A GUIDED TOUR OF AI: FROM FOUNDATIONS TO LATEST APPLICATION

Introduction to linear algebra

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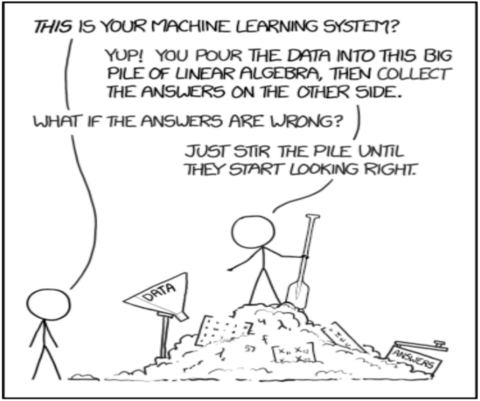
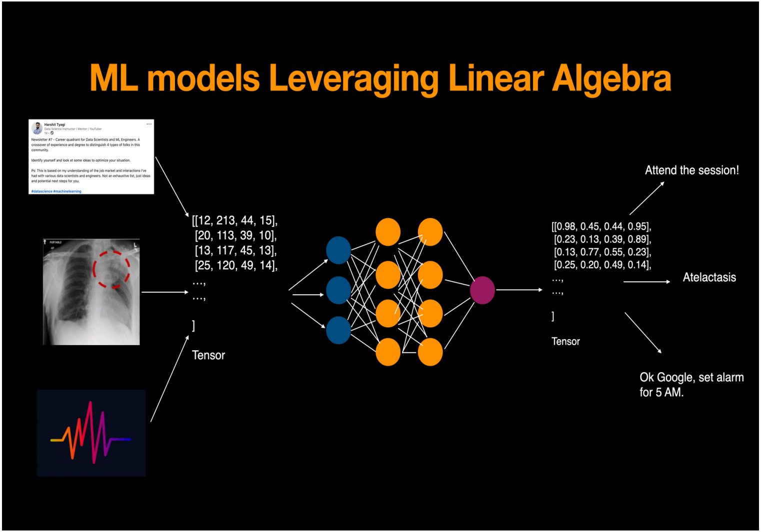
Importance in ML: We convert input vectors $(x_1, .., x_n)$ into outputs by a series of linear transformations.



But what is RIGHT? And is that enough? (Image: Machine Learning, XKCD)


ML models Leveraging Linear Algebra
[[12, 213, 44, 15],
[20, 113, 39, 10],
[13, 117, 45, 13],
[25, 120, 49, 14],
...,
...,
]
Tensor
[[0.98, 0.45, 0.44, 0.95],
[0.23, 0.13, 0.39, 0.89],
[0.13, 0.77, 0.55, 0.23],
[0.25, 0.20, 0.49, 0.14],
...,
...,
]
Tensor
Attend the session!
Atelactasis
Ok Google, set alarm for 5 AM.

Outline

1. Introduction to matrices
   - Model a real-life situation using a system of linear equations
   - Introduction to matrices
   - Matrix algebra

2. Eigenvalues, eigenvectors

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1. Introduction to matrices
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2. Eigenvalues, eigenvectors

Model a real-life situation using a system of linear equations

Problem 1: With one unknown value

If all the cows in a pasture have 124 legs, how many cows are in the pasture?

We can model this real-life situation using an equation with one unknown value, represented by one variable, to say x.

1. Identify the unknown and define your variable:
   - the unknown value is the number of cows,
   - set x to be the number of cows.
2. Analyse the problem and write your equation accordingly:
   - every cow has four legs,
   - $4\times$ *the number of cows* $= 124$ legs
   - $\boxed{4\,x = 124}$

Model a real-life situation using a system of linear equations

Problem 1: With one unknown value

If all the cows in a pasture have 124 legs, how many cows are in the pasture?

$$4\,x = 124 \quad \Longleftrightarrow \quad x = 124/4 = 31$$

There are 31 cows in the pasture ☺

Model a real-life situation using a system of linear equations

Problem 2: With two unknown values

If all the cows and chickens in a pasture have 35 heads and 110 legs, how many cows are in the pasture?

We can model this real-life situation using a **system of linear equations**.

Model a real-life situation using a system of linear equations

Let's review some important vocabulary:

- A system of equations is a set of two or more equations with the same set of unknown values, which are represented by the same variables.
- They are also called **simultaneous equations**

For example:

$$\begin{cases} y - \frac{1}{2}x = 2 \\ y + x = -1 \end{cases}$$

- Notice that both equations have the same two variables x and y
- The brace on the left is written to show that the equations are simultaneous, that is, the variables x and y represent the same unknowns in both equations

Let's review some important vocabulary:

- A **solution** of a system (when it exists), is a **set of values** that makes both equations in the system **true** at the same time.

For example: The solution of the system

$$\begin{cases} y - \frac{1}{2}x = 2 \\ y + x = -1 \end{cases}$$

is the ordered pair $(-2, 1)$.

- We can check this by substituting the values in this solution for the variables in the equations in the system and get true statements:

Model a real-life situation using a system of linear equations

Let's review some important vocabulary:

- A **solution** of a system (when it exists), is a **set of values** that makes both equations in the system **true** at the same time.

For example: The solution of the system

$$\begin{cases} 1 - \frac{1}{2}(-2) = 2 \\ 1 + (-2) = -1 \end{cases}$$

is the ordered pair $(-2, 1)$.

- We can check this by substituting the values in this solution for the variables in the equations in the system and get true statements:

Model a real-life situation using a system of linear equations

Let's review some important vocabulary:

- A **solution** of a system (when it exists), is a **set of values** that makes both equations in the system **true** at the same time.

For example: The solution of the system

$$\begin{cases} 1 + 1 = 2 \\ 1 - 2 = -1 \end{cases}$$

is the ordered pair $(-2, 1)$.

- We can check this by substituting the values in this solution for the variables in the equations in the system and get true statements:

Let's review some important vocabulary:

- A **solution** of a system (when it exists), is a **set of values** that makes both equations in the system **true** at the same time.

For example: The solution of the system

$$\begin{cases} 2 = 2 & \textit{TRUE}! \\ -1 = -1 & \textit{TRUE}! \end{cases}$$

is the ordered pair $(-2, 1)$.

- We can check this by substituting the values in this solution for the variables in the equations in the system and get true statements:

Problem 2: With two unknown values

If all the cows and chickens in a pasture have 35 heads and 110 legs, how many cows are in the pasture?

We can model this real-life situation using a **system of linear equations**.

- To begin writing appropriate equations to any system, we must first define our variables:

"What unknown values will I need to find for this system?"

Model a real-life situation using a system of linear equations

Problem 2: With two unknown values

If all the cows and chickens in a pasture have 35 heads and 110 legs, how many cows are in the pasture?

We can model this real-life situation using a **system of linear equations**.

- To begin writing appropriate equations to any system, we must first define our variables:

"The number of cows and the number of chickens "

Model a real-life situation using a system of linear equations

Problem 2: With two unknown values

> If all the cows and chickens in a pasture have 35 heads and 110 legs, how many cows are in the pasture?

