Precalculus advanced

Chapter 1:

Vectors In The Plane & In Three Dimensions

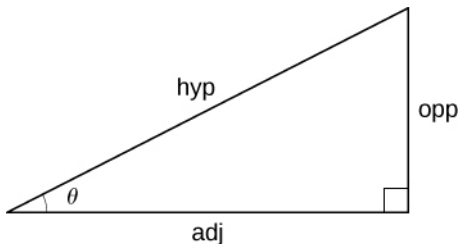Foundation Year - 2022/2023
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Formulas from Trigonometry

Right-Angle Trigonometry

- $\sin \theta = \dfrac{\text{opp}}{\text{hyp}}$

- $\cos \theta = \dfrac{\text{adj}}{\text{hyp}}$

- $\tan \theta = \dfrac{\text{opp}}{\text{adj}}$

Trigonometric Functions of Important Angles:

θ	Radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
$0°$	0	0	1	0
$30°$	$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
$45°$	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$60°$	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$90°$	$\pi/2$	1	0	$-$

Fundamental Identities

$$\sin^2\theta + \cos^2\theta \;=\; 1 \qquad\qquad \tan(\pi - \theta) \;=\; -\tan\theta$$

$$\sin(\pi - \theta) \;=\; \sin\theta \qquad\qquad \cos(\pi - \theta) \;=\; -\cos\theta$$

$$\sin(\pi + \theta) \;=\; -\sin\theta \qquad\qquad \cos(\pi + \theta) \;=\; -\cos\theta$$

$$\sin\left(\tfrac{\pi}{2} - \theta\right) \;=\; \cos\theta \qquad\qquad \sin(-\theta) \;=\; -\sin\theta$$

$$\cos\left(\tfrac{\pi}{2} - \theta\right) \;=\; \sin\theta \qquad\qquad \cos(-\theta) \;=\; \cos\theta$$

$$\sin(2\pi + \theta) \;=\; \sin\theta \qquad\qquad \tan(-\theta) \;=\; -\tan\theta$$

$$\cos(2\pi + \theta) \;=\; \cos\theta \qquad\qquad \tan(\theta + \pi) \;=\; \tan\theta$$
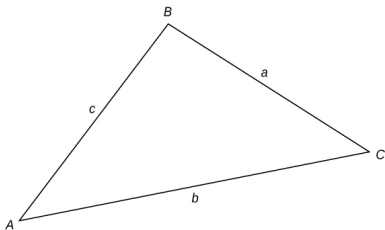
Law of Sines: $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

Law of Cosines:

$$
\begin{aligned}
a^2 &= b^2 + c^2 - 2bc\cos A \\
b^2 &= a^2 + c^2 - 2ac\cos B \\
c^2 &= a^2 + b^2 - 2ab\cos C
\end{aligned}
$$

- When describing the movement of an airplane in flight, it is important to communicate two pieces of information: the direction in which the plane is traveling and the plane's speed.

- When measuring a force, such as the thrust of the plane's engines, it is important to describe not only the strength of that force, but also the direction in which it is applied.

- Some quantities, such as velocity or force, are defined in terms of both size (also called magnitude) and direction.

- A quantity that has magnitude and direction is called a vector. Hereafter, vectors are named with lowercase letters in bold type or by drawing an arrow over their name, such as $\vec{v}$ or **v** (We shall mainly see vectors named with lowercase letters in bold type in the figures).

DEFINITION

A vector is a quantity that has both magnitude and direction.
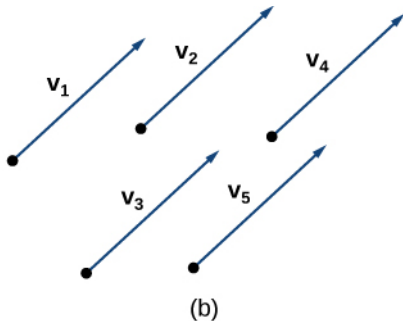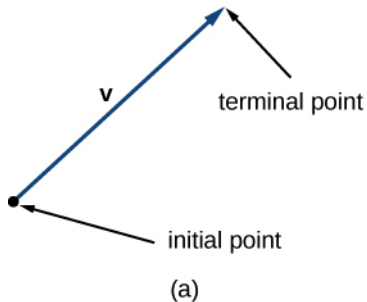
Vector Representation

- A vector in a plane is represented by a directed line segment (an arrow).
- The endpoints of the segment are called the **initial point** and the **terminal point** of the vector.
- An arrow from the initial point to the terminal point indicates the **direction** of the vector. The length of the line segment represents its **magnitude** (sometimes we call it **modulus**).

- We use the notation $\|\vec{v}\|$ to denote the magnitude of the vector $\vec{v}$.

- A vector with an initial point and terminal point that are the same is called the **zero vector**, denoted **0**. The zero vector is the only vector without a direction, and by convention can be considered to have any direction convenient to the problem at hand.

- Vectors with the same magnitude and direction are called **equivalent vectors**. We treat equivalent vectors as equal, even if they have different initial points. Thus, if $\vec{v}$ and $\vec{w}$ are equivalent, we write $\vec{v} = \vec{w}$.

Definition

Vectors are said to be equivalent vectors if they have the same magnitude and direction.

(a) A vector is represented by a directed line segment from its initial point to its terminal point. (b) Vectors $\vec{v}_1$ through $\vec{v}_5$ are equivalent.

☺ Two vectors are said to be **parallel** (or **collinear**) if they have the same or opposite directions.

• A vector is defined by its magnitude and direction, regardless of where its initial point is located.

• When a vector has initial point P and terminal point Q, the notation $\overrightarrow{PQ}$ is useful because it indicates the direction and location of the vector.

EXAMPLE

Sketch a vector in the plane from initial point $P(1,1)$ to terminal point $Q(8,5)$.

Solution
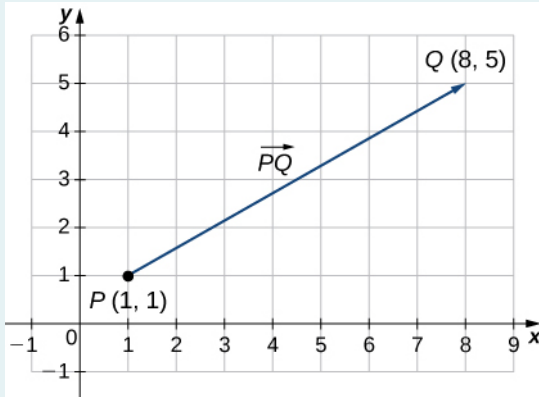
Example

Sketch the vector $\overrightarrow{ST}$ where S is point $(3, -1)$ and T is point $(-2, 3)$.

- Vectors have many real-life applications, including situations involving force or velocity. For example, consider the forces acting on a boat crossing a river. The boat's motor generates a force in one direction, and the current of the river generates a force in another direction. Both forces are vectors. We must take both the magnitude and direction of each force into account if we want to know where the boat will go.

- A second example that involves vectors is a quarterback throwing a football. The quarterback does not throw the ball parallel to the ground; instead, he aims up into the air. The velocity of his throw can be represented by a vector. If we know how hard he throws the ball (magnitude—in this case, speed), and the angle (direction), we can tell how far the ball will travel down the field.

- A real number is often called a **scalar** in mathematics and physics.
- Unlike vectors, scalars are generally considered to have a magnitude only, but no direction.
- Multiplying a vector by a scalar changes the vector's magnitude. This is called scalar multiplication.
- Note that changing the magnitude of a vector does not indicate a change in its direction. For example, wind blowing from north to south might increase or decrease in speed while maintaining its direction from north to south.

DEFINITION

The product $k\vec{v}$ of a vector $\vec{v}$ and a scalar k is a vector with a magnitude that is $|k|$ times the magnitude of $\vec{v}$, and with a direction that is the same as the direction of $\vec{v}$ if $k > 0$, and opposite the direction of $\vec{v}$ if $k < 0$. This is called **scalar multiplication**. If $k = 0$ or $\vec{v} = 0$, then $k\vec{v} = 0$.

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- As you might expect, if $k = -1$, we denote the product $k\vec{v}$ as

$$k\vec{v} = (-1)\vec{v} = -\vec{v}.$$

- Note that $-\vec{v}$ has the same magnitude as $\vec{v}$, but has the opposite direction as we shall see in the following figure.

- $\vec{u}$ and $\vec{v}$ are collinear if $\vec{u} = k\vec{v}$, where k is a scalar
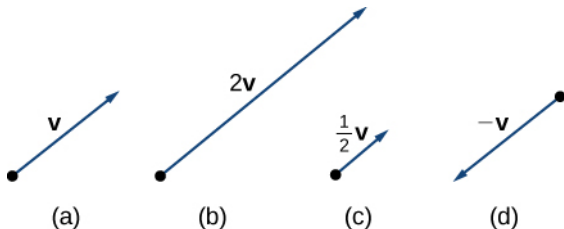
(a) The original vector $\vec{v}$ has length *n* units. (b) The length of $2\vec{v}$ equals 2*n* units. (c) The length of $\vec{v}/2$ is *n/2units*. (d) The vectors $\vec{v}$ and $-\vec{v}$ have the same length but opposite directions.
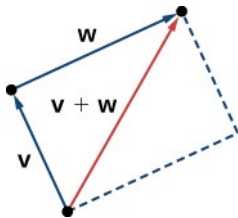
Another operation we can perform on vectors is to add them together in vector addition, but because each vector may have its own direction, the process is different from adding two numbers.

