INTRODUCTION TO LINEAR ALGEBRA

Chapter 4:

MATRIX TRANSFORMATIONS EIGENVALUES & EIGENVECTORS

Foundation Year - 2023/2024 Dr. Grace Younes



- 1. MATRIX TRANSFORMATIONS
- 2. One-to-one and Onto Transformations
- 3. Eigenvalues & Eigenvectors
- 4. The Characteristic Polynomial



2. One-to-one and Onto Transformations

3. Eigenvalues & Eigenvectors

4. The Characteristic Polynomial

• Informally, a function is a <u>rule</u> that accepts inputs and produces outputs. For instance, $f(x) = x^2$ is a function that accepts one number x as its input, and outputs the square of that number: f(2) = 4. In this section, we interpret matrices as functions.

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- Hereafter, when there is no ambiguity, we may simply denote a vector like any variable, i.e., without accenting it by a right arrow, that is simply v instead of \vec{v} .
- Let A be a matrix with m rows and n columns. Consider the matrix equation b = Ax (we write it this way instead of Ax = b to be reminded of the notation y = f(x)). If we vary x, then b will also vary; in this way, we think of A as a function with independent variable x and dependent variable b.
 - The independent variable (the input) is x, which is a vector in \mathbb{R}^n .
 - The dependent variable (the output) is b, which is a vector in \mathbb{R}^m .
- The set of all possible output vectors are the vectors b such that Ax = b has some solution;

this is the same as the column space of A

EXAMPLE (PROJECTION ONTO THE xy-PLANE)

Let

$$A = \left(\begin{array}{rrrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right).$$

Describe the function b = Ax geometrically.

Solution: In the equation Ax = b, the input vector x and the output vector b are both in \mathbb{R}^3 . First we multiply A by a vector to see what it does:

$$A\left(\begin{array}{c}x\\y\\z\end{array}\right) = \left(\begin{array}{ccc}1&0&0\\0&1&0\\0&0&0\end{array}\right)\left(\begin{array}{c}x\\y\\z\end{array}\right) = \left(\begin{array}{c}x\\y\\0\end{array}\right)$$

Multiplication by A simply sets the z-coordinate equal to zero: it projects vertically onto the xy-plane.

EXAMPLE (REFLECTION)

Let

$$\mathcal{A} = \left(\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right).$$

Describe the function b = Ax geometrically.

Solution: In the equation Ax = b, the input vector x and the output vector b are both in \mathbb{R}^2 . First we multiply A by a vector to see what it does:

$$A\left(\begin{array}{c}x\\y\end{array}\right)=\left(\begin{array}{c}-1&0\\0&1\end{array}\right)\left(\begin{array}{c}x\\y\end{array}\right)=\left(\begin{array}{c}-x\\y\end{array}\right).$$

Multiplication by A negates the x-coordinate: it reflects over the y-axis.



EXAMPLE (DILATION)

Let

$$A = \left(\begin{array}{cc} 1.5 & 0\\ 0 & 1.5 \end{array}\right).$$

Describe the function b = Ax geometrically.

Solution: In the equation Ax = b, the input vector x and the output vector b are both in \mathbb{R}^2 . First we multiply A by a vector to see what it does:

$$A\left(\begin{array}{c} x\\ y\end{array}\right) = \left(\begin{array}{c} 1.5 & 0\\ 0 & 1.5\end{array}\right) \left(\begin{array}{c} x\\ y\end{array}\right) = \left(\begin{array}{c} 1.5x\\ 1.5y\end{array}\right) = 1.5 \left(\begin{array}{c} x\\ y\end{array}\right).$$

Multiplication by A is the same as scalar multiplication by 1.5: it scales or dilates the plane by a factor of 1.5.





EXAMPLE (IDENTITY)

Let

$$A = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right).$$

Describe the function b = Ax geometrically.

