent.

Application: Linear Independence, Linear Dependence

Theorem

Let A be a square $n \times n$ matrix. Then the following are equivalent:

1. *A is invertible.*
2. $\det(A) \neq 0$.
3. *A has n pivots (there is pivot in every row and column.)*
4. *The columns of A are linearly independent.*
5. *Every vector in $\mathbb{R}^n$ is a linear combination of the columns of A (In other words, the columns of A span $\mathbb{R}^n$.).*
6. *The set of the columns of A is a basis of $\mathbb{R}^n$.*

Example

Verify if $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ is a basis of $\mathbb{R}^3$

Solution:

Method 1:

Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 0 \\ 3 & 2 & -1 \end{pmatrix}$. Then

$$\det(A) = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 0 & 0 \\ 3 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 3 & 2 \end{vmatrix}$$

$$= 0 - 2 + 4 = 2 \neq 0$$

Therefore, by the previous theorem, S is a basis of $\mathbb{R}^3$.

Solution:

Method 2:

Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 0 \\ 3 & 2 & -1 \end{pmatrix}$. Then

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 0 \\ 3 & 2 & -1 \end{pmatrix} \xrightarrow[R_3:=R_3-3R_1]{R_2:=R_2-2R_1} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 5 & -4 \end{pmatrix}$$

$$\xrightarrow{R_3:=R_3-\frac{5}{2}R_2} \begin{pmatrix} \boxed{1} & -1 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & \boxed{1} \end{pmatrix}$$

A has 3 pivots. Therefore, by the previous theorem, S is a basis of $\mathbb{R}^3$.

Outline

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6. Subspaces

Definition

1. A set is the mathematical model for a collection of different things;
2. A set contains elements or members, which can be mathematical objects of any kind: numbers, symbols, points in space, lines, other geometrical shapes, variables, or even other sets.
3. The set with no element is the empty set and is denoted by $\varnothing$;
4. A set with a single element is a singleton.
5. A set may have a finite number of elements or be an infinite set.
6. Two sets are equal if they have precisely the same elements.

There are many ways to define a set. One of them is the **Roster** (or **Enumeration notation**):

- it defines a set by listing its elements between curly brackets,

$$A = \{4, 2, 1, 3\}$$
$$B = \{\text{blue}, \text{white}, \text{red}\}.$$

- the elements in roster notation is irrelevant For example, $\{1, 3, 5\}$ and $\{3, 1, 5\}$ represent the same set.
- For sets with many elements, especially those following an implicit pattern, the list of members can be abbreviated using an ellipsis .... For instance, the set of the first thousand positive integers may be specified in roster notation as

$$\{1, 2, 3, \ldots, 1000\}.$$

Membership:

- If B is a set and x is an element of B, this is written in shorthand as $x \in B$, which can also be read as "x belongs to B", or "x is in B".
- The statement "y is not an element of B" is written as $y \notin B$, which can also be read as "y is not in B".
- For example, with respect to the sets $A = \{1, 2, 3, 4\}$ and $B = \{\text{blue}, \text{white}, \text{red}\}$, $4 \in A$, $12 \notin A$, green $\notin B$.

Definition

1. If every element of set A is also in B, then A is described as being a subset of B, or contained in B, written $A \subseteq B$ or $B \supseteq A$. The latter notation may be read B contains A, B includes A, or B is a superset of A.
2. The relationship between sets established by $\subseteq$ is called inclusion or containment. Two sets are equal if they contain each other: $A \subseteq B$ and $B \subseteq A$is equivalent to $A = B$.
3. If A is a subset of B, but A is not equal to B, then A is called a proper subset of B. This can be written $A \subsetneq B$ or simply $A \subset B$. Likewise, $B \supsetneq A$ or $B \supset A$ mean B is a proper superset of A, i.e. B contains A, and is not equal to A.

EXAMPLE

- The set of all humans is a proper subset of the set of all mammals.
- $\{1,3\} \subset \{1,2,3,4\}$.
- $\{1,2,3,4\} \subseteq \{1,2,3,4\}$.
- The empty set is a subset of every set, and every set is a subset of itself:
  - $\varnothing \subseteq A$.
  - $A \subseteq A$.

Definition

A subset of $\mathbb{R}^n$ is any collection of points of $\mathbb{R}^n$.

- For instance, the unit circle

$$C = \left\{(x, y) \text{ in } \mathbb{R}^2 \mid x^2 + y^2 = 1\right\}$$

is a subset of $\mathbb{R}^2$.

Remark: Above we expressed C in **set builder notation**: in English, it reads " C is the set of all ordered pairs (x, y) in $\mathbb{R}^2$ such that $x^2 + y^2 = 1$."

Definition

A subspace of $\mathbb{R}^n$ is a subset V of $\mathbb{R}^n$ satisfying:

1. **Non-emptiness:** The zero vector is in V.
2. **Closure under addition:** If $\vec{u}$ and $\vec{v}$ are in V, then $\vec{u}+\vec{v}$ is also in V.
3. **Closure under scalar multiplication:** If $\vec{v}$ is in V and c is in $\mathbb{R}$, then $c\vec{v}$ is also in V.