We can model this real-life situation using a **system of linear equations**.

- Set x = number of cows and y = number of chickens

Model a real-life situation using a system of linear equations

Problem 2: With two unknown values

If all the cows and chickens in a pasture have 35 heads and 110 legs, how many cows are in the pasture?

We can model this real-life situation using a **system of linear equations**.

- Set x = number of cows and y = number of chickens

1. Heads equation: $x + y = 35$
2. Legs equation: $4x + 2y = 110$

Problem 2: With two unknown values

If all the cows and chickens in a pasture have 35 heads and 110 legs, how many cows are in the pasture?

We can model this real-life situation using a **system of linear equations**.

- Set x = number of cows and y = number of chickens

1. Heads equation: $x + y = 35$
2. Legs equation: $4x + 2y = 110$

- And here we go!

$$\begin{cases} x + y = 35 \\ 4x + 2y = 110 \end{cases}$$

Problem 2: With two unknown values

If all the cows and chickens in a pasture have 35 heads and 110 legs, how many cows are in the pasture?

$$\begin{cases} x + y = 35 \\ 4\,x + 2\,y = 110 \end{cases}$$

"How do we solve it?"

$$\begin{cases} x + y = 35 & \text{eq. 1} \\ 4\,x + 2\,y = 110 & \text{eq. 2} \end{cases}$$

Since we are familiar with solving an equation with one variable, we can proceed as follows:

1. **Eliminate** one variable, i.e., using eq. 1 and eq. 2, find a new equation depending only on 1 variable (to say y).
2. Solve the obtained equation (for y).
3. Substitute the obtained value for its corresponding variable in either eq. 1 or eq. 2 and then solve for the other variable (x).

$$\begin{cases} \boxed{x} + y = 35 & \text{eq. 1} \\ \boxed{4\,x} + 2\,y = 110 & \text{eq. 2} \end{cases}$$

- **Eliminate** one variable, i.e., using eq. 1 and eq. 2, find a new equation depending only on 1 variable (to say y)

$$\begin{cases} \boxed{x} + y = 35 & \text{eq. 1} \\ \boxed{4\,x} + 2\,y = 110 & \text{eq. 2} \end{cases}$$

- **Eliminate** one variable, i.e., using eq. 1 and eq. 2, find a new equation depending only on 1 variable (to say y)

$$\begin{cases} \boxed{-4\,x} + (-4)y = (-4)35 & -4 \times \text{eq. 1} \\ \boxed{4\,x} + 2\,y = 110 & \text{eq. 2} \end{cases}$$

$$\begin{cases} \boxed{x} + y = 35 & \text{eq. 1} \\ \boxed{4\,x} + 2\,y = 110 & \text{eq. 2} \end{cases}$$

- **Eliminate** one variable, i.e., using eq. 1 and eq. 2, find a new equation depending only on 1 variable (to say y)

$$\begin{cases} \boxed{-4\,x} + (-4)y = (-4)35 & -4 \times \text{eq. 1} \\ \boxed{4\,x} + 2\,y = 110 & \text{eq. 2} \end{cases}$$

$$\begin{cases} x + y = 35 & \text{eq. 1} \\ \boxed{4\,x} + 2\,y + \boxed{(-4)\,x} + (-4)y = 110 + (-4)35 & \text{eq. 2} + (-4) \times \text{eq. 1} \end{cases}$$

$$\begin{cases} \boxed{x} + y = 35 & \text{eq. } 1 \\ \boxed{4\,x} + 2\,y = 110 & \text{eq. } 2 \end{cases}$$

- **Eliminate** one variable, i.e., using eq. 1 and eq. 2, find a new equation depending only on 1 variable (to say y)

$$\begin{cases} x + y = 35 & \text{eq. } 1 \\ \boxed{4\,x} + 2\,y + \boxed{(-4)\,x} + (-4)y = 110 + (-4)35 & \text{eq. } 2 \ + \ (-4) \times \text{eq. } 1 \end{cases}$$

$$\begin{cases} x + y = 35 \\ \boxed{2\,y - 4y = 110 - 4 \times 35} \end{cases}$$

$$\begin{cases} x + y = 35 & \text{eq. 1} \\ 4\,x + 2\,y = 110 & \text{eq. 2} \end{cases}$$

- Solve the obtained equation (for y):

$$\begin{cases} x + y = 35 \\ \boxed{2\,y - 4y = 110 - 4 \times 35} \end{cases} \iff \begin{cases} x + y = 35 \\ \boxed{-2\,y = -30} \end{cases}$$

$$\begin{cases} x + y = 35 & \text{eq. 1} \\ 4\,x + 2\,y = 110 & \text{eq. 2} \end{cases}$$

- Solve the obtained equation (for y):

$$\begin{cases} x + y = 35 \\ -2\,y = -30 \end{cases} \iff \begin{cases} x + y = 35 \\ \boxed{y = 15} \end{cases}$$

$$\begin{cases} x + y = 35 & \text{eq. 1} \\ 4\,x + 2\,y = 110 & \text{eq. 2} \end{cases}$$

- Substitute the obtained value for its corresponding variable in either eq. 1 or eq. 2 and then solve for the other variable.

$$\begin{cases} x + y = 35 \\ \boxed{y = 15} \end{cases} \iff \begin{cases} x + 15 = 35 \\ \boxed{y = 15} \end{cases}$$

$$\begin{cases} x + y = 35 & \text{eq. 1} \\ 4\,x + 2\,y = 110 & \text{eq. 2} \end{cases}$$

$$\begin{cases} \boxed{x = 20} \\ \boxed{y = 15} \end{cases}$$

Problem 2: With two unknown values

If all the cows and chickens in a pasture have 35 heads and 110 legs, how many cows are in the pasture?

$$\begin{cases} x + y = 35 \\ 4\,x + 2\,y = 110 \end{cases} \iff \begin{cases} \boxed{x = 20} \\ \boxed{y = 15} \end{cases}$$

"There are 20 cows in the pasture"

- We notice while solving this system that what really matter in the operations done ($-4\times$ equ. 1 and equ. 2 $-4\times$ equ. 1) (during the elimination) are the numbers attached to the variables (the coefficients).
- We can represent the same system in an equivalent way, but without any letters (without the variables x and y)
- In this case we obtain the following **rectangular table of numbers** (let's call it **matrix**)

$$\begin{cases} 1x + 1y = 35 \\ 4\,x + 2\,y = 110 \end{cases} \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 35 \\ 4 & 2 & 110 \end{array}\right)$$