Solution: In the equation Ax = b, the input vector x and the output vector b are both in \mathbb{R}^2 . First we multiply A by a vector to see what it does:

$$A\left(\begin{array}{c}x\\y\end{array}\right)=\left(\begin{array}{c}1&0\\0&1\end{array}\right)\left(\begin{array}{c}x\\y\end{array}\right)=\left(\begin{array}{c}x\\y\end{array}\right).$$

Multiplication by A does not change the input vector at all: it is the identity transformation which does nothing.



EXAMPLE (ROTATION)

Let

$$\mathcal{A} = \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right).$$

Describe the function b = Ax geometrically.

Solution: In the equation Ax = b, the input vector x and the output vector b are both in \mathbb{R}^2 . First we multiply A by a vector to see what it does:

$$A\left(\begin{array}{c}x\\y\end{array}\right)=\left(\begin{array}{c}0&-1\\1&0\end{array}\right)\left(\begin{array}{c}x\\y\end{array}\right)=\left(\begin{array}{c}-y\\x\end{array}\right).$$

We substitute a few test points in order to understand the geometry of the transformation:

Multiplication by A is counterclockwise rotation by 90° .





EXAMPLE (SHEAR)

Let

$$A = \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right).$$

Describe the function b = Ax geometrically.

Solution: In the equation Ax = b, the input vector x and the output vector b are both in \mathbb{R}^2 . First we multiply A by a vector to see what it does:

$$A\left(\begin{array}{c}x\\y\end{array}\right) = \left(\begin{array}{c}1&1\\0&1\end{array}\right)\left(\begin{array}{c}x\\y\end{array}\right) = \left(\begin{array}{c}x+y\\y\end{array}\right).$$

Multiplication by A adds the y-coordinate to the x-coordinate; this is called a shear in the x-direction.



DEFINITION

A transformation from \mathbb{R}^n to \mathbb{R}^m is a rule *T* that assigns to each vector *x* in \mathbb{R}^n a vector T(x) in \mathbb{R}^m .

- \mathbb{R}^n is called the **domain** of T.
- \mathbb{R}^m is called the **codomain** of T.
- For x in \mathbb{R}^n , the vector T(x) in \mathbb{R}^m is the **image** of x under T.

• The set of all images $\{T(x) \mid x \text{ in } \mathbb{R}^n\}$ is the **range** of T. The notation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ means "T is a transformation from \mathbb{R}^n to \mathbb{R}^m "

• Now we specialize the general notions and vocabulary from the previous frame to the functions defined by matrices that we considered in the first frame.

DEFINITION

Let A be an $m \times n$ matrix. The matrix transformation associated to A is the transformation

 $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defined by T(x) = Ax.

This is the transformation that takes a vector x in \mathbb{R}^n to the vector Ax in \mathbb{R}^m .

- If A has n columns, then it only makes sense to multiply A by vectors with n entries. This is why the domain of T(x) = Ax is \mathbb{R}^n .
- If A has n rows, then Ax has m entries for any vector x in \mathbb{R}^n ; this is why the codomain of T(x) = Ax is \mathbb{R}^m .

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The definition of a matrix transformation T tells us how to evaluate T on any given vector: we multiply the input vector by a matrix. For instance, let

$$A = \left(\begin{array}{rrrr} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right)$$

and let $T(\boldsymbol{x})=A\boldsymbol{x}$ be the associated matrix transformation. Then

$$T\left(\begin{array}{c}-1\\-2\\-3\end{array}\right) = A\left(\begin{array}{c}-1\\-2\\-3\end{array}\right) = \left(\begin{array}{c}1&2&3\\4&5&6\end{array}\right)\left(\begin{array}{c}-1\\-2\\-3\end{array}\right) = \left(\begin{array}{c}-14\\-32\end{array}\right).$$

• Suppose that A has columns v_1, v_2, \ldots, v_n . If we multiply A by a general vector x, we get

$$Ax = \begin{pmatrix} | & | & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1v_1 + x_2v_2 + \cdots + x_nv_n.$$

This is just a general linear combination of v_1, v_2, \ldots, v_n . Therefore, the outputs of T(x) = Ax are exactly the linear combinations of the columns of A:

The range of T is the column space of A.

