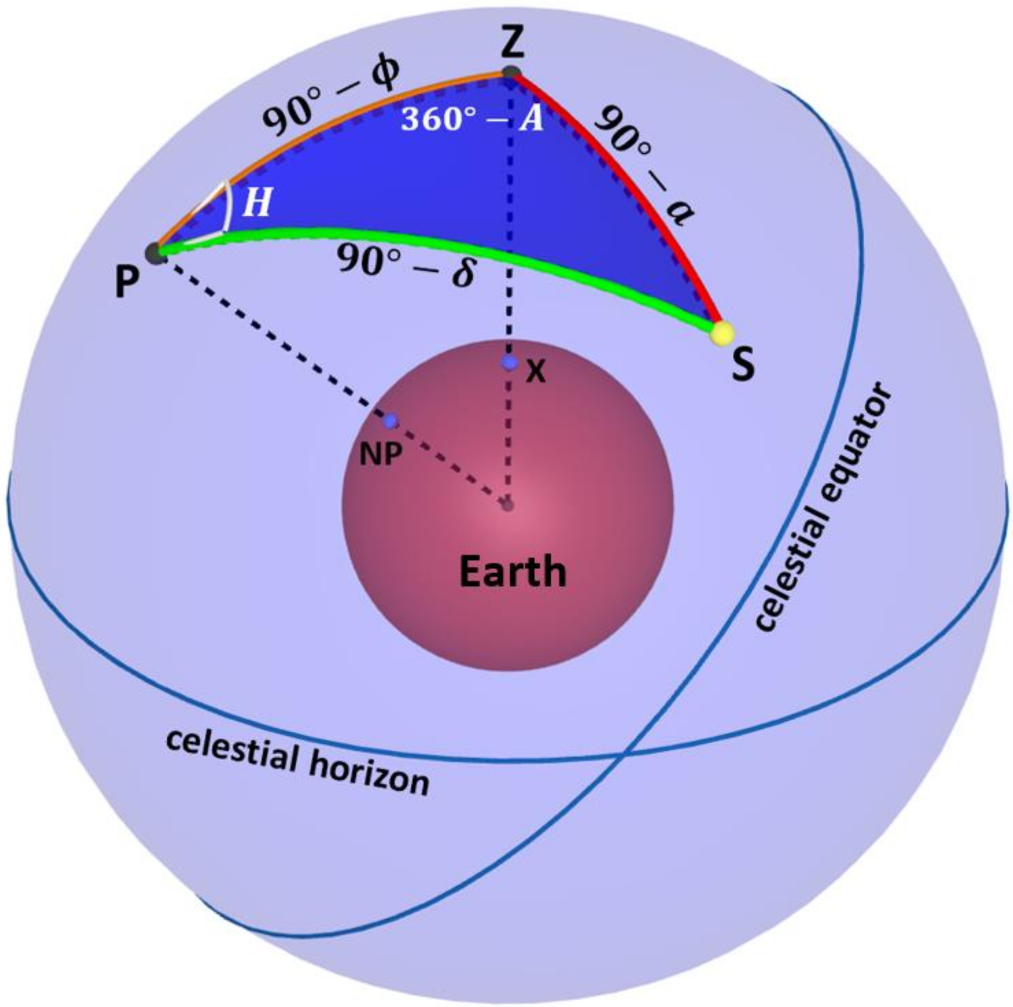
Lecture for Thursday, June 27

The celestial sphere is the imaginary sphere where the stars live. The connection of the celestial sphere and spherical trigonometry comes from the astronomical triangle PZS as seen in the figure below.



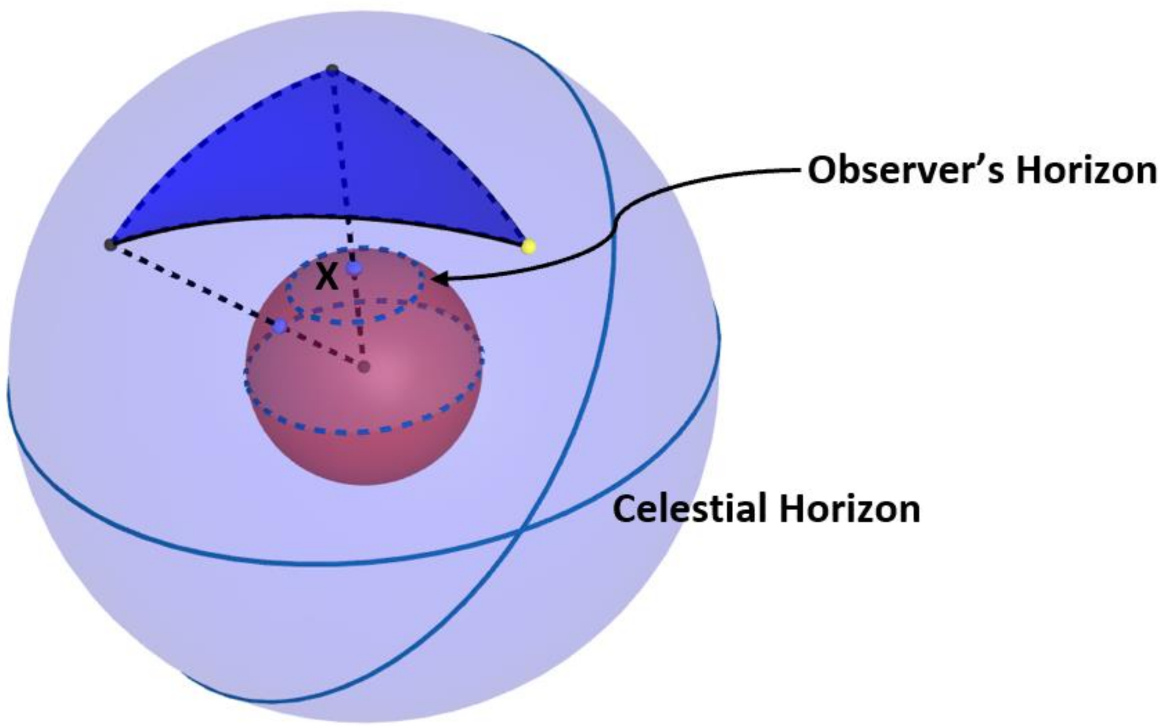
Celestial Sphere

The point P (**north celestial pole**) is located by drawing a line from the center of the Earth, through the North Pole (NP) and continuing up to the celestial sphere.

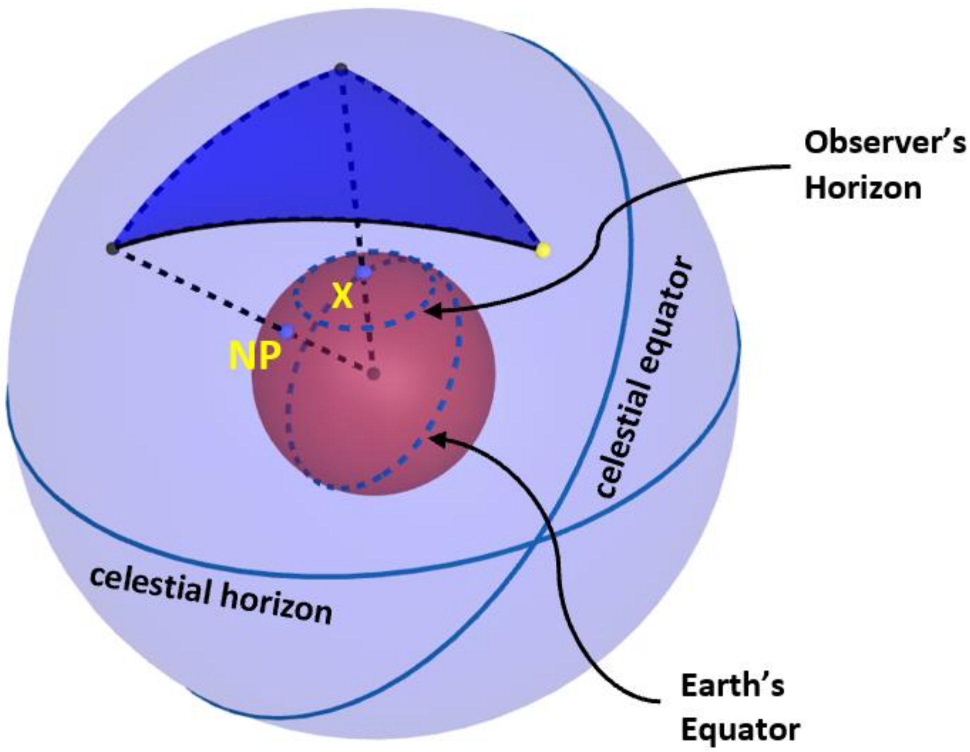
The point Z (**observer's zenith**) is located by drawing a line from the center of the Earth, through the point X (the observer's location) and continuing up to the celestial sphere.

The point S can be any star, but in these notes it will always be our Sun.

To understand the celestial horizon, suppose our observer at point **X** is on a tiny little island sitting in a vast ocean. Then our **observer's horizon** would be the border of the limited region they could see because of the curvature of the Earth. (On a clear day the observer's horizon would be a circle with a radius of about 2.5 miles.)

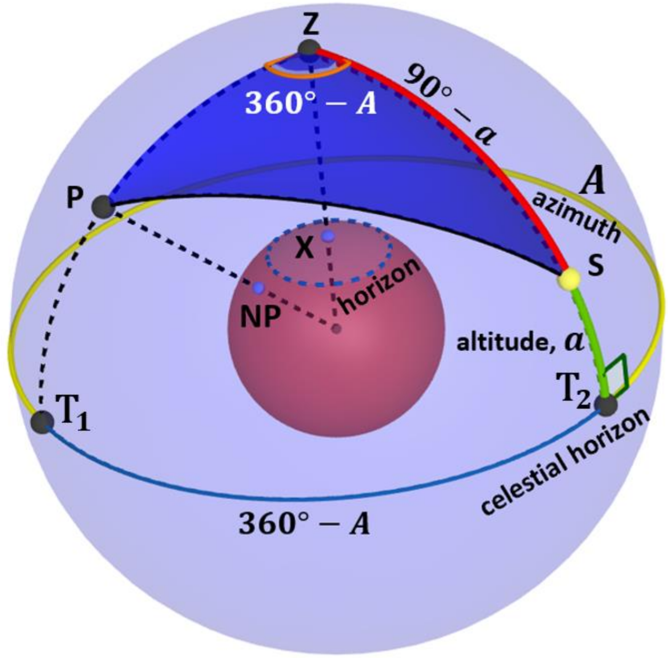


The **celestial horizon** is that imaginary circle created by drawing a line from **X** through every point on your earthly horizon and continuing onward to the celestial sphere.



The **celestial equator** is that imaginary circle created by drawing a line from the center of the Earth through every point on the Earth's equator and continuing onward to the celestial sphere.

