Chapter 0.01: The Chain Rule

What is the Chain Rule?

In calculus, we often encounter functions that are compositions of two or more functions. That is, a function like $y = f(g(x))$, where the output of one function becomes the input of another. To differentiate such functions, we use the **Chain Rule**.

The Chain Rule Formula

Chain Rule

If $y = f(u)$ and $u = g(x)$, then:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Or, for compositions:

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

Why It Works: An Intuitive View

Imagine turning a crank that spins a gear, which turns another gear. The speed of the final gear depends on how fast each gear in the chain spins. The Chain Rule expresses this compounding of rates.

Examples

Example 1

Differentiate $y = (3x + 2)^5$
Let $u = 3x + 2$, then $y = u^5$.

$$\frac{dy}{dx} = 5u^4 \cdot 3 = 15(3x + 2)^4$$

Example 2

Differentiate $y = \sin(5x^2)$
Let $u = 5x^2$, then:

$$\frac{dy}{dx} = \cos(5x^2) \cdot 10x$$

Example 3

Differentiate $y = \sqrt{1 + \tan x}$
Let $u = 1 + \tan x$, then:

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1 + \tan x}} \cdot \sec^2 x$$

Example 4

Differentiate $y = \cos^3(x)$
Let $u = \cos x$, then $y = u^3$:

$$\frac{dy}{dx} = 3u^2 \cdot (-\sin x) = -3\cos^2 x \sin x$$

Application: Volume of Inflating Balloon

A spherical balloon is being inflated. The volume is given by:

$$V = \frac{4}{3}\pi r^3$$

Suppose the radius increases at a rate of 2 cm/min. Find the rate of change of the volume when $r = 5$ cm.

$$\begin{aligned}\frac{dV}{dt} &= \frac{dV}{dr} \cdot \frac{dr}{dt} \\ &= 4\pi r^2 \cdot \frac{dr}{dt} \\ &= 4\pi(5)^2 \cdot 2 \\ &= 200\pi \text{ cm}^3/\text{min} \approx 628 \text{ cm}^3/\text{min}\end{aligned}$$

Example 5

Differentiate $y = e^{\tan(x)}$

$$u = \tan(x) \quad \Rightarrow \quad \frac{du}{dx} = \sec^2(x)$$
$$y = e^u \quad \Rightarrow \quad \frac{dy}{du} = e^u$$
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= e^{\tan(x)} \cdot \sec^2(x) \\ &= e^{\tan(x)} \sec^2(x)\end{aligned}$$

Practice Problems

Differentiate the following:

1. $y = (2x^2 + 1)^6$
2. $y = \ln(3x^2 + 2x)$
3. $y = e^{\sin x}$
4. $y = \tan(4x^3)$
5. $y = \sqrt{5x + 7}$
6. $y = \sec^2(2x)$
7. $y = \cos^4(x^2)$
8. $y = \sin(\ln x)$
9. $y = \ln(\sqrt{1 + x^2})$
10. $y = (1 + 2x)^5 \cdot \sin(3x)$

Solutions

1. $$\frac{dy}{dx} = 6(2x^2+1)^5 \cdot 4x = 24x(2x^2+1)^5$$

2. $$\frac{dy}{dx} = \frac{1}{3x^2+2x} \cdot (6x+2) = \frac{6x+2}{3x^2+2x}$$

3. $$\frac{dy}{dx} = e^{\sin x} \cdot \cos x$$

4. $$\frac{dy}{dx} = \sec^2(4x^3) \cdot 12x^2$$

5. $$\frac{dy}{dx} = \frac{5}{2\sqrt{5x+7}}$$

6. $$\frac{dy}{dx} = 4\sec^2(2x)\tan(2x)$$

7. $$\frac{dy}{dx} = -8x\cos^3(x^2)\sin(x^2)$$

8. $$\frac{dy}{dx} = \cos(\ln x) \cdot \frac{1}{x}$$

9. $$\frac{dy}{dx} = \frac{x}{1+x^2}$$

10. $$\frac{dy}{dx} = 10(1+2x)^4\sin(3x) + 3(1+2x)^5\cos(3x)$$