Let A be an $m\times n$ matrix, and let T(x)=Ax be the associated matrix transformation.

- The domain of T is \mathbb{R}^n , where n is the number of columns of A.
- The codomain of T is \mathbb{R}^m , where m is the number of rows of A.
- The range of T is the column space of A.

EXAMPLE

Let

$$A = \left(\begin{array}{rrr} 1 & -1 & 2 \\ -2 & 2 & -4 \end{array}\right)$$

and define T(x) = Ax. The domain of T is \mathbb{R}^3 , and the codomain is \mathbb{R}^2 . The range of T is the column space; since all three columns are collinear, the range is a line in \mathbb{R}^2 .

EXAMPLE

Let

$$A = \left(\begin{array}{rrr} 1 & 0\\ 0 & 1\\ 1 & 0 \end{array}\right)$$

and define T(x) = Ax. The domain of T is \mathbb{R}^2 , and the codomain is \mathbb{R}^3 . The range of T is the column space; since A has two columns which are not collinear, the range is a plane in \mathbb{R}^3 .

EXAMPLE (PROJECTION ONTO THE xy-PLANE)

Let

$$A = \left(\begin{array}{rrrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

and let T(x) = Ax. What are the domain, the codomain, and the range of T ?

Solution: Geometrically, the transformation T projects a vector directly "down" onto the *xy*-plane in \mathbb{R}^3 . The inputs and outputs have three entries, so the domain and codomain are both \mathbb{R}^3 . The possible outputs all lie on the *xy*-plane, and every point on the *xy* plane is an output of T (with itself as the input), so the range of T is the *xy*-plane.

Be careful not to confuse the codomain with the range here. The range is a plane, but it is a plane in \mathbb{R}^3 , so the codomain is still \mathbb{R}^3 . The outputs of \mathcal{T} all have three entries; the last entry is simply always zero.

2. One-to-one and Onto Transformations

3. EIGENVALUES & EIGENVECTORS

4. The Characteristic Polynomial

DEFINITION (ONE-TO-ONE TRANSFORMATIONS)

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if, for every vector *b* in \mathbb{R}^m , the equation T(x) = b has at most one solution x in \mathbb{R}^n .

Remark: Another word for one-to-one is injective.

Here are some equivalent ways of saying that T is one-to-one:

- For every vector b in \mathbb{R}^m , the equation T(x) = b has zero or one solution x in \mathbb{R}^n .
- Different inputs of T have different outputs.

• If
$$T(u) = T(v)$$
 then $u = v$.

Here are some equivalent ways of saying that T is not one-to-one:

- There exists some vector b in \mathbb{R}^m such that the equation T(x) = b has more than one solution x in \mathbb{R}^n .
- There are two different inputs of T with the same output.
- There exist vectors u, v such that $u \neq v$ but T(u) = T(v).

Theorem (One-to-one matrix transformations)

Let A be an $m \times n$ matrix, and let T(x) = Ax be the associated matrix transformation. The following statements are equivalent:

- 1 T is one-to-one.
- 2 For every b in \mathbb{R}^m , the equation T(x) = b has at most one solution.
- **3** For every *b* in \mathbb{R}^m , the equation Ax = b has a unique solution or is inconsistent.
- 4 Ax = 0 has only the trivial solution.
- **5** The columns of A are linearly independent.
- 6 A has a pivot in every column.

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EXAMPLE

Let A be the matrix

$$A = \left(\begin{array}{rrr} 1 & 0\\ 0 & 1\\ 1 & 0 \end{array}\right)$$

and define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by T(x) = Ax. Is T one-to-one?

Solution: The reduced row echelon form of *A* is

$$\left(\begin{array}{rrr}
1 & 0\\
0 & 1\\
0 & 0
\end{array}\right).$$

Hence A has a pivot in every column, so T is one-to-one.

EXAMPLE

Let

$$A = \left(\begin{array}{rrrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

and define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by T(x) = Ax. Is T one-to-one? If not, find two different vectors u, v such that T(u) = T(v).

