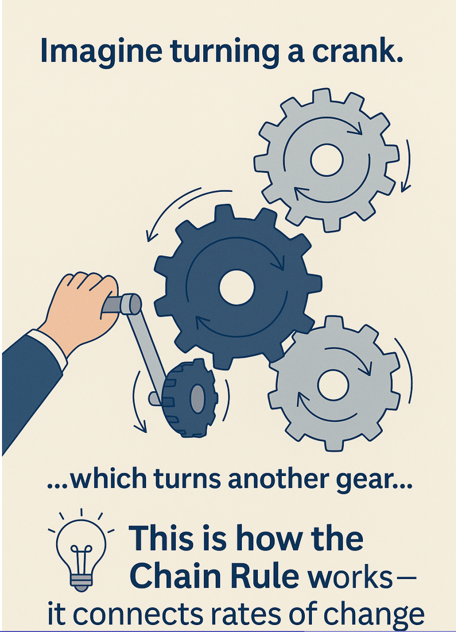
Introduction to the Chain Rule

Ayak David Chol

May 29, 2025

What is the speed of the final gear?



Imagine turning a crank
that spins a gear,
which turns another gear.

The speed of the final gear depends on how fast each gear in the chain spins.

The Chain Rule expresses this *compounding of rates.*

Learning Objectives: Chain Rule

By the end of this lesson, students will be able to:

- State the Chain Rule in both Leibniz and function notation.

Learning Objectives: Chain Rule

By the end of this lesson, students will be able to:

- State the Chain Rule in both Leibniz and function notation.
- Identify composite functions where the Chain Rule is applicable.

Learning Objectives: Chain Rule

By the end of this lesson, students will be able to:

- State the Chain Rule in both Leibniz and function notation.
- Identify composite functions where the Chain Rule is applicable.
- Differentiate composite functions using the Chain Rule.

Learning Objectives: Chain Rule

By the end of this lesson, students will be able to:

- State the Chain Rule in both Leibniz and function notation.
- Identify composite functions where the Chain Rule is applicable.
- Differentiate composite functions using the Chain Rule.
- Combine the Chain Rule with other differentiation rules.

Learning Objectives: Chain Rule

By the end of this lesson, students will be able to:

- State the Chain Rule in both Leibniz and function notation.
- Identify composite functions where the Chain Rule is applicable.
- Differentiate composite functions using the Chain Rule.
- Combine the Chain Rule with other differentiation rules.
- Interpret $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ in the context of composition and substitution.

Learning Objectives: Chain Rule

By the end of this lesson, students will be able to:

- State the Chain Rule in both Leibniz and function notation.
- Identify composite functions where the Chain Rule is applicable.
- Differentiate composite functions using the Chain Rule.
- Combine the Chain Rule with other differentiation rules.
- Interpret $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ in the context of composition and substitution.
- Apply the Chain Rule to solve real-world problems involving related rates.

What is the Chain Rule?

In calculus, the chain rule provides a way to find the derivative of a composition of two differentiable functions, f and g.

What is the Chain Rule?

In calculus, the chain rule provides a way to find the derivative of a composition of two differentiable functions, f and g.

Definition and Notation

Function Notation: For all values of x for which the derivative is defined; if $y = f(g(x))$, then

What is the Chain Rule?

In calculus, the chain rule provides a way to find the derivative of a composition of two differentiable functions, f and g.

Definition and Notation

Function Notation: For all values of x for which the derivative is defined; if $y = f(g(x))$, then

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

What is the Chain Rule?

In calculus, the chain rule provides a way to find the derivative of a composition of two differentiable functions, f and g.

Definition and Notation

Function Notation: For all values of x for which the derivative is defined; if $y = f(g(x))$, then

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

Leibniz Notation: If y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

What is the Chain Rule?

In calculus, the chain rule provides a way to find the derivative of a composition of two differentiable functions, f and g.

