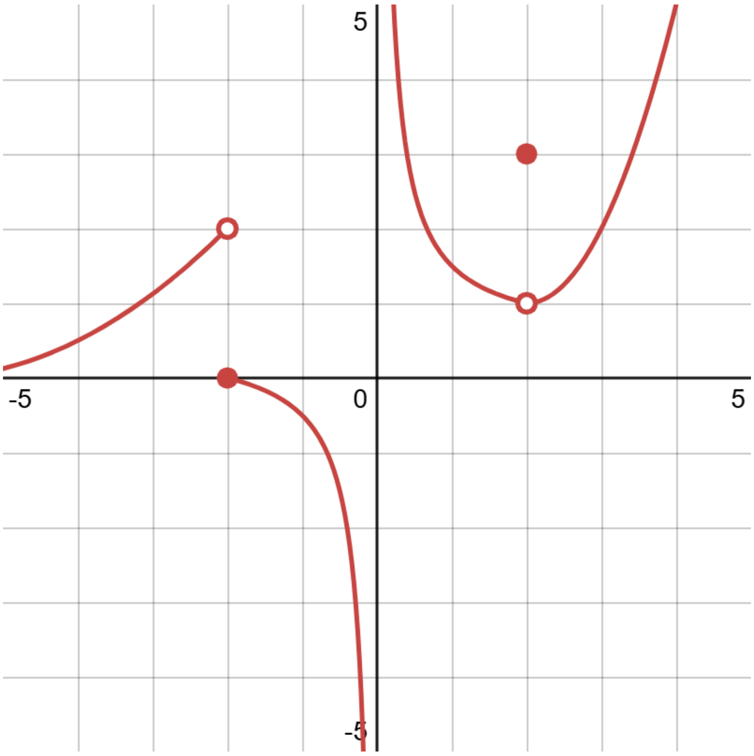
Problems 1-35

1. The graph of $f(x)$ is given;



find each limit:

a) $\lim_{x\to -2^-} f(x) = 2$

b) $\lim_{x\to -2^+} f(x) = 0$ $\quad \therefore \lim_{x\to -2} f(x) =$ DNE, since $\lim_{x\to -2^-} f(x) \neq \lim_{x\to -2^+} f(x)$

c) $\lim_{x\to 0^+} f(x) = \infty$ and $\lim_{x\to 0^-} f(x) = -\infty$

d) $\lim_{x\to 0} f(x) =$ DNE since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

e) $\lim_{x\to 2} f(x) = 1$ since $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 1$

2. For $f(x) = \dfrac{4x+1}{2x-1}$, give the following:

a. Domain:
The domain is all x except where the denominator is zero:

$$2x - 1 \neq 0 \implies x \neq \frac{1}{2}.$$

$$\text{Domain: } \left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$$

b. Range:
Solve for x in terms of y; because range of $f(x)$ is always the domain of $f^{-1}(x)$.

$$y = \frac{4x+1}{2x-1} \implies y(2x-1) = 4x+1 \implies x = \frac{1+y}{2y-4} \qquad 2y-4 \neq 0 \implies y \neq 2.$$

Range excludes $y = 2$ (denominator becomes zero):

$$\text{Range: } (-\infty, 2) \cup (2, \infty)$$

c. y-intercept:
Set $x = 0$:

$$f(0) = \frac{4(0)+1}{2(0)-1} = -1.$$

$$y\text{-intercept: } (0, -1)$$

d. x-intercept:
Set $f(x) = 0$, numerator is zero:

$$4x + 1 = 0 \implies x = -\frac{1}{4}.$$

$$x\text{-intercept: } \left(-\frac{1}{4}, 0\right)$$

e. Vertical asymptote:
Denominator is zero:

$$2x - 1 = 0 \implies x = \frac{1}{2}.$$

$$\text{Vertical asymptote: } x = \frac{1}{2}$$

f. Horizontal asymptote:
As $x \to \pm\infty$, degrees of numerator and denominator are the same.
Divide leading coefficients:Horizontal asymptote $\Rightarrow \boxed{y = 2}$.

For 3-6, find the limits

3. Find: $\lim_{x \to -1} \left(3x^2 - 6x + 9\right)$

$$\lim_{x \to -1} \left(3x^2 - 6x + 9\right) = 3(-1)^2 - 6(-1) + 9 = \boxed{18}$$

4. Find: $\lim_{x \to -3} \frac{x^2 - 9}{x^2 + 2x - 3}$

Factor numerator and denominator and cancel common terms:

$$\lim_{x \to -3} \frac{x^2 - 9}{x^2 + 2x - 3} = \lim_{x \to -3} \frac{(x-3)\cancel{(x+3)}}{(x-1)\cancel{(x+3)}} = \lim_{x \to -3} \frac{x-3}{x-1} = \frac{-3-3}{-3-1} = \frac{-6}{-4} = \boxed{\frac{3}{2}}$$