ONE-TO-ONE TRANSFORMATIONS

Solution: The matrix A is already in reduced row echelon form. It does not have a pivot in every column, so T is not one-to-one. Therefore, we know from the theorem that Ax = 0has nontrivial solutions. If v is a nontrivial (i.e., nonzero) solution of Av = 0, then T(v) = Av = 0 = A0 = T(0), so 0 and vare different vectors with the same output. For instance,

$$\mathcal{T}\left(\begin{array}{c} 0\\ 0\\ 1\end{array}\right) = \left(\begin{array}{cc} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{array}\right) \left(\begin{array}{c} 0\\ 0\\ 1\end{array}\right) = 0 = \mathcal{T}\left(\begin{array}{c} 0\\ 0\\ 0\\ 0\end{array}\right).$$

Geometrically, T is projection onto the *xy*-plane. Any two vectors that lie on the same vertical line will have the same projection. For *b* on the *xy*-plane, the solution set of T(x) = bis the entire vertical line containing *b*. In particular, T(x) = bhas infinitely many solutions.

EXAMPLE

Let A be the matrix

$$A=\left(\begin{array}{rrr}1&1&0\\0&1&1\end{array}\right),$$

and define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by T(x) = Ax. Is T one-to-one? If not, find two different vectors u, v such that T(u) = T(v).

ONE-TO-ONE TRANSFORMATIONS

Solution: The reduced row echelon form of A is

$$\left(\begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array} \right)$$

There is not a pivot in every column, so T is not one-to-one. Therefore, we know from the theorem that Ax = 0 has nontrivial solutions. If v is a nontrivial (i.e., nonzero) solution of Av = 0, then T(v) = Av = 0 = A0 = T(0), so 0 and v are different vectors with the same output. In order to find a nontrivial solution, we find the parametric form of the solutions of Ax = 0 using the reduced matrix above:

$$\begin{cases} x-z = 0\\ y+z = 0 \end{cases} \implies \begin{cases} x=z\\ y=-z \end{cases}$$

The free variable is z. Taking z=1 gives the nontrivial solution

$$\mathcal{T}\left(\begin{array}{c}1\\-1\\1\end{array}\right) = \left(\begin{array}{cc}1&1&0\\0&1&1\end{array}\right) \left(\begin{array}{c}1\\-1\\1\end{array}\right) = 0 = \mathcal{T}\left(\begin{array}{c}0\\0\\0\end{array}\right)$$
Wide matrices do not have one-to-one transformations. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a one-to-one matrix transformation, what can we say about the relative sizes of n and m?

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ONE-TO-ONE TRANSFORMATIONS

- The matrix associated to T has n columns and m rows. Each row and each column can only contain one pivot, so in order for A to have a pivot in every column, it must have at least as many rows as columns: $n \le m$.
- 2 This says that, for instance, \mathbb{R}^3 is "too big" to admit a one-to-one linear transformation into \mathbb{R}^2 .

Note that there exist tall matrices that are not one-to-one: for example,

1	1	0	0	
	0	1	0	
	0	0	0	
	0	0	0	J

does not have a pivot in every column.

DEFINITION (ONTO TRANSFORMATIONS)

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if, for every vector *b* in \mathbb{R}^m , the equation T(x) = b has at least one solution *x* in \mathbb{R}^n .

Remark: Another word for onto is surjective.

Here are some equivalent ways of saying that T is onto:

- The range of T is equal to the codomain of T.
- Every vector in the codomain is the output of some input vector.

Here are some equivalent ways of saying that T is not onto:

- The range of T is smaller than the codomain of T.
- There exists a vector b in \mathbb{R}^m such that the equation T(x) = b does not have a solution.
- There is a vector in the codomain that is not the output of any input vector.

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THEOREM (ONTO MATRIX TRANSFORMATIONS)

Let A be an $m \times n$ matrix, and let T(x) = Ax be the associated matrix transformation. The following statements are equivalent:

- 1 T is onto.
- 2 T(x) = b has at least one solution for every b in \mathbb{R}^m .
- **3** Ax = b is consistent for every b in \mathbb{R}^m .
- **4** The columns of A span \mathbb{R}^m .
- 6 A has a pivot in every row.