Definition and Notation

Function Notation: For all values of x for which the derivative is defined; if $y = f(g(x))$, then

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

Leibniz Notation: If y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Interpretation: Derivative of the *outer function* with respect to the *inner variable* times the derivative of the *inner function*.

Example 1: Power Function

Differentiate $y = (3x + 2)^5$

Example 1: Power Function

Differentiate $y = (3x+2)^5$

Solution

$$u = 3x + 2 \quad \Rightarrow \quad \frac{du}{dx} = 3$$

$$y = u^5 \quad \Rightarrow \quad \frac{dy}{du} = 5u^4$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && \text{Apply the Chain Rule} \\
&= 5u^4 \cdot 3 && \text{and the Power Rule} \\
&= 5(3x+2)^4 \cdot 3 && \text{substitute } u \text{ and simplify} \\
&= \boxed{15(3x+2)^4}
\end{aligned}$$

Example 2: Logarithmic Function

Differentiate: $y = \ln(5x^2 + 1)$

Example 2: Logarithmic Function

Differentiate: $y = \ln(5x^2 + 1)$

Solution

$$\begin{aligned} u &= 5x^2 + 1 \quad && \Rightarrow \quad \frac{du}{dx} = 10x \\ y &= \ln(u) \quad && \Rightarrow \quad \frac{dy}{du} = \frac{1}{u} \\ \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && \text{Apply the Chain Rule} \\ &= \frac{1}{5x^2 + 1} \cdot 10x && \text{substitute and simplify} \\ &= \boxed{\frac{10x}{5x^2 + 1}} \end{aligned}$$

Example 3: Exponential and Trigonometric Functions

Differentiate $y = e^{\tan(x)}$

Example 3: Exponential and Trigonometric Functions

Differentiate $y = e^{\tan(x)}$

Solution

$$u = \tan(x) \quad \Rightarrow \quad \frac{du}{dx} = \sec^2(x)$$

$$y = e^u \quad \Rightarrow \quad \frac{dy}{du} = e^u$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && \text{Apply the Chain Rule} \\ &= e^{\tan(x)} \cdot \sec^2(x) && \text{substitute } u \text{ and } \frac{du}{dx} \\ &= \boxed{e^{\tan(x)} \sec^2(x)} \end{aligned}$$

Example 4: Rising Balloon (Related Rates)

A spherical balloon is being inflated. The volume is given by:

$$V = \frac{4}{3}\pi r^3$$

Example 4: Rising Balloon (Related Rates)

A spherical balloon is being inflated. The volume is given by:

$$V = \frac{4}{3}\pi r^3$$

Suppose the radius increases at a rate of 2 cm/min.

Example 4: Rising Balloon (Related Rates)

A spherical balloon is being inflated. The volume is given by:

$$V = \frac{4}{3}\pi r^3$$

Suppose the radius increases at a rate of 2 cm/min. Find the rate of change of the volume when $r = 5$ cm.

Example 4: Rising Balloon (Related Rates)

A spherical balloon is being inflated. The volume is given by:

$$V = \frac{4}{3}\pi r^3$$

Suppose the radius increases at a rate of 2 cm/min. Find the rate of change of the volume when $r = 5$ cm.

Solution

$$\begin{aligned}
\frac{dV}{dt} &= \frac{dV}{dr} \cdot \frac{dr}{dt} \\
&= 4\pi r^2 \cdot \frac{dr}{dt} \\
&= 4\pi (5)^2 \cdot 2 \\
&= \boxed{200\pi \text{ cm}^3/\text{min} \approx 628 \text{ cm}^3/\text{min}}
\end{aligned}$$

Chain Rule Summary and Extensions

Chain Rule

- Differentiates composite functions.
- Function Notation: $\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$
- Leibniz Notation: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Combined with Other Rules

- **Power Rule:** $\dfrac{d}{dx}[(g(x))^n] = n(g(x))^{n-1} \cdot g'(x)$
- **Product Rule:** $\dfrac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$
- **Quotient Rule:** $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$