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# Lorentz Invariance and Free-Particle Equations

**Introduction:** Lorentz invariance is a fundamental symmetry in special relativity and modern physics. It requires that the form of physical laws remain the same in all inertial frames moving at constant relative velocity. In practice, this means quantities like the spacetime interval and the energy–momentum relation are *invariant* (unchanged) under Lorentz boosts. For example, the combination $E^2 - (p\,c)^2$ remains equal to $(m_0 c^2)^2$ in every frame, defining the particle's rest mass $m_0$. Lorentz symmetry underlies the fact that the speed of light is constant in all frames, and it has been tested to extraordinary precision in experiments. (For instance, current analyses show CPT/Lorentz invariance holds to better than one part in $10^{20}$ [1].) This report reviews how Lorentz invariance is built into free-particle equations, the associated mathematics, and what this implies for motion and field equations.

## Basics of Lorentz Transformations and Invariants

- **Lorentz Transformations:** Coordinates $(t, x, y, z)$ and energy–momentum $(E, p_x, p_y, p_z)$ between two inertial frames are related by Lorentz boosts so that the spacetime interval $c^2t^2 - x^2 - y^2 - z^2$ and the 4-momentum invariant $E^2 - (p\,c)^2$ remain constant. In practical terms, a boost with velocity $v$ mixes time and space (e.g. $t' = \gamma(t - vx/c^2)$, $x' = \gamma(x - vt)$) and similarly mixes energy and momentum. This ensures formulas like $E^2 = p^2c^2 + (m_0c^2)^2$ are valid in every frame.
- **Invariance of the Speed of Light:** Under a Lorentz boost, observers always measure light in vacuum traveling at speed $c$. Physically, changing frames alters the observed momenta of particles (impulses) but leaves $c$ fixed. In fact, one can show the pair $(p\,c, E)$ transforms under Lorentz boosts exactly like $(x, ct)$ [2]. This symmetry means that even though different frames disagree about a particle's momentum, the relation $E = pc$ for photons (massless particles) remains true everywhere.
- **Key Invariant:** The free-particle energy–momentum invariant $E^2 - (p\,c)^2 = (m_0c^2)^2$ follows from these transformation rules. It sets the particle's rest mass $m_0$ and is unchanged by boosts. In other words, all observers agree on the value of $m_0$.

## Lorentz-Invariant Action and Free-Particle Motion

Physically, a free particle (no forces) should move in a straight line at constant velocity. This geodesic motion in flat spacetime follows from a Lorentz-invariant action or Lagrangian. One can show [3]:
- The **relativistic Lagrangian** for a free particle of rest mass $m_0$ is $L = -m_0c^2\sqrt{1 - v^2/c^2}$ (with $v = dx/dt$), whose action $S = L\,t$ can be rewritten as the invariant dot-product $-E\,t + \mathbf{p}\cdot\mathbf{x}$. Here $E$ and $\mathbf{p}$ are the particle's energy and momentum, and $\mathbf{x}$ and $t$ are position and time taken as independent variables. In fact, one finds

$$S = -E\,t + \mathbf{p}\cdot\mathbf{x},$$

which is explicitly Lorentz-invariant [3].
- **Geodesic (Straight-Line) Motion:** Applying Lagrange's equations to this action gives linear (inertial) motion. In special relativity, this means the particle follows a straight world-line (geodesic in flat spacetime)

at constant velocity [3]. In other words, enforcing the Lorentz-invariant action automatically yields the expected free-particle dynamics.
- **Momentum from Lagrangian:** In Lagrangian mechanics, momentum is $p = \partial L/\partial v$. With $A = L\,t$ as the action integral (treating $x, t$ independently), one finds $p = \partial A/\partial x$. This produces the term $\mathbf{p}\cdot\mathbf{x}$ in the action. Ruggeri notes that $A = Lt$ and $p = dA/dx$ make the $\mathbf{p}\cdot\mathbf{x}$ term "consistent with Lagrangian theory and a Lorentz invariant" [4]. Thus the dot-product form naturally emerges from the Lagrangian formulation.

## Lorentz-Invariant Quantum Wave Equations

In quantum mechanics and field theory, free particles are often described by plane-wave solutions. The fundamental building block is the wavefunction (or field)

$$\Psi(\mathbf{x}, t) = \exp(-iEt + i\,\mathbf{p}\cdot\mathbf{x}),$$

which we can call a "complex probability" amplitude. However, as Ruggeri emphasizes [5], this plane-wave alone does not fully encode all particle properties: it contains no information about the particle's rest mass or intrinsic spin, nor about whether it carries fields (like the photon's electromagnetic field). To capture the complete physics, one constructs a **Lorentz-scalar (invariant) equation** that these plane waves satisfy, analogous to the Dirac or Maxwell equations.

