Tensor Computing
A Thought on the Limitations of Digital Computers

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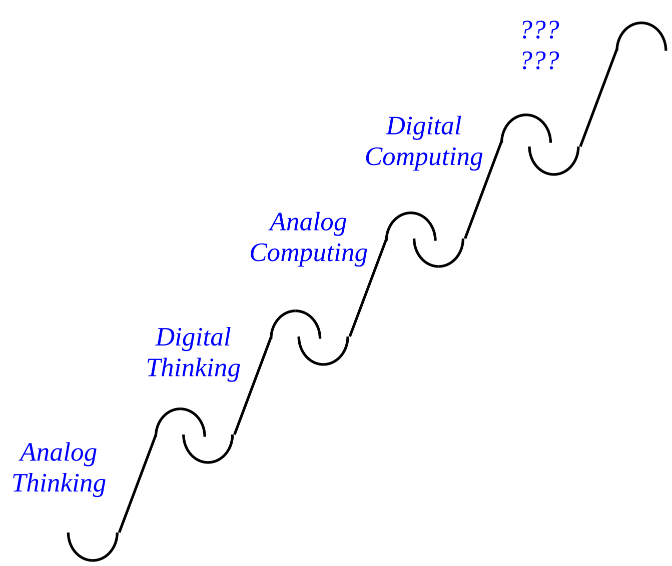
The Challenge

The internet opens the world to massive "any-to-any" communications. The rapidly dropping cost of information drives the accessibility and adoption of computing technologies in areas that were previously infeasible and unthinkable. This parallel and convergent phenomenon of massively distributed and interconnected computing will quickly push the limits of our current information processing paradigm – digital technologies that deal with linear constructs in discrete time representations.

What are the limits? How small can silicon be etched? How fast can microprocessors cycle? How many connections can digital switches make and manage with precision and predictability? How many packets of digital information can be crammed through fiber optic networks? When will we hit the limits of digital information processing at which our connected world implodes?



The only technical advancement that can be forecast with any certainty is that the limits of digital computing will be reached far ahead of any rational expectations we have today. Technical obsolescence is an ever accelerating factor in our environment of irrational, stratospheric values given to high technology stocks in today's capital markets. Our own drive for value creation and wealth building activities forces the innovative and clever implementations of digital technologies into a constant treadmill of research, development, application, obsolescence, and discard. Hitting the wall on digital information processing is an inevitable phenomenon. The question is not "if" but "how soon".

So what do we do? Are we destined to hit a technical discontinuity as we drop off the trailing edge of the digital s-curve? What is the next paradigm for information processing technology?

The Conjecture

Analog computing came and went with the advent of transistors, binary codes, and microprocessors.

Digital computing is how we run the world today. Object oriented technologies and programming methods are all the rage. Thin client applications speed up the network computing phenomenon, but only in the context of client-server/objected oriented and digital network architectures.

Objected oriented architectures have their inherent limits. They assume a relatively static relationship between the components of a system or process to be modeled and automated. Object oriented structures rely on rigid taxonomies and rules to define attributes and relationships. At their most advanced implementations, object orientation leads to a multitude of matrices that classify and categorize static definitions of a system and its attributes. Rules drive the elements within each matrix and its cells.

There have been attempts at rules and taxonomies that adapt dynamically, yet even these solutions rely on rules of adaptation which are themselves imbedded in tables and matrices. Matrices beget more matrices

Our current information and communications platforms reflect a relatively static view of the world based on a digital paradigm.

