Precalculus advanced

Chapter 2:

Introduction to matrices

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Outline

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Definition

- A **matrix** (plural matrices) is a rectangular array or table of numbers, symbols, or expressions, arranged in rows and columns, which is used to represent a mathematical object or a property of such an object.
- The **dimensions** of a matrix tells its size: the number of rows and columns of the matrix, in that order.

For example:

$$\begin{pmatrix} 2 & 3 & 5 \\ 4 & 3 & 2 \end{pmatrix}$$

is a matrix with two rows and three columns. This is often referred to as a "two by three matrix", a "2×3-matrix", or a matrix of dimension 2×3.

Matrix Elements:

- A **matrix element** is simply a matrix entry. Each element in a matrix is identified by naming the row and column in which it appears, i.e., each entry is referred to as $a_{i,j}$, such that i represents the row and j represents the column.

For example, consider the matrix M:

$$\begin{pmatrix} 24 & 15 & -6 \\ 31 & -5 & 78 \\ 1 & -1 & 2 \end{pmatrix}$$

The element $a_{2,1}$ is the entry in the second row and the first column.

$$\begin{pmatrix} 24 & 15 & -6 \\ 31 & -5 & 78 \\ 1 & -1 & 2 \end{pmatrix}$$

In this case, $a_{2,1} = 31$.

As long as the dimensions of two matrices are the same, we can add and subtract them much like we add and subtract numbers.

Let's take a closer look!

Adding matrices:

- Given $A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}$, let's find $A + B$.
- We can find the sum simply by adding the corresponding entries in matrices A and B:

$$\begin{aligned} A + B &= \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix} \\ &= \begin{pmatrix} 1+3 & 2+4 \\ 3+5 & 0+6 \\ 4+7 & 3+8 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 6 \\ 8 & 6 \\ 11 & 11 \end{pmatrix} \end{aligned}$$

Subtracting matrices:

- Given $A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}$, let's find $A - B$.
- Similarly, we can find the $A - B$ simply by subtracting the corresponding entries in matrices A and B:

$$\begin{aligned} A - B &= \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix} \\ &= \begin{pmatrix} 1-3 & 2-4 \\ 3-5 & 0-6 \\ 4-7 & 3-8 \end{pmatrix} \\ &= \begin{pmatrix} -2 & -2 \\ -2 & -6 \\ -3 & -5 \end{pmatrix} \end{aligned}$$

Adding and subtracting matrices

- Given matrices A and B of like dimensions, addition and subtraction of A and B will produce matrix C or matrix D of the same dimension.

$$A + B = C \text{ such that } a_{i,j} + b_{i,j} = c_{i,j}$$

$$A - B = D \text{ such that } a_{i,j} - b_{i,j} = d_{i,j}$$

- Matrix addition is commutative.

$$A + B = B + A$$

- It is also associative.

$$(A + B) + C = A + (B + C)$$

Multiplying matrices by scalars

Scalar multiplication:

- Given $A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 3 \end{pmatrix}$, consider the scalar 3 and let's find $3A$.
- This scalar multiplication can be seen as repeated addition: $3A = A + A + A$
- In this case, we have

$$
\begin{aligned}
3A = A + A + A &= \begin{pmatrix} 1+1+1 & 2+2+2 \\ 3+3+3 & 0+0+0 \\ 4+4+4 & 3+3+3 \end{pmatrix} \\
&= \begin{pmatrix} 3\cdot 1 & 3\cdot 2 \\ 3\cdot 3 & 3\cdot 0 \\ 3\cdot 4 & 3\cdot 3 \end{pmatrix}
\end{aligned}
$$

- In general, in scalar multiplication, each entry in the matrix is multiplied by the given scalar.

Scalar multiplication

- Scalar multiplication involves finding the product of a constant by each entry in the matrix. Given

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$$

the scalar multiple cA is

$$cA = c\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = \begin{pmatrix} ca_{1,1} & ca_{1,2} \\ ca_{2,1} & ca_{2,2} \end{pmatrix}$$

- Scalar multiplication is distributive. For the matrices A,B, and C with scalars a and b,

$$a(A+B) = aA + aB, \quad (a+b)A = aA + bA$$

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \end{pmatrix}$$

We recall that

- A is a 2×3 matrix.
- The element $a_{2,1}$ is the entry in the second row and the first column of matrix A, that is $a_{2,1} = 2$.