5. Find: $\lim\limits_{h\to 0} \dfrac{(h-1)^3+1}{h}$

Expand the numerator, simplify, cancel common terms and take the limit:

$$\begin{aligned}\lim_{h\to 0} \frac{(h-1)^3+1}{h} &= \lim_{h\to 0} \frac{h^3-3h^2+3h-1+1}{h} \\ &= \lim_{h\to 0} h^2-3h+3 \\ &= 0^2-3(0)+3 = \boxed{3}.\end{aligned}$$

6. Find: $\lim\limits_{t\to 0} \left(\dfrac{1}{t} - \dfrac{1}{t^2+t}\right)$

Simplify, cancel common terms and take the limit:

$$\lim_{t\to 0} \left(\frac{1}{t} - \frac{1}{t^2+t}\right) = \lim_{t\to 0} \left(\frac{\cancel{t}}{\cancel{t}(t+1)}\right) = \lim_{t\to 0} \frac{1}{t+1} = \frac{1}{0+1} = \boxed{1}$$

7. True or False: If the domain of $f(x)$ is $(-\infty,\infty)$ and $\lim\limits_{x\to 2} f(x) = 6$, then $f(2) = 6$.

$\boxed{\text{False}}$, since $f(x)$ might have a discontinuity at $x = 2$.

8. Find the equation of the tangent line to $y = 3x^2+2$ at $(1,5)$:

$$\frac{dy}{dx} = 6x\Big|_{x=1} \implies \frac{dy}{dx} = 6(1) = 6.$$

Using the point slope formula : $y - y_1 = m(x - x_1)$

$$y - 5 = 6(x-1) \implies \boxed{y = 6x - 1}$$

For #9-15, find the derivative.

9. Find the derivative: $f(x) = (5x^3-2x+1)^3$

Using the chain rule: $\dfrac{d}{dx}\Big[f\big(g(x)\big)\Big] = f'(g(x))\cdot g'(x)$

$$f'(x) = 3(5x^3-2x+1)^2(15x^2-2) \implies \boxed{f'(x) = (45x^2-6)(5x^3-2x+1)^2}$$

10. Find the derivative of $r(x) = \sqrt{x} + \dfrac{1}{\sqrt[3]{x^4}}$.

Rewrite the function as: $r(x) = x^{1/2} + x^{-4/3}$

$$r'(x) = \frac{1}{2}x^{-1/2} - \frac{4}{3}x^{-7/3} \implies \boxed{r'(x) = \frac{1}{2\sqrt{x}} - \frac{4}{3\sqrt[3]{x^7}} \quad \Rightarrow \quad \frac{1}{2\sqrt{x}} - \frac{4}{3x^2\sqrt[3]{x}}}$$

11. Find the derivative of $f(x) = 2x \ln x$.
    Using the product rule: $(uv)' = u'v + uv'$.
    Let $u = 2x$, $u' = 2$ and $v = \ln x$, $v' = \dfrac{1}{x}$

$$f'(x) = 2\ln x + 2\cancel{x} \times \frac{1}{\cancel{x}} \Longrightarrow \boxed{f'(x) = 2\ln x + 2}$$

12. Find the derivative of $g(x) = \dfrac{x}{1-x^2}$.
    Using the quotient rule: $\left(\dfrac{u}{v}\right)' = \dfrac{vu' - uv'}{v^2}$

$$g'(x) = \frac{(1-x^2) - (-2x)(x)}{(1-x^2)^2} = \frac{1+x^2}{(1-x^2)^2} \Longrightarrow \boxed{g'(x) = \frac{1+x^2}{(1-x^2)^2}}$$

13. Find the derivative of $f(x) = \sqrt{3x^2+4}$.
    Using the chain rule:

$$f'(x) = \frac{1}{2\sqrt{3x^2+4}} \cdot 6x \Longrightarrow \boxed{f'(x) = \frac{3x}{\sqrt{3x^2+4}}}$$

14. Find the derivative of $h(x) = e^{5-x^2}$.
    Using the chain rule:

$$h'(x) = e^{5-x^2} \cdot (-2x) \Longrightarrow \boxed{h'(x) = -2xe^{5-x^2}}$$

15. Find the derivative of $f(t) = \ln(3t^4)$.
    Logarithmic differentiation, using the chain rule:

$$f'(t) = \frac{1}{3t^4} \cdot 12t^3 \Longrightarrow \boxed{f'(t) = \frac{4}{t}}$$

16. Find the tangent line to $x^2 + 4xy + y^2 = 13$ at $(2, 1)$.
    Implicit differentiation:

$$2x + 4\left(x\frac{dy}{dx} + y\right) + 2y\frac{dy}{dx} = 2x + 4x\frac{dy}{dx} + 4y + 2y\frac{dy}{dx} = 0.$$

Solve for $\dfrac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{2x+4y}{4x+2y}.$$

At $(2, 1)$:

$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,1)} = -\frac{2(2)+4(1)}{4(2)+2(1)} = -\frac{4+4}{8+2} = -\frac{8}{10} = -\frac{4}{5}.$$

The equation of the tangent line is: $y - 1 = -\dfrac{4}{5}(x-2) \Longrightarrow \boxed{y = -\dfrac{4}{5}x + \dfrac{13}{5}}$

17. Analyze $f(x) = x^3 - 6x^2 - 15x + 4$

   (a) Intervals of Increase/Decrease

   - Find the first derivative and solve $f'(x) = 0$:

   $$f'(x) = 3x^2 - 12x - 15 = 0$$

   $$3x^2 - 12x - 15 = 0 \implies x^2 - 4x - 5 = 0 \implies (x-5)(x+1) = 0$$

   Critical points: $x = -1$ and $x = 5$.