Altitude and Azimuth



Consider the great arc $\widehat{ZS}$ continued on till it reaches the celestial horizon at a right angle at the point T_2. The measure in degrees of the green arc $\widehat{ST_2}$ is called the **altitude** of S (and in our case S is our Sun).

The length (in degrees) of the great arc from Z down to the celestial horizon at T_2 equals 90°.

Therefore, the red arc $\widehat{ZS}$ has length $90° - a$. $\widehat{ZS}$ is sometimes called the **co-altitude.**

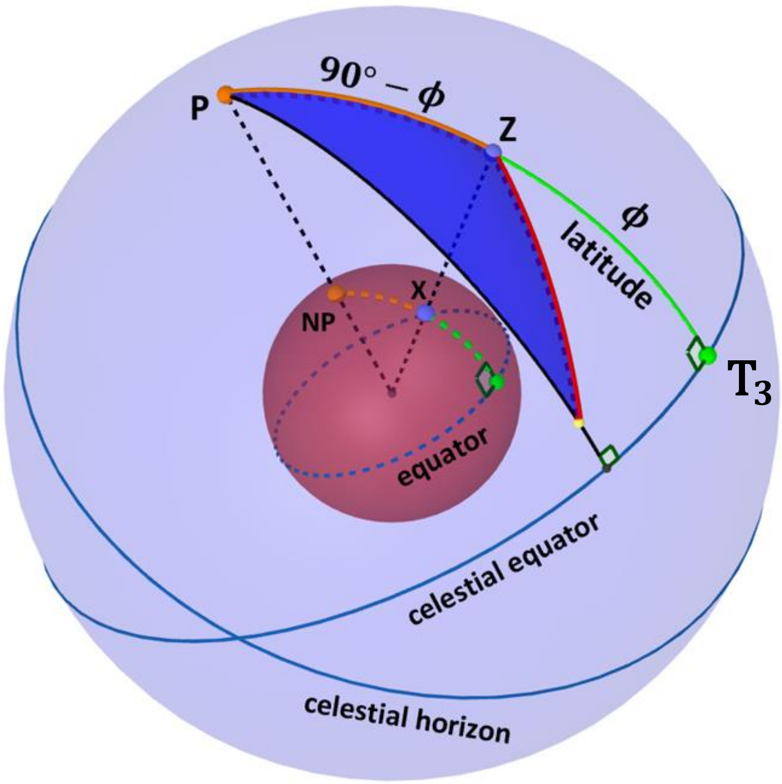
Consider the great arc $\widehat{ZP}$ continued on till it reaches the celestial horizon at a right angle at the point T_1.

The measure $\boldsymbol{A}$ of the great arc $\widehat{T_1T_2}$ in degrees going ***clockwise*** is called the **azimuth** of S.

It follows that the measure of the great arc $\widehat{T_1T_2}$ in degrees going ***counter-clockwise*** measures $360° - A$.

Furthermore, this tells us that the spherical angle $\sphericalangle PZS$ measure $360° - A$.

Celestial Latitude



The angular measure ϕ of the great arc connecting the observer's location on Earth at X to the earth's equator at a right angle is how latitude on Earth is defined.

It follows that the angular measure of the great arc $\widehat{ZT_3}$ connecting the zenith point Z to the celestial equator at a right angle will also equal ϕ.

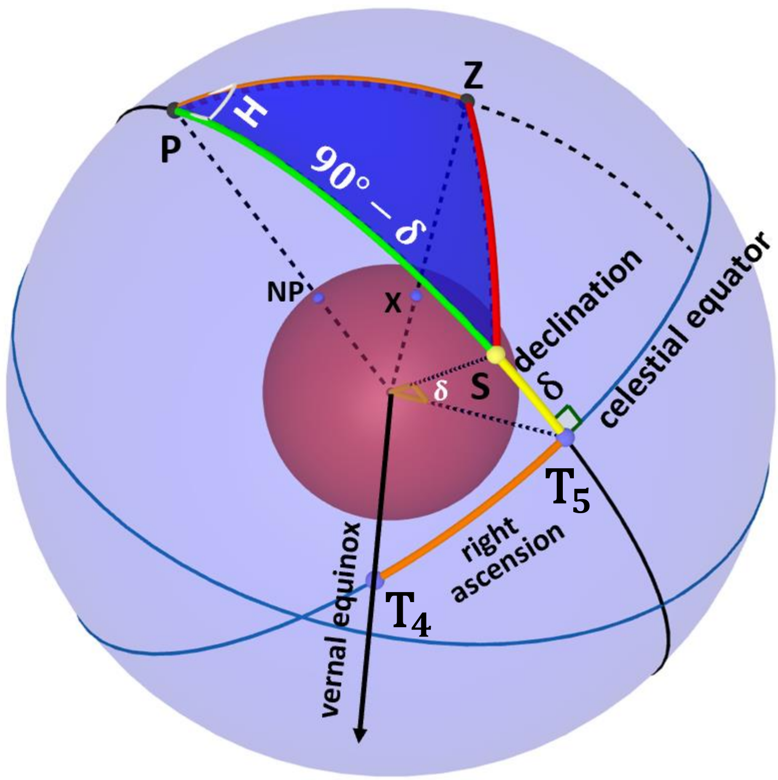
And because the angular measure of the great arc $\widehat{PZ_3}$ equals 90°, it follows that

$$\widehat{PZ} = 90° - \phi.$$

meridian). Clearly, $45 = 3 \cdot 15$, so in the above diagram, the sun is 3 hours past solar noon.

Declination and Right Ascension

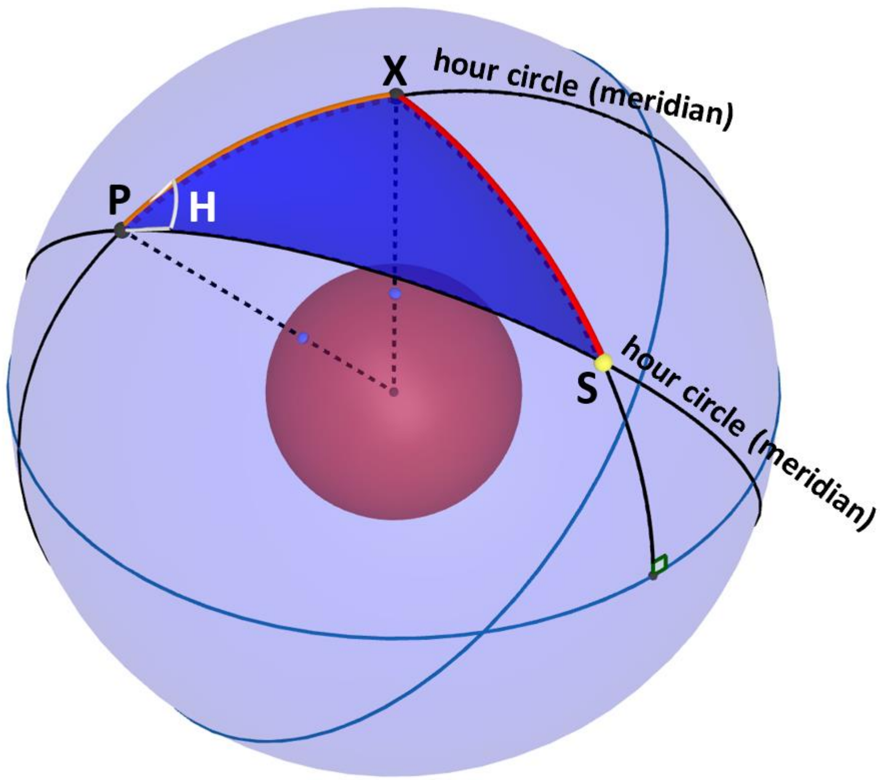
The measure δ of the great arc $\widehat{ST_5}$ connecting the star S to the celestial equator at a right angle is called the **declination** of the star S (which in our case will be the declination of the sun).



The point T_4 on the celestial equator where $\delta = 0°$ in the spring (about March 20) is called the vernal equinox.

The measure of the great arc connecting the point T_4 to the point T_5 going counterclockwise is called the **right ascension** of the star (sun) S.

By contrast on the Winter and Summer Solstice (respectively, the shortest and longest days of the year), the declination $\delta = 23.45$, its maximum value and $\delta = -23.45$, its minimum value.



Hour Angle H

Great circles on the celestial sphere that go through the two poles of the celestial sphere are called ***hour circles*** (and sometimes called meridians).

Hour circles on the celestial sphere are analogous to meridians (or longitudes) on Earth (terrestrial sphere) which are great circles on Earth that go through both the North Pole and South Pole.

The ***hour angle***, $\boldsymbol{H}$, is the angle between the hour circle that goes through X (the observer) and the hour circle that goes through S (the sun).

The hour circle that goes through X (the observer) is generally called the ***local meridian***.

The west moving angular displacement of the sun (due to rotation of the earth on its axis at $15°$ per hour) with morning being negative and afternoon being positive. That is, the **hour angle is changing at the rate of $\mathbf{15°}$ per hour**.

- When the sun is east of the local meridian (*i.e.* the morning), H, the hour angle, is negative.
- When the sun is on the local meridian then $H = 0°$. $H = 0°$ exactly at solar noon, when the sun achieves its maximum altitude (*i.e.* the highest point in the sky.
- When the sun is west of the local meridian (*i.e.* afternoon), H is positive.

In the above figure the hour angle $H = 45°$ (positive because the sun is west of the local meridian). Clearly, $45 = 3 \cdot 15$, so in the above diagram, the sun is 3 hours past solar noon.