How to find the product of two matrices? For example, find

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix}$$

- Up until now, you may have found operations with matrices fairly intuitive. For example
  - when you add two matrices, you add the corresponding entries,
  - in scalar multiplication, each entry in the matrix is multiplied by the given scalar.
- But things do not work as you'd expect them to work with multiplication. To multiply two matrices, we **cannot** simply multiply the corresponding entries.

Matrices and vectors:

- When multiplying matrices, it's useful to think of each matrix row and column as a vector.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

- In this matrix, denote
  1. row 1 by $\vec{r}_1 = \begin{pmatrix} 1 & 2 \end{pmatrix}$
  2. row 2 by $\vec{r}_2 = \begin{pmatrix} 3 & 4 \end{pmatrix}$
  3. column 1 by $\vec{c}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$
  4. column 2 by $\vec{c}_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

Matrix multiplication: The entry in the product matrix located in the i^{th} row and j^{th} column, is the dot product of the i^{th} row in the first matrix and the j^{th} column in the second matrix. For instance,

given $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix}$. Let's find $C = AB$

- denote
  1. row 1 of A by $\vec{a_1}$,
  2. row 2 of A by $\vec{a_2}$,
  3. column 1 of B by $\vec{b_1}$,
  4. column 2 of B by $\vec{b_2}$.
- then we have

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} \vec{a_1} \cdot \vec{b_1} & \vec{a_1} \cdot \vec{b_2} \\ \vec{a_2} \cdot \vec{b_1} & \vec{a_2} \cdot \vec{b_2} \end{pmatrix}$$

Matrix multiplication:

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 3 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ \begin{pmatrix} 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 3 \end{pmatrix} & \begin{pmatrix} 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix}$$

Matrix multiplication:

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 3 \end{pmatrix} & \begin{pmatrix} 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ \begin{pmatrix} 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 3 \end{pmatrix} & \begin{pmatrix} 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \times (-3) + 2 \times 3 & 1 \times 1 + 2 \times (-1) \\ 2 \times (-3) + 4 \times 3 & 2 \times 1 + 4 \times (-1) \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix}$$

Generally speaking, in matrix multiplication, the entry in the product matrix located in the i^{th} row and j^{th} column, is the dot product of the i^{th} row in the first matrix and the j^{th} column in the second matrix.

"But when are we allowed to multiply two matrices?"
"What are the properties of this operation?"

1. **When is matrix multiplication defined?** In order for matrix multiplication to be defined, the number of columns in the first matrix must be equal to the number of rows in the second matrix.

$$(m \times \underbrace{n) \cdot (n}_{\text{product is defined}} \times k)$$

2. **What about dimensions the obtained matrix product?** When the matrix multiplication is defined, then the resulting matrix product has the number of lines of the first matrix and the number of comumns of the second matrix.

$$(m \times \underbrace{n) \cdot (n}_{\text{product is defined}} \times k) \quad = \quad (m \times k)$$

- A matrix that has the same number of rows and columns is called **square matrix**.
- The entries of a matrix that lie on the i^{th} row and the i^{th} column form the so-called **diagonal** of a matrix. For example, the diagonal of the following matrix is given by the blue entries

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

- The **square matrix** where the entries on the diagonal from the upper left to the bottom right are all 1's, and all other entries are 0 is called **identity matrix**, and is denoted by I_n where n is the number of rows (and columns) of the matrix. For example

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- The product of any square matrix and the appropriate identity matrix is always the original matrix, regardless of the order in which the multiplication was performed!
- In other words, for a square matrix A we have

$$A \cdot I = I \cdot A = A.$$

- Let A, B, and C be $(n \times n)$ matrices and I the $(n \times n)$ identity matrix. then we have:
  1. $AB \neq BA$ (Check it!)
  2. $(AB)C = A(BC)$
  3. $A(B + C) = AB + AC$
  4. $(B + C)A = BA + CA$
- If $AB = BA = I$, then we say that A is the **inverse** of B (or even B is the **inverse** of A)

Example

When possible, multiply matrix A and matrix B (compute AB and BA),

1. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$
2. $A = \begin{pmatrix} 1 & -2 \\ 2 & 4 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 6 & 0 \\ 7 & 8 & 1 \end{pmatrix}$
3. $A = \begin{pmatrix} 3 & -1 \\ -2 & 0 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix}$
4. $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 0 & 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$

Determinant of a 2×2 matrix

- The determinant is a special number that can be calculated from the entries of a matrix. The matrix has to be square (same number of rows and columns).
- The determinant of a (2×2) matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $|A| = ad - cb$. It is simply obtained by cross multiplying the elements starting from the top left, then subtracting the products.
- For example, if $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then the determinant $|A| = 1 \times 4 - 3 \times 2 = 4 - 6 = -2$.