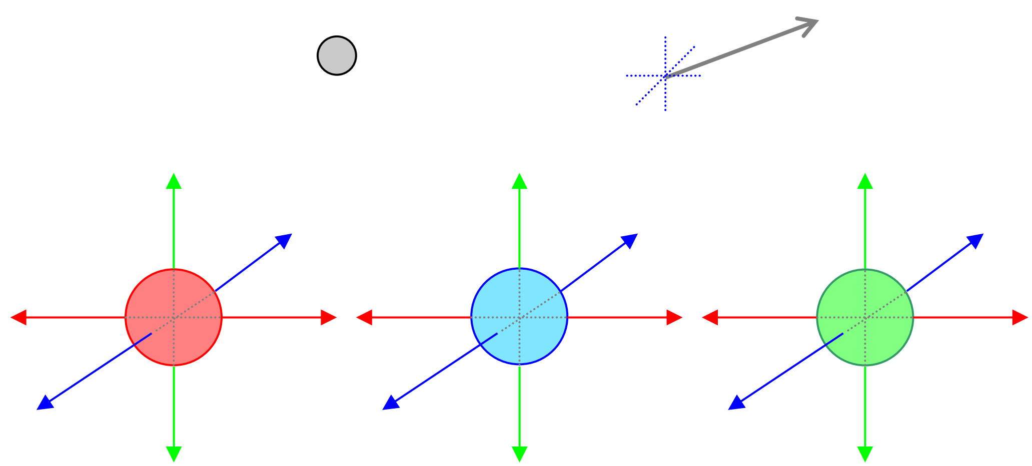
An Alternative

One word: tensors.

Tensors are a mathematical construct developed in the domain of continuum mechanics. These are the mechanics of solids and fluids which describe the physical world in which we live. Tensor equations model not only static attributes of a system but also its dynamic flux and the complex interrelationships between its components at all levels of interaction, from the purely mechanical to the gravitational, from the electromagnetic to the subatomic, from the micro to the macro.

The Definition of Tensors

Tensors are a generalized mathematical methodology with a higher order of abstraction than matrix or linear algebra. Scalars and vectors are simple, special cases of tensors. Traditional dimensional analysis applies to scalars and we generally use tables and static matrices to capture the results. Vector analysis extends the method to capture the magnitude and direction of physical entities in 3-dimensional space (n-dimensions if dealing with state space constructs). Tensors are the next level of abstraction beyond the dimensional analysis of scalars and the vector analysis of state space. Tensor equations do everything a scalar does (capture dimensionality) and everything a vector does (capture magnitude and direction). The additional benefit of tensors is that they also capture the interaction and interrelations between components of a system, where system has its broadest definition.



The Advantage of Tensors

Based upon these theorems, tensors present some unique advantages as a systems methodology.

Tensors are transformable from one frame of reference to another. Tensors can be operated upon. Tensor equations are transformable. Tensors follow dynamic behaviors.

If a tensor equation can be established in one orthogonal coordinate system that spans the state space of interest, then it must hold for all coordinate systems obtained by admissable transforms. A tensor equation can have general validity in any frame of reference if and only if every term in the equation has the same tensor characteristics and behaviors. Conversely, if a change in coordinate system changes the characteristics of the tensor equation, the original representation is invalid.

In other words, tensor equations are valid system representations if allowable transformations from one frame of reference to another do not change the characteristics of the system. This is a powerful concept as it tells you when you have properly captured and modeled the minimal and essential properties that define a system, its components, their attributes, their magnitude and direction in the frame of reference, and the components' interrelationships.

The transformation of tensors from one paradigm to another (from one frame of reference to another, from one perspective to another, or from one abstract construct to another) yield proper representations of the system, its characteristics, and its behaviors. The tensor equation representation of a system is thus indifferent to the reference model used.

Due to the nature of the transformation laws, tensor equations and their related theorems are aligned to the laws of physics. For example, a tensor of rank 0 is defined in accordance with the physical idea of a scalar, a tensor of rank 1 is defined in accordance with the physical idea of a vector, and so forth.