- The most common graphical method for adding two vectors is to place the initial point of the second vector at the terminal point of the first, as in
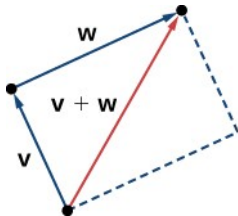
Combining Vectors

- To see why this makes sense, suppose, for example, that both vectors represent displacement. If an object moves first from the initial point to the terminal point of vector $\vec{v}$, then from the initial point to the terminal point of vector $\vec{w}$, the overall displacement is the same as if the object had made just one movement from the initial point to the terminal point of the vector $\vec{v} + \vec{w}$. For obvious reasons, this approach is called the **triangle method**.
- We notice that if we had switched the order, so that $\vec{w}$ was our first vector and $\vec{v}$ was our second vector, we would have ended up in the same place. Thus, $\vec{v} + \vec{w} = \vec{w} + \vec{v}$.

- A second method for adding vectors is called the **parallelogram method**. With this method, we place the two vectors so they have the same initial point, and then we draw a parallelogram with the vectors as two adjacent sides, as in
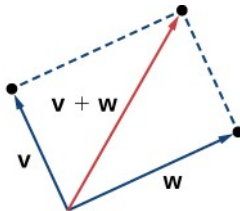
- The diagonal of the parallelogram is the sum.

The vector $\vec{v} + \vec{w}$ is called the **vector sum**.

DEFINITION

The sum of two vectors $\vec{v}$ and $\vec{w}$ can be constructed graphically by placing the initial point of $\vec{w}$ at the terminal point of $\vec{v}$. Then, the vector sum, $\vec{v} + \vec{w}$, is the vector with an initial point that coincides with the initial point of $\vec{v}$ and has a terminal point that coincides with the terminal point of $\vec{w}$. This operation is known as **vector addition**.

- We define $\vec{v} - \vec{w}$ as $\vec{v} + (-\vec{w}) = \vec{v} + (-1)\vec{w}$.
- The vector $\vec{v} - \vec{w}$ is called the **vector difference**.
- Graphically, the vector $\vec{v} - \vec{w}$ is depicted by drawing a vector from the terminal point of $\vec{w}$ to the terminal point of $\vec{v}$.
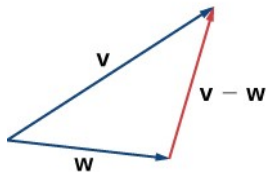
Figure: The vector difference $\vec{v} - \vec{w}$ is depicted by drawing a vector from the terminal point of $\vec{w}$ to the terminal point of $\vec{v}$.
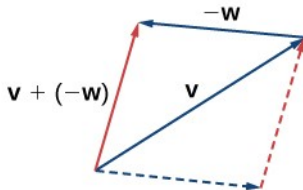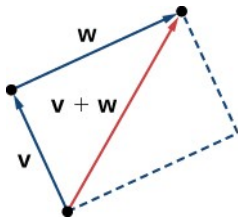

Figure: The vector $\vec{v} - \vec{w}$ is equivalent to the vector $\vec{v} + (-\vec{w})$.

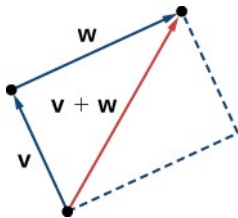- In this image we see that the initial point of $\vec{v} + \vec{w}$ is the initial point of $\vec{v}$. The terminal point of $\vec{v} + \vec{w}$ is the terminal point of $\vec{w}$.

- These three vectors form the sides of a triangle. It follows that the length of any one side is less than the sum of the lengths of the remaining sides. So we have

$$\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|.$$

- These three vectors form the sides of a triangle. It follows that the length of any one side is less than the sum of the lengths of the remaining sides. So we have

$$\|\vec{v} + \vec{w}\| \le \|\vec{v}\| + \|\vec{w}\|.$$

- This is known more generally as the **triangle inequality**.

- There is one case, however, when the resultant vector $\vec{u} + \vec{v}$ has the same magnitude as the sum of the magnitudes of $\vec{u}$ and $\vec{v}$. This happens only when $\vec{u}$ and $\vec{v}$ have the same direction.

Example

Given the vectors $\vec{v}$ and $\vec{w}$ shown in
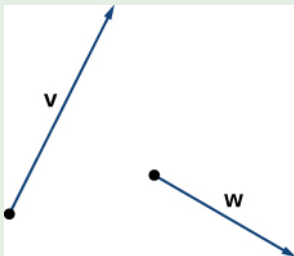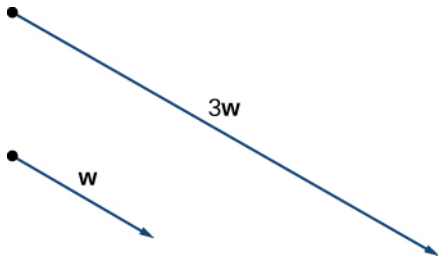

sketch the vectors

1. $3\vec{w}$
2. $\vec{v} + \vec{w}$
3. $2\vec{v} - \vec{w}$

Solution:

1. The vector $3\vec{w}$ has the same direction as $\vec{w}$; it is three times as long as $\vec{w}$. Vector $3\vec{w}$ has the same direction as $\vec{w}$ and is three times as long.

SOLUTION:

2. Use either addition method to find $\vec{v} + \vec{w}$.
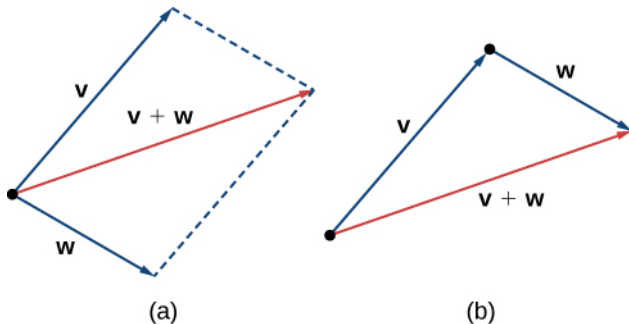
FIGURE: To find $\vec{v} + \vec{w}$, align the vectors at their initial points or place the initial point of one vector at the terminal point of the other. (a) The vector $\vec{v} + \vec{w}$ is the diagonal of the parallelogram with sides $\vec{v}$ and $\vec{w}$. (b) The vector $\vec{v} + \vec{w}$ is the third side of a triangle formed with $\vec{w}$ placed at the terminal point of $\vec{v}$.

SOLUTION:

3. To find $2\vec{v} - \vec{w}$, we can first rewrite the expression as $2\vec{v} + (-\vec{w})$. Then we can draw the vector $-\vec{w}$, then add it to the vector $2\vec{v}$.

FIGURE: To find $2\vec{v} - \vec{w}$, simply add $2\vec{v} + (-\vec{w})$.

- Working with vectors in a plane is easier when we are working in a coordinate system.
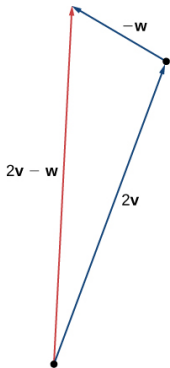- When the initial points and terminal points of vectors are given in Cartesian coordinates, computations become straightforward.

VECTOR COMPONENTS

- We have seen how to plot a vector when we are given an initial point and a terminal point.

- However, because a vector can be placed anywhere in a plane, it may be easier to perform calculations with a vector when its initial point coincides with the origin.

- We call a vector with its initial point at the origin a **standard-position vector**.

- Because the initial point of any vector in standard position is known to be $(0, 0)$, we can describe the vector by looking at the coordinates of its terminal point.

- Thus, if vector $\vec{v}$ has its initial point at the origin and its terminal point at (x, y), we write the vector in component form as

$$v = \langle x, y \rangle \quad \equiv \quad \begin{pmatrix} x \\ y \end{pmatrix} \quad \equiv \quad \begin{pmatrix} x & y \end{pmatrix}$$

When a vector is written in component form like

$$v = \langle x, y \rangle \quad \text{or} \quad \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x & y \end{pmatrix},$$

the scalars x and y are called the **components** of $\vec{v}$.

Definition

The vector with initial point $(0, 0)$ and terminal point (x, y) can be written in component form as

$$v = \langle x, y \rangle.$$

The scalars x and y are called the components of $\vec{v}$.

- We have learned that we can name a vector by its component form, with the coordinates of its terminal point in angle brackets (when its a standard-position vector).

- However, when writing the component form of a vector, it is important to distinguish between $\langle x, y \rangle$ and (x, y). The first ordered pair uses angle brackets to describe a vector, whereas the second uses parentheses to describe a point in a plane.

- The initial point of $\langle x, y \rangle$ is $(0, 0)$; the terminal point of $\langle x, y \rangle$ is (x, y).

"But what if we have a vector not already in standard position?"

> When we have a vector not already in standard position, we can determine its component form in one of two ways.

- We can use a geometric approach, in which we sketch the vector in the coordinate plane, and then sketch an equivalent standard-position vector.

- Or, we can find it algebraically, using the coordinates of the initial point and the terminal point. To find it algebraically, we subtract the x-coordinate of the initial point from the x-coordinate of the terminal point to get the x component, and we subtract the y-coordinate of the initial point from the y-coordinate of the terminal point to get the y component.