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Let A be the matrix

$$A = \left(\begin{array}{rrr} 1 & 1 & 0 \\ 0 & 1 & 1 \end{array}\right),$$

and define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by T(x) = Ax. Is T onto?

Solution: The reduced row echelon form of A is $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$

Hence A has a pivot in every row, so T is onto.

Let A be the matrix

$$\mathsf{A} = \left(\begin{array}{rrr} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{array} \right)$$

and define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by T(x) = Ax. Is T onto? If not, find a vector b in \mathbb{R}^3 such that T(x) = b has no solution.

ONTO TRANSFORMATIONS

Solution: The reduced row echelon form of A is

$$\left(\begin{array}{cc}1&0\\0&1\\0&0\end{array}\right)$$

Hence A does not have a pivot in every row, so T is not onto. In fact, since

$$T\left(\begin{array}{c} x\\ y\end{array}\right) = \left(\begin{array}{c} 1 & 0\\ 0 & 1\\ 1 & 0\end{array}\right) \left(\begin{array}{c} x\\ y\end{array}\right) = \left(\begin{array}{c} x\\ y\\ x\end{array}\right),$$

we see that for every output vector of \mathcal{T} , the third entry is equal to the first. Therefore,

$$b = (1, 2, 3)$$

is not in the range of T.

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Tall matrices do not have onto transformations. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is an onto matrix transformation, what can we say about the relative sizes of n and m?

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- **1** The matrix associated to T has n columns and m rows. Each row and each column can only contain one pivot, so in order for A to have a pivot in every row, it must have at least as many columns as rows: $m \le n$.
- 2 This says that, for instance, \mathbb{R}^2 is "too small" to admit an onto linear transformation to \mathbb{R}^3 .

Note that there exist wide matrices that are not onto: for example,

$$\left(\begin{array}{ccc} 1 & -1 & 2 \\ -2 & 2 & -4 \end{array}\right)$$

does not have a pivot in every row.

Let A be an $m \times n$ matrix, and $T : \mathbb{R}^n \to \mathbb{R}^m$ is the matrix transformation T(x) = Ax.

T is one-to-oneT is ontoT(x) = b has at most one solution
for every bT(x) = b has at least one solution
for every bThe columns of A are linearly
independentThe columns of A span \mathbb{R}^m .A has a pivot in every column.A has a pivot in every row.

One-to-one is the same as onto for square matrices.

- a square matrix has a pivot in every row if and only if it has a pivot in every column.
- Therefore, a matrix transformation \mathcal{T} from \mathbb{R}^n to itself is one-to-one if and only if it is onto: in this case, the two notions are equivalent.
- Conversely, by this note and this note, if a matrix transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is both one-to-one and onto, then m = n.

Note that in general, a transformation T is both one-to-one and onto if and only if T(x) = b has exactly one solution for all b in \mathbb{R}^m .

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DEFINITION

Let A be an $n \times n$ matrix.

- **1** An eigenvector of A is a nonzero vector v in \mathbb{R}^n such that $Av = \lambda v$, for some scalar λ
- 2 An eigenvalue of A is a scalar λ such that the equation $Av = \lambda v$ has a nontrivial solution.

If $Av = \lambda v$ for $v \neq 0$, we say that λ is the eigenvalue for v, and that v is an eigenvector for λ .

EIGENVALUES AND EIGENVECTORS

Remarks:

- The German prefix "eigen" roughly translates to "self" or "own". An eigenvector of A is a vector that is taken to a multiple of itself by the matrix transformation T(x) = Ax, which perhaps explains the terminology. On the other hand, "eigen" is often translated as "characteristic"; we may think of an eigenvector as describing an intrinsic, or characteristic, property of A.
- Eigenvalues and eigenvectors are only for square matrices.
- Eigenvectors are by *definition nonzero*. Eigenvalues may be equal to zero. In fact, we do not consider the zero vector to be an eigenvector: since $A0 = 0 = \lambda 0$ for *every* scalar λ , the associated eigenvalue would be undefined.