Tensors are scalable from the subatomic level to the universal plane. Tensors can be used to model systems and the relevant interactions between their components. Traditionally, in continuum mechanics context, tensors represent stress. This stress can be the stress between subatomic particles in an atom (strong and weak forces), the stress between two atoms in a crystal lattice structure (chemical bonds), the stress between two steel beams in a cantilever bridge (mechanical stress and strain), the stress between planetary bodies (gravitational forces), or the stress between humans (emotion).

Clearly, tensors have widespread and powerful applications in the material and abstract worlds of mechanical, electromagnetic, and human interactions. Tensors describe the world so they necessarily combine analog and digital paradigms. You cannot fully describe the world without integrating both.

Tensors are non-linear. Linearity is a special case of non-linearity. Tensors interact in real time. A discrete, or digital, time representation is a special case of real time. Tensors are powerful abstract constructs in that they represent both linear and non-linear characteristics as well as discrete and continuous time system properties.

	Non-Linear Paradigm	**Linear Paradigm**	**Integrated Paradigm**
Continuous Time	*Analog Computing*		
Discrete Time		*Digital Computing*	
Integrated Time			*Tensor Computing*

The Answer

My conjecture is that tensor analysis and tensor equations represent a methodology that could help bridge the gap between the analog world in which we live and the digital paradigm in which we currently process and fulfill our information, data, and communication needs.

Do I have answers to how exactly tensor analysis of systems and the subsequent tensor models will be implemented in the information processing world? No.

However, I do know that tensors go way beyond matrices, object oriented methods, and potentially, could help us go beyond the digital paradigm of our current technologies.

There is also the question of where in the information processing world to begin to apply tensor approaches. Applications? Middleware? Operating systems? Systems management? Network management? All of the above? None of the above? There are no answers yet.

Theoretical and explorative research is required to discover these answers. Who knows? Even looking for solutions in our current architectural confines presupposes solutions are embedded in the digitally-built machines of today. To begin looking through a digital set of lenses is mind-limiting and perhaps a dangerous path. Further thinking and stretching is required to even begin to know we are asking the right questions.

The Conclusion (The Beginning)

I believe a new abstract paradigm will be required to breakthrough the limits of digital technologies. Whether analog computing makes a come back (with the obvious precedents of further advances in solid state physics and materials science), tensors find an adequate digital implementation, or a hybrid, integrated analog-digital technology emerges is currently irrelevant.

What does matter is that we take on the challenge of continuously searching and experimenting with new paradigms of information processing that can surpass the limits of the digital world before our globally distributed and connected systems self-destruct from the speed, complexity, throughput, and processing demands placed on them by the weight of large numbers.

The answer is in the search for the answer.

Keywords

tensors; tensor equations; applied mathematics; computing; information science; systems theory; advanced technology; transformation; reengineering; state space models; hybrid systems; s-curves; chaos theory; non-linear systems

Note:

The Analytical Definition of Tensors

Let *n* be the number of orthogonal dimensions in the state space system to be modeled. *n* , then, also represents the span of the system.

Let *r* be the rank of the tensor field in question.

The number of components in this system is n^r.

A tensor is said to be an orthogonal tensor if it is defined in a system of coordinates that are orthogonal.

For example, in rectangular, Cartesian coordinates, $n = 3$, with the typical three dimensions: (x, y, z) or (x_1, x_2, x_3). In this case, a tensor in the Cartesian coordinate system is an orthogonal tensor.

If the system (or entity) in question can be described by a single component, it follows that its tensor field is of rank zero ($n^0 = 1$). Tensor fields of rank 0 in Cartesian coordinates are called *scalars* if they follow the following transformation law:

$$\mathbf{\Phi}(x_1, x_2, x_3) = \underline{\mathbf{\Phi}}(\underline{x}_1, \underline{x}_2, \underline{x}_3)$$

where $\mathbf{\Phi}$ represents the single component of the system with variables x_1, x_2, and x_3. $\underline{\mathbf{\Phi}}(\underline{x}_1, \underline{x}_2, \underline{x}_3)$ denotes the scalar transformed into an alternate frame of reference.