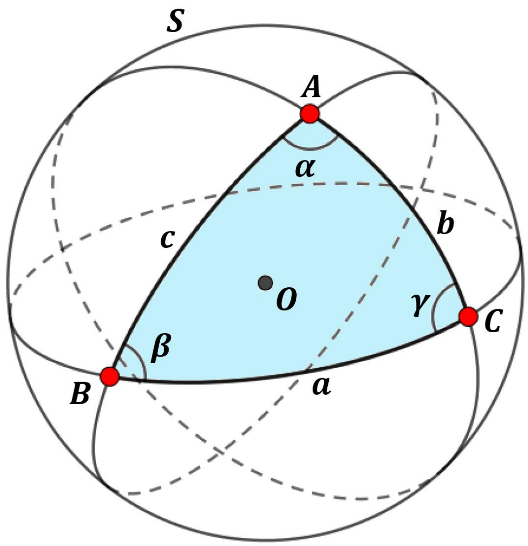
Summary

$\boldsymbol{\phi}$	**Observers Latitude**	$\boldsymbol{P}$	**Celestial Pole**
$\boldsymbol{a}$	**Altitude of the Sun**	$\boldsymbol{Z}$	**Observer's Zenith**
$\boldsymbol{\delta}$	**Declination of the Sun**	$\boldsymbol{S}$	**Sun**
$\boldsymbol{H}$	**Solar Hour Angle (negative in mornings, positive in afternoons)**	$\boldsymbol{NP}$	**North Pole**
$\boldsymbol{A}$	**Azimuth**	$\boldsymbol{X}$	**Observer**
$\boldsymbol{\omega_0}$	**The solar hour angle at either sunrise or sunset, negative for sunrise and positive for sunset.**		

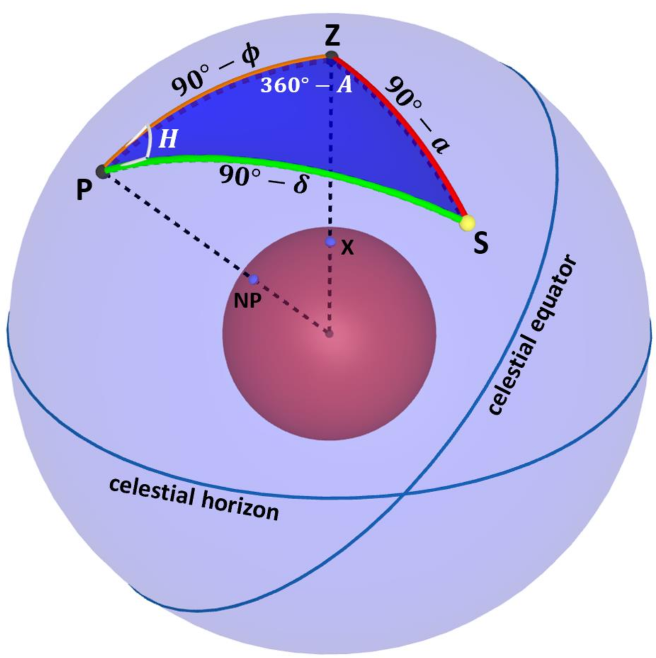
Relating Latitude, Declination and Maximum Altitude

The spherical law of cosines for sides states that

$$\cos(a) = \cos(b)\cos(c) + \sin(b)\sin(c)\cos(\alpha)$$



Applied to the astronomical triangle we find



$$\cos(90 - a) = \cos(90 - \phi)\cos(90 - \delta) + \sin(90 - \phi)\sin(90 - \delta)\cos(H)$$

or

$$\sin(a) = \sin(\phi)\sin(\delta) + \cos(\phi)\cos(\delta)\cos(H).$$

It follows that

$$a = \sin^{-1}\left(\sin(\phi)\sin(\delta) + \cos(\phi)\cos(\delta)\cos(H)\ \right).$$

Solar noon is by definition the point during a day when the sun reaches its maximum altitude and at solar noon $H = 0°$.

Thus, on any given day

$$\max(a) =$$

$$\max\left(\sin^{-1}\left(\sin(\phi)\sin(\delta) + \cos(\phi)\cos(\delta)\cos(H)\ \right)\right)$$

$$= \sin^{-1}\left(\sin(\phi)\sin(\delta) + \cos(\phi)\cos(\delta)\cos(H = 0°)\right)$$

$$= \sin^{-1}(\sin(\phi)\sin(\delta) + \cos(\phi)\cos(\delta))$$

$$= \sin^{-1}\left(\cos(\delta - \phi)\right).$$

But we also know that

$$\cos(x) = \sin(90° + x).$$

So

$$\max(a) = \sin^{-1}\left(\cos(\delta - \phi)\right)$$

$$= \sin^{-1}\left(\sin\,(90° + \delta - \phi\right)$$

$$= 90° + \delta - \phi.$$

Theorem

$$\text{maximum altitude of the sun on a given day} = 90° + \delta - \text{latitude}$$

So, over the course of a year

$$\begin{aligned}&\text{maximum altitude of the sun during the year}\\ &= 90° - \text{latitude} + \text{maximum declination of the sun during that year}\\ &= 90° - \text{latitude} + 23.45°\end{aligned}$$

Problem 1.

When does the sun rise on the longest day of the year (at which time $\delta = +23°27'$) in Boston ($\phi = 42°21'$), and what is the length of the day from sunrise to sunset?

Solution

The altitude of the sun at sunrise is not 0 because of two factors, refraction and the fact that sunrise is

From

$$\sin(a) = \sin(\phi)\sin(\delta) + \cos(\phi)\cos(\delta)\cos(H)$$

we have

$$\cos(H) = \frac{\sin(a) - \sin(\phi)\sin(\delta)}{\cos(\phi)\cos(\delta)}.$$

$$\cos(H) = \frac{\sin\left(-\left((15+34)/60\right)°\right) - \sin(42°21')\sin(23°27')}{\cos(42°21')\cos(23°27')}$$

$$= \frac{\sin(-0.8333333333) - \sin(42.35)\sin(23.45)}{\cos(42.35)\cos(23.45)}$$

$$= -0.4168485326$$

Therefore,

$$H = 114.6357814°.$$

Notice that we took the altitude of the sun at sunrise to be

$$a = -\left(\frac{15 + 34}{60}\right)^{\circ}$$

instead of $a = 0$. Why?

There are two reasons.

The first reason is refraction of sunlight. Refraction at sunrise causes the sun to appear $34/60$ degrees higher than it really is. So we have to subtract $(34/60)°$.

The second reason is that this equation

$$\sin(a) = \sin(\phi)\sin(\delta) + \cos(\phi)\cos(\delta)\cos(H)$$

assumes the sun is a point in the sky (like other stars). This point represents the center of the sun. But sunrise is defined as when we first see the sun break the horizon. The center of the sun is $15/60$ degrees below that.

So we have to subtract an additional $(15/60)°$ from the assumed 0° for sunrise.

This gives an altitude of

$$a = -\left(\frac{15 + 34}{60}\right)^{\circ}$$

for the sun at sunrise.

The hour angle at a given moment is the angular distance between the sun and solar noon, the position of the sun at its maximum altitude.

The sun travels $15°$ every hour. So, it takes

$$\frac{114.6357814}{15} = 7.642385427 \text{ hours}$$

$$= 7 \text{ hours } 38 \text{ minutes } 32 \text{ seconds}$$

under the given parameters the sun rises 7 hours 38 minutes 32 seconds before solar noon.

However solar noon (when the sun is at its maximum height) is not the same as noon on our clocks. The difference between “clock noon” and “solar noon” is called the ***equation of time*** and changes from day to day through a complicated cycle.

Fortunately, this cycle repeats itself every year and the daily equation of time is recorded for each day of the year.

Checking nautical tables (and/or the web) the equation of time correction for June 20, 2024 is $-1.71 = -(1 \text{ min } 43 \text{ sec.})$.

This means that we must add 1 minute and 43 seconds to our clock’s noon ($12{:}00$ p.m.) to get solar noon (which is also the time midway between sunrise and sunset).

So, on June 20, 2004 solar noon in Boston occurs at $12{:}01{:}43$ p.m.

From our calculations we know that sunrise at Boston on June 20, 2024 occurs 7 hours 38 minutes 32 seconds before solar noon.

Therefore, sunrise is at

$$12{:}01{:}43 \text{ p.m.} - 7 \text{ hours } 38 \text{ minutes } 32 \text{ seconds} = 4{:}23{:}11 \text{ a.m.}$$

and by symmetry, sunset is at

$$12{:}03{:}40 + 7 \text{ hours } 38 \text{ minutes } 32 \text{ seconds} = 19{:}41{:}72 = 7{:}42{:}12 \text{ p.m.}$$

But these times are not taking time zones or daily light savings time into account.

The longitude of Boston is $71.0589°$W which is $3.9411°$ before the beginning of the Eastern time zone (75 meridian)

Converting from degrees to time have that

$$3.9411 \text{ degrees} = 3.9411° \cdot \frac{24 \text{ hours}}{360°}$$
$$= 0.26274 \text{ hours} = 15 \text{ min } 46 \text{ seconds.}$$

So, we have to subtract 15 minutes 46 seconds from the non-time zone corrected time of sunrise in Boston.

$$4{:}\,23{:}\,11 \text{ a.m.} \;-\; 15 \text{ min } 46 \text{ seconds} \;=\; 4{:}\,07{:}\,25 \text{ a.m.}$$

Finally, we have to adjust for daily savings time by adding one hour. So sunrise on June 20, 2004 occurs at $5{:}\,07{:}\,25 \text{ a.m.}$ in Boston.

How much daylight was there on this day in Boston? We determined that the sun rose 7 hours 38 minutes 32 seconds before solar noon. And we know that solar noon is the midpoint in time of the hours of daylight.

Therefore, there were

$$7 \text{ hours } 38 \text{ minutes } 32 \text{ seconds} + \; 7 \text{ hours } 38 \text{ minutes } 32 \text{ seconds}$$

$$= 15 \text{ hours } 7 \text{ minutes } 4 \text{ seconds}$$

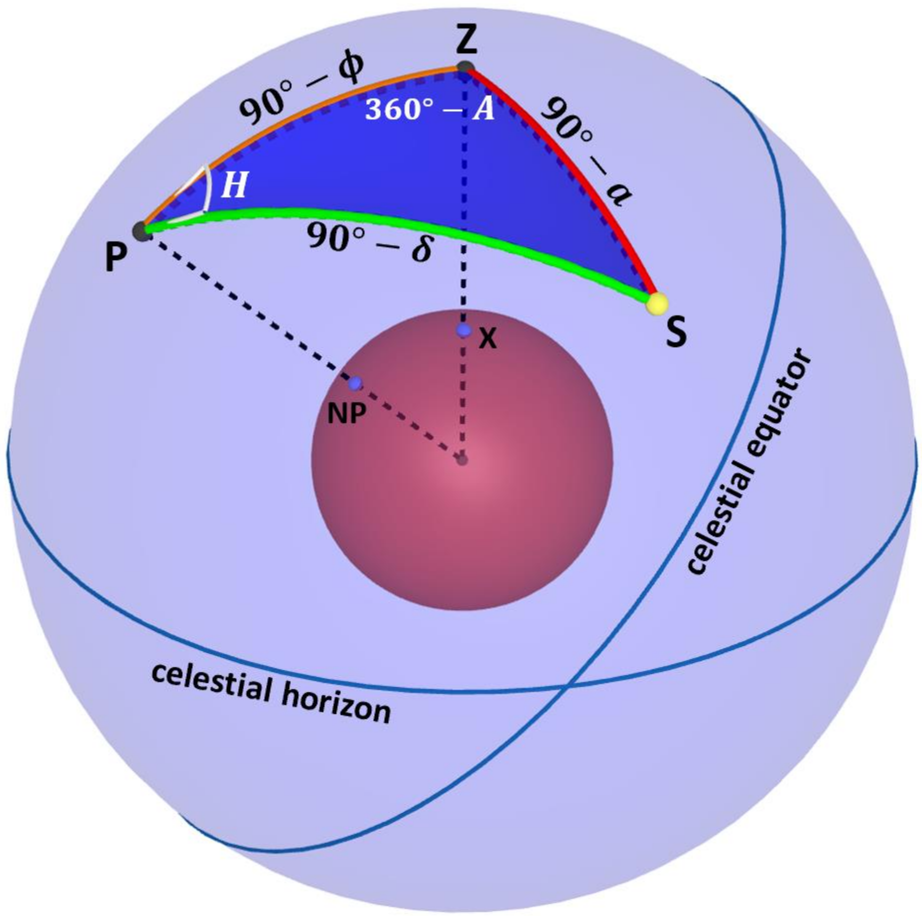
of daylight on June 20, 2004 in Boston.