   - Test intervals:
     - For $x \in (-\infty, -1)$, $f'(-2) = 21 > 0$ $\Rightarrow f'(x) > 0$: Increasing.
     - For $x \in (-1, 5)$, $f'(0) = -5 < 0$ $\Rightarrow f'(x) < 0$: Decreasing.
     - For $x \in (5, \infty)$, $f'(6) = 21 > 0$ $\Rightarrow f'(x) > 0$: Increasing.
   - Thus; $f(x)$ is increasing on $\boxed{(-\infty, -1) \cup (5, \infty)}$ and decreasing on $\boxed{(-1, 5)}$.

   (b) Local Maxima/Minima

   - Evaluate $f(x)$ at critical points:

   $$f(-1) = (-1)^3 - 6(-1)^2 - 15(-1) + 4 = 12$$
   $$f(5) = (5)^3 - 6(5)^2 - 15(5) + 4 = -96$$

   - Nature of extrema:
     - Local maximum at $\boxed{(-1, 12)}$ (sign change $f'(x) : + \to -$).
     - Local minimum at $\boxed{(5, -96)}$ (sign change $f'(x) : - \to +$).

   (c) Concavity and Inflection Points

   - Find the second derivative:

   $$f''(x) = 6x - 12$$

   - Solve $f''(x) = 0$:

   $$6x - 12 = 0 \implies x = 2$$

   - Test intervals:
     - For $x \in (-\infty, 2)$, $f''(x) < 0$: Concave downward.
     - For $x \in (2, \infty)$, $f''(x) > 0$: Concave upward.
   - Inflection point: At $x = 2$,

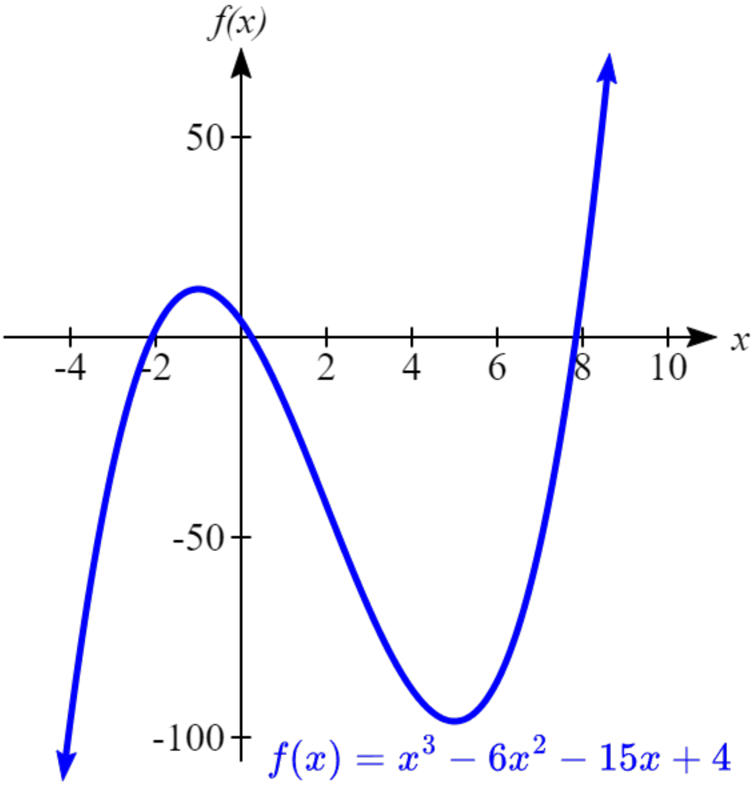
   $$f(2) = (2)^3 - 6(2)^2 - 15(2) + 4 = -42$$

   Inflection point: $\boxed{(2, -42)}$.

(d) Asymptotes

- Since $f(x)$ is a polynomial, it has no vertical, horizontal, or oblique asymptotes.

(e) Sketch the curve for $f(x) = x^3 - 6x^2 - 15x + 4$



18. Analyze $f(x) = \dfrac{1}{1-x^2}$

(a) Intervals of Increase/Decrease:

- Find the first derivative of $f(x)$:

$$f'(x) = \frac{2x}{(1-x^2)^2}$$

- Solve $f'(x) = 0$: $\Rightarrow$ Critical points are where $f'(x) = 0$ or undefined.

$$\frac{2x}{(1-x^2)^2} = 0 \implies 2x = 0 \implies x = 0$$

$f'(x)$ is undefined at $x = \pm 1$, which corresponds to the vertical asymptotes.