VECTOR COMPONENTS

> ### GENERAL RULE: COMPONENT FORM OF A VECTOR
>
> Let $\vec{v}$ be a vector with initial point (x_1, y_1) and terminal point (x_2, y_2).
>
>> Then we can express $\vec{v}$ in component form as
>>
>> $$v = \langle x_2 - x_1, y_2 - y_1 \rangle \quad \equiv \quad \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} \quad \equiv \quad \begin{pmatrix} x_2 - x_1 & y_2 - y_1 \end{pmatrix}$$
>> .

Example

Express vector $\vec{v}$ with initial point $(-3, 4)$ and terminal point $(1, 2)$ in component form.

VECTOR COMPONENTS

GEOMETRIC SOLUTION

❶ Sketch the vector in the coordinate plane(check it in the next slide)

❷ The terminal point is 4 units to the right and 2 units down from the initial point.

❸ Find the point that is 4 units to the right and 2 units down from the origin.

❹ In standard position, this vector has initial point $(0,0)$ and terminal point $(4,-2)$:

$$\vec{v} = \langle 4, -2 \rangle \quad \equiv \quad \begin{pmatrix} 4 \\ -2 \end{pmatrix} \quad \equiv \quad \begin{pmatrix} 4 & -2 \end{pmatrix}.$$

FIGURE: Equivalent vectors

Vector Components
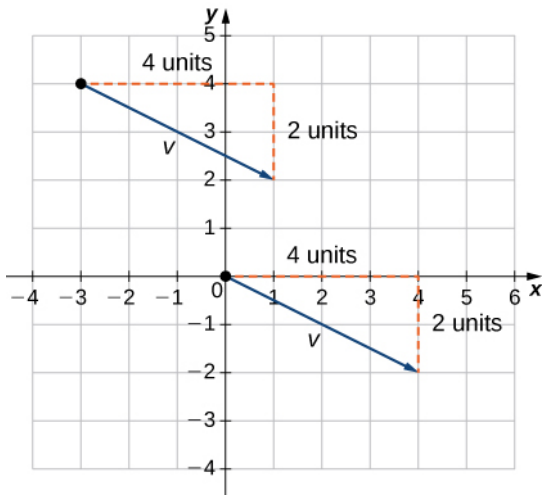
Algebraic solution

1 In the geometric solution, we used a sketch of the vector to see that the terminal point lies 4 units to the right. We can accomplish this algebraically by finding the difference of the x-coordinates:

$$x_2 - x_1 = 1 - (-3) = 4.$$

2 Similarly, the difference of the y-coordinates shows the vertical length of the vector.

$$y_2 - y_1 = 2 - 4 = -2.$$

3 So, in component form,

$$\begin{aligned} v &= \langle x_2 - x_1 , y_2 - y_1 \rangle \\ &= \langle 1 - (-3) , 2 - 4 \rangle \\ &= \langle 4 , -2 \rangle. \end{aligned}$$

- To find the magnitude of a vector, we calculate the distance between its initial point and its terminal point.
- The magnitude of vector $\vec{v} = \langle x, y \rangle$ is denoted $\|\vec{v}\|$, or $|\vec{v}|$, and can be computed using the formula

$$\|\vec{v}\| = \sqrt{x^2 + y^2}$$

- ⋆ Note that because this vector is written in component form, it is equivalent to a vector in standard position, with its initial point at the origin and terminal point (x, y).

- Thus, it suffices to calculate the magnitude of the vector in standard position. Using the distance formula to calculate the distance between initial point $(0, 0)$ and terminal point (x, y), we have

$$\begin{aligned} \|\vec{v}\| &= \sqrt{(x - 0)^2 + (y - 0)^2} \\ &= \sqrt{x^2 + y^2} \end{aligned}$$

- Using the distance formula to calculate the distance between initial point $(0,0)$ and terminal point (x, y), we have

$$\begin{aligned} \|\vec{v}\| &= \sqrt{(x-0)^2 + (y-0)^2} \\ &= \sqrt{x^2 + y^2} \end{aligned}$$

- Based on this formula, it is clear that for any vector $\vec{v}$, $\|\vec{v}\| \geq 0$, and $\|\vec{v}\| = 0$ if and only if $\vec{v} = 0$.

- We have defined scalar multiplication and vector addition geometrically.
- Expressing vectors in component form allows us to perform these same operations algebraically.

DEFINITION

Let $v = \langle x1, y1 \rangle$ and $w = \langle x2, y2 \rangle$ be vectors, and let k be a scalar.

- **Scalar multiplication**: $k\vec{v} = \langle kx_1 , ky_1 \rangle$
- **Vector addition**:

$$\vec{v} + \vec{w} = \langle x_1 , y_1 \rangle + \langle x_2 , y_2 \rangle = \langle x_1 + x_2 , y_1 + y_2 \rangle$$
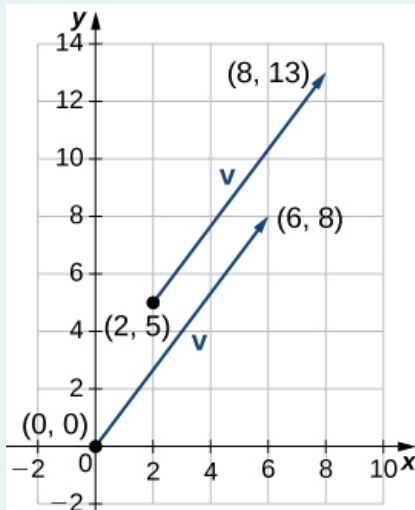
Example

Let $\vec{v}$ be the vector with initial point $(2, 5)$ and terminal point $(8, 13)$, and let $\vec{w} = \langle -2, 4 \rangle$.

1. Express $\vec{v}$ in component form and find $\|\vec{v}\|$. Then, using algebra, find

2. $\vec{v} + \vec{w}$,

3. $3\vec{v}$, and

4. $\vec{v} - 2\vec{w}$

Solution

1. To place the initial point of $\vec{v}$ at the origin, we must translate the vector 2 units to the left and 5 units down

SOLUTION

1. Using the algebraic method, we can express $\vec{v}$ as
   $\vec{v} = \langle 8 - 2 , \; 13 - 5 \rangle = \langle 6, 8 \rangle$:

$$\|\vec{v}\| = \sqrt{6^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10$$

2. To find $\vec{v} + \vec{w}$, add the x-components and the y-components separately:

$$\vec{v} + \vec{w} = \langle 6, 8 \rangle + \langle -2, 4 \rangle = \langle 4, 12 \rangle.$$

3. To find $3\vec{v}$, multiply $\vec{v}$ by the scalar $k = 3$:

$$3\vec{v} = 3 \cdot \langle 6, 8 \rangle = \langle 3 \cdot 6, 3 \cdot 8 \rangle = \langle 18, 24 \rangle.$$

4. To find $v - 2w$, find $-2w$ and add it to $\vec{v}$:

$$\vec{v} - 2\vec{w} = \langle 6, 8 \rangle - 2 \cdot \langle -2, 4 \rangle = \langle 6, 8 \rangle + \langle 4, -8 \rangle = \langle 10, 0 \rangle$$

Now that we have established the basic rules of vector arithmetic, we can state the properties of vector operations. We will prove two of these properties. The others can be proved in a similar manner.

THEOREM

Properties of Vector Operations:

Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in a plane. Let r and s be scalars.

1.	$\vec{u} + \vec{v} = \vec{v} + \vec{u}$	*Commutative property*
2.	$\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$	*Associative property*
3.	$\vec{u} + \vec{0} = \vec{u}$	*Additive identity property*
4.	$\vec{u} + (-\vec{u}) = \vec{0}$	*Additive inverse property*
5.	$r(s\vec{u}) = (rs)\vec{u}$	*Ass. of scalar multiplication*
6.	$(r + s)\vec{u} = r\vec{u} + s\vec{u}$	*Distributive property*
7.	$r(\vec{u} + \vec{v}) = r\vec{u} + r\vec{v}$	*Distributive property*
8.	$1\vec{u} = \vec{u}, \; 0\vec{u} = 0$	*Identity & zero peoperties*

Proof of Commutative Property:

Let $\vec{u} = \langle x_1, y_1 \rangle$ and $\vec{v} = \langle x_2, y_2 \rangle$. Apply the commutative property for real numbers:

$$
\begin{aligned}
\vec{u} + \vec{v} &= \langle x_1 + x_2, y_1 + y_2 \rangle \\
&= \langle x_2 + x_1, y_2 + y_1 \rangle \\
&= \vec{v} + \vec{u}.
\end{aligned}
$$

Proof of Distributive Property:

Apply the distributive property for real numbers:

$$
\begin{aligned}
r(\vec{u} + \vec{v}) &= r \cdot \langle x_1 + x_2, y_1 + y_2 \rangle \\
&= \langle r(x_1 + x_2), r(y_1 + y_2) \rangle \\
&= \langle rx_1 + rx_2, ry_1 + ry_2 \rangle \\
&= \langle rx_1, ry_1 \rangle + \langle rx_2, ry_2 \rangle \\
&= r\vec{u} + r\vec{v}.
\end{aligned}
$$

- We have found the components of a vector given its initial and terminal points. In some cases, we may only have the magnitude and direction of a vector, not the points. For these vectors, we can identify the horizontal and vertical components using trigonometry:
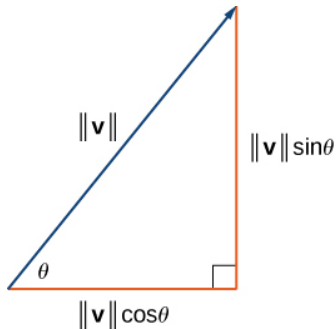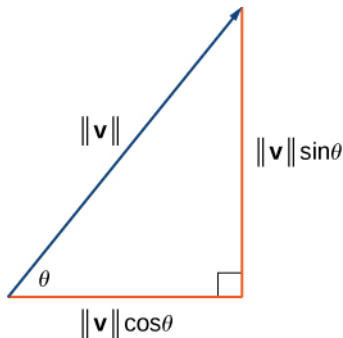

FIGURE: The components of a vector form the legs of a right triangle, with the vector as the hypotenuse.