Consider the matrix

$$A = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \quad \text{and vectors} \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad w = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Which are eigenvectors? What are their eigenvalues?

EIGENVALUES AND EIGENVECTORS

Solution:

• We have

$$A_{V} = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4_{V}.$$

Hence, v is an eigenvector of A, with eigenvalue $\lambda=4.$

• On the other hand,

$$Aw = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}.$$

which is not a scalar multiple of w. Hence, w is not an eigenvector of A.

Use this link to view the online demo

Consider the matrix

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \quad \text{and vectors} \quad v = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} \quad w = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

Which are eigenvectors? What are their eigenvalues?

EIGENVALUES AND EIGENVECTORS

Solution:

• We have

$$Av = \begin{pmatrix} 0 & 6 & 8\\ \frac{1}{2} & 0 & 0\\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 16\\ 4\\ 1 \end{pmatrix} = \begin{pmatrix} 32\\ 8\\ 2 \end{pmatrix} = 2v.$$

Hence, ν is an eigenvector of A, with eigenvalue $\lambda=2.$ • On the other hand,

$$A_{W} = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 28 \\ 1 \\ 1 \end{pmatrix},$$

which is not a scalar multiple of w. Hence, w is not an eigenvector of A.

Let

$$A = \left(\begin{array}{cc} 1 & 3 \\ 2 & 6 \end{array}\right) \quad v = \left(\begin{array}{c} -3 \\ 1 \end{array}\right).$$

Is v an eigenvector of A? If so, what is its eigenvalue?

Solution: The product is

$$A_{V} = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0_{V}.$$

Hence, v is an eigenvector with eigenvalue zero. As noted above, an eigenvalue is allowed to be zero, but an eigenvector is not.

Use this link to view the online demo

FACT: EIGENVECTORS WITH DISTINCT EIGENVALUES ARE LINEARLY INDEPENDENT

- Let v_1, v_2, \ldots, v_k be eigenvectors of a matrix A, and suppose that the corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct (all different from each other).
- Then $\{v_1, v_2, \dots, v_k\}$ is linearly independent.

EIGENSPACES

• Let A be an $n \times n$ matrix, and let λ be a scalar. The eigenvectors with eigenvalue λ , if any, are the nonzero solutions of the equation $Av = \lambda v$. We can rewrite this equation as follows:

$$Av = \lambda v$$

$$\iff Av - \lambda v = 0$$

$$\iff Av - \lambda I_n v = 0$$

$$\iff (A - \lambda I_n) v = 0.$$

- Therefore, the eigenvectors of A with eigenvalue λ , if any, are the nontrivial solutions of the matrix equation $(A \lambda I_n) v = 0$, i.e., the nonzero vectors in Nul $(A \lambda I_n)$.
- If the equation $(A \lambda I_n) v = 0$ has no nontrivial solutions, then λ is not an eigenvalue of A.

• In other words, λ is an eigenvalue of Aif and only if $(A - \lambda I_n) v = 0$ has non trivial solutions if and only if $\det (A - \lambda I_n) = 0$

DEFINITION

Let A be an $n \times n$ matrix, and let λ be an eigenvalue of A. The λ eigenspace of A is the solution set of vectors such that

$$(A-\lambda I_n)v=0$$

i.e., the subspace Nul $(A - \lambda I_n)$.

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DEFINITION

Let A be an $n \times n$ matrix. The characteristic polynomial of A is the function $f(\lambda)$ given by

$$f(\lambda) = \det \left(A - \lambda I_n \right).$$

Find the characteristic polynomial of the matrix

$$A = \left(\begin{array}{cc} 5 & 2\\ 2 & 1 \end{array}\right)$$

Solution: We have

$$f(\lambda) = \det (A - \lambda I_2) = \det \left(\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right)$$
$$= \det \left(\begin{array}{c} 5 - \lambda & 2 \\ 2 & 1 - \lambda \end{array} \right)$$
$$= (5 - \lambda)(1 - \lambda) - 2 \cdot 2 = \lambda^2 - 6\lambda + 1.$$