- Test intervals:
  - For $x \in (-\infty, -1)$: Choose $x = -2$:

$$f'(-2) = \frac{2(-2)}{(1-(-2)^2)^2} = \frac{-4}{(1-4)^2} = \frac{-4}{9} < 0$$

$f(x)$ is decreasing.

– For $x \in (-1, 0)$: Choose $x = -\frac{1}{2}$:

$$f'\left(-\frac{1}{2}\right) = \frac{2(-\frac{1}{2})}{(1-(-\frac{1}{2})^2)^2} = \frac{-1}{(1-\frac{1}{4})^2} = \frac{-1}{\frac{9}{16}} = -\frac{16}{9} < 0$$

$f(x)$ is decreasing.

– For $x \in (0, 1)$: Choose $x = \dfrac{1}{2}$:

$$f'\left(\frac{1}{2}\right) = \frac{2(\frac{1}{2})}{(1-(\frac{1}{2})^2)^2} = \frac{1}{(1-\frac{1}{4})^2} = \frac{1}{\frac{9}{16}} = \frac{16}{9} > 0$$

$f(x)$ is increasing.

– For $x \in (1, \infty)$: Choose $x = 2$:

$$f'(2) = \frac{2(2)}{(1-(2)^2)^2} = \frac{4}{(1-4)^2} = \frac{4}{9} > 0$$

$f(x)$ is increasing.

Thus; $f(x)$ is decreasing on $\boxed{(-\infty, -1) \cup (-1, 0)}$, and increasing on $\boxed{(0, 1) \cup (1, \infty)}$.

(b) Local Maxima/Minima Analyze

i. **Extrema:**

- Find critical points by solving $f'(x) = 0$:

$$f'(x) = \frac{2x}{(1-x^2)^2}, \quad \frac{2x}{(1-x^2)^2} = 0 \quad \Longrightarrow \quad x = 0$$

- Test the nature of the critical point using $f''(x)$:

$$f''(0) = \frac{2(1+3(0)^2)}{(1-0^2)^3} = 2 > 0$$

Since $f''(0) > 0$, $f(x)$ has a local minimum at $x = 0$.

- Evaluate $f(x)$ at $x = 0$:

$$f(0) = \frac{1}{1-(0)^2} = 1$$

Thus; a local minimum at $\boxed{(0, 1)}$.

ii. **Concavity:**

- Find the second derivative:

$$f''(x) = \frac{2(1+3x^2)}{(1-x^2)^3}, \quad x \neq \pm 1$$

- Test intervals for $f''(x)$:

– For $x \in (-\infty, -1)$, choose $x = -2$:

$$f''(-2) = \frac{2(1+3(-2)^2)}{(1-(-2)^2)^3} = \frac{2(1+12)}{(1-4)^3} = \frac{26}{-27} < 0$$

$f(x)$ is concave downward.

– For $x \in (-1, 1)$, choose $x = 0$:

$$f''(0) = \frac{2(1+3(0)^2)}{(1-0^2)^3} = \frac{2(1)}{1^3} = 2 > 0$$

$f(x)$ is concave upward.

– For $x \in (1, \infty)$, choose $x = 2$:

$$f''(2) = \frac{2(1+3(2)^2)}{(1-(2)^2)^3} = \frac{2(1+12)}{(1-4)^3} = \frac{26}{-27} < 0$$

$f(x)$ is concave downward.

Therefore; concave downward on $\boxed{(-\infty, -1) \cup (1, \infty)}$, concave upward on $\boxed{(-1, 1)}$.

iii. **Inflection Points:**

- $f''(x) = 0$ when $1 + 3x^2 = 0$, which has no real solutions.
- $f''(x)$ is undefined at $x = \pm 1$, corresponding to vertical asymptotes.
- Thus; there are no inflection points, but the function changes concavity at $x = \pm 1$.

(c) Vertical Asymptotes:

- The function $f(x) = \dfrac{1}{1-x^2}$ is undefined when the denominator $1 - x^2 = 0$.

$$1 - x^2 = 0 \quad \Longrightarrow \quad x^2 = 1 \quad \Longrightarrow \quad x = \pm 1$$

- Thus; the function has vertical asymptotes at:

$$x = 1 \quad \text{and} \quad x = -1$$

(d) Horizontal Asymptote:

- To find the horizontal asymptote, evaluate the behavior of $f(x)$ as $x \to \pm\infty$:

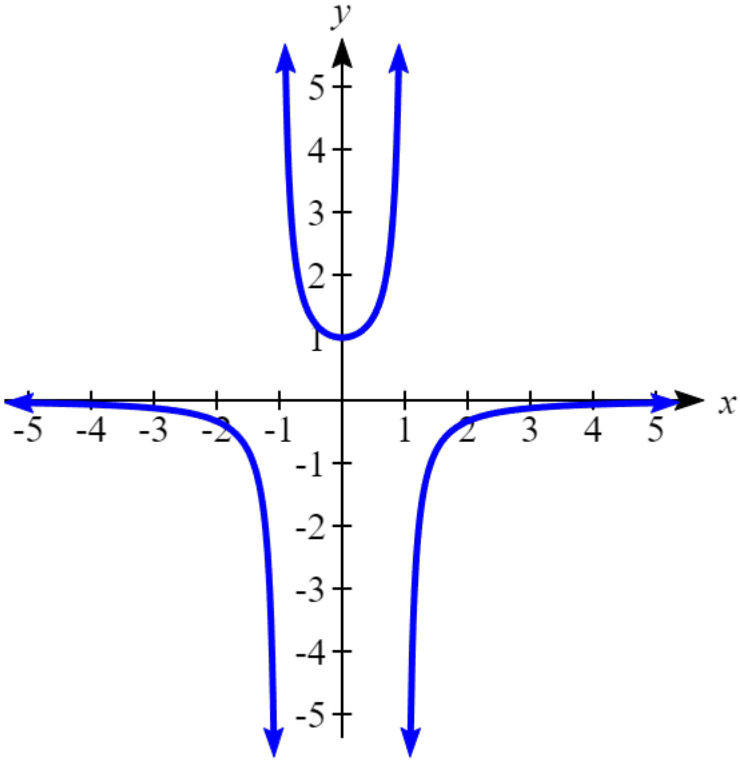
$$f(x) = \frac{1}{1-x^2}$$

As $x \to \pm\infty$, $x^2 \to \infty$, so $1 - x^2 \to -\infty$ and:

$$f(x) \to 0$$

- Thus; the horizontal asymptote is: $\boxed{y = 0}$

(e) Sketch the graph of $f(x) = \dfrac{1}{1-x^2}$



For #19-28, Integrate.

19. Evaluate: $\displaystyle\int_1^2 (8x^3 + 3x^2)\,dx.$

$$\int (8x^3 + 3x^2)\,dx = 2x^4 + x^3\Big|_1^2 = (32+8) - (2+1) = \boxed{37}$$

20. Evaluate: $\displaystyle\int_1^9 (\sqrt{u} - 2u^2)\,du. \quad \Longrightarrow \quad \int_1^9 u^{1/2}du - \int_1^9 2u^2du.$

$$\int_1^9 u^{1/2}\,du - \int_1^9 2u^2\,du = \frac{2}{3}u^{3/2} - \frac{2}{3}u^3\Big|_1^9 = \left(\frac{2}{3}(27) - \frac{2}{3}(729)\right) - \left(\frac{2}{3}(1) - \frac{2}{3}(1)\right) = \boxed{-76}$$

21. Evaluate: $\displaystyle\int \left(x^4 - \frac{6}{x^3}\right)\,dx. \quad \Longrightarrow \quad \int (x^4 - 6x^{-3})\,dx.$

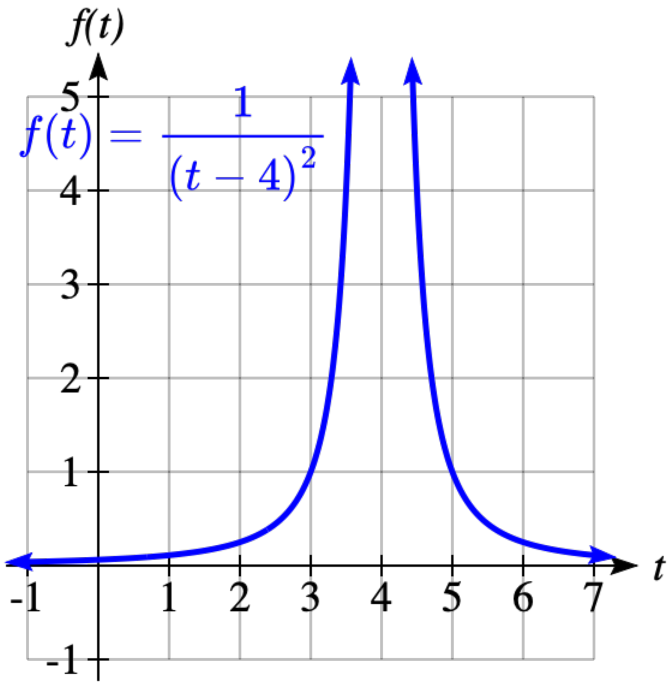
$$\int x^4dx - \int 6x^{-3}\,dx = \frac{x^5}{5} + 2x^{-2} + C = \boxed{\frac{x^5}{5} + \frac{2}{x^2} + C}$$