- Consider the angle θ formed by the vector $\vec{v}$ and the positive x-axis. We can see from the triangle that the components of vector $\vec{v}$ are $\langle \|\vec{v}\| \cos\theta, \|\vec{v}\| \sin\theta \rangle$. Therefore, given an angle and the magnitude of a vector, we can use the cosine and sine of the angle to find the components of the vector.

Example

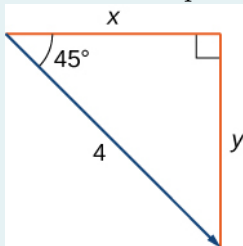Find the component form of a vector with magnitude 4 that forms an angle of −45° with the x-axis.

Solution:

Let x and y represent the components of the vector

Then $x = 4\cos(-45°) = 2\sqrt{2}$ and $y = 4\sin(-45°) = -2\sqrt{2}$. The component form of the vector is

$$\langle 2\sqrt{2}, -2\sqrt{2} \rangle$$

- A **unit vector** is a vector with magnitude 1. For any nonzero vector $\vec{v}$, we can use scalar multiplication to find a unit vector $\vec{u}$ that has the same direction as $\vec{v}$.

- To do this, we multiply the vector by the reciprocal of its magnitude:

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$$

- Recall that when we defined scalar multiplication, we noted that $\|k\vec{v}\| = |k| \cdot \|\vec{v}\|$. For $\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$, it follows that

$$\|\vec{u}\| = \frac{1}{\|\vec{v}\|}\left(\|\vec{v}\|\right) = 1.$$

- We say that u is the unit vector in the direction of v.

- Recall that when we defined scalar multiplication, we noted that $\|k\vec{v}\| = |k| \cdot \|\vec{v}\|$. For $\vec{u} = \dfrac{1}{\|\vec{v}\|}\vec{v}$, it follows that

  $\|\vec{u}\| = \dfrac{1}{\|\vec{v}\|}(\|\vec{v}\|) = 1$.

- We say that u is the unit vector in the direction of v:

- The process of using scalar multiplication to find a unit vector with a given direction is called **normalization**.

Example

Let $\vec{v} = \langle 1, 2 \rangle$.

1. Find a unit vector with the same direction as v.
2. Find a vector $\vec{w}$ with the same direction as v such that $\|\vec{w}\| = 7$

Solution

❶ First, find the magnitude of $\vec{v}$, then divide the components of $\vec{v}$ by the magnitude:

$$\|\vec{v}\| = \sqrt{1^2 + 2^2} = \sqrt{1 + 4} = \sqrt{5}.$$

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\sqrt{5}}\langle 1, 2\rangle = \Big\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\Big\rangle.$$

❷ The vector $\vec{u}$ is in the same direction as $\vec{v}$ and $\|\vec{u}\| = 1$. Use scalar multiplication to increase the length of $\vec{u}$ without changing direction:

$$\vec{w} = 7\vec{u} = 7\Big\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\Big\rangle = \Big\langle \frac{7}{\sqrt{5}}, \frac{7 \times 2}{\sqrt{5}}\Big\rangle = \Big\langle \frac{7}{\sqrt{5}}, \frac{14}{\sqrt{5}}\Big\rangle$$

- We have seen how convenient it can be to write a vector in component form.
- Sometimes, though, it is more convenient to write a vector as a sum of a horizontal vector and a vertical vector.
- To make this easier, let's look at **standard unit vectors**.
- The standard unit vectors are the vectors

$$\vec{i} = \langle 1, 0 \rangle \quad \text{and} \quad \vec{j} \langle 0, 1 \rangle$$

equivalently

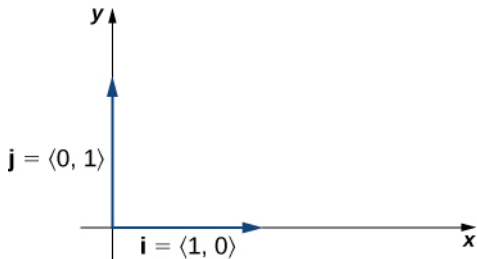$$\vec{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{j} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

FIGURE: The standard unit vectors $\vec{i}$ and $\vec{j}$.

- By applying the properties of vectors, it is possible to express any vector in terms of $\vec{i}$ and $\vec{j}$ in what we call a *linear combination*:
$$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = x\vec{i} + y\vec{j}$$

- Thus, $\vec{v}$ is the sum of a horizontal vector with magnitude x, and a vertical vector with magnitude y, as in the following figure.
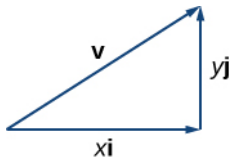
The vector $\vec{v}$ is the sum of $x\vec{i}$ and $y\vec{j}$.

UNIT VECTORS

> **EXAMPLE**
>
> 1. Express the vector $\vec{w} = \langle 3, -4 \rangle$ in terms of standard unit vectors.
>
> 2. Vector $\vec{u}$ is a unit vector that forms an angle of 60° with the positive x-axis. Use standard unit vectors to describe $\vec{u}$.
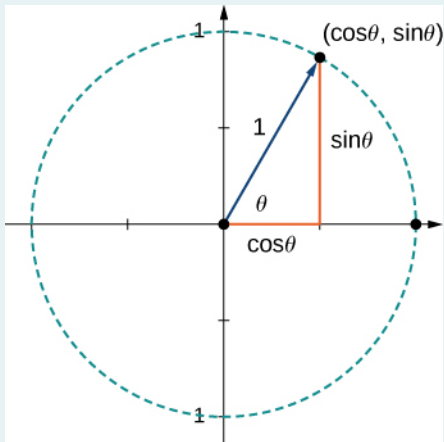
Solution

1. Resolve vector $\vec{w}$ into a vector with a zero y-component and a vector with a zero x-component:

$$\vec{w} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -4 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 4\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3\vec{i} - 4\vec{j}$$

2. Because $\vec{u}$ is a unit vector, the terminal point lies on the unit circle when the vector is placed in standard position:

Solution

The terminal point of $\vec{u}$ lies on the unit circle $(\cos\theta, \sin\theta)$.

Solution

1. Resolve vector $\vec{w}$ into a vector with a zero y-component and a vector with a zero x-component:

$$\vec{w} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -4 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 4\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3\vec{i} - 4\vec{j}$$

2. Because $\vec{u}$ is a unit vector, the terminal point lies on the unit circle when the vector is placed in standard position:

$$\begin{aligned} \vec{u} &= \langle \cos 60°, \sin 60° \rangle \\ &= \langle \tfrac{1}{2}, \tfrac{\sqrt{3}}{2} \rangle \\ &= \tfrac{1}{2}\vec{i} + \tfrac{\sqrt{3}}{2}\vec{j} \end{aligned}$$

- ⋆ Vectors are useful tools for solving two-dimensional problems. Life, however, happens in three dimensions.

- ⋆ To expand the use of vectors to more realistic applications, it is necessary to create a framework for describing three-dimensional space.

- ⋆ For example, although a two-dimensional map is a useful tool for navigating from one place to another, in some cases the topography of the land is important. Does your planned route go through the mountains? Do you have to cross a river? To appreciate fully the impact of these geographic features, you must use three dimensions.

This section presents a natural extension of the two-dimensional Cartesian coordinate plane into three dimensions.

- ⋆ As we have learned, the two-dimensional rectangular coordinate system contains two perpendicular axes: the horizontal *x*-axis and the vertical *y*-axis.

- ⋆ We can add a third dimension, the *z*-axis, which is perpendicular to both the *x*-axis and the *y*-axis.

- ⋆ We call this system the three-dimensional rectangular coordinate system. It represents the three dimensions we encounter in real life.
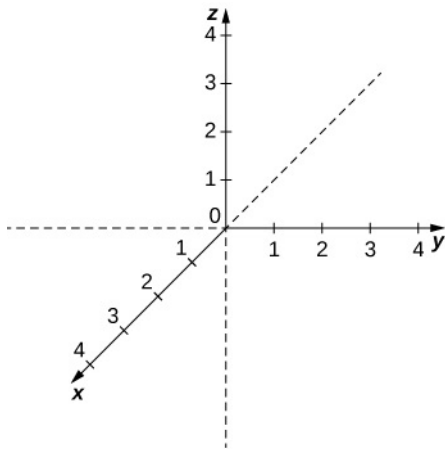
Definition

* The **three-dimensional rectangular coordinate system** consists of three perpendicular axes: the x-axis, the y-axis, the z-axis, and an origin at the point of intersection (0) of the axes.