As a consequence of these properties, we see:

- If $\vec{v}$ is a vector in V, then all scalar multiples of $\vec{v}$ are in V by the third property. In other words the line through any nonzero vector in V is also contained in V.
- If $\vec{u}, \vec{v}$ are vectors in V and c, d are scalars, then $c\vec{u}, d\vec{v}$ are also in V by the third property, so $c\vec{u} + d\vec{v}$ is in V by the second property. Therefore, all of $\mathsf{Span}\{\vec{u}, \vec{v}\}$ is contained in V
- Similarly, if $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are all in V, then $\mathsf{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is contained in V. In other words, *a subspace contains the span of any vectors in it.*

Example

- The set R^n is a subspace of itself: indeed, it contains zero, and is closed under addition and scalar multiplication.
- The set $\{0\}$ containing only the zero vector is a subspace of R^n : it contains zero, and if you add zero to itself or multiply it by a scalar, you always get zero.
- A line L through the origin is a subspace of $\mathbb{R}^n$. Indeed, L contains zero, and is easily seen to be closed under addition and scalar multiplication.
- A plane P through the origin is a subspace. Indeed, P contains zero; the sum of two vectors in P is also in P; and any scalar multiple of a vector in P is also in P.
- A line L (or any other subset) that does not contain the origin is not a subspace. It fails the first defining property: every subspace contains the origin by definition.

Theorem

(Spans are Subspaces and Subspaces are Spans) If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ *are any vectors in* R^n*, then* $\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ *is a subspace of* R^n*. Moreover, any subspace of* R^n *can be written as a span of a set of* p *linearly independent vectors in* R^n *for* $p \leq n$*.*

Proof Part 1.

- To show that $\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is a subspace, we have to verify the three defining properties.
  1. The zero vector $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_p$ is in the span.
  2. If $\vec{u} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_p\vec{v}_p$ and $\vec{v} = b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_p\vec{v}_p$ are in $\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$, then

     $$\vec{u} + \vec{v} = (a_1 + b_1)\,\vec{v}_1 + (a_2 + b_2)\,\vec{v}_2 + \cdots + (a_p + b_p)\,\vec{v}_p$$

     is also in $\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$.
  3. If $\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_p\vec{v}_p$ is in $\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ and c is a scalar, then

     $$c\vec{v} = ca_1\vec{v}_1 + ca_2\vec{v}_2 + \cdots + ca_p\vec{v}_p$$

     is also in $\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$

  Since $\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ satisfies the three defining properties of a subspace, it is a subspace.

□

Proof Part 2.

- Now let V be a subspace of $\mathbb{R}^n$. If V is the zero subspace, then it is the span of the empty set, so we may assume V is nonzero. Choose a nonzero vector $\vec{v}_1$ in V. If $V = \text{Span}\{\vec{v}_1\}$, then we are done. Otherwise, there exists a vector $\vec{v}_2$ that is in V but not in $\text{Span}\{\vec{v}_1\}$. Then the set $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent and $\text{Span}\{\vec{v}_1, \vec{v}_2\}$ is contained in V. If $V = \text{Span}\{\vec{v}_1, \vec{v}_2\}$ then we are done. Otherwise, we continue in this fashion until we have written $V = \text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ for some linearly independent set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$. This process terminates after at most n steps.

□

If $V = \text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$, we say that V is the subspace **spanned by** or **generated by** the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$. We call $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ a **spanning set** for V.

Any matrix naturally gives rise to two subspaces.

Definition

Let A be an $m \times n$ matrix.

- The **column space** of A is the subspace of $\mathbb{R}^m$ spanned by the columns of A. It is written $\mathrm{Col}(A)$.
- The **null space** of A is the subspace of $\mathbb{R}^n$ consisting of all solutions of the homogeneous equation $Ax = 0$:

$$\mathrm{Nul}(A) = \{x \text{ in } \mathbb{R}^n \mid Ax = 0\}$$

1. The column space is defined to be a span, so it is a subspace by the above theorem.
2. We need to verify that the null space is really a subspace. But we have already saw that the set of solutions of $Ax = 0$ is always a span, so the fact that the null spaces is a subspace should not come as a surprise and we leave its proof for an exercise.

Example

Describe the column space and the null space of

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Solution:

- The column space is the span of the columns of A :

$$\mathsf{Col}(A) = \mathsf{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\} = \mathsf{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$$

This is a line in R^3.

Solution:

- The null space is the solution set of the homogeneous system $Ax = 0$. To compute this, we need to row reduce A. Its reduced row echelon form is

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This gives the equation $x + y = 0$, or

$$\begin{cases} x = -y \\ y = y \end{cases} \xrightarrow{\text{parametric vector form}} \begin{pmatrix} x \\ y \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Hence the null space is $\mathrm{Span}\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$, which is a line in $\mathbb{R}^2$.