If the system (or entity) in question must be described by three (3) components, it follows that its tensor field is of rank one ($n^1 = 3$). Tensor fields of rank 1 in Cartesian coordinates are called *vectors* if they follow the following transformation law:

$$\underline{\boldsymbol{\beta}}_k (\underline{x}_1, \underline{x}_2, \underline{x}_3) = \boldsymbol{\beta}_i (x_1, x_2, x_3)\, M_{ik}$$
$$\boldsymbol{\beta}_i (x_1, x_2, x_3) = \underline{\boldsymbol{\beta}}_k (\underline{x}_1, \underline{x}_2, \underline{x}_3)\, M_{ki}$$

where $\boldsymbol{\beta}_i$ represents the three components of the system with variables x_1, x_2, and x_3. $\underline{\boldsymbol{\beta}}_k (\underline{x}_1, \underline{x}_2, \underline{x}_3)\, M_{ki}$ denotes the vector, $\boldsymbol{\beta}_I$, transformed into an alternate frame of reference by M_{ik}.

Tensors of rank two, in Cartesian coordinates, thus are entities that follow the following transformation law:

$$\underline{\boldsymbol{\tau}}_{ij} (\underline{x}_1, \underline{x}_2, \underline{x}_3) = \boldsymbol{\tau}_{mn} (x_1, x_2, x_3)\, M_{im}\, M_{jn}$$
$$\boldsymbol{\tau}_{ij} (x_1, x_2, x_3) = \underline{\boldsymbol{\tau}}_{mn} (\underline{x}_1, \underline{x}_2, \underline{x}_3)\, M_{mi}\, M_{nj}$$

The extension of tensors into a generalized form with n-dimensions and r-rank follows as:

$$\underline{\boldsymbol{\tau}}_{\underline{1}\,\underline{2}\,\underline{3}\ldots\underline{r}}(\underline{x}_1, \underline{x}_2, \underline{x}_3, \ldots \underline{x}_n) = \boldsymbol{\tau}_{1\,2\,3\ldots r}(x_1, x_2, x_3, \ldots x_n)\, M_{\underline{1}1}\, M_{\underline{2}2}\, M_{\underline{3}3} \ldots M_{\underline{r}r}$$

$$\boldsymbol{\tau}_{1\,2\,3\ldots r}(x_1, x_2, x_3, \ldots x_n) = \underline{\boldsymbol{\tau}}_{\underline{1}\,\underline{2}\,\underline{3}\ldots\underline{r}}(\underline{x}_1, \underline{x}_2, \underline{x}_3, \ldots \underline{x}_n)\, M_{1\underline{1}}\, M_{2\underline{2}}\, M_{3\underline{3}} \ldots M_{r\underline{r}}$$

Here, we go beyond Cartesian space, beyond the three orthogonal dimensions that define physical space. We step into state space constructs of n-dimensions. The only restriction is that the n-dimensions be orthogonal and span the state space of interest. The n-dimensions represent the minimal state space dimensions of our system of interest.

Therefore, tensors describe the minimal and essential properties and characteristics of a system of components.

A Note About Matrices

Matrices are not tensors. Matrices capture information and attributes of particular entities but simply as categorizations and classifications of attributes in table format. They are an aggregation of scalar properties and lend themselves to digitization quite well. Scalar properties are also known as attributes which define objects at the dimensional level only.

Some Useful Theorems Involving Tensors

Theorem 1:

If all components of an orthogonal tensor vanish in one coordinate system, then they vanish in all other orthogonal coordinate systems.

Theorem 2:

The sum or difference of two orthogonal tensors of the same rank is again a tensor of the same rank

Theorem 3:

If a tensor equation is true in one orthogonal coordinate system, then it is true in all orthogonal coordinate systems.

Theorem 4:

When only orthogonal coordinates are considered, the partial derivatives of any tensor field behave like the components of an orthogonal tensor.