22. Evaluate: $\displaystyle\int_0^1 y\left(y^2+1\right)^5\,dy.$

Let $u = y^2 + 1$, so $du = 2ydy$. Limits change as: $u(0) = 1, \quad u(1) = 2.$

$$\int_0^1 y\left(y^2+1\right)^5 dy = \frac{1}{2}\int_1^2 u^5du = \frac{1}{2}\cdot\frac{u^6}{6}\Big|_1^2 = \frac{1}{12}\left(2^6 - 1^6\right) = \frac{1}{12}(64-1) = \frac{63}{12} = \boxed{\frac{21}{4}}$$

23. Evaluate: Evaluate $\displaystyle\int_1^5 \frac{dt}{(t-4)^2}$. Improper integral; split the integral at $t = 4$, integrate and then take the limit.



$$\begin{aligned}
\int_1^5 \frac{dt}{(t-4)^2} &= \lim_{b\to 4^-} \int_1^b \frac{dt}{(t-4)^2} + \lim_{b\to 4^+} \int_b^5 \frac{dt}{(t-4)^2} \\
&= \lim_{b\to 4^-} \left(-\frac{1}{t-4}\right)\Bigg|_1^b + \lim_{b\to 4^+} \left(-\frac{1}{t-4}\right)\Bigg|_b^5. \\
&= \lim_{b\to 4^-} -\frac{1}{(b-4)} - \frac{1}{3} \quad \Longrightarrow \text{diverges.}
\end{aligned}$$

Since the first part of the integral diverges, the entire integral diverges and the Fundamental Theorem of Calculus does not apply to the integral, since it *requires the function to be continuous on a closed interval.*

24. Evaluate: $\displaystyle\int e^{6x-1}\,dx$.

Let $u = 6x - 1$, so $du = 6dx$:

$$\int e^{6x-1}\,dx = \frac{1}{6}\int e^u\,du = \frac{1}{6}e^u + C = \boxed{\frac{1}{6}e^{6x-1} + C}$$

25. Evaluate: $\displaystyle\int_0^4 \frac{3x}{3x^2+1}\,dx$.

Let $u = 3x^2 + 1$, so $du = 6xdx$, $\quad x = 0 \Rightarrow u = 1, \quad x = 4 \Rightarrow u = 49$:

$$\int_0^4 \frac{3x}{3x^2+1}\,dx = \frac{1}{2}\int_1^{49} \frac{1}{u}\,du = \frac{1}{2}\ln u\Big|_1^{49} = \frac{1}{2}\ln 49 - \frac{1}{2}\ln 1 = \frac{1}{2}\ln 49 = \boxed{\frac{1}{2}\ln 49 = \frac{1}{2}\ln 7^2 = \ln 7}$$

26. Evaluate: $\displaystyle\int 4xe^{4x}\,dx$.

Using integration by parts: $\implies \displaystyle\int u\,dv = uv - \int v\,du$: choice for u: $L.i.A.T.E$[1]

$$u = 4x, \quad dv = e^{4x}dx, \quad du = 4dx, \quad v = \frac{1}{4}e^{4x}.$$

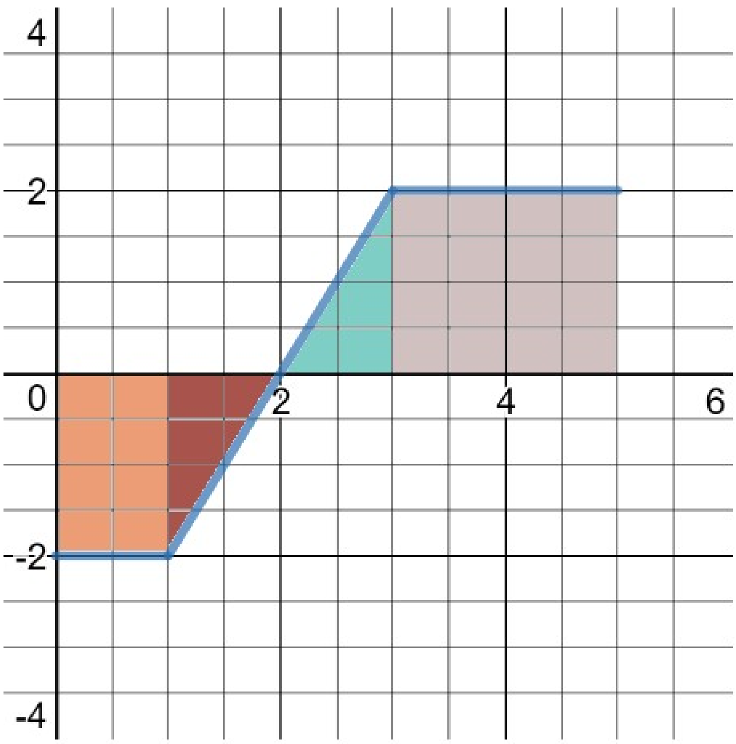
$$\int 4xe^{4x}\,dx = 4x\cdot\frac{1}{4}e^{4x} - \int\frac{1}{4}e^{4x}\cdot 4\,dx = \boxed{xe^{4x} - \frac{1}{4}e^{4x} + C}$$