- Let R_1 and R_2 be respectively the first and second row of our matrix.
- By following the same operations done in the elimination step previously, that is replacing equ.2 by equ.2 $-4\times$ equ.1, we can, in an equivalent way, replace R_2 by $R_2 - 4 \times R_1$ in the matrix (that is an **elementary matrix row operation**) to obtain

$$\begin{cases} 1x + 1y = 35 \\ 4\,x + 2\,y = 110 \end{cases} \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 35 \\ 4 & 2 & 110 \end{array}\right)$$

$$\xrightarrow{\mathrm{R_2 \leftarrow R_2 - 4 \times R_1}} \left(\begin{array}{cc|c} 1 & 1 & 35 \\ 0 & -2 & -30 \end{array}\right)$$

$$\begin{cases} 1x + 1y = 35 \\ 4\,x + 2\,y = 110 \end{cases} \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 35 \\ 4 & 2 & 110 \end{array}\right)$$

$$\xrightarrow{R_2 \leftarrow R_2 - 4 \times R_1} \left(\begin{array}{cc|c} 1 & 1 & 35 \\ 0 & -2 & -30 \end{array}\right)$$

- The number 0 represents the variable x being eliminated in a new equation of 1 variable y: $-2\,y = -4$.
- In this case, we can immediately see again the new equivalent system

$$\begin{cases} 1x + 1y = 35 \\ 4\,x + 2\,y = 110 \end{cases} \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 35 \\ 4 & 2 & 110 \end{array}\right)$$

$$\begin{cases} 1x + 1y = 35 \\ 0\,x + -2\,y = -30 \end{cases} \longleftarrow \left(\begin{array}{cc|c} 1 & 1 & 35 \\ 0 & -2 & -30 \end{array}\right)$$

$$\begin{cases} 1x + 1y = 35 \\ 4\,x + 2\,y = 110 \end{cases} \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 35 \\ 4 & 2 & 110 \end{array}\right)$$

$$\xrightarrow{R_2 \leftarrow R_2 - 4 \times R_1} \left(\begin{array}{cc|c} 1 & 1 & 35 \\ 0 & -2 & -30 \end{array}\right)$$

- The number 0 represents the variable x being eliminated in a new equation of 1 variable y: $-2\,y = -4$.
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$$\begin{cases} 1x + 1y = 35 \\ 4\,x + 2\,y = 110 \end{cases} \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 35 \\ 4 & 2 & 110 \end{array}\right)$$

$$\begin{cases} x + y = 35 \\ -2\,y = -30 \end{cases} \longleftarrow \left(\begin{array}{cc|c} 1 & 1 & 35 \\ 0 & -2 & -30 \end{array}\right)$$

$$\begin{cases} 1x + 1y = 35 \\ 4\,x + 2\,y = 110 \end{cases} \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 35 \\ 4 & 2 & 110 \end{array}\right)$$

$$\begin{cases} x + y = 35 \\ -2\,y = -30 \end{cases} \longleftarrow \left(\begin{array}{cc|c} 1 & 1 & 35 \\ 0 & -2 & -30 \end{array}\right)$$

$$\begin{cases} \boxed{x = 20} \\ \boxed{y = 15} \end{cases}$$

More precisely:

- A **matrix** (plural matrices) is a rectangular array or table of numbers, symbols, or expressions, arranged in rows and columns, which is used to represent a mathematical object or a property of such an object.
- The **dimensions** of a matrix tells its size: the number of rows and columns of the matrix, in that order.

For example:

$$\begin{pmatrix} 2 & 3 & 5 \\ 4 & 3 & 2 \end{pmatrix}$$

is a matrix with two rows and three columns. This is often referred to as a "two by three matrix", a "2×3-matrix", or a matrix of dimension 2×3.

Matrix Elements:

- A **matrix element** is simply a matrix entry. Each element in a matrix is identified by naming the row and column in which it appears.

For example, consider the matrix M:

$$\begin{pmatrix} 24 & 15 & -6 \\ 31 & -5 & 78 \\ 1 & -1 & 2 \end{pmatrix}$$

The element $m_{2,1}$ is the entry in the second row and the first column.

$$\begin{pmatrix} 24 & 15 & -6 \\ 31 & -5 & 78 \\ 1 & -1 & 2 \end{pmatrix}$$

In this case, $m_{2,1} = 31$.

- A system of equations can be represented by an **augmented matrix**.
- In an augmented matrix, each row represents one equation in the system and each column represents a variable or the constant terms.
- Augmented matrices are a shorthand way of writing systems of equations. The organization of the numbers into the matrix makes it unnecessary to write various symbols like x, y, and $=$, yet all of the information is still there!

For example:

$$\begin{cases} -4x + 5y = -1 \\ 35\,x + 7\,y = 123 \end{cases} \longrightarrow \left(\begin{array}{cc|c} -4 & 5 & -1 \\ 35 & 7 & 123 \end{array}\right)$$

EXAMPLE

Write the following system of equations as an augmented matrix.

$$\begin{cases} 3x - 2y &= 4 \\ x + 5z &= 3 \\ -4x - y + 3z &= 0 \end{cases}$$

Example

Write the following system of equations as an augmented matrix.

$$\begin{cases} 3x - 2y & = & 4 \\ x + 5z & = & 3 \\ -4x - y + 3z & = & 0 \end{cases}$$

Solution:

- First, in order to make things easier, we can rewrite the system in a way where we can see each of the coefficients clearly.
- P.S.: If a variable term is not written in an equation, it means that the coefficient is 0.

Example

Write the following system of equations as an augmented matrix.

$$\begin{cases} 3x + (-2)y + 0z = 4 \\ 1x + 0y + 5z = 3 \\ -4x + (-1)y + 3z = 0 \end{cases}$$

Solution:

- First, in order to make things easier, we can rewrite the system in a way where we can see each of the coefficients clearly.
- P.S.: If a variable term is not written in an equation, it means that the coefficient is 0.

Example

Write the following system of equations as an augmented matrix.

$$\begin{cases} 3x + (-2)y + 0z = 4 \\ 1x + 0y + 5z = 3 \\ -4x + (-1)y + 3z = 0 \end{cases}$$

Solution:

$$\left(\begin{array}{ccc|c} 3 & -2 & 0 & 4 \\ 1 & 0 & 5 & 3 \\ -4 & -1 & 3 & 0 \end{array}\right)$$

Remark

In general, before converting a system into an augmented matrix, be sure that the variables appear in the same order in each equation, and that the constant terms are isolated on one side.

There are three elementary matrix row operations:

1. Switch any two rows: for example

$$\begin{pmatrix} 1 & 3 & 5 \\ 7 & 7 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 7 & 7 & 0 \\ 1 & 3 & 5 \end{pmatrix}$$

2. Multiply a row by a nonzero constant: for example

$$\begin{pmatrix} 1 & 3 & 5 \\ 7 & 7 & 0 \end{pmatrix} \xrightarrow{R_1 \leftarrow 2 \times R_1} \begin{pmatrix} 2 \times 1 & 2 \times 3 & 2 \times 5 \\ 7 & 7 & 0 \end{pmatrix}$$

3. Add a multiple of one row to another: for example

$$\begin{pmatrix} 1 & 3 & 5 \\ 7 & 7 & 0 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + 3 \times R_1} \begin{pmatrix} 1 & 3 & 5 \\ 7 + 3 \times 1 & 7 + 3 \times 3 & 0 + 3 \times 5 \end{pmatrix}$$

Elementary matrix row operations

Systems of equations and matrix row operations:

- Recall that in an augmented matrix, each row represents one equation in the system and each column represents a variable or the constant terms.
- For example, the system on the left corresponds to the augmented matrix on the right.

$$\text{System:} \begin{cases} 3x + y = 5 \\ x + 2y = 6 \end{cases} \qquad \text{Matrix:} \left(\begin{array}{cc|c} 3 & 1 & 5 \\ 1 & 2 & 6 \end{array}\right)$$

When working with augmented matrices, we can perform any of the **matrix row operations** to create a new augmented matrix that produces an equivalent system of equations. Why?