Problem 2.

Find the latitude of an observer who finds that the altitude of the sun is $24°15'$ at $2{:}\,15$ local solar time, on a day when the sun's declination is $-12°20'$.

Solution

$$\sin(a) = \sin(\phi)\sin(\delta) + \cos(\phi)\cos(\delta)\cos(H).$$



$$\sin(a) = \sin(\phi)\sin(\delta) + \cos(\phi)\cos(\delta)\cos(H)$$

$$\sin(24°15') = \sin(\phi)\sin(-12°20') + \cos(\phi)\cos(-12°20'\delta)\cos(2 \text{ hours } 15 \text{ minutes})$$

Find the latitude of an observer who finds that the altitude of the sun is $24°15'$ at $2{:}15$ local solar time, on a day when the sun's declination is $-12°20'$.

$$\sin(24°15') = \sin(\phi)\sin(-12°20') + \cos(\phi)\cos(-12°20'\delta)\cos(2 \text{ hours } 15 \text{ minutes})$$

$$\sin(24.25°) = \sin(\phi)\sin(-12.33333333°) + \cos(\phi)\cos(-12.33333333°)\cos(33.75°)$$

$$\boxed{0.8122805184\cos(\phi) + (-0.2135987722)\sin(\phi) = 0.4107188526}$$

So we need to find the latitude ϕ that satisfies this equation. We could resort to graphing but its actually possible to solve this equation analytically.

(And solving this analytically has been given on previous MSHSML exams.)

$$0.8122805184\cos(\phi) + (-0.2135987722)\sin(\phi)$$

$$= \sqrt{0.8122805184^2 + (-0.2135987722)^2} \cdot \cos(\phi - \theta)$$

$$= 0.8398952768 \cdot \cos(\phi - \theta)$$

$$\tan(\theta) = \frac{b}{a} = \frac{-0.2135987722}{0.8122805184} = -0.2629618307$$

$$\theta = \tan^{-1}(-0.2629618307) = -14.73305626$$

θ in fourth quadrant

$$\cos(\theta) = \frac{a}{\sqrt{a^2 + b^2}} \text{ and } \sin(\theta) = \frac{b}{\sqrt{a^2 + b^2}}$$

$$0.8122805184\cos(\phi) + (-0.2135987722)\sin(\phi) = 0.4107188526$$

$$0.8398952768 \cdot \cos(\phi + 14.73305626) = 0.4107188526$$

$$\cos(\phi + 14.73305626) = \frac{0.4107188526}{0.8398952768} = 0.4890119804 \text{ or } -0.4890119804$$

$$\phi + 14.73305626 = \cos^{-1}(0.4890119804) = 60.72433742 \text{ or } -60.72433742$$

$$\phi = 60.72433742 - 14.73305626 = 45.99128116 = 45°59'28.6''$$

or

$$\phi = -60.72433742 - 14.73305626 = -75°27'.$$

Useful Identity:

$$a\cos(x) + b\sin(x) = \sqrt{a^2 + b^2} \cdot \cos(x - \theta)$$

where θ is that angle in $0 \leq \theta \leq 2\pi$ such that

$$\cos(\theta) = \frac{a}{\sqrt{a^2 + b^2}} \text{ and } \sin(\theta) = \frac{b}{\sqrt{a^2 + b^2}}.$$

Note: We can determine which of the four quadrants θ belongs in according to which combination of positive and negative $\cos(\theta)$ and $\sin(\theta)$ are.

To understand why this works, notice that with these substitutions we have

$$a\cos(x) + b\sin(x) = \cos(x)\cos(\theta) + \sin(x)\sin(\theta) = \cos(x - \theta).$$

Furthermore, these are valid substitutions because with these substitutions we have $-1 \leq \cos(\theta) \leq 1$, $-1 \leq \sin(\theta) \leq 1$ and $\sin^2(\theta) + \cos^2(\theta) = 1$.

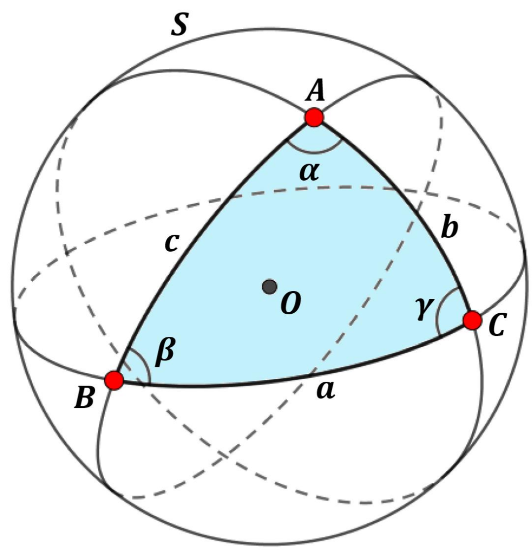
Problem 3.

In what latitude is the shortest day of the year just one hour long?

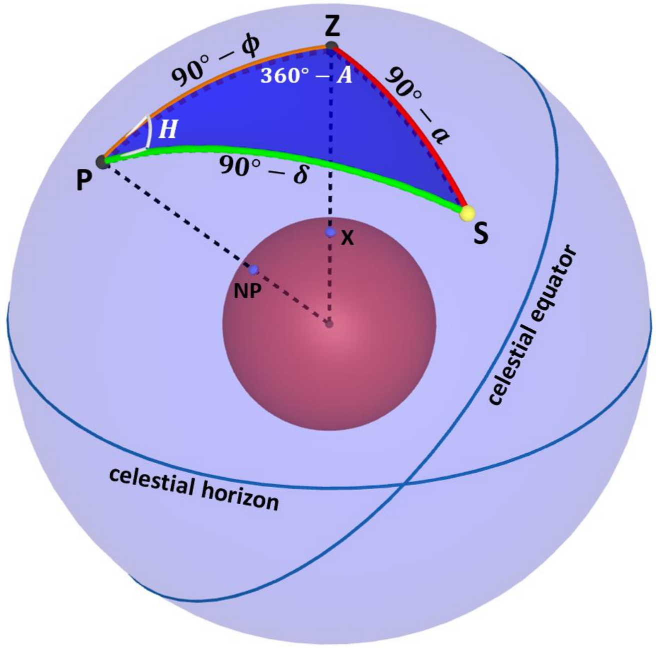
<u>Solution</u>

The spherical law of cosines for sides states that

$$\cos(a) = \cos(b)\cos(c) + \sin(b)\sin(c)\cos(\alpha)$$



Applying to the astronomical triangle



$$\cos(90 - a) = \cos(90 - \phi)\cos(90 - \delta) + \sin(90 - \phi)\sin(90 - \delta)\cos(H)$$

or

$$\sin(a) = \sin(\phi)\sin(\delta) + \cos(\phi)\cos(\delta)\cos(H).$$

On the shortest day of the year $\delta = -23.44°$. Because daylight is only 1 hour long, H at sunrise equals $1/2$ hour. Converted to an angle, at $15°$ per hour (remember, H is taken as negative in the morning) we have

$$H = -\frac{15°}{2}\text{degrees} = -7.5°.$$

Again using the factor that $a = -50/60$ degrees (and not $0°$) at sunrise we have

$$\sin\left(-\frac{50}{60}\right) = \sin(\phi)\sin(-23.44) + \cos(\phi)\cos(-23.44)\cos(-7.5°)$$

$$-0.01454389765 = \sin(\phi) \cdot (-0.3977885074) + \cos(\phi)(0.9174771405)(0.9914448614)$$

$$(0.9096279964)\cos(\phi) + (-0.3977885074)\sin(\phi) = -0.01454389765.$$

Notice this is another problem of the form

$$a\cos(\phi) + b\sin(\phi) = c$$

to solve.

$$\sqrt{(0.9096279964)^2 + (-0.3977885074)^2} \cdot \cos(\phi - \theta) = -0.01454389765$$

$$\tan(\theta) = \frac{b}{a} = \frac{-0.3977885074}{0.9096279964} = -0.4373089977$$

$$\theta = \arctan(-0.4373089977) = -23.62019162$$

$\cos(\theta) > 0, \sin(\theta) < 0$ so in 4th quadrant

$$\sqrt{(0.9096279964)^2 + (-0.3977885074)^2}\cos\big(\phi - (-23.62019162)\big) = -0.01454389765$$

$$\cos(\phi + 23.62019162) = \frac{-0.01454389765}{\sqrt{(0.9096279964)^2 + (-0.3977885074)^2}}$$

$$= \frac{-0.01454389765}{0.9928034994} = -0.0146493215$$

$$\phi + 23.62019162 = \cos^{-1}(-0.0146493215)$$

$$\phi = \cos^{-1}(-0.0146493215) - 23.62019162 = 67.2191827°$$
$$= 67°13'9''$$

Useful Identity:

$$a\cos(x) + b\sin(x) = \sqrt{a^2 + b^2} \cdot \cos(x - \theta)$$

where θ is that angle in $0 \leq \theta \leq 2\pi$ such that

$$\cos(\theta) = \frac{a}{\sqrt{a^2 + b^2}} \text{ and } \sin(\theta) = \frac{b}{\sqrt{a^2 + b^2}}.$$

Note: We can determine which of the four quadrants θ belongs in according to which combination of positive and negative $\cos(\theta)$ and $\sin(\theta)$ are.