27. Evaluate: $\displaystyle\int_1^\infty \frac{4}{x^3}\,dx$

Improper integral: Needs to be evaluated using a limit. Rewrite the integral as a limit.

$$\int_1^\infty \frac{4}{x^3}\,dx = \lim_{b\to\infty}\int_1^b 4x^{-3}\,dx = \lim_{b\to\infty} -\frac{2}{x^2}\Bigg|_1^b = \lim_{b\to\infty}\left(-\frac{2}{b^2}+2\right) = 0+2 = \boxed{2}$$

28. Evaluate: $\displaystyle\int_0^5 g(x)\,dx$.

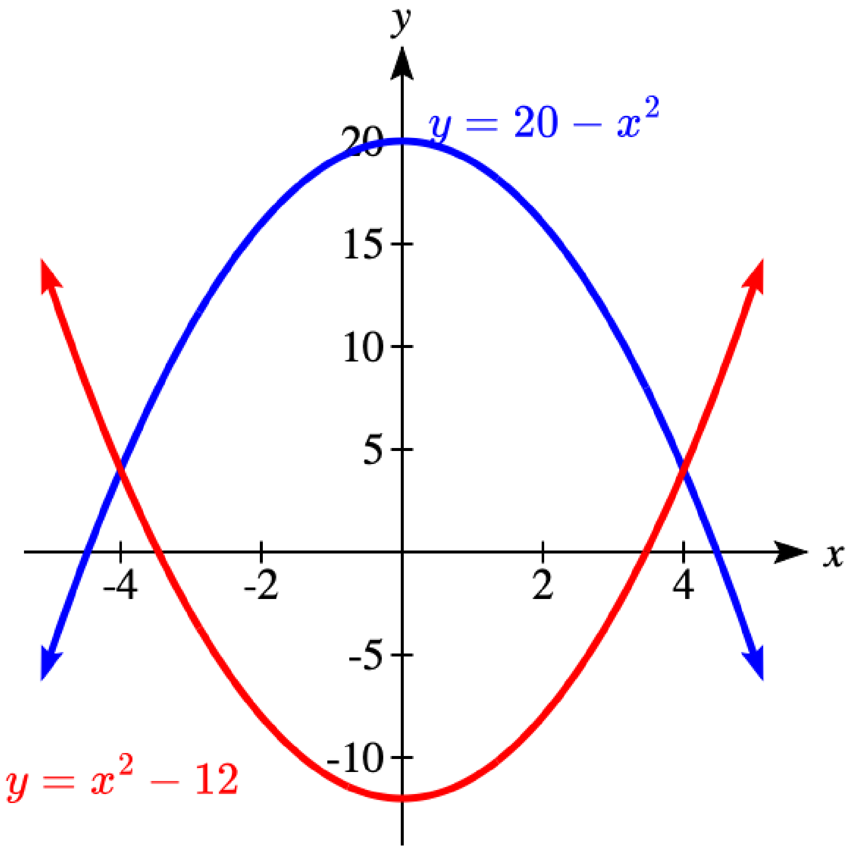


Sum the areas under the curve. Note that areas below the x-axis are negative and those above the x-axis are positive; for a triangle $A = \frac{1}{2}bh$, rectangle $A = \ell\times w$ and square $A = s\times s$.

$$\int_0^5 g(x)\,dx = \underbrace{-(1\times 2)}_{\text{Orange rectangle}} - \underbrace{\left(\frac{1}{2}\times 1\times 2\right)}_{\text{Brown triangle}} + \underbrace{\left(\frac{1}{2}\times 1\times 2\right)}_{\text{Teal triangle}} + \underbrace{(2\times 2)}_{\text{Gray square}} = -2-1+1+4 = \boxed{2}$$

[1] $L.i.A.T.E$: L = Logarithmic, i = inverse function, A = Algebraic, T = Trigonometric, E = Exponential

29. Find the area of the region bounded by the curves $y = 20 - x^2$ and $y = x^2 - 12$.
To find the area of the region, we first determine where the two curves intersect by setting $20 - x^2 = x^2 - 12$: $\Rightarrow 2x^2 = 32 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$



The area between the curves from $x = -4$ to $x = 4$ is: $\Rightarrow \displaystyle\int_a^b \Big(\text{Top} - \text{Bottom}\Big)\, dx$

$$\begin{aligned}
\int_{-4}^{4} \left[(20 - x^2) - (x^2 - 12)\right]\, dx &= \int_{-4}^{4} (32 - 2x^2)\, dx \\
&= 32x - \frac{2x^3}{3}\Bigg|_{-4}^{4} \\
&= 32 \cdot 4 - 32 \cdot (-4) - \left(\frac{2 \cdot 4^3}{3} - \frac{2 \cdot (-4)^3}{3}\right) \\
&= 128 + 128 - \left(\frac{2 \cdot 64}{3} - \frac{2 \cdot (-64)}{3}\right) = \frac{512}{3} \\
&= \boxed{\frac{512}{3}}
\end{aligned}$$