- To evaluate the determinant of a 3×3 matrix, we must be able to evaluate the minor of an entry in the determinant.
- The minor of an entry is the 2×2 determinant found by eliminating the row and column in the 3×3 determinant that contains the entry.
- For example, to find the minor of entry a_1, we eliminate the row and column which contain it. So, we eliminate the first row and first column. Then we write the 2×2 determinant that remains.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{minor of } a_1 \quad \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

- To find the minor of entry b_2, we eliminate the row and column that contain it. So, we eliminate the second row and second column. Then we write the 2×2 determinant that remains.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{minor of } b_2 \quad \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$$

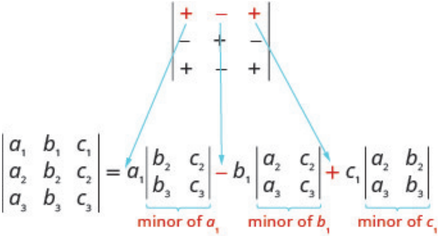
Strategy for evaluating the determinant of a 3×3 matrix:

- To evaluate a 3×3 determinant we can expand by minors using any row or column. Choosing a row or column other than the first row sometimes makes the work easier.
- When we expand by any row or column, we must be careful about the sign of the terms in the expansion. To determine the sign of the terms, we use the following sign pattern chart.

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Expanding by minors along the first row to evaluate a 3×3 determinant.

- To evaluate a 3×3 determinant by expanding by minors along the first row, we use the following pattern:



NOTE: We can evaluate the determinant of a matrix by expanding minors along any row or column. When a row or a column has a zero entry, expanding by that row or column results in less calculations.

Example

Compute the determinant of

1. $A = \begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix}$
2. $B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \\ -2 & 0 & 2 \end{pmatrix}$

- We know that the multiplicative inverse of a real number a is a^{-1}, and $aa^{-1} = a^{-1}a = (\frac{1}{a})a = 1$.
- For example, $2^{-1} = \frac{1}{2}$ and $(\frac{1}{2})2 = 1$.
- The multiplicative inverse of a matrix is similar in concept, except that the product of matrix A and its inverse A^{-1} equals the identity matrix.
- We recall that the identity matrix is a square matrix containing ones down the main diagonal and zeros everywhere else. We identify identity matrices by I_n where n represents the dimension of the matrix.
- A matrix that has a multiplicative inverse is called an invertible matrix.
- Only a square matrix may have a multiplicative inverse, as the reversibility, $AA^{-1} = A^{-1}A = I$, is a requirement.
- Not all square matrices have an inverse, but if A is invertible, then A^{-1} is unique.

- So far we have defined the inverse matrix without giving any strategy for computing it. We do so now, beginning with the special case of 2×2 matrices. Then we will give a recipe for the $n \times n$ case in the future chapter.

Proposition

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

- If $\det(A) \neq 0$, then A is invertible, and

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Proof

- Suppose that $\det(A) \neq 0$.
- Define $B = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
- Then

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = I_2$$

- We can check as well that $BA = I_2$, so A is invertible and $B = A^{-1}$

EXAMPLE

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $a, b \in \mathbb{R}$.

1. Compute $A\begin{pmatrix} a \\ b \end{pmatrix}$
2. Compute $\det(A)$.
3. Verify if A is invertible and, if so, compute A^{-1}.
4. Deduce the values of a and b such that $A\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Kuttler, K.,
Elementary linear algebra. The Saylor Foundation, 2012.

Strang, G.,
Linear algebra and its applications. Belmont, CA: Thomson, Brooks/Cole, 2006.

OpenStax,
Calculus Volumes 1, 2, and 3

Margalit, D., Rabinoff, J. and Rolen, L.,
"Interactive linear algebra." Georgia Institute of Technology (2017).