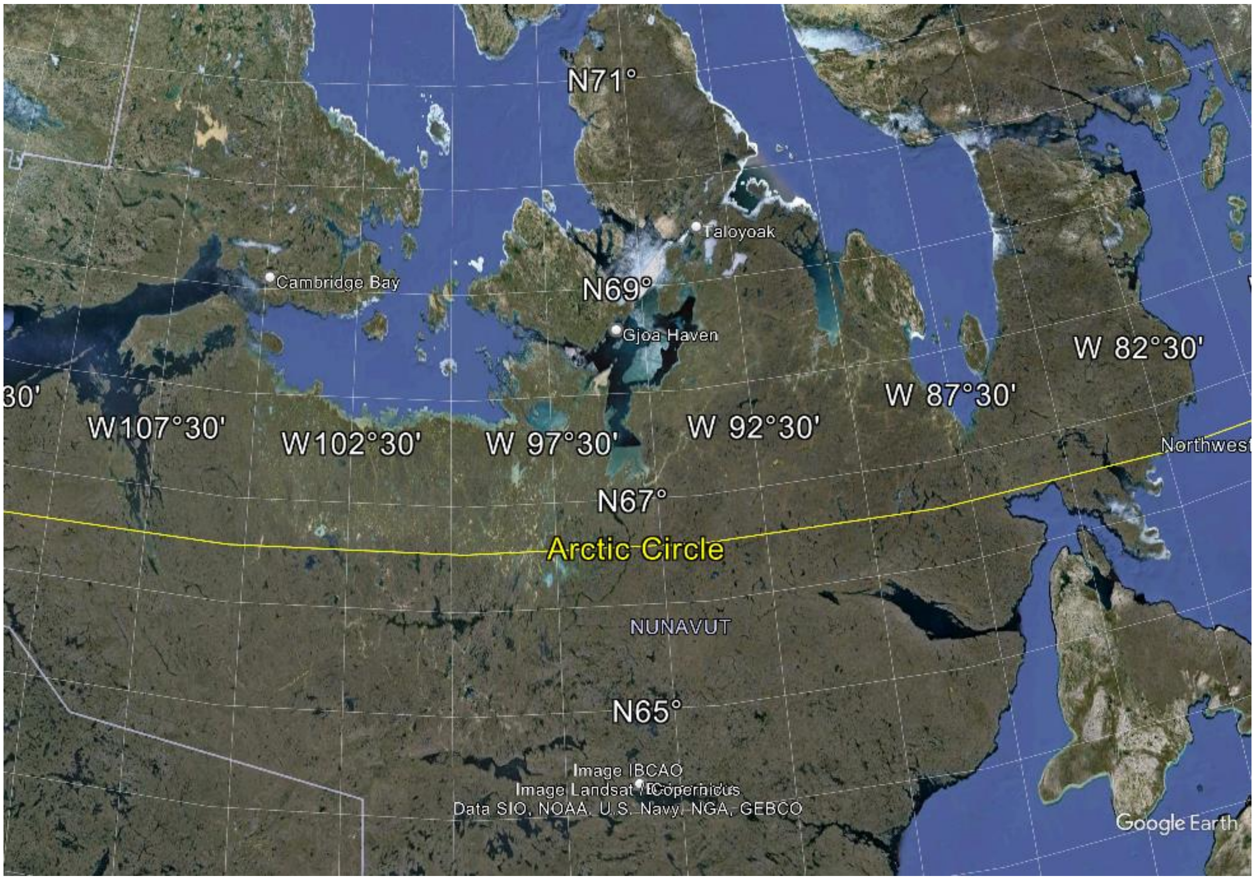
To understand why this works, notice that with these substitutions we have

$$a\cos(x) + b\sin(x) = \cos(x)\cos(\theta) + \sin(x)\sin(\theta) = \cos(x - \theta).$$

Furthermore, these are valid substitutions because with these substitutions we have $-1 \leq \cos(\theta) \leq 1$, $-1 \leq \sin(\theta) \leq 1$ and $\sin^2(\theta) + \cos^2(\theta) = 1$.

Where is latitude $\phi = 67°13'9''$?


Antimeridian
N75°
W 170°
W 30°
Arctic Circle
W 150°
N65°
W 50°
W 130°
W 70°
W 110°
W 90°
N55°
N45°
Data SIO, NOAA, U.S. Navy, NGA, GEBCO
Image Landsat / Copernicus
Image IBCAO
Image U.S. Geological Survey
Google Earth



Note the Arctic Circle is roughly at 66° 33' 49.83"N.

The imaginary circle round the earth, parallel to the equator, at latitude 66° 32′ N; it marks *the northernmost point at which the sun appears above the level of the horizon on the winter solstice*.

Problem 4

Find an observer's position if the sun's greatest altitude is $57°7'$ and is obtained at $3:43$ p.m. Greenwich time, on a day when the sun's declination is $14°13'$.

Your turn!

Hint:

$$\text{maximum altitude of the sun on a given day} = 90° + \delta - \text{latitude}$$

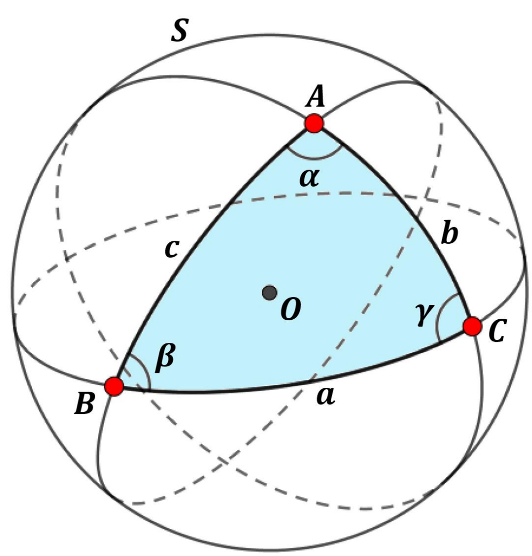
Problem 5.

What is the latitude of the place at which the sun rises exactly in the northeast on the longest day of the year?

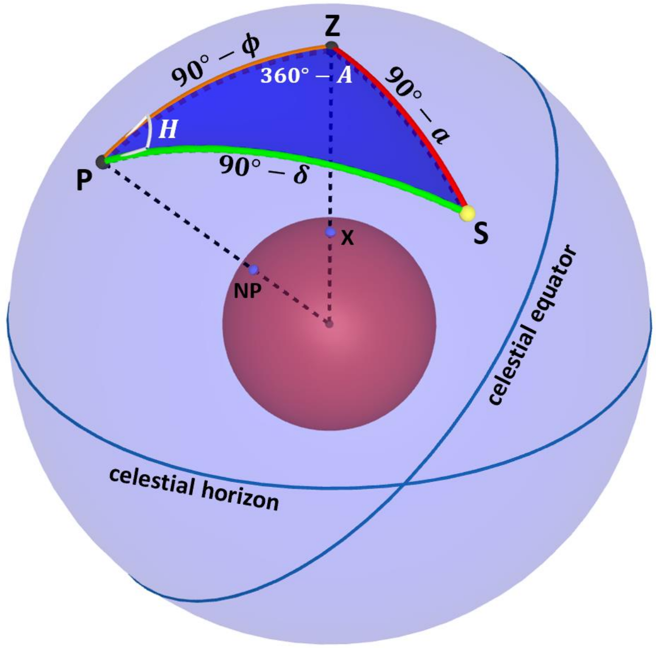
Solution

The spherical law of cosines for sides states that

$$\cos(a) = \cos(b)\cos(c) + \sin(b)\sin(c)\cos(\alpha)$$



Applying to the astronomical triangle



$$\cos(90 - \delta) = \cos(90 - \phi)\cos(90 - a) + \sin(90 - \phi)\sin(90 - a)\cos(360 - A)$$

If the sun rises in exactly the Northeast then it rises $45°$ east of north. Thus, the azimuth, A equals $45°$.

Remember that on the longest day of the year $\delta = 23.44$ and that in general, at sunrise $a = -\frac{50}{60}$.

Now we can use the fact that $A = 45°$.

$$\sin(23.44) = \sin(\phi)\sin\left(-\frac{50}{60}\right) + \cos(\phi)\cos\left(-\frac{50}{60}\right)\cos(360 - 45)$$

$$0.3977885074 = \sin(\phi)\,(-0.01454389765) + \cos(\phi)\,(0.9998942319)(0.7071067812)$$

$$= (0.7070319919)\cos(\phi) + (-0.01454389765)\sin(\phi)$$

$$= \sqrt{(0.7070319919)^2 + (-0.01454389765)^2} \cdot \cos(\phi - \theta)$$

where

$$\tan(\theta) = \frac{-0.01454389765}{0.7070319919} = -0.02057035299$$

$$\theta = -0.02057035299$$

Cos pos, sin neg, 4th quadrant

$$0.7071815626 \cdot \cos\big(\phi - (-0.02057035299)\big) = 0.3977885074$$

$$\cos(\phi + 0.02057035299) = \frac{0.3977885074}{0.7071815626} = 0.562498414$$

$$\phi + 0.02057035299 = \cos^{-1}(0.562498414) = 55.77124358$$

$$\phi = 55.77124358 - 0.02057035299 = 55.75067323$$

$$= 55°45'2''$$

Useful Identity:

$$a\cos(x) + b\sin(x) = \sqrt{a^2 + b^2} \cdot \cos(x - \theta)$$

where θ is that angle in $0 \leq \theta \leq 2\pi$ such that

$$\cos(\theta) = \frac{a}{\sqrt{a^2 + b^2}} \text{ and } \sin(\theta) = \frac{b}{\sqrt{a^2 + b^2}}.$$

Note: We can determine which of the four quadrants θ belongs in according to which combination of positive and negative $\cos(\theta)$ and $\sin(\theta)$ are.

To understand why this works, notice that with these substitutions we have

$$a\cos(x) + b\sin(x) = \cos(x)\cos(\theta) + \sin(x)\sin(\theta) = \cos(x - \theta).$$

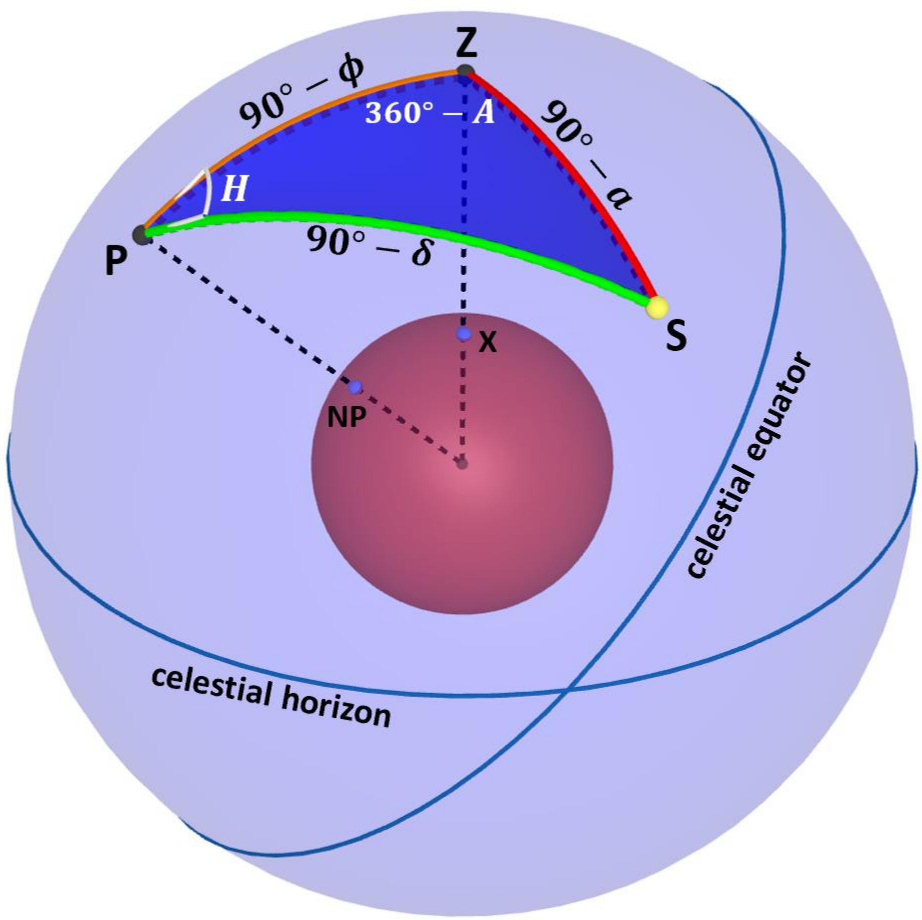
Furthermore, these are valid substitutions because with these substitutions we have $-1 \leq \cos(\theta) \leq 1$, $-1 \leq \sin(\theta) \leq 1$ and $\sin^2(\theta) + \cos^2(\theta) = 1$.