We say that a matrix is in **row-echelon form** if it meets the following two requirements:

1. For each row, the first (leftmost) nonzero entry (called a **leading coefficient** or **pivot**) is to the right of the one above it.
2. Any non-zero rows are always above rows with all zeros.

Example of row-echelon form matrices:

$$\begin{pmatrix} 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & \boxed{5} & 0 \\ 0 & 0 & 0 & \boxed{-5} \end{pmatrix}, \quad \begin{pmatrix} \boxed{1} & 1 & -1 & 0 \\ 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \boxed{1} & 0 \\ 0 & \boxed{3} \end{pmatrix}$$

Example of matrices that are not in row echelon form:

$$\begin{pmatrix} 1 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} \boxed{1} & 0 & 2 & 2 \\ 0 & 0 & \boxed{1} & 3 \\ 0 & \boxed{1} & 0 & 4 \end{pmatrix}$$

- Any matrix can be transformed to reduced row echelon form using one or more of the row operations
  1. Interchange one row with another.
  2. Multiply one row by a non-zero constant.
  3. Replace one row with: one row, plus a constant, times another row.
- In addition, it isn't enough just to know the rules, you have to be able to look at the matrix and make a logical decision about which rule you're going to use and when.

$$\begin{pmatrix} 0 & \boxed{1} & 3 & 4 \\ 0 & 0 & \boxed{2} & 3 \\ 0 & 0 & \boxed{4} & 1 \end{pmatrix} \longrightarrow \text{Not echelon form}$$

$$\begin{pmatrix} 0 & \boxed{1} & 3 & 4 \\ 0 & 0 & \boxed{2} & 3 \\ 0 & 0 & \boxed{4} & 1 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 - 2.R_2} \begin{pmatrix} 0 & \boxed{1} & 3 & 4 \\ 0 & 0 & \boxed{2} & 3 \\ 0 & 0 & 0 & \boxed{-5} \end{pmatrix} \longrightarrow \text{Echelon form}$$

Gaussian Elimination

- **Gaussian Elimination** is a set of well-defined instructions to solve a system of linear equations.
- It consists of a sequence of **elementary row operations** performed on the corresponding **augmented matrix**, in order to get in a **row-echelon form**.
- Then the system of linear equations corresponding to the row-echelon form is said to be **triangular**.

Example

- Supposing it exists and is unique, find the solution to the following system of linear equations:

$$\begin{cases} x + 2y + 3z &= 2 \\ 2x + y + 4z &= 5 \\ x + 3y + 2z &= 1 \end{cases}$$

Example

- Supposing it exists and is unique, find the solution to the following system of linear equations:

$$\begin{cases} x + 2y + 3z &= 2 \\ 2x + y + 4z &= 5 \\ x + 3y + 2z &= 1 \end{cases}$$

- Solution:

The table below is the **Gaussian Elimination** process applied simultaneously to the system of equations and its associated augmented matrix.

- The Guassian Elimination procedure may be summarized as follows:
  - eliminate x from all equations below R_1, and then eliminate y from all equations below R_2. This will put the system into **triangular form**,
  - then, using back-substitution, each unknown can be solved for.

System of equations	Row operations	Augmented matrix
$\begin{cases} x+2y+3z = 2 \\ 2x+y+4z = 5 \\ x+3y+2z = 1 \end{cases}$		$\left(\begin{array}{ccc\|c} 1 & 2 & 3 & 2 \\ 2 & 1 & 4 & 5 \\ 1 & 3 & 2 & 1 \end{array}\right)$
$\begin{cases} x+2y+3z = 2 \\ 0x-3y-2z = 1 \\ 0x+y-z = -1 \end{cases}$	$R_2 \leftarrow R_2 - 2R_1$ $R_3 \leftarrow R_3 - R_1$	$\left(\begin{array}{ccc\|c} 1 & 2 & 3 & 2 \\ 0 & -3 & -2 & 1 \\ 0 & 1 & -1 & -1 \end{array}\right)$
$\begin{cases} x+2y+3z = 2 \\ 0x-3y-2z = 1 \\ 0x+0y-\frac{5}{3}z = -\frac{2}{3} \end{cases}$	$R_3 \leftarrow R_3 + \frac{1}{3}R_2$	$\left(\begin{array}{ccc\|c} 1 & 2 & 3 & 2 \\ 0 & -3 & -2 & 1 \\ 0 & 0 & -\frac{5}{3} & -\frac{2}{3} \end{array}\right)$
The matrix now in echelon form and the system is triangular		

System of equations	Row operations	Augmented matrix
$\begin{cases} x+2y+3z = 2 \\ 2x+y+4z = 5 \\ x+3y+2z = 1 \end{cases}$		$\left(\begin{array}{ccc\|c} 1 & 2 & 3 & 2 \\ 2 & 1 & 4 & 5 \\ 1 & 3 & 2 & 1 \end{array}\right)$
$\begin{cases} x+2y+3z = 2 \\ 0x-3y-2z = 1 \\ 0x+y-z = -1 \end{cases}$	$R_2 \leftarrow R_2 - 2R_1$ $R_3 \leftarrow R_3 - R_1$	$\left(\begin{array}{ccc\|c} 1 & 2 & 3 & 2 \\ 0 & -3 & -2 & 1 \\ 0 & 1 & -1 & -1 \end{array}\right)$
$\begin{cases} x+2y+3z = 2 \\ 0x-3y-2z = 1 \\ 0x+0y-\frac{5}{3}z = -\frac{2}{3} \end{cases}$	$R_3 \leftarrow R_3 + \frac{1}{3}R_2$	$\left(\begin{array}{ccc\|c} 1 & 2 & 3 & 2 \\ 0 & -3 & -2 & 1 \\ 0 & 0 & -\frac{5}{3} & -\frac{2}{3} \end{array}\right)$
The matrix now in echelon form and the system is triangular		

1. From R_3: $-\frac{5}{3}z = -\frac{2}{3} \iff \boxed{z = \frac{2}{5}}$.
2. From R_2: $-3y - 2z = 1 \overset{z=\frac{2}{5}}{\iff} -3y = \frac{9}{5} \iff \boxed{y = -\frac{3}{5}}$.
3. From R_1: $x + 2y + 3z = 2 \overset{z=\frac{2}{5},\, y=-\frac{3}{5}}{\iff} x - \frac{6}{5} + \frac{6}{5} = 2 \iff \boxed{x = 2}$.

Reduced row echelon form

Reduced row echelon form is a type of matrix used to solve systems of linear equations. Reduced row echelon form has four requirements:

- The first non-zero number in the first row (the leading coefficient) is the number 1.
- The second row also starts with the number 1, which is further to the right than the leading coefficient in the first row. For every subsequent row, the number 1 must be further to the right.
- The leading coefficient in each row must be the only non-zero number in its column.
- Any non-zero rows are placed at the bottom of the matrix.

Reduced row echelon form

Reduced row echelon form is a type of matrix used to solve systems of linear equations. For example

$$\begin{pmatrix} \boxed{1} & 0 & 2 & 0 & 4 \\ 0 & \boxed{1} & -3 & 0 & 7 \\ 0 & 0 & 0 & \boxed{1} & 12 \end{pmatrix}$$

Row-echelon form & Gaussian elimination

System of equations	Row operations	Augmented matrix
$\begin{cases} x+2y+3z &= 2 \\ 2x+y+4z &= 5 \\ x+3y+2z &= 1 \end{cases}$		$\left(\begin{array}{ccc\|c} 1 & 2 & 3 & 2 \\ 2 & 1 & 4 & 5 \\ 1 & 3 & 2 & 1 \end{array}\right)$
$\begin{cases} x+2y+3z &= 2 \\ 0x-3y-2z &= 1 \\ 0x+y-z &= -1 \end{cases}$	$R_2 \leftarrow R_2 - 2R_1$ $R_3 \leftarrow R_3 - R_1$	$\left(\begin{array}{ccc\|c} 1 & 2 & 3 & 2 \\ 0 & -3 & -2 & 1 \\ 0 & 1 & -1 & -1 \end{array}\right)$
$\begin{cases} x+2y+3z &= 2 \\ 0x-3y-2z &= 1 \\ 0x+0y-\frac{5}{3}z &= -\frac{2}{3} \end{cases}$	$R_3 \leftarrow R_3 + \frac{1}{3}R_2$	$\left(\begin{array}{ccc\|c} 1 & 2 & 3 & 2 \\ 0 & -3 & -2 & 1 \\ 0 & 0 & -\frac{5}{3} & -\frac{2}{3} \end{array}\right)$
The matrix now in echelon form and the system is triangular		
$\begin{cases} x+2y+3z &= 2 \\ 0x+y+\frac{2}{3}z &= -\frac{1}{3} \\ 0x+0y+z &= \frac{2}{5} \end{cases}$	$R_3 \leftarrow -\frac{3}{5}R_3$ $R_2 \leftarrow -\frac{1}{3}R_2$	$\left(\begin{array}{ccc\|c} 1 & 2 & 3 & 2 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{2}{5} \end{array}\right)$
$\begin{cases} x+2y+0z &= \frac{4}{5} \\ 0x+y+0z &= -\frac{3}{5} \\ 0x+0y+z &= 0 \end{cases}$	$R_2 \leftarrow R_2 - \frac{2}{3}R_3$ $R_1 \leftarrow R_1 - 3R_3$	$\left(\begin{array}{ccc\|c} 1 & 2 & 0 & \frac{4}{5} \\ 0 & 1 & 0 & -\frac{3}{5} \\ 0 & 0 & 1 & \frac{2}{5} \end{array}\right)$
$\begin{cases} x+0y+0z &= 2 \\ 0x+y+0z &= -\frac{3}{5} \\ 0x+0y+z &= \frac{2}{5} \end{cases}$	$R_1 \leftarrow R_1 - 2.R_2$	$\left(\begin{array}{ccc\|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)$

We recall that

- a matrix is a rectangular arrangement of numbers into rows and columns,
- each number in a matrix is referred to as a **matrix element** or **entry**.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \end{pmatrix} \quad \longrightarrow \quad \text{2 rows, 3 columns},$$

- the **dimensions** of a matrix give the number of rows and columns of the matrix in that order. Since matrix A has 2 rows and 3 columns, it is called a 2×3 matrix.

As long as the dimensions of two matrices are the same, we can add and subtract them much like we add and subtract numbers.

Let's take a closer look!

Adding matrices:

- Given $A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}$, let's find $A + B$.
- We can find the sum simply by adding the corresponding entries in matrices A and B:

$$\begin{aligned} A + B &= \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix} \\ &= \begin{pmatrix} 1+3 & 2+4 \\ 3+5 & 0+6 \\ 4+7 & 3+8 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 6 \\ 8 & 6 \\ 11 & 11 \end{pmatrix} \end{aligned}$$

Subtracting matrices:

- Given $A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}$, let's find $A - B$.
- Similarly, we can find the $A - B$ simply by subtracting the corresponding entries in matrices A and B:

$$
\begin{aligned}
A - B &= \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix} \\
&= \begin{pmatrix} 1-3 & 2-4 \\ 3-5 & 0-6 \\ 4-7 & 3-8 \end{pmatrix} \\
&= \begin{pmatrix} -2 & -2 \\ -2 & -6 \\ -3 & -5 \end{pmatrix}
\end{aligned}
$$

Scalar multiplication:

- Given $A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 3 \end{pmatrix}$, consider the scalar 3 and let's find $3A$.
- This scalar multiplication can be seen as repeated addition: $3A = A + A + A$
- In this case, we have

$$3A = A + A + A = \begin{pmatrix} 1+1+1 & 2+2+2 \\ 3+3+3 & 0+0+0 \\ 4+4+4 & 3+3+3 \end{pmatrix} = \begin{pmatrix} 3 \cdot 1 & 3 \cdot 2 \\ 3 \cdot 3 & 3 \cdot 0 \\ 3 \cdot 4 & 3 \cdot 3 \end{pmatrix}$$

- In general, in scalar multiplication, each entry in the matrix is multiplied by the given scalar.

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \end{pmatrix}$$

We recall that

- A is a 2×3 matrix.
- the element $a_{2,\,1}$ is the entry in the second row and the first column of matrix A, that is $a_{2,\,1} = 2$.

How to find the product of two matrices? For example, find

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix}$$

- Up until now, you may have found operations with matrices fairly intuitive. For example
  - when you add two matrices, you add the corresponding entries,
  - in scalar multiplication, each entry in the matrix is multiplied by the given scalar.
- But things do not work as you'd expect them to work with multiplication. To multiply two matrices, we **cannot** simply multiply the corresponding entries.

- Before studying **matrix multiplication**, let's first understand how to find the **dot product** of two ordered lists of numbers, which can help us tremendously in this quest!

n-tuples and the dot product:

- We are familiar with ordered pairs, for example $(1, 2)$, $(-1, 0), \ldots$ and perhaps even ordered triples, for example $(1, 1, 2)$, $(-1, 0, 2), \ldots$
- An ***n*-tuple** is a generalization of this. It is an ordered list of n numbers.
- We can find the **dot product** of two n-tuples of equal length by summing the products of corresponding entries.
- For example, to find the dot product of two ordered pairs, we multiply the first coordinates and the second coordinates and add the results.

$$\begin{aligned}(2,3)\cdot(1,4) &= 2\cdot 1+3\cdot 4\\ &= 2+12\\ &= 14\end{aligned}$$

- Ordered n-tuples are often indicated by a variable with an arrow on top. For example, we can let $\vec{a} = (1, 2, -1)$ and $\vec{b} = (3, 4, 1)$. The expression $\vec{a} \cdot \vec{b}$ indicates the dot product of these two ordered triples and can be found as follows:

$$\begin{aligned} \vec{a} \cdot \vec{b} &= (1, 2, -1) \cdot (3, 4, 1) \\ &= 1 \cdot 3 + 2 \cdot 4 + (-1) \cdot 1 \\ &= 3 + 8 - 1 \\ &= 10 \end{aligned}$$

Notice that the dot product of two n-tuples of equal length is always a single real number (called scalar).

Matrices and n-tuples:

- When multiplying matrices, it's useful to think of each matrix row and column as an n-tuple.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

- In this matrix, denote
  1. row 1 by $\vec{r}_1 = (1, 2)$
  2. row 2 by $\vec{r}_2 = (3, 4)$
  3. column 1 by $\vec{c}_1 = (1, 3)$
  4. column 2 by $\vec{c}_2 = (2, 4)$

Matrix multiplication: Now we are ready to answer our question

Given $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix}$. Let's find $C = AB$

- denote
  1. row 1 of A by $\vec{a}_1$,
  2. row 2 of A by $\vec{a}_2$,
  3. column 1 of A by $\vec{b}_1$,
  4. column 2 of A by $\vec{b}_2$.
- then we have

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} \vec{a}_1 \cdot \vec{b}_1 & \vec{a}_1 \cdot \vec{b}_2 \\ \vec{a}_2 \cdot \vec{b}_1 & \vec{a}_2 \cdot \vec{b}_2 \end{pmatrix}$$

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Given $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix}$. Let's find $C = AB$

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  1. row 1 of A by $\vec{a}_1$,
  2. row 2 of A by $\vec{a}_2$,
  3. column 1 of A by $\vec{b}_1$,
  4. column 2 of A by $\vec{b}_2$.
- then we have

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} (1,2)\cdot(-3,3) & (1,2)\cdot(1,-1) \\ (2,4)\cdot(-3,3) & (2,4)\cdot(1,-1) \end{pmatrix}$$

Matrix multiplication: Now we are ready to answer our question

Given $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix}$. Let's find $C = AB$

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  2. row 2 of A by $\vec{a_2}$,
  3. column 1 of A by $\vec{b_1}$,
  4. column 2 of A by $\vec{b_2}$.
- then we have

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 \times (-3) + 2 \times 3 & 1 \times 1 + 2 \times (-1) \\ 2 \times (-3) + 4 \times 3 & 2 \times 1 + 4 \times (-1) \end{pmatrix}$$

Matrix multiplication: Now we are ready to answer our question

Given $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix}$. Let's find $C = AB$

- denote
  1. row 1 of A by $\vec{a}_1$,
  2. row 2 of A by $\vec{a}_2$,
  3. column 1 of A by $\vec{b}_1$,
  4. column 2 of A by $\vec{b}_2$.
- then we have

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix}$$

Generally speaking, in matrix multiplication, the entry in the product matrix located in the i^{th} row and j^{th} column, is the dot product of the i^{th} row in the first matrix and the j^{th} column in the second matrix.

"But when are we allowed to multiply two matrices?"
"What are the properties of this operation?"

1. **When is matrix multiplication defined?** In order for matrix multiplication to be defined, the number of columns in the first matrix must be equal to the number of rows in the second matrix.

$$(m \times \underbrace{n) \cdot (n}_{\text{product is defined}} \times k)$$

2. **What about dimensions the obtained matrix product?** When the matrix multiplication is defined, then the resulting matrix product has the number of lines of the first matrix and the number of comumns of the second matrix.

$$(m \times \underbrace{n) \cdot (n}_{\text{product is defined}} \times k) \quad = \quad (m \times k)$$

- A matrix that has the same number of rows and columns is called **square matrix**.
- The entries of a matrix that lie on the i^{th} row and the i^{th} column form the so-called **diagonal** of a matrix. For example, the diagonal of the following matrix is given by the blue entries

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

- The **square matrix** where the entries on the diagonal from the upper left to the bottom right are all 1's, and all other entries are 0 is called **identity matrix**, and is denoted by I_n where n is the number of rows (and columns) of the matrix. For example

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- The product of any square matrix and the appropriate identity matrix is always the original matrix, regardless of the order in which the multiplication was performed!
- In other words, for a square matrix A we have

$$A \cdot I = I \cdot A = A.$$

- Let A, B, and C be $(n \times n)$ matrices and I the $(n \times n) identity matrix.$ then we have:
  1. $AB \neq BA$ (Check it!)
  2. $(AB)C = A(BC)$
  3. $A(B+C) = AB + AC$
  4. $(B+C)A = BA + CA$
- If $AB = BA = I$, then we say that A is the **inverse** of B (or even B is the **inverse** of A)

- The determinant of a (2×2) matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $|A| = ad - cb$. It is simply obtained by cross multiplying the elements starting from the top left, then subtracting the products.
- Fro example, if $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then the determinant $|A| = 1 \times 4 - 3 \times 2 = 4 - 6 = -2$.

- We know that the inverse of a square matrix A is a square matrix B that verifies $AB = BA = I$, where I is the identity matrix.
- When such a matrix B exists, we say A is **invertible**.
- So let's state a very important rule!

A matrix is **invertible** if and only if **its determinant is <u>nonzero</u>**

Example

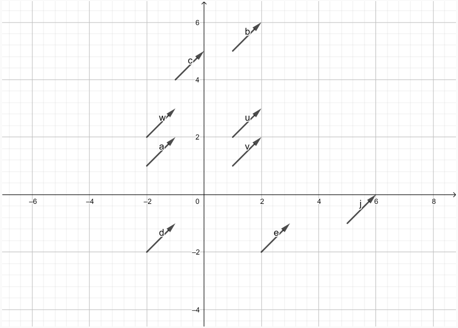
Are the following matrices invertible?

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \quad \begin{pmatrix} 4 & -2 \\ 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} 5 & 6 \\ -5 & 1 \end{pmatrix}$$

Outline

1. A **vector** is an n-tuple, or simply an ordered list of numbers. For example, a vector of the flat plane is an ordered pair, for instance $(9, 2)$, $(-1, 2)$, etc..
2. A vector of the flat plane can also be written like $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$, etc..
3. In general, a vector of n components can be seen as an $n \times 1$ or $1 \times n$ matrix.

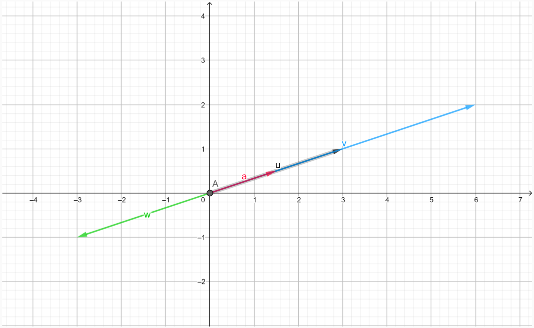
1. A vector can also be seen geometrically as an arrow pointing in space. In this case, what define a given vector are its length and the direction it's pointing in.
2. Two vectors that have same length and direction are considered to be the same!



Vectors

- Multiplying a vector by a number (**scalar**) would scale this vector, i.e., either stretching it or squishing it.
- Consider for example the vector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ then the vectors

$$2\begin{pmatrix} 1 \\ 3 \end{pmatrix},\ -1\begin{pmatrix} 1 \\ 3 \end{pmatrix},\ \frac{1}{2}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$



- In general, vectors such that one is the scale of the other are called **colinear vectors**.
- Hereafter, we focus on vectors in the flat coordinate plane (the (x, y)-plane), where a vector is given by $\begin{pmatrix} a \\ b \end{pmatrix}$
- In this case, we think of a vector as an arrow in the (x, y)-plane, having its tail at the origin and its tip at the point (a, b).

- See for instnace the vector $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ of tail A at the origin, and tip B of coordinates $(4, 2)$.
- And let's simply denote this vector $\overrightarrow{AB}$ (symbolizing a trajectory from A to B).

- But as we can see, going from A to B is equivalent to going from A to C and then from C to B.

- This means that $\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$.

- The vector $\overrightarrow{AC}$ starts from the origin and ends at the point C of coordinates $(4, 0)$, thus

$$\overrightarrow{AC} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 4\vec{i}$$

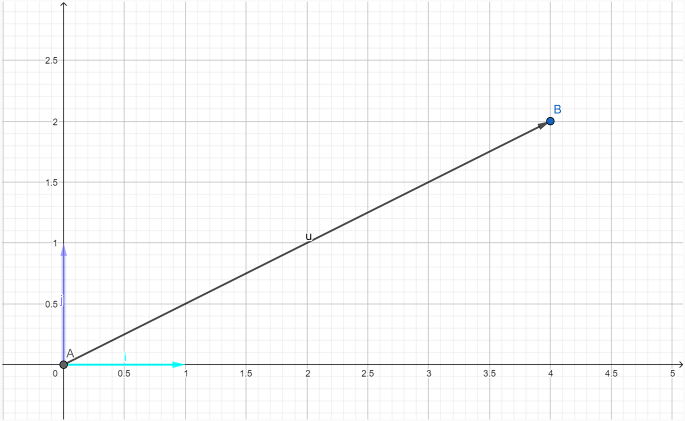
- The vector $\overrightarrow{CB}$ doesn't start from the origin. But having the same length and direction, $\overrightarrow{CB}$ is the same vector as $\overrightarrow{AE}$, that starts at the origin, and ends at the point E of coordinates $(0, 2)$.

- Thus

$$\overrightarrow{CB} = \overrightarrow{AE} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2\vec{j}$$

- Finally,

$$\overrightarrow{AB} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 4\vec{i} + 2\vec{j}$$



- In general, any vector $\overrightarrow{AB} = \begin{pmatrix} a \\ b \end{pmatrix}$ in the coordinate plane can be written as

$$\overrightarrow{AB} = \begin{pmatrix} a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} = a\vec{i} + b\vec{j}$$

- In this case, we say that $\overrightarrow{AB}$ is a linear combination of the vectors $\vec{i}$ and $\vec{j}$.
- This property, along with the fact that $\vec{i}$ and $\vec{j}$ are not colinear, make of the couple $(\vec{i}, \vec{j})$ a so-called **basis** of the coordinate flat plane, but more precisely, its **the canonical basis**.
- In this case, we say that a and b are the coordinates of the vector $\overrightarrow{AB}$ in the canonical basis $(\vec{i}, \vec{j})$.

- In general, if we chose any two vectors in the coordinate flat plane that are not colinear, these two vectors form a basis.
- This means, in the (x, y)-plane, if two vectors $\vec{u}$ and $\vec{v}$ are not colinear, then they form a basis, that is, any vector $\overrightarrow{AB}$ can be written as a linear combination of $\vec{u}$ and $\vec{v}$.
- This means, that we can find two scalars c and d such that $\overrightarrow{AB} = c\vec{u} + d\vec{v}$.
- In this case, we say c and d are the coordinates of $\overrightarrow{AB}$ in the basis $(\vec{u}, \vec{v})$.

- For example, if we consider the non-colinear vectors $\vec{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, we know they form a basis.
- In this case,

$$\begin{pmatrix} 1 \\ 6 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 2\vec{u} + \vec{v}$$

- In this case, we can say that the vector $\begin{pmatrix} 1 \\ 6 \end{pmatrix}$:
  1. Is of coordinates $(1, 6)$ in the canonical basis
  2. Is of coordinates $(2, 1)$ in the basis $(\vec{u}, \vec{v})$.

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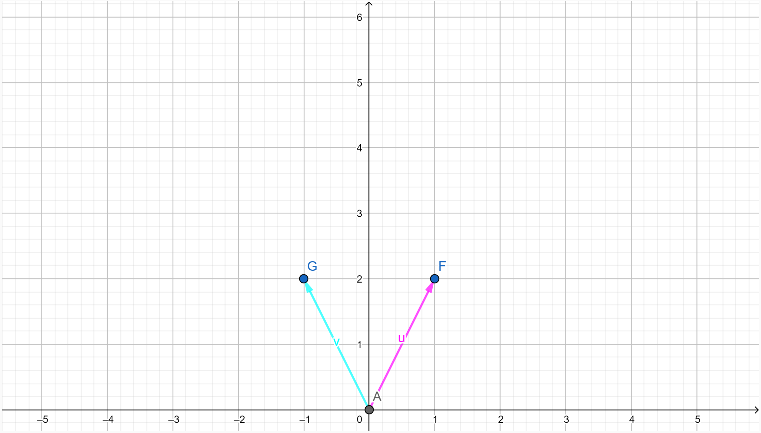

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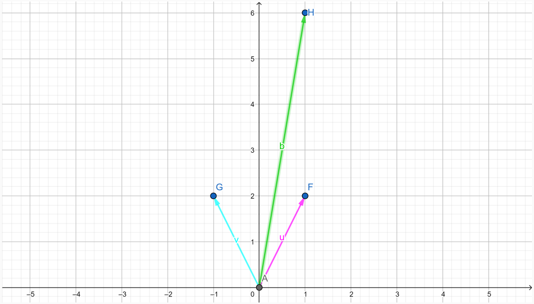

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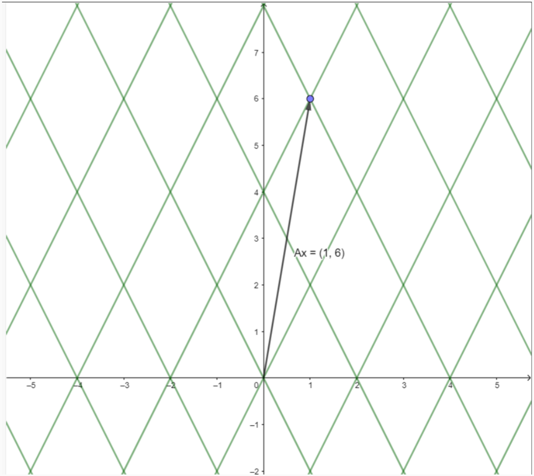
  2. Is of coordinates $(2, 1)$ in the basis $(\vec{u}, \vec{v})$.

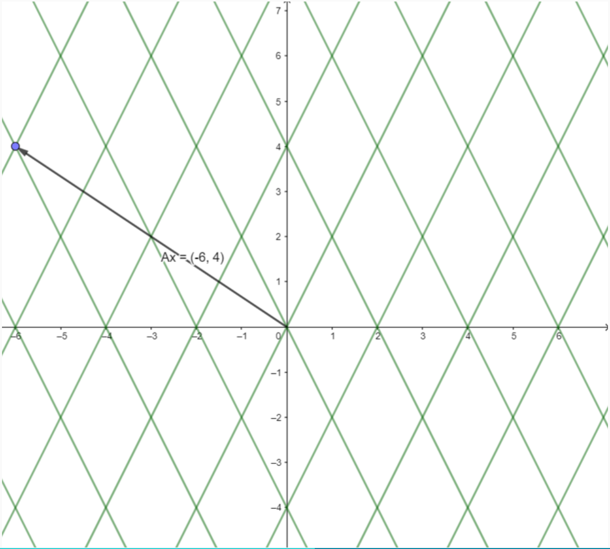


- Now take again the non-colinear vectors $\vec{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and then consider the matrix M having as columns the vectors $\vec{u}$ and $\vec{v}$

$$\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}.$$

- Can you imagine how the vector of coordinates $(-2, 4)$ in the basis $(\vec{u}, \vec{v})$ would look like?
- Of course you can, but you don't have to, you can simply **multiply** M by the vector $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$ and it will take straight to the vector you are searching for, that is the vector of coordinates $(-2, 4)$ in the basis $(\vec{u}, \vec{v})$.

$$\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -6 \\ 4 \end{pmatrix} \iff -2\vec{u} + 4\vec{v} = \begin{pmatrix} -6 \\ 4 \end{pmatrix}$$



This means that the matrix $M = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}$ transforms the vector $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$ to the vector $\begin{pmatrix} -6 \\ 4 \end{pmatrix}$:

- $\begin{pmatrix} -6 \\ 4 \end{pmatrix}$ is of coordinates $(-6, 4)$ in the canonical basis.
- $\begin{pmatrix} -6 \\ 4 \end{pmatrix}$ is of coordinates $(-2, 4)$ in the basis $(\vec{u}, \vec{v})$.

- For some matrices, there is some special vectors such that after the transformation, the new obtained vector in the new basis, is simply a scale of the original one.
- In other words, we can have a matrix M and a vector $\vec{v}$ such that

$$M \cdot \vec{v} = \text{a number} \cdot \vec{v}, \text{ that is, } \boxed{M \cdot \vec{v} = \lambda \cdot \vec{v}}$$

Example

- Consider the matrix $M = \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix}$
- Compute $\begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$
- Compute $\begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

- For some matrices, there is some special vectors such that after the transformation, **the new obtained vector in the new basis, is simply a scale of the original one**.
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Solution:

- Consider the matrix $M = \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix}$
- $\begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -7 + 2 \times 2 \\ 4 + 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix} = 3 \begin{pmatrix} -1 \\ 2 \end{pmatrix}$
- $\begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -7 + 2 \\ 4 + 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \end{pmatrix} = 5 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

- For some matrices, there is some special vectors such that after the transformation, **the new obtained vector in the new basis, is simply a scale of the original one**.
- In other words, we can have a matrix M and a vector $\vec{v}$ such that

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- In this case, λ is called an *eigenvalue* of the matrix M.
- The vector $\vec{v}$ is called an *eigenvector* of the matrix M.

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- The vector $\vec{v}$ is called an *eigenvector* of the matrix M.

For instance, the matrix $M = \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix}$ has

- an eigenvector $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ associated to the eigenvalue 3,
- an eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ associated to the eigenvalue 5,

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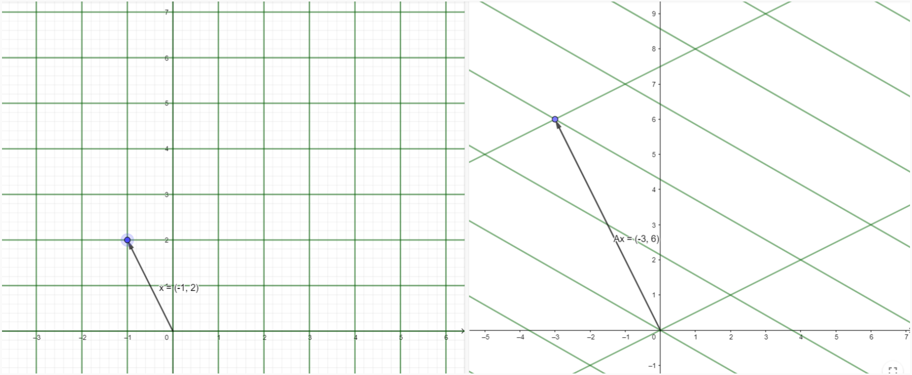

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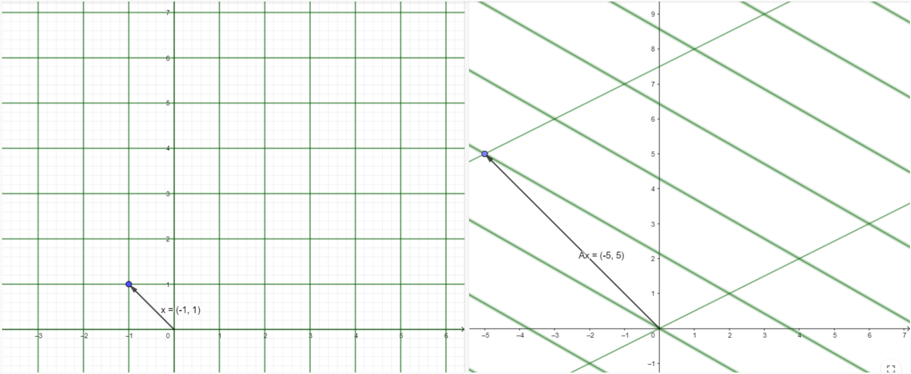
- an eigenvector $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ associated to the eigenvalue 3,
- an eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ associated to the eigenvalue 5,



We can visualize more examples on Geogebra

Thank you!