30. The average cost of a gallon of milk in 2016 was \$2.60, and the average cost of a gallon of milk in 2018 was \$2.88. If we assume the cost of a gallon of milk will grow according to the continuous growth model, what will the average cost of a gallon of milk be in the year 2030?
The continuous growth model is:
$$A(t) = A_0 e^{kt}.$$
Using $A_0 = 2.60$, $A(2) = 2.88$, solve for k:
$$2.88 = 2.60e^{2k} \implies e^{2k} = \frac{2.88}{2.60}.$$
$$2k = \ln\left(\frac{2.88}{2.60}\right) \implies k = \frac{1}{2}\ln\left(\frac{2.88}{2.60}\right).$$
Find the cost in 2030 ($t = 14$):
$$A(14) = 2.60e^{14k}.$$
$$A(14) = 2.60e^{14\cdot\frac{1}{2}\ln\left(\frac{2.88}{2.60}\right)} = 2.60\left(\frac{2.88}{2.60}\right)^7 \approx \boxed{\$5.32}$$

31. The cost function for producing x pairs of sunglasses is:
$$C(x) = 640 + 1.5x + 0.5x^2.$$
If sold for \$38.50, find x to maximize profit.
Solution:
Profit is:
$$P(x) = R(x) - C(x) = 38.5x - (640 + 1.5x + 0.5x^2).$$
Simplify:
$$P(x) = -0.5x^2 + 37x - 640.$$
Take the derivative:
$$P'(x) = -x + 37.$$
Set $P'(x) = 0$ to find critical points:
$$-x + 37 = 0 \implies x = 37.$$
Since $P''(x) = -1 < 0$, so $\boxed{\text{37 pairs}}$ maximizes profit.

32. A hockey team plays in an arena with a seating capacity of 15,000 spectators. With the ticket price set at \$12, average attendance at a game has been 11,000. A market survey indicates that for each dollar the ticket price is lowered, average attendance will increase by 1,000. How should the owners of the team set the ticket price to maximize their revenue from ticket sales?
Solution: Revenue:
$$R(p) = p(11,000 + 1,000(12 - p)).$$
Simplify:
$$R(p) = p(23,000 - 1,000p) = -1,000p^2 + 23,000p.$$
Take the derivative:
$$R'(p) = -2,000p + 23,000.$$
Set $R'(p) = 0$:
$$-2,000p + 23,000 = 0 \implies p = 11.50.$$
Since $R''(p) = -2,000 < 0$, $\boxed{\$11.50}$ is the maximum.

33. The volume of a cube is increasing at a rate of $10\,\text{cm}^3/\text{min}$. Find the rate of increase of surface area when the edge length is 30 cm.
Solution: Volume:
$$V = s^3 \implies \frac{dV}{dt} = 3s^2\frac{ds}{dt}.$$
Solve for $\frac{ds}{dt}$:
$$10 = 3(30)^2\frac{ds}{dt} \implies \frac{ds}{dt} = \frac{10}{2,700} = \frac{1}{270}.$$
Surface area:
$$A = 6s^2 \implies \frac{dA}{dt} = 12s\frac{ds}{dt}.$$
At $s = 30$:
$$\frac{dA}{dt} = 12(30)\left(\frac{1}{270}\right) = \frac{360}{270} = \boxed{\frac{4}{3}\,\text{cm}^2/\text{min}}$$

34. Find the future value of a continuous money flow of \$500/month invested at 4% continuous interest for 2 years.
Solution: Future value:
$$FV = e^{kT}\int_0^T R(t)e^{-kt}\,dt.$$
Here $R(t) = 500,\ k = 0.04,\ T = 2$:
$$FV = e^{0.04\cdot 2}\cdot\int_0^2 500e^{-0.04t}dt = e^{0.04\cdot 2}\int_0^2 500e^{-0.04t}\,dt = -\frac{500}{0.04}e^{0.04\cdot 2}\times e^{-0.04t}\Big|_0^2$$
$$= \frac{500}{0.04}e^{0.04\cdot 2}\left[1 - e^{-0.04\cdot 2}\right] = 12,500\cdot\left(1 - e^{-0.08}\right) \approx \boxed{\$20,146.21}$$

35. The marginal profit for producing x fidget-spinners is:
$$P'(x) = 4x\sqrt{4x^2+21}.$$
Find the total profit from selling 1,000 units.
Solution: Profit:
$$P(x) = \int_0^{1,000} 4x\sqrt{4x^2+21}\,dx.$$
Let $u = 4x^2 + 21$, so $du = 8xdx$:
$$P(x) = \int \frac{1}{2}\sqrt{u}\,du = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(4x^2+21)^{3/2} + C.$$
Evaluate at limits:
$$\frac{1}{3}(4x^2+21)^{3/2}\Big|_0^{1,000} = \frac{1}{3}(4\times 1000^2+21)^{3/2} - \frac{1}{3}(4\times 0^2+21)^{3/2} \approx \boxed{\$2,666,687,635}$$

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