Problem 6.

On April 13th, 1937, the shortest shadow of a pole was exactly equal to its height, and pointed due north. This happened when a BBC British radio program timed for 12: 10 p.m. began. Where were these observations made?

<u>Solution</u>

The standard direction we need to take in celestial sphere problems is to figure out what all the parameters in the diagram of the astronomical sphere equal.

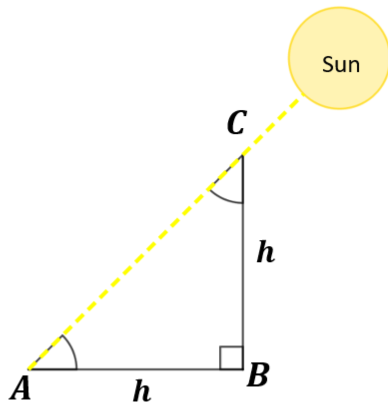


On any given day, the shortest shadow of an object occurs at solar noon. If the shadow points due north at solar noon then the direction of the sun must be due south and furthermore we must be in the northern hemisphere.

It follows that at solar noon the Azimuth $A = 180°$.

From declination tables we know that on April 13th, $\delta = 8°\ 50' = 8.833333333°$.

If the (vertical) pole $\overline{BC}$ has the same length as the shadow $\overline{BA}$ then ΔABC is a right isosceles triangle.



It follows that the altitude angle α of the sun at solar noon at this location equals $45°$.

H, the hour angle, measures the time between the observer's current time and solar noon. So in this problem $H = 0$.

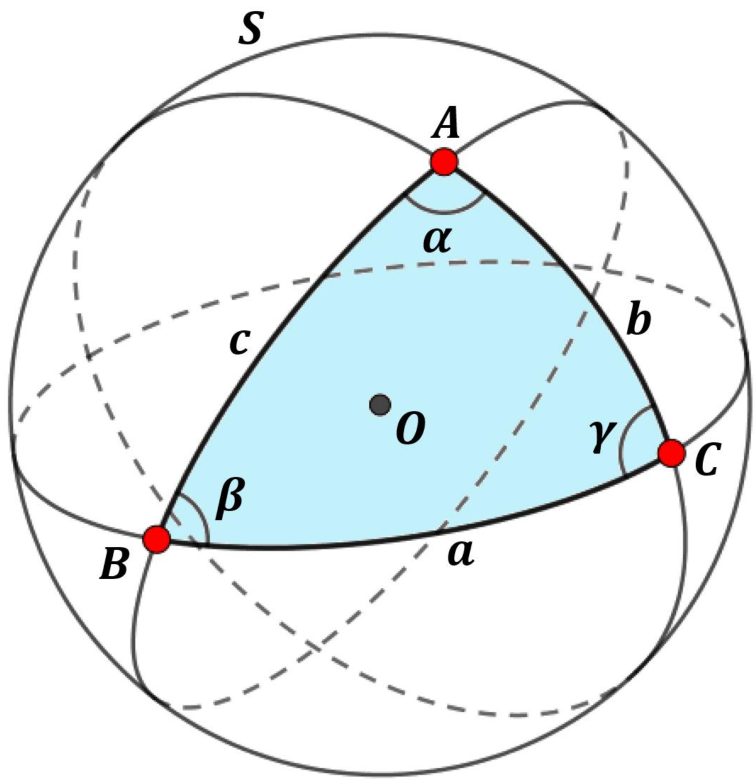
There is one more thing we can glean from this problem. We can find the clock time of solar noon. We will see how that will allow us to find the longitude of our observer.

But first we will concentrate on finding the latitude of the observer.

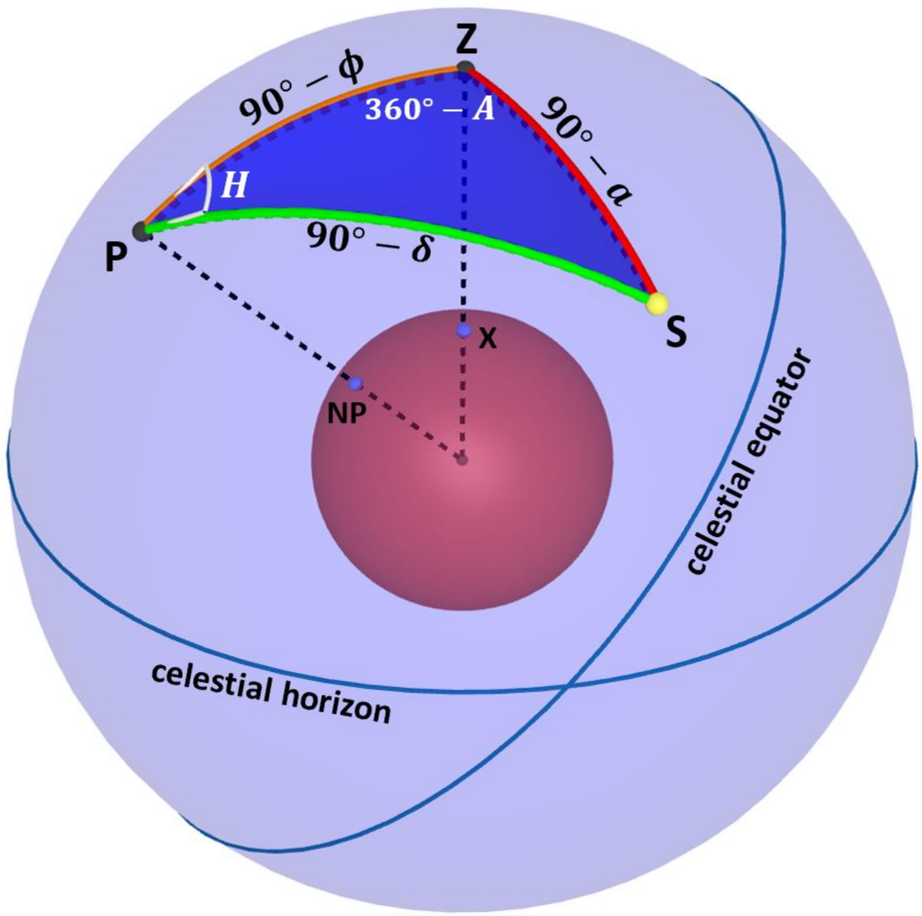
Now let's apply all of this information.

The spherical law of cosines for sides states that

$$\cos(a) = \cos(b)\cos(c) + \sin(b)\sin(c)\cos(\alpha)$$



Applying the spherical law of cosines to the astronomical triangle



we have

$$\cos(90 - \delta) = \cos(90 - \phi)\cos(90 - a) + \sin(90 - \phi)\sin(90 - a)\cos(360 - A)$$

which we can simplify to

$$\sin(\delta) = \sin(\phi)\sin(a) + \cos(\phi)\cos(a)\cos(360 - A)$$

Plugging in all the parameter values we have

$$\sin(8.833333333) = \sin(\phi)\sin(45) + \cos(\phi)\cos(45)\cos(360 - 180)$$

$$0.1535607383 = \left(\frac{1}{\sqrt{2}}\right)\sin(\phi) + \left(\frac{1}{\sqrt{2}}\right)(-1)\cos(\phi)$$

$$= \left(\frac{-1}{\sqrt{2}}\right)\cos(\phi) + \left(\frac{1}{\sqrt{2}}\right)\sin(\phi).$$

We could continue to solve problems of this type analytically, but at some point one gets tired and resorts to graphing!

Graphing we find the only feasible solution is $\phi = 53.8333°$.

Alternatively, we could solve for ϕ through the angle $H = 0$.

$$\cos(90 - a) = \cos(90 - \phi)\cos(90 - \delta) + \sin(90 - \phi)\sin(90 - \delta)\cos(H)$$
$$\sin(a) = \sin(\phi)\sin(\delta) + \cos(\phi)\cos(\delta)\cos(0)$$

$$\sin(45) = \sin(\phi)\,(0.1535607383) + (0.9881392107)$$

$$\frac{1}{\sqrt{2}} = \sin(\phi)\,(0.1535607383) + \cos(\phi)\,(0.9881392107)$$

Graphing we will once again find the only feasible solution is $\phi = 53.83333°$.

How do we find the longitude?

From the website https://planetcalc.com/9198/ the **equation of time** for April 13, 1937 is -0.603404 minutes. Recall that

Equation of Time = Solar Noon − Standard Clock Noon.

We know that in solar time, solar noon is just $12:00$ p.m.

So, we can use the above result to find the clock time of solar noon.

$$-0.603404 \text{ minutes} = 12:00 - \text{Standard Clock Noon}$$

or

$$\text{Standard Clock Noon} = 12:00:00 + 0.603404 \text{ minutes}$$
$$= 12:00:00 + 36.2024 \text{ seconds}$$
$$\approx 12:00:36$$

So, on April 13, 1937 solar noon at the observer's location occurs at $12:00:36$ p.m. and at that exact same moment it is $12:10:00$ at the BBC Broadcasting House in Westminster (a borough of London) with coordinates $(51° \, 31' \, 6.8'' \text{N} \;\; 00° \, 8' \, 37.8'' \text{W})$.

As BBC radio programs all come from their Broadcasting House based in London, then its 12:10 pm program starting time is 9 minutes 24 seconds $= 9.4$ minutes after my time.

As the (apparent) movement of the sun is from east to west, I must be west of the prime meridian.

Let my longitude be x.

That is $12:00:36$ at longitude x equals $12:10:00$ at longitude $00° \, 8' \, 37.8''\text{W}$. The sun moves $15°$ per hour or equivalently $0.25°$ per minute.