* Because each axis is a number line representing all real numbers in $\mathbb{R}$ (the symbol of the set of real numbers), the three-dimensional system is often denoted by $\mathbb{R}^3$.
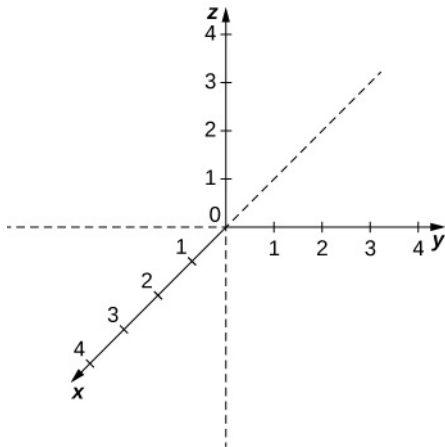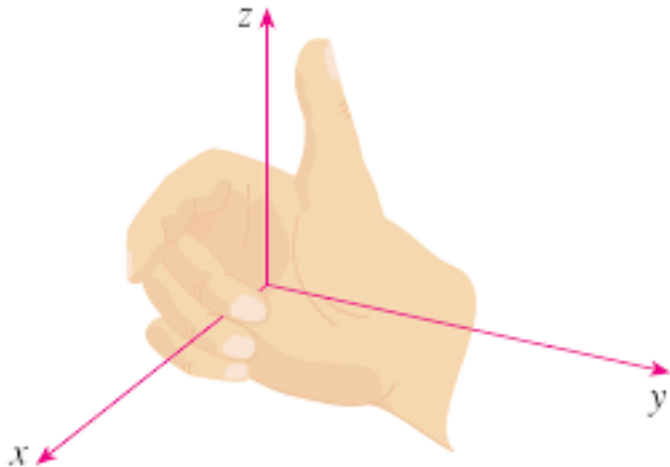
☺ In this figure, the positive z-axis is shown above the plane containing the x- and y-axes.

☺ The positive x-axis appears to the left and the positive y-axis is to the right.

A natural question to ask is: How was arrangement determined? The system displayed follows the right-hand rule.

The right-hand rule is used to determine the placement of the coordinate axes in the standard Cartesian plane.

☺ If we take our right hand and align the fingers with the positive *x*-axis, then curl the fingers so they point in the direction of the positive *y*-axis, our thumb points in the direction of the positive *z*-axis.

☺ Some systems do follow a left-hand rule, but the right-hand rule is considered the standard representation.

- ⋆ In two dimensions, we describe a point in the plane with the coordinates (x, y). Each coordinate describes how the point aligns with the corresponding axis.
- ⋆ In three dimensions, a new coordinate, z, is appended to indicate alignment with the z-axis: (x, y, z).
- ⋆ A point in space is identified by all three coordinates.
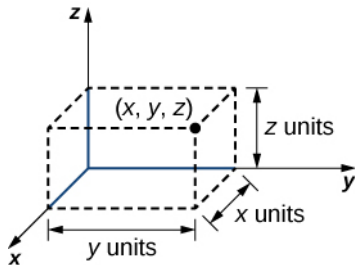
- ⋆ To plot the point (x, y, z), go x units along the x-axis, then y units in the direction of the y-axis, then z units in the direction of the z-axis.
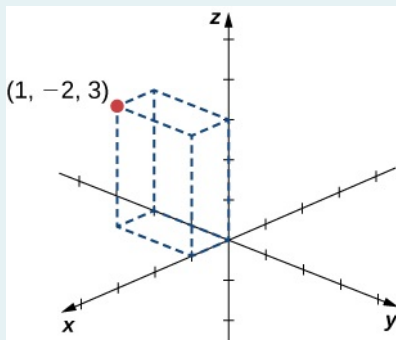
EXAMPLE

Sketch the point $(1, -2, 3)$ in three-dimensional space.

Solution

To sketch a point, start by sketching three sides of a rectangular prism along the coordinate axes: one unit in the positive x direction, 2 units in the negative y direction, and 3 units in the positive z direction.

- In two-dimensional space, the coordinate plane is defined by a pair of perpendicular axes. These axes allow us to name any location within the plane.

- In three dimensions, we define coordinate planes by the coordinate axes, just as in two dimensions.

- There are three axes now, so there are three intersecting pairs of axes. Each pair of axes forms a coordinate plane: the xy-plane, the xz-plane, and the yz-plane.

# THREE-DIMENSIONAL COORDINATE SYSTEMS
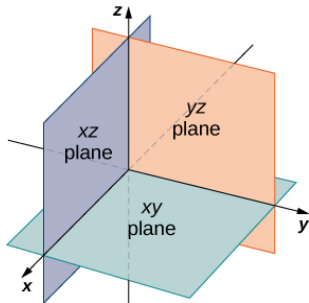
> ### THEOREM
>
> *The distance d between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the formula*
>
> $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z^2 - z_1)^2} \tag{1}$$

- Before moving on to the next section, let's have a look to how $\mathbb{R}^3$ differs from $\mathbb{R}^2$. For example, in $\mathbb{R}^2$, lines that are not parallel must always intersect.

- This is not the case in $\mathbb{R}^3$.

- For example, consider the line shown in the figure. These two lines are not parallel, nor do they intersect.

Working with Vectors in $\mathbb{R}^3$
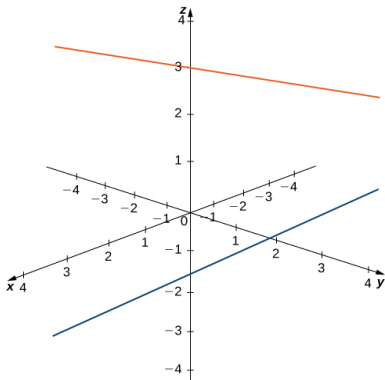
- Just like two-dimensional vectors, three-dimensional vectors are quantities with both magnitude and direction, and they are represented by directed line segments (arrows). With a three-dimensional vector, we use a three-dimensional arrow.

- Three-dimensional vectors can also be represented in component form. The notation $\vec{v} = \langle x, y, z \rangle$ is a natural extension of the two-dimensional case, representing a vector with the initial point at the origin, $(0, 0, 0)$, and terminal point (x, y, z).

- The zero vector is $\vec{0} = \langle 0, 0, 0 \rangle$.

- So, for example, the three dimensional vector $\vec{v} = \langle 2, 4, 1 \rangle$ is represented by a directed line segment from point $(0, 0, 0)$ to point $(2, 4, 1)$ (as in the figure in the next slide)
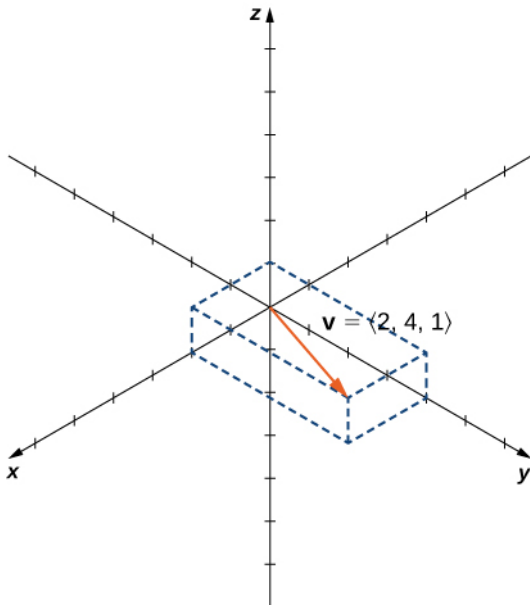
- Vector addition and scalar multiplication are defined analogously to the two-dimensional case.
- If $\vec{v} = \langle x_1, y_1, z_1 \rangle$ and $\vec{w} = \langle x_2, y_2, z_2 \rangle$ are vectors, and k is a scalar, then

$$\vec{v} + \vec{w} = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle \quad \text{and} \quad k\vec{v} = \langle kx_1, ky_1, kz_1 \rangle.$$

- If $k = -1$, then $k\vec{v} = (-1)\vec{v}$ is written as $-\vec{v}$, and vector subtraction is defined by
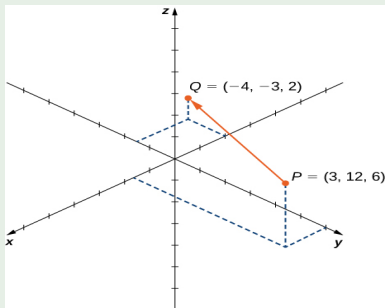
$$\vec{v} - \vec{w} = \vec{v} + (-\vec{w}) = \vec{v} + (-1)\vec{w}.$$

- The standard unit vectors extend easily into three dimensions as well – $\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$, and $k = \langle 0, 0, 1 \rangle$ – and we use them in the same way we used the standard unit vectors in two dimensions.
- Thus, we can represent a vector in $\mathbb{R}^3$ in the following ways:

$$\vec{v} = \langle x, y, z \rangle = x\vec{i} + y\vec{j} + z\vec{k}.$$

EXAMPLE

Let $\overrightarrow{PQ}$ be the vector with initial point $P = (3, 12, 6)$ and terminal point $Q = (-4, -3, 2)$ as shown in

Express $\overrightarrow{PQ}$ in both component form and using standard unit vectors.
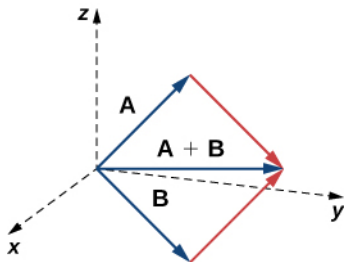
Solution

- In component form,

$$
\begin{aligned}
\overrightarrow{PQ} &= \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \\
&= \langle -4 - 3, -3 - 12, 2 - 6 \rangle \\
&= \langle -7, -15, -4 \rangle
\end{aligned}
$$

- In standard unit form,

$$
\overrightarrow{PQ} = -7\vec{i} - 15\vec{j} - 4\vec{k}.
$$

- As described earlier, vectors in three dimensions behave in the same way as vectors in a plane.
- The geometric interpretation of vector addition, for example, is the same in both two- and three-dimensional space

☺ We have already seen how some of the algebraic properties of vectors, such as vector addition and scalar multiplication, can be extended to three dimensions. Other properties can be extended in similar fashion. They are summarized in the next slide for our reference.