Key points for building such an equation include: - The operators $i\partial/\partial t$ and $-i\nabla$ acting on $\Psi$ yield the eigenvalues $E$ and $\mathbf{p}$. Any relativistic wave equation should use these operators. But to be Lorentz-invariant, the equation must couple time and space in a symmetric way. In other words, one needs an operator that forms a 4-vector dot-product with $-i\nabla$.
- **Adding Geometry via Matrices:** Ruggeri shows that to make a Lorentz-scalar equation from the plane-wave basis, one introduces a set of matrices that act on the spinor or vector components of the wavefunction [6]. In practical terms, this is exactly what is done in the Dirac equation for spin-½ particles or the Proca/Maxwell equations for spin-1 particles: one uses gamma matrices or tensor operators to form the invariant dot-products. These matrices inject the necessary "geometry" into the equation, relating the energy and momentum components $(E, p_x, p_y, p_z)$ in a covariant way. Ruggeri notes that this added structure is *hidden in how E and p components are interrelated* [6], and it allows the resulting wave equation to describe properties like rest mass and electromagnetic fields.
- **Plane-wave as Basis:** Formally, one postulates there is a Lorentz-scalar equation built from functions like $\exp(-iEt + i\mathbf{p}\cdot\mathbf{x})$ [5]. The standard relativistic wave equations (Klein-Gordon, Dirac, Maxwell) all fit this scheme: each plane wave is a solution characterized by a 4-momentum $(E, \mathbf{p})$ satisfying the invariant relation, and the full equation (with the introduced matrices) ensures Lorentz symmetry is maintained.

## Key Points and Takeaways

- **Invariance of Physical Laws:** Lorentz symmetry forces physical laws (and their mathematical equations) to take the same form in all inertial frames. This underlies both classical relativity and modern field theories.
- **Free-Particle Action:** The action $S = \int L\,dt$ for a free particle can be written in the invariant form $-Et + \mathbf{p}\cdot\mathbf{x}$. From this, one recovers geodesic (straight-line) motion and the standard energy–momentum relation [3] [4].

- **Quantum Equations:** The basic plane-wave $\exp(-iEt + i\mathbf{p} \cdot \mathbf{x})$ is an eigenstate of energy/momentum operators, but alone it lacks information about mass and spin. A Lorentz-invariant wave equation is obtained by extending this plane wave with matrix operators (as in Dirac/Maxwell theories) that respect Lorentz symmetry [6].
- **Experimental Confirmation:** Lorentz invariance is not just theoretical – it's experimentally verified to extremely high accuracy. Analyses of particle physics and astrophysical data find no violation down to better than one part in $10^{20}$ [1], meaning any Lorentz-violating effect must be smaller than current detection limits.
- **Unified Description:** Taken together, the Lorentz-invariant action for particles and the Lorentz-covariant wave equations provide a complete and consistent description of free particles and fields in relativistic physics. All observable predictions (trajectories, spectra, field behavior) follow from these principles and agree with experiments.

## Conclusion

Lorentz invariance deeply constrains the form of physical equations. For free particles, it leads to a simple but powerful structure: the invariant action $-Et + \mathbf{p} \cdot \mathbf{x}$ and straight-line (geodesic) motion [3] [4]. In quantum terms, it requires that plane-wave states be embedded in a Lorentz-symmetric equation, typically by introducing appropriate matrices or tensors [6]. These constructions ensure that concepts like energy, momentum, and mass are treated consistently across all inertial frames. Modern high-precision tests confirm that nature indeed honors these rules with extraordinary fidelity [1]. In summary, Lorentz invariance is a cornerstone of physics: it demands certain mathematical forms (geometric dot-products, invariant scalars) and yields the straight-line, constant-velocity behavior of free particles, as well as the form of fundamental wave equations, thereby unifying kinematics and dynamics in a single framework.

**Sources:** Key ideas and formulations are drawn from recent preprints and summaries on Lorentz-invariant equations and free-particle Lagrangians [3] [4] [5] [6] [1], which explore these topics in depth.

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[1] MNT-END-Will Remain | JREMNT
https://jremnt.com/mnt-end-will-remain

[2] Speed in Lorentz Transformation Directly Linked to Increased Momentum Hence Impulse?
https://zenodo.org/records/15480454

[3] [4] Free Particle Action, Lorentz Invariants and Geodesics
https://zenodo.org/records/7117739

[5] [6] Spin, Lorentz Invariant Equation for Free Probability and Lorentz Invariant Physical Equation
https://zenodo.org/records/17329846