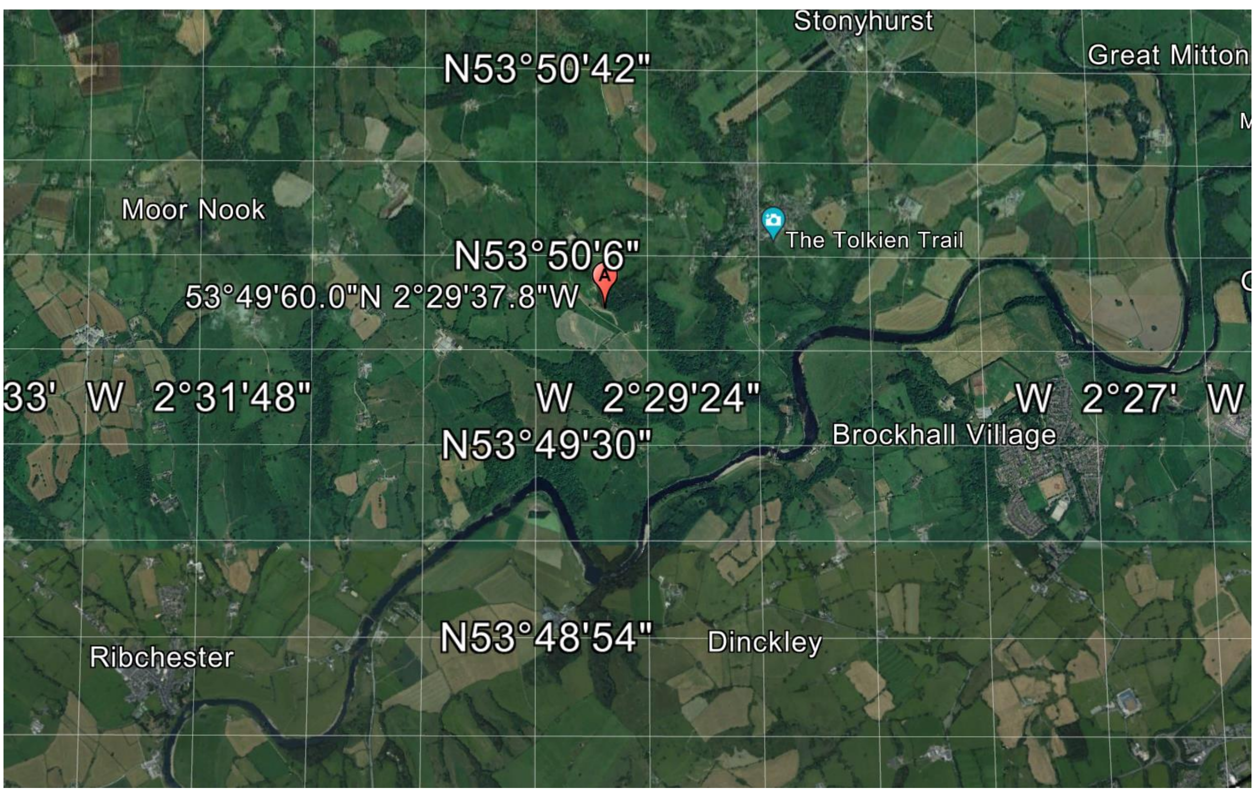
$$9.4 \text{ minutes} = 9.4 \text{ minutes} \cdot \frac{0.25°}{1 \, minute} = 2.35°.$$

So my longitude is $2.35°$ west of the meridian of the BBC. That is, my longitude is

$$00° \, 8' \, 37.8'' + \; 2.35° = 0.1438333333° + 2.35°$$
$$= 2.493833333 \text{ W}$$

So my coordinates (latitude, longitude) are $(53.83333° \text{ N}, 2.493833333 \text{ W})$.

Where is this? Google it!



So it looks like I'm in a wooded area not too far from *The Tolkien Trail*. Could this be Fangorn forest? Might I meet some Ents along the trail?

Problem 7

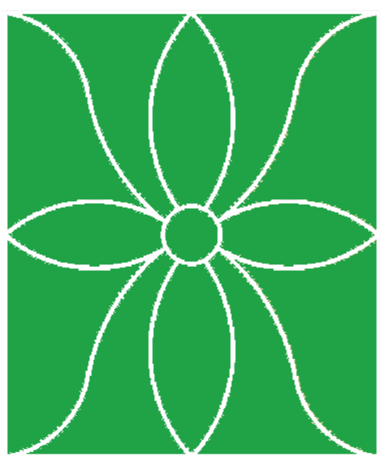
Sunrise and Sunset in Murphy Square

What are the times for sunrise and sunset in the center of Murphy Square at Augsburg University in Minneapolis, Minnesota on **Thursday, June 27, 2024?**

Murphy Square

Latitude	44.96527688701874° N
Longitude	93.2401503765309° W

Make sure to check out historic Murphy Square during your stay at Augsburg. It happens to be the oldest park in Minneapolis. Look carefully for the "hidden" flower made by the sidewalks. Perhaps a case of not being able to see the forsythia for the trees?



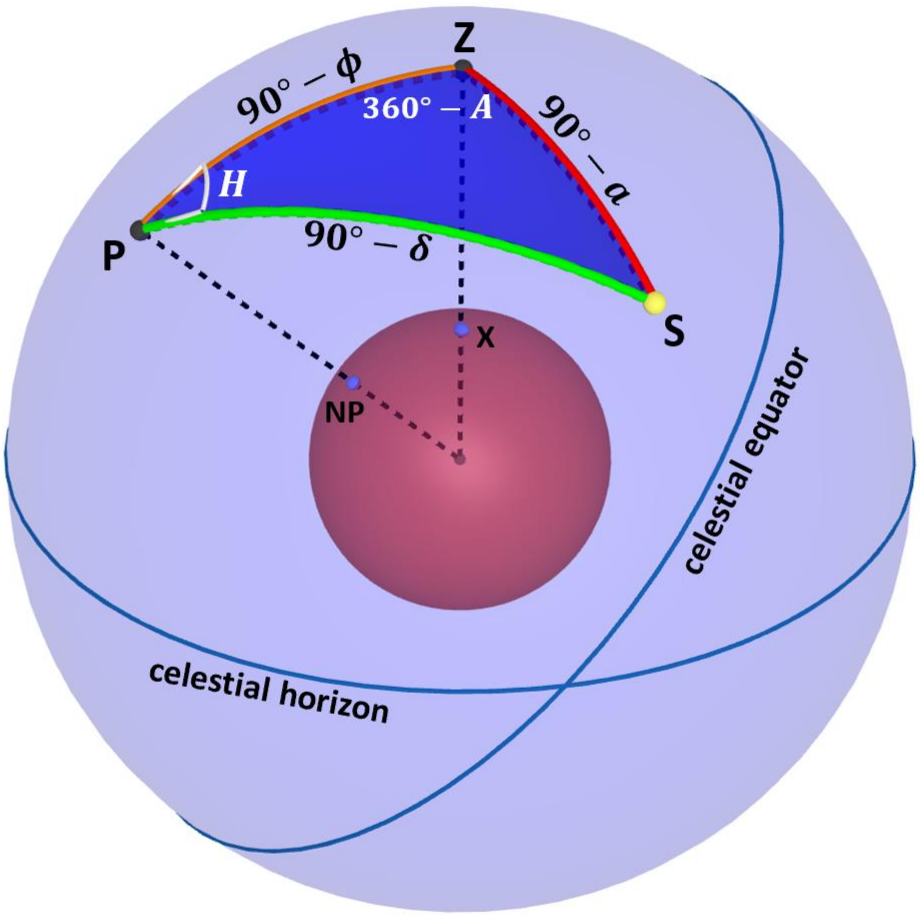
Your Turn Again.

Problem 8

Find the length of the day and the bearing of the rising sun at Minneapolis, latitude $45°\mathrm{N}$, on the shortest day of the year.

Solution

$$\sin(a) = \sin(\phi)\sin(\delta) + \cos(\phi)\cos(\delta)\cos(H).$$



Remember that on the shortest day of the year $\delta = -23.44°$ and when the sun is rising $a = -50/60$.

$$\sin\left(-\frac{50°}{60}\right) = \sin(45)\sin(-23.44) + \cos(45)\cos(-23.44)\cos(H)$$

$$\cos(H) = \frac{\sin\left(-\frac{50°}{60}\right) - \sin(45)\sin(-23.44)}{\cos(45)\cos(-23.44)} = 0.4111495681$$

$$H = 65.72293065°$$

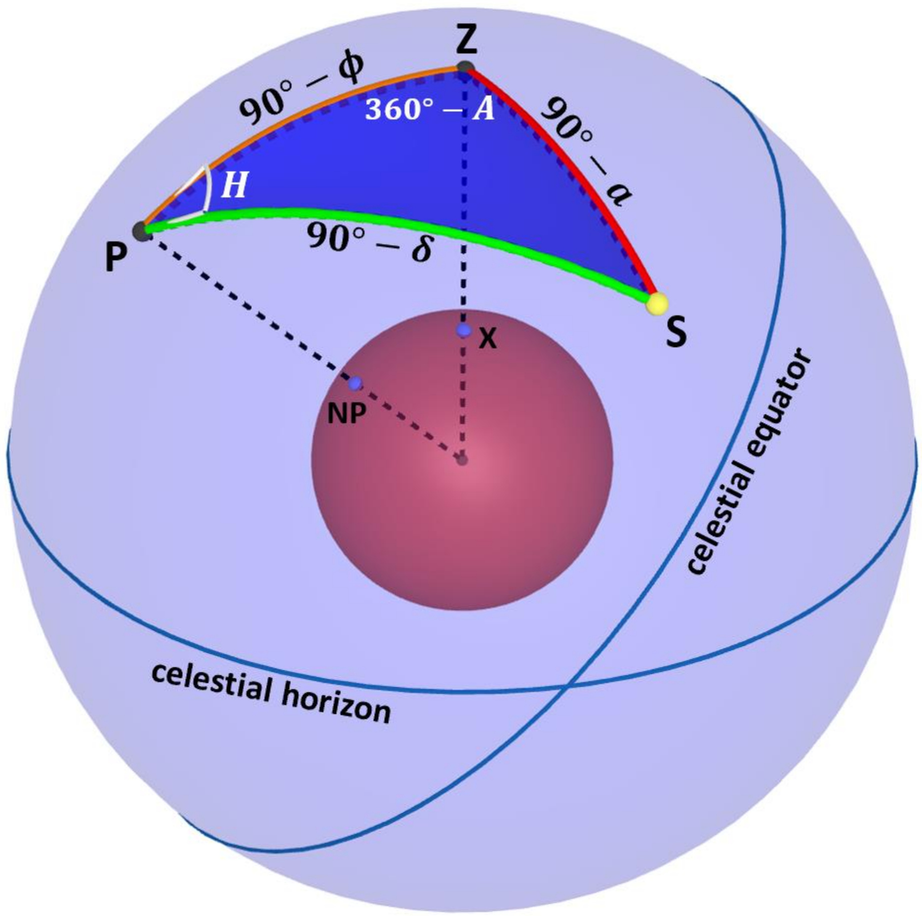
But this is in the morning so $H = -65.72293065°$.

$$\frac{65.72293065}{15} \cdot 2 = 8.76305742\ hours$$
$$= 8 \text{ hours } 45 \text{ minutes}$$

Do I need to make any adjustments for the Equation of Time, Time Zones or Daylight Savings Time?