Find the characteristic polynomial of the matrix

$$A = \left(\begin{array}{rrrr} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{array}\right)$$

Solution: We compute the determinant by expanding cofactors along the third column:

$$f(\lambda) = \det (A - \lambda I_3) = \det \begin{pmatrix} -\lambda & 6 & 8\\ \frac{1}{2} & -\lambda & 0\\ 0 & \frac{1}{2} & -\lambda \end{pmatrix}$$
$$= 8 \left(\frac{1}{4} - 0 \cdot -\lambda\right) - \lambda \left(\lambda^2 - 6 \cdot \frac{1}{2}\right)$$
$$= -\lambda^3 + 3\lambda + 2.$$

THEOREM (EIGENVALUES ARE ROOTS OF THE CHARACTERISTIC POLYNOMIAL)

Let A be an $n \times n$ matrix, and let $f(\lambda) = \det (A - \lambda I_n)$ be its characteristic polynomial. Then a number λ_0 is an eigenvalue of A if and only if $f(\lambda_0) = 0$.

THE CHARACTERISTIC POLYNOMIAL

Proof. The matrix equation $(A - \lambda_0 I_n) x = 0$ has a nontrivial solution if and only if det $(A - \lambda_0 I_n) = 0$. Therefore,

 $\lambda_0 \text{ is an eigenvalue of } A \iff Ax = \lambda_0 x \text{ has a nontrivial solution}$ $\iff (A - \lambda_0 I_n) x = 0 \text{ has a nontrivial solution}$ $\iff A - \lambda_0 I_n \text{ is not invertible}$ $\iff \det (A - \lambda_0 I_n) = 0$ $\iff f(\lambda_0) = 0$

Find the eigenvalues and eigenvectors of the matrix

$$A = \left(\begin{array}{cc} 5 & 2\\ 2 & 1 \end{array}\right)$$

Solution: In the above example we computed the characteristic polynomial of A to be $f(\lambda) = \lambda^2 - 6\lambda + 1$. We can solve the equation $\lambda^2 - 6\lambda + 1 = 0$ using the quadratic formula:

$$\lambda = \frac{6 \pm \sqrt{36 - 4}}{2} = 3 \pm 2\sqrt{2}.$$

Therefore, the eigenvalues are $3 + 2\sqrt{2}$ and $3 - 2\sqrt{2}$. To compute the eigenvectors, we solve the homogeneous system of equations $(A - \lambda I_2) x = 0$ for each eigenvalue λ . When $\lambda = 3 + 2\sqrt{2}$, we have
THE CHARACTERISTIC POLYNOMIAL

Solution:

$$A - (3 + \sqrt{2})I_2 = \begin{pmatrix} 2 - 2\sqrt{2} & 2\\ 2 & -2 - 2\sqrt{2} \end{pmatrix}$$

$$\stackrel{R_1 = R_1 \times (2 + 2\sqrt{2})}{\longrightarrow} \begin{pmatrix} -4 & 4 + 4\sqrt{2}\\ 2 & -2 - 2\sqrt{2} \end{pmatrix}$$

$$\stackrel{R_2 = R_2 + R_1/2}{\longrightarrow} \begin{pmatrix} -4 & 4 + 4\sqrt{2}\\ 0 & 0 \end{pmatrix}$$

$$\stackrel{R_1 = R_1/-4}{\longrightarrow} \begin{pmatrix} 1 & -1 - \sqrt{2}\\ 0 & 0 \end{pmatrix}$$

The parametric form of the general solution is $x = (1 + \sqrt{2})y$, so the $(3 + 2\sqrt{2})$ eigenspace is the line spanned by $\begin{pmatrix} 1 + \sqrt{2} \\ 1 \end{pmatrix}$. We compute in the same way that the $(3 - 2\sqrt{2})$ -eigenspace is the line spanned by $\begin{pmatrix} 1 - \sqrt{2} \\ 1 \end{pmatrix}$.