Working with Vectors in $\mathbb{R}^3$

Rule: Properties of vectors in space

Let $\vec{v} = \langle x_1, y_1, z_1 \rangle$ and $\vec{w} = \langle x_2, y_2, z_2 \rangle$ be vectors, and let k be a scalar.

- **Scalar multiplication**: $k\vec{v} = \langle kx_1, ky_1, kz_1 \rangle$

- **Vector addition**:
  $\vec{v} + \vec{w} = \langle x_1, y_1, z_1 \rangle + \langle x_2, y_2, z_2 \rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$

- **Vector subtraction**:
  $\vec{v} - \vec{w} = \langle x_1, y_1, z_1 \rangle - \langle x_2, y_2, z_2 \rangle = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle$

- **Vector magnitude**: $\|\vec{v}\| = \sqrt{x_1^2 + y_1^2 + z_1^2}$

- **Unit vector in the direction of $\vec{v}$**:

$$\frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\|\vec{v}\|}\langle x_1, y_1, z_1 \rangle = \langle \frac{x_1}{\|\vec{v}\|}, \frac{y_1}{\|\vec{v}\|}, \frac{z_1}{\|\vec{v}\|} \rangle, \ \ \text{if } \vec{v} \neq 0$$

- We have seen that vector addition in two dimensions satisfies the commutative, associative, and additive inverse properties.

- These properties of vector operations are valid for three-dimensional vectors as well.

- Scalar multiplication of vectors satisfies the distributive property, and the zero vector acts as an additive identity.

- The proofs to verify these properties in three dimensions are straightforward extensions of the proofs in two dimensions.

We have already learned how to add and subtract vectors. In this section, we investigate one type of vector multiplication. It is called the dot product.

DEFINITION

The **dot product** of vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is given by the sum of the products of the components

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3. \tag{1}$$

☺ Note that if $\vec{u}$ and $\vec{v}$ are two-dimensional vectors, we calculate the dot product in a similar fashion. Thus,

☺ the dot product of vectors $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$ is given by the sum of the products of the components

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2. \tag{2}$$

- ★ When two vectors are combined under addition or subtraction, the result is a vector.
- ★ When two vectors are combined using the dot product, the result is a scalar.
- ★ For this reason, the dot product is often called the scalar product. It may also be called the inner product.

THE DOT PRODUCT

EXAMPLE

1. Find the dot product of $\vec{u} = \langle 3, 5, 2 \rangle$ and $\vec{v} = \langle -1, 3, 0 \rangle$.
2. Find the scalar product of $\vec{p} = 10\vec{i} - 4\vec{j} + 7\vec{k}$ and $\vec{q} = -2\vec{i} + \vec{j} + 6\vec{k}$.

SOLUTION:

1. Substitute the vector components into the formula for the dot product:

$$\begin{aligned} \vec{u} \cdot \vec{v} &= u_1 v_1 + u_2 v_2 + u_3 v_3 = 3(-1) + 5(3) + 2(0) \\ &= -3 + 15 + 0 = 12. \end{aligned}$$

2. The calculation is the same if the vectors are written using standard unit vectors. We still have three components for each vector to substitute into the formula for the dot product:

$$\begin{aligned} \vec{p} \cdot \vec{q} &= u_1 v_1 + u_2 v_2 + u_3 v_3 \\ &= 10(-2) + (-4)(1) + (7)(6) = -20 - 4 + 42 = 18. \end{aligned}$$

Like vector addition and subtraction, the dot product has several algebraic properties. We prove three of these properties and leave the rest as exercises.

Theorem

Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors, and let c be a scalar.

1. **Commutative property:** $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. **Distributive property:** $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
3. **Associative property:** $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$
4. **Property of magnitude:** $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

Proof: Commutative property

Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$. Then

$$
\begin{aligned}
\vec{u} \cdot \vec{v} &= \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\
&= u_1 v_1 + u_2 v_2 + u_3 v_3 = v_1 u_1 + v_2 u_2 + v_3 u_3 \\
&= \langle v_1, v_2, v_3 \rangle \cdot \langle u_1, u_2, u_3 \rangle \\
&= \vec{v} \cdot \vec{u}.
\end{aligned}
$$

PROPERTIES OF THE DOT PRODUCT

PROOF: ASSOCIATIVE PROPERTY

Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$. The associative property looks like the associative property for real-number multiplication, but pay close attention to the difference between scalar and vector objects:

$$
\begin{aligned}
c(\vec{u} \cdot \vec{v}) &= c(u_1 v_1 + u_2 v_2 + u_3 v_3) \\
&= c(u_1 v_1) + c(u_2 v_2) + c(u_3 v_3) \\
&= (cu_1)v_1 + (cu_2)v_2 + (cu_3)v_3 \\
&= \langle cu_1, cu_2, cu_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\
&= c\langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\
&= (c\vec{u}) \cdot \vec{v}.
\end{aligned}
$$

The proof that $c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$ is similar.

Proof: Property of magnitude

Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$. The property of magnitude shows the relationship between the magnitude of a vector and its dot product with itself:

$$
\begin{aligned}
\vec{v} \cdot \vec{v}) &= \langle v_1, v_2, v_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\
&= v_1 v_1 + v_2 v_2 + v_3 v_3 \\
&= v_1^2 + v_2^2 + v_3^2 \\
&= \left(\sqrt{v_1^2 + v_2^2 + v_3^2} \right)^2 \\
&= \|\vec{v}\|^2.
\end{aligned}
$$

- Note that the definition of the dot product yields $0 \cdot \vec{v} = 0$. By property 4., if $\vec{v} \cdot \vec{v} = 0$, then $\vec{v} = 0$.

EXAMPLE

Let $\vec{a} = \langle 1, 2, -3 \rangle$, $\vec{b} = \langle 0, 2, 4 \rangle$, and $\vec{c} = \langle 5, -1, 3 \rangle$. Find each of the following products.

1. $(\vec{a} \cdot \vec{b})\vec{c}$
2. $\vec{a} \cdot (2\vec{c})$
3. $\|\vec{b}\|^2$

SOLUTION

1. Note that this expression asks for the scalar multiple of $\vec{c}$ by $\vec{a} \cdot \vec{b}$:

$$
\begin{aligned}
(\vec{a} \cdot \vec{b})\vec{c} &= (\langle 1, 2, -3 \rangle \cdot \langle 0, 2, 4 \rangle)\langle 5, -1, 3 \rangle \\
&= (1(0) + 2(2) + (-3)(4))\langle 5, -1, 3 \rangle \\
&= -8\langle 5, -1, 3 \rangle \\
&= \langle -40, 8, -24 \rangle.
\end{aligned}
$$

2. This expression is a dot product of vector $\vec{a}$ and scalar multiple 2c:

$$
\begin{aligned}
\vec{a} \cdot (2\vec{c}) &= 2(\vec{a} \cdot \vec{c}) \\
&= 2(\langle 1, 2, -3 \rangle \cdot \langle 5, -1, 3 \rangle) \\
&= 2(1(5) + 2(-1) + (-3)(3)) \\
&= 2(-6) \\
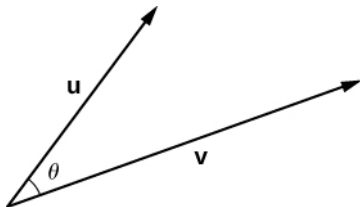&= -12.
\end{aligned}
$$

SOLUTION

3. Simplifying this expression is a straightforward application of the dot product:

$$\|\vec{b}\|^2 = \vec{b} \cdot \vec{b} = \langle 0, 2, 4 \rangle \cdot \langle 0, 2, 4 \rangle = 0^2 + 2^2 + 4^2 = 0 + 4 + 16 = 20.$$

☺ When two nonzero vectors are placed in standard position, whether in two dimensions or three dimensions, they form an angle (let's say θ) between them ($0 \leq \theta \leq \pi$)

☺ The dot product provides a way to find the measure of this angle.

☺ This property is a result of the fact that we can express the dot product in terms of the cosine of the angle formed by two vectors.