No. Why not?

The second part of this problem is to find the bearing of the sun.



$$\cos(90-\delta) = \cos(90-\phi)\cos(90-a) + \sin(90-\phi)\sin(90-a)\cos(360-A)$$

$$\delta = -23.44°, a = -50/60, \phi = 45°$$

Find the length of the day and the bearing of the rising sun at Minneapolis, latitude 45°N, on the shortest day of the year.

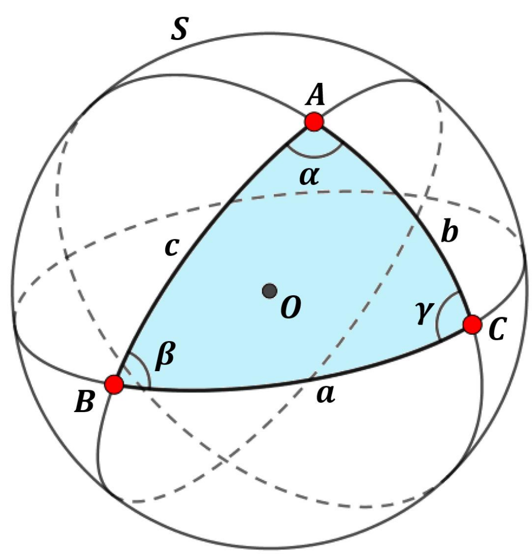
$$\sin(-23.44) = \sin(45)\sin\left(-\frac{50}{60}\right) + \cos(45)\cos\left(-\frac{50}{60}\right)\cos(360-A)$$

$$\cos(360 - A) = \frac{\sin(-23.44) - \sin(45)\sin\left(-\frac{50}{60}\right)}{\cos(45)\cos\left(-\frac{50}{60}\right)}$$

$$= -0.548071973$$

$$360 - A = 123.2348425°$$

$$A = 236.7651575° \text{ or } S\ 56.7651575°\ E = S\ 56°45.9'E$$



Let r equal the radius of sphere S. Then

$$\cos(a) = \cos(b)\cos(c) + \sin(b)\sin(c)\cos(\alpha)$$

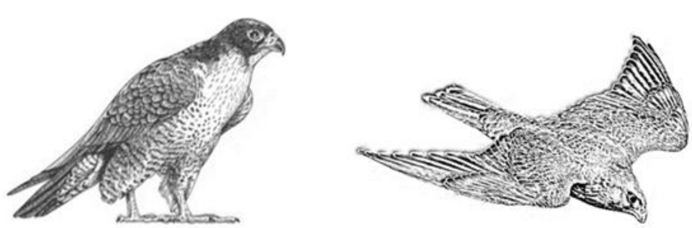
Problem 9

Virtual Geocache

The following is an entry from a sea captain's log dating from 1910. Your virtual geocache challenge is to find the name of the lighthouse mentioned in the historic log and to determine its latitude and longitude.

February 22. We sailed up the coastline for most of the day. By midafternoon we reached a cape jutting out into the Atlantic. The cape was appealing so we dropped anchor at the bottom south side and rowed our skiff ashore. We took time to explore and soon chanced upon an unobstructed view of a beautiful sunset. That night we slept on deck to enjoy the sea air which never got below fifty degrees [Fahrenheit].

February 23. We rose early that morning to the sun rising over the ocean. We opted for a fresh look of the lighthouse we had spotted the day before. We were struck by the familiar and pleasing aroma of the old growth yellow pine wide plank stairs of the lighthouse. At the top of the tower, from a height of one hundred and ten feet, we could study the marshes and wildlife. We recorded having seen any number of great egrets, blue herons and white ibises and a smaller number of raptors including peregrine falcons and merlins.

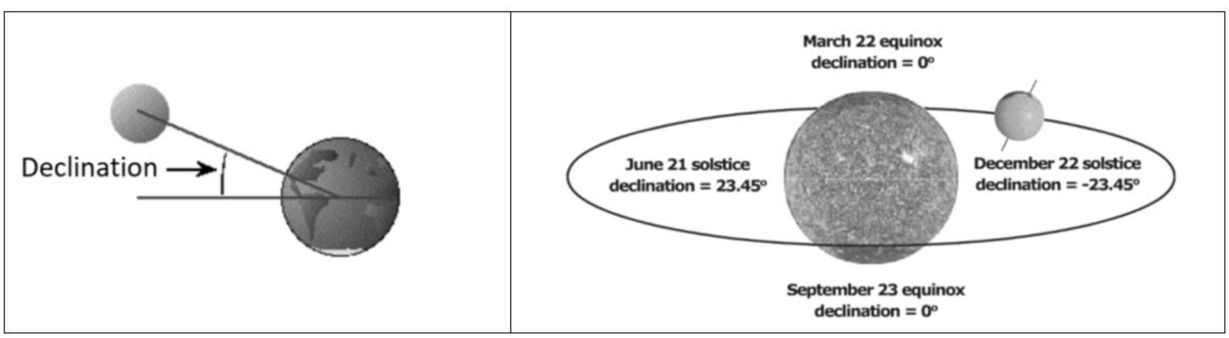


Sketches of Peregrin Falcon from 2/23/1910

We took photographs yesterday and today, one taken only moments before sunset on our first day and the other of the lighthouse we visited.

For the log, sunset occurred at 6:02:00 p.m., sunrise at 6:48:48 a.m. and from the ephemeris tables the declination of the sun was $-10.11°$.

Recall from astronomy that sunset and sunrise are defined as the moments when the upper limb of the sun has just disappeared below or appeared above horizon. Also recall that declination is the angle δ formed by the line connecting the centers of the Earth and Sun and the Earth's equator. Declination varies seasonal between -23.45 and 23.45. Declination angles are tabled in nautical almanacs.



https://www.pveducation.org/pvcdrom/properties-of-sunlight/declination-angle

Solution

The sunrise equation is

$$\cos(\omega_o) = \frac{\sin(a) - \sin(\phi)\sin(\delta)}{\cos(\phi)\cos(\delta)}$$

where

$\boldsymbol{\omega_0}$ **is the solar hour angle** at either sunrise or sunset, negative for sunrise and positive for sunset.

One way to picture ***solar hour angle*** is to image a clock with two hour hands (and no minute hands) where the two hour hands point to different times. Let one hour hand point to the time when the sun rises and let the other hour hand point to the time (called ***solar noon***) when the sun is at its highest point in the sky on a given day. The angle between these two hour hands is the solar hour angle for that day.

It is useful to know that the time when the sun is at solar noon for that day is necessarily halfway *in time* between sunrise and sunset for that day. So, by symmetry, the angle between between solar noon and sunrise equals the angle between solar noon and sunset *in absolute value*. But definition the angle between solar noon and sunrise is taken to be negative (and the angle between solar noon and sunset is taken to be positive).

Also, to convert from hours to degrees remember that $360° = 24$ hours so

$$x \text{ hours} = \frac{360x}{24} = 15x \text{ degrees.}$$

So, in this problem we see that

$$\begin{aligned} \text{sunset} - \text{sunrise} &= 6{:}02{:}00 \text{ p.m.} - 6{:}48{:}48 \text{ a.m.} \\ &= 6{:}02 + (\text{noon} - 6{:}48{:}48) \\ &= 6{:}02 + 5{:}11{:}12 = 11{:}13{:}12 \text{ hours} \\ &= 11 + \frac{13}{60} + \frac{12}{60^2} = 11.22 \text{ hours.} \end{aligned}$$

Therefore,

$$\begin{aligned} \omega_0 &= \frac{11.22}{2} = 5.61 \text{ hours} \\ &= 15(5.61)° = 84.15°. \end{aligned}$$

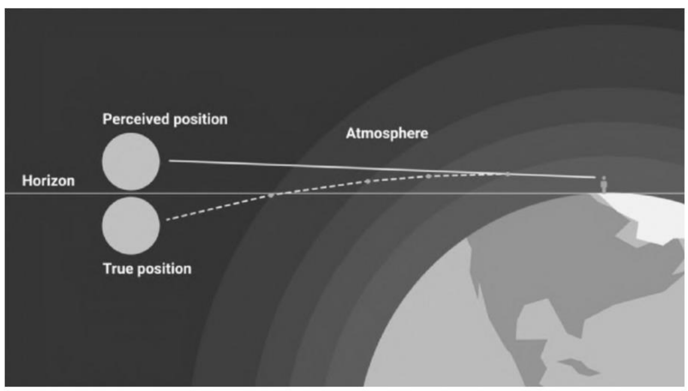
Note: Solar noon does *not* often occur at the same time as “noon” on your clock. When and if they do agree depends on many factors including longitude, time zones, daylight saving time among other factors.

ϕ **is the latitude** of the observer on Earth

δ **is the declination of the sun**

a **is the *altitude angle*** of the *center* point of the sun at sunrise. When the upper limb of the sun is on the horizon the center point will be $(16/60)°$ below the horizon. Altitude angle is also affected by refraction. The average correction for refraction is about $(34/60)°$. Together, these two corrects make $a = -\big((15+34)/60\big)°$ at sunrise instead of $0°$ as you might initially think.

> Recall that the density of Earth's atmosphere decreases with increasing altitude. Refraction is the bending of light according to Snell's Law as it passes through our atmosphere with varying density. Refraction makes celestial objects appear higher in the sky than they really are. This means that we start to see the Sun a few minutes *before* it actually rises above the horizon.



https://www.timeanddate.com/astronomy/refraction.html

Solving for Latitude

Notice that in this problem we know the altitude angle $a = -\big((15+34)/60\big)°$ at sunrise, we are given δ, the declination of the sun and we can calculate ω_0, the solar hour angle at sunrise, because we are the times of sunrise and sunset.

Thus, the only unknown in the sunrise equation

$$\cos(\omega_o) = \frac{\sin(a) - \sin(\phi)\sin(\delta)}{\cos(\phi)\cos(\delta)}$$

is ϕ, the latitude of our observer, which in theory means we can solve for ϕ. We start by rewriting the sunrise equation in the form

$$\Big(\sin(\delta)\Big)\sin(\phi) + \Big(\cos(\omega_0)\cos(\delta)\Big)\cos(\phi) = \sin(a).$$

Graphing

$$\Big(\sin(-10.11°)\Big)\sin(\phi) + \Big(\cos(84.15°)\cos(-10.11°)\Big)\cos(\phi) = \sin\left(-\frac{16+34}{60}\right)$$

we can use the zoom feature to determine that