DEFINITION

The trace of a square matrix A is the number Tr(A) obtained by summing the diagonal entries of A:

$$\mathsf{Tr}\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1n}\\a_{21} & a_{22} & \cdots & a_{2,n-1} & a_{2n}\\\vdots & \vdots & \ddots & \vdots & \vdots\\a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} & a_{n-1,n}\\a_{n1} & a_{n2} & \cdots & a_{n,n-1} & a_{nn}\end{pmatrix} = a_{11} + a_{22} + \cdots + a_{nn}.$$

THEOREM

Let A be an $n \times n$ matrix, and let $f(\lambda) = \det (A - \lambda I_n)$ be its characteristic polynomial. Then $f(\lambda)$ is a polynomial of degree n. Moreover, $f(\lambda)$ has the form

$$f(\lambda) = (-1)^n \lambda^n + (-1)^{n-1} \operatorname{Tr}(A) \lambda^{n-1} + \dots + \det(A)$$

In other words, the coefficient of λ^{n-1} is $\pm \operatorname{Tr}(A)$, and the constant term is det(A) (the other coefficients are just numbers without names).

THE CHARACTERISTIC POLYNOMIAL

Proof.

First we notice that

$$f(0) = \det(A - 0I_n) = \det(A),$$

so that the constant term is always det(A). We will prove the rest of the theorem only for 2×2 matrices; We can write a 2×2 matrix as $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; then

$$f(\lambda) = \det (A - \lambda I_2) = \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = (a - \lambda)(d - \lambda) - bc$$
$$= \lambda^2 - (a + d)\lambda + (ad - bc) = \lambda^2 - \operatorname{Tr}(A)\lambda + \det(A)$$

Recipe: The characteristic polynomial of a 2×2 matrix. When n = 2, the previous theorem tells us all of the coefficients of the characteristic polynomial:

$$f(\lambda) = \lambda^2 - \operatorname{Tr}(A)\lambda + \det(A)$$

This is generally the fastest way to compute the characteristic polynomial of a 2×2 matrix

EXAMPLE

Find the characteristic polynomial of the matrix

$$A = \left(\begin{array}{cc} 5 & 2\\ 2 & 1 \end{array}\right)$$

Solution: We have

$$f(\lambda) = \lambda^2 - \operatorname{Tr}(A)\lambda + \det(A) = \lambda^2 - (5+1)\lambda + (5\cdot 1 - 2\cdot 2) = \lambda^2 - 6\lambda + 1,$$

as in the above example.

Remark: By the above theorem, the characteristic polynomial of an $n \times n$ matrix is a polynomial of degree n. Since a polynomial of degree n has at most n roots, this gives a proof of the fact that an $n \times n$ matrix has at most n eigenvalues.

Eigenvalues of a triangular matrix It is easy to compute the determinant of an upper- or lower-triangular matrix; this makes it easy to find its eigenvalues as well.

COROLLARY

If A is an upper- or lower-triangular matrix, then the eigenvalues of A are its diagonal entries.

THE CHARACTERISTIC POLYNOMIAL

Proof.

Suppose for simplicity that A is a 3×3 upper-triangular matrix:

$$A = \left(\begin{array}{rrrr} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{array}\right)$$

Its characteristic polynomial is

$$f(\lambda) = \det(A - \lambda I_3) = \det\begin{pmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{pmatrix}$$

This is also an upper-triangular matrix, so the determinant is the product of the diagonal entries:

$$f(\lambda) = (a_{11} - \lambda) (a_{22} - \lambda) (a_{33} - \lambda).$$

The zeros of this polynomial are exactly a_{11}, a_{22}, a_{33} .

EXAMPLE

Find the eigenvalues of the matrix

$$A = \left(\begin{array}{rrrrr} 1 & 7 & 2 & 4 \\ 0 & 1 & 3 & 11 \\ 0 & 0 & \pi & 101 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Solution: The eigenvalues are the diagonal entries $1, \pi, 0$. (The eigenvalue 1 occurs twice, so it is said to be of multiplicity 2)