THEOREM

The dot product of two vectors is the product of the magnitude of each vector and the cosine of the angle between them:
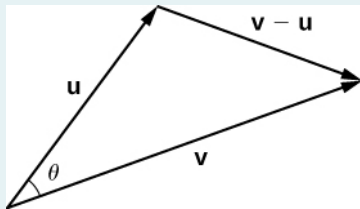
$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos \theta \tag{1}$$

Proof

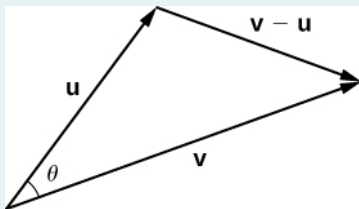Place vectors $\vec{u}$ and $\vec{v}$ in standard position and consider the vector $\vec{v} - \vec{u}$

These three vectors form a triangle with side lengths $\|\vec{u}\|$, $\|\vec{v}\|$, and $\|\vec{v} - \vec{u}\|$.

PROOF

Recall from trigonometry that the law of cosines describes the relationship among the side lengths of the triangle and the angle θ. Applying the law of cosines here gives

$$\|\vec{v} - \vec{u}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\| \, \|\vec{v}\| \cos \theta.$$

The dot product provides a way to rewrite the left side of this equation:

PROOF

Recall from trigonometry that the law of cosines describes the relationship among the side lengths of the triangle and the angle θ. Applying the law of cosines here gives

$$\|\vec{v} - \vec{u}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\,\|\vec{v}\|\cos\theta.$$

The dot product provides a way to rewrite the left side of this equation:

$$\begin{aligned}
\|\vec{v} - \vec{u}\|^2 &= (\vec{v} - \vec{u}) \cdot (\vec{v} - \vec{u}) \\
&= (\vec{v} - \vec{u}) \cdot \vec{v} - (\vec{v} - \vec{u}) \cdot \vec{u} \\
&= \vec{v} \cdot \vec{v} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} + \vec{u} \cdot \vec{u} \\
&= \vec{v} \cdot \vec{v} - \vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{u} \\
&= \|\vec{v}\|^2 - 2\vec{u} \cdot \vec{v} + \|\vec{u}\|^2.
\end{aligned}$$

Proof

$$\begin{aligned}
\|\vec{v} - \vec{u}\|^2 &= (\vec{v} - \vec{u}) \cdot (\vec{v} - \vec{u}) \\
&= (\vec{v} - \vec{u}) \cdot \vec{v} - (\vec{v} - \vec{u}) \cdot \vec{u} \\
&= \vec{v} \cdot \vec{v} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} + \vec{u} \cdot \vec{u} \\
&= \vec{v} \cdot \vec{v} - \vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{u} \\
&= \|\vec{v}\|^2 - 2\vec{u} \cdot \vec{v} + \|\vec{u}\|^2.
\end{aligned}$$

Substituting into the law of cosines yields

$$\begin{aligned}
\|\vec{v} - \vec{u}\|^2 &= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\,\|\vec{v}\| \cos\theta \\
\|\vec{v}\|^2 - 2\vec{u} \cdot \vec{v} + \|\vec{u}\|^2 &= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\,\|\vec{v}\| \cos\theta \\
-2\vec{u} \cdot \vec{v} &= -2\|\vec{u}\|\,\|\vec{v}\| \cos\theta \\
\vec{u} \cdot \vec{v} &= \|\vec{u}\|\,\|\vec{v}\| \cos\theta
\end{aligned}$$

(end of proof)

* We can use this form of the dot product to find the measure of the angle between two nonzero vectors:

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|} \tag{1}$$

* Using this equation, we can find the cosine of the angle between two nonzero vectors.

* Since we are considering the smallest angle between the vectors, we assume $0° \leq \theta \leq 180°$ (or $0 \leq \theta \leq \pi$ if we are working in radians).

* The inverse cosine is unique over this range, so we are then able to determine the measure of the angle θ.

EXAMPLE

Find the measure of the angle between each pair of vectors.

1. $\vec{i} + \vec{j} + \vec{k}$ and $2\vec{i} - \vec{j} - 3\vec{k}$
2. $\langle 2, 5, 6 \rangle$ and $\langle -2, -4, 4 \rangle$

SOLUTION

1. To find the cosine of the angle formed by the two vectors, substitute the components of the vectors into the equation $\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$:

$$
\begin{aligned}
\cos\theta &= \frac{(\vec{i} + \vec{j} + \vec{k}) \cdot (2\vec{i} - \vec{j} - 3\vec{k})}{\|\vec{i} + \vec{j} + \vec{k}\| \cdot \|2\vec{i} - \vec{j} - 3\vec{k}\|} \\[2mm]
&= \frac{1(2) + 1(-1) + 1(-3)}{\sqrt{1^2 + 1^2 + 1^2}\,\sqrt{2^2 + (-1)^2 + (-3)^2}} \\[2mm]
&= \frac{-2}{\sqrt{3}\sqrt{14}} = \frac{-2}{\sqrt{42}}
\end{aligned}
$$

Therefore,

$$
\boxed{\theta = \arccos\frac{-2}{\sqrt{42}}\text{rad}}.
$$

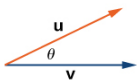### SOLUTION

2. Start by finding the value of the cosine of the angle between the vectors:
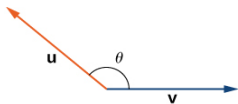
$$\cos \theta = \frac{\langle 2, 5, 6 \rangle \cdot \langle -2, -4, 4 \rangle}{\|\langle 2, 5, 6 \rangle\| \cdot \|\langle -2, -4, 4 \rangle\|}$$

$$= \frac{2(-2) + 5(-4) + 6(4)}{\sqrt{2^2 + 5^2 + 6^2}\sqrt{(-2)^2 + (-4)^2 + (4)^2}}$$

$$= \frac{0}{\sqrt{65}\sqrt{36}} = 0$$

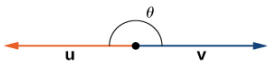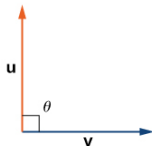Now, $\cos \theta = 0$ and $0 \leq \theta \leq \pi$, so $\theta = \pi/2$.

(a) An **acute angle** has $0 < \cos\theta < 1$. (b) An **obtuse angle** has $-1 < \cos\theta < 0$. (c) A **straight line** has $\cos\theta = -1$. (d) If the vectors have the same direction, $\cos\theta = 1$. (e) If the vectors are orthogonal (perpendicular), $\cos\theta = 0$.

THEOREM

The nonzero vectors $\vec{u}$ and $\vec{v}$ are orthogonal vectors if and only if $\vec{u} \cdot \vec{v} = 0$.

PROOF

- Let $\vec{u}$ and $\vec{v}$ be nonzero vectors, and let θ denote the angle between them. First, assume $\vec{u} \cdot \vec{v} = 0$. Then

$$\|\vec{u}\| \, \|\vec{v}\| \, \cos\theta = 0$$

However, $\|\vec{u}\| \neq 0$ and $\|\vec{v}\| \neq 0$, so we must have $\cos\theta = 0$. Hence, $\theta = 90°$, and the vectors are orthogonal.

- Now assume $\vec{u}$ and $\vec{v}$ are orthogonal. Then $\theta = 90°$ and we have

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \, \cos\theta = \|\vec{u}\| \, \|\vec{v}\| \, \cos 90° = \|\vec{u}\| \, \|\vec{v}\| \, (0) = 0$$

(end of proof)

- ☺ The terms *orthogonal*, *perpendicular*, and *normal* each indicate that mathematical objects are intersecting at right angles.

- ☺ The use of each term is determined mainly by its context.

- ☺ We say that vectors are orthogonal and lines are perpendicular. The term normal is used most often when measuring the angle made with a plane or other surface.
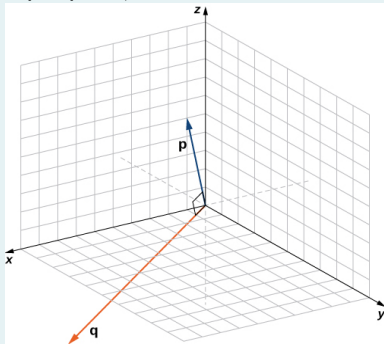
EXAMPLE

Determine whether $\vec{p} = \langle 1, 0, 5 \rangle$ and $\vec{q} = \langle 10, 3, -2 \rangle$ are orthogonal vectors.

SOLUTION

Using the definition, we need only check the dot product of the vectors:

$$\vec{p} \cdot \vec{q} = 1(10) + (0)(3) + (5)(-2) = 10 + 0 - 10 = 0.$$

Because $\vec{p} \cdot \vec{q} = 0$, the vectors are orthogonal

EXAMPLE

Let $\vec{v} = \langle 2, 3, 3 \rangle$. Find the measures of the angles formed by the following vectors.

1. $\vec{v}$ and $\vec{i}$
2. $\vec{v}$ and $\vec{j}$
3. $\vec{v}$ and $\vec{k}$

SOLUTION

1. Let α be the angle formed by $\vec{v}$ and $\vec{i}$:

$$\cos\alpha = \frac{\vec{v} \cdot \vec{i}}{\|\vec{v}\|\,\|\vec{i}\|}$$

$$= \frac{\langle 2,3,3\rangle \cdot \langle 1,0,0\rangle}{\|\langle 2,3,3\rangle\|\,\|\langle 1,0,0\rangle\|}$$

$$= \frac{2}{\sqrt{2^2+3^2+3^2}\,\sqrt{1^2+0^2+0^2}} = \frac{2}{\sqrt{22}}$$

$$\alpha = \arccos\frac{2}{\sqrt{22}} \approx 1.130\mathrm{rad}$$

SOLUTION

2. *Let β be the angle formed by $\vec{v}$ and $\vec{j}$:*

$$\begin{aligned}
\cos\beta &= \frac{\vec{v} \cdot \vec{j}}{\|\vec{v}\|\,\|\vec{j}\|} \\
&= \frac{\langle 2,3,3\rangle \cdot \langle 0,1,0\rangle}{\|\langle 2,3,3\rangle\|\,\|\langle 0,1,0\rangle\|} \\
&= \frac{3}{\sqrt{2^2+3^2+3^2}\,\sqrt{0^2+1^2+0^2}} = \frac{3}{\sqrt{22}}
\end{aligned}$$

$$\beta = \arccos\frac{3}{\sqrt{22}} \approx 0.877\mathrm{rad}$$

SOLUTION

2. *Let γ be the angle formed by $\vec{v}$ and $\vec{k}$:*

$$
\begin{aligned}
\cos\gamma &= \frac{\vec{v} \cdot \vec{k}}{\|\vec{v}\|\,\|\vec{k}\|} \\
&= \frac{\langle 2,3,3 \rangle \cdot \langle 0,0,1 \rangle}{\|\langle 2,3,3 \rangle\|\,\|\langle 0,0,1 \rangle\|} \\
&= \frac{3}{\sqrt{2^2 + 3^2 + 3^2}\,\sqrt{0^2 + 0^2 + 1^2}} = \frac{3}{\sqrt{22}}
\end{aligned}
$$

$$
\gamma = \arccos\frac{3}{\sqrt{22}} \approx 0.877\mathrm{rad}
$$

- We recall that any vector $\vec{w} = \begin{pmatrix} x \\ y \end{pmatrix}$ in the flat coordinate plane can be written as

$$\vec{w} = \begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = x\vec{i} + y\vec{j}$$

- In this case, we say that $\vec{w}$ is a linear combination of the vectors $\vec{i}$ and $\vec{j}$. Since any vector in $\mathbb{R}^2$ can be written as linear combination of $\vec{i}$ and $\vec{j}$, then we say that $\{\vec{i}, \vec{j}\}$ spans $\mathbb{R}^2$.

- This property, along with the fact that $\vec{i}$ and $\vec{j}$ are not collinear (we say they are linearly independent), make of the couple $(\vec{i}, \vec{j})$ a so-called **basis** of the two dimensional space, but more precisely, its **the canonical basis**.

- In this case, we say that x and y are the components of the vector $\vec{w} = \begin{pmatrix} x \\ y \end{pmatrix}$ in the canonical basis $(\vec{i}, \vec{j})$.

- In general, if we chose any two vectors in the two dimensional space <u>that are not collinear</u>, these two vectors form <u>a basis</u>.

- This means, in the (x, y)-plane, if two vectors $\vec{u}$ and $\vec{v}$ are not collinear, then they form a basis, that is, <u>any</u> vector $\vec{w}$ can be written as a linear combination of $\vec{u}$ and $\vec{v}$.

- This means, that we can find two scalars c and d such that $\vec{w} = c\,\vec{u} + d\,\vec{v}$.

- In this case, we say c and d are the components (or coordinates) of $\vec{w}$ in the basis $(\vec{u}, \vec{v})$.
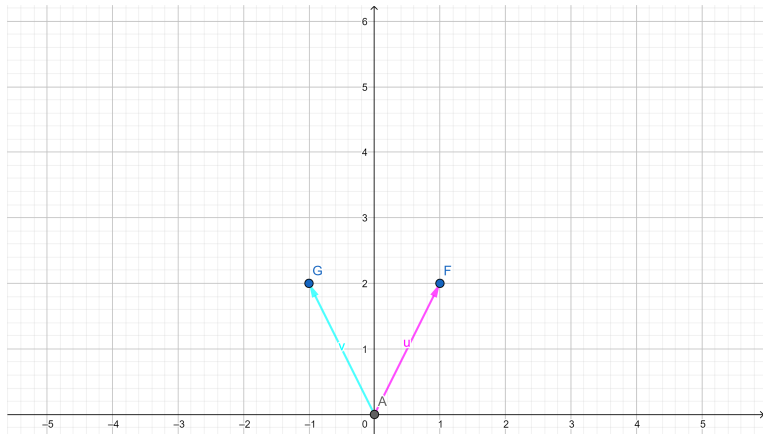
- For example, if we consider the non-collinear vectors
  $\vec{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, we know they form a basis.

- In this case,

$$\begin{pmatrix} 1 \\ 6 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 2\vec{u} + \vec{v}$$

- In this case, we can say that the vector $\begin{pmatrix} 1 \\ 6 \end{pmatrix}$:

  1. Is of components $(1, 6)$ in the canonical basis
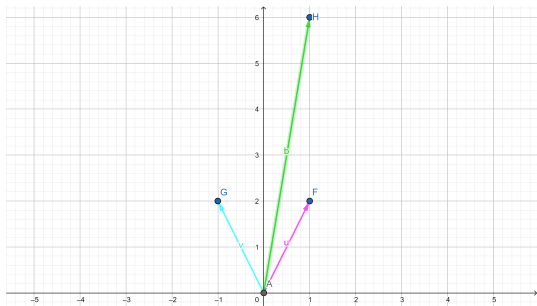 2. Is of components $(2, 1)$ in the basis $(\vec{u}, \vec{v})$.

- For example, if we consider the non-collinear vectors $\vec{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, we know they form a basis.
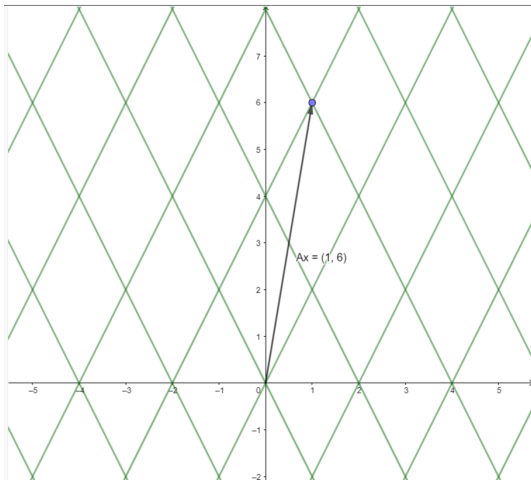
- For example, if we consider the non-collinear vectors $\vec{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, we know they form a basis.

- In this case,

$$\begin{pmatrix} 1 \\ 6 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1\begin{pmatrix} -1 \\ 2 \end{pmatrix} = 2\vec{u} + \vec{v}$$

- In this case, we can say that the vector $\begin{pmatrix} 1 \\ 6 \end{pmatrix}$:

  ❶ Is of coordinates $(1, 6)$ in the canonical basis
  ❷ Is of coordinates $(2, 1)$ in the basis $(\vec{u}, \vec{v})$.

Ax = (1, 6)

LINEAR COMBINATION, LINEAR DEPENDENCE, LINEAR INDEPENDENCE

Linear Combinations:

DEFINITION

- Let $c_1, c_2, \ldots, c_k$ be scalars, and let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be vectors in $\mathbb{R}^n$, i.e., vectors of n components in $\mathbb{R}$.

- The vector in $\mathbb{R}^n$

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_k \vec{v}_k$$

is called a linear combination of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$, with weights or coefficients $c_1, c_2, \ldots, c_k$.

☺ Geometrically, a linear combination is obtained by stretching / shrinking the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ according to the coefficients, then adding them together using the parallelogram method.

EXAMPLE

Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $\vec{v}_3 = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$

1. Propose 3 different linear combinations of $\vec{v}_1$ and $\vec{v}_2$.
2. Propose 3 different linear combinations of $\vec{v}_1$.
3. Propose 3 different linear combinations of $\vec{v}_1$ and $\vec{v}_3$.

Linear Combination, Linear Dependence, Linear Independence

Spans:

Definition

- Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be vectors in $\mathbb{R}^n$. The **span** of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ is the collection of all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$, and is denoted $\mathrm{Span}\{\vec{v}_1, v_2, \ldots, \vec{v}_k\}$.

- In this case, we say that a vector $\vec{b}$ is **in the span of** $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ if and only if $\vec{b}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$.

Linear Combination, Linear Dependence, Linear Independence

Definition

- A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is **linearly dependent** if there exist numbers $x_1, x_2, \ldots, xk$, not all equal to zero, such that

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + c_k \vec{v}_k = 0,$$

that is, one special linear combination of $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ "will cancel out everything".

- Otherwise, the set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is **linearly independent**, that is,

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + c_k \vec{v}_k = 0 \iff x_1 = x_2 = \cdots = x_k = 0$$

- In chapter 3, we will learn how to check whether a set of vectors is linearly independent or not.

Linear Combination, Linear Dependence, Linear Independence

Definition

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be vectors in $\mathbb{R}^n$.

The set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is said to be a basis of $\mathbb{R}^n$ if

1. $\mathbb{R}^n = \operatorname{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$,

2. $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is linearly independent.

In this case, for any vector $\vec{v}$ in $\mathbb{R}^n$, we can find a set of scalars $\{x_1, x_2, \ldots, x_n\}$ such that

$$\vec{v} = x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n.$$

We call $x_1, x_2, \ldots, x_n$ the components (or coordinates) of $\vec{v}$ in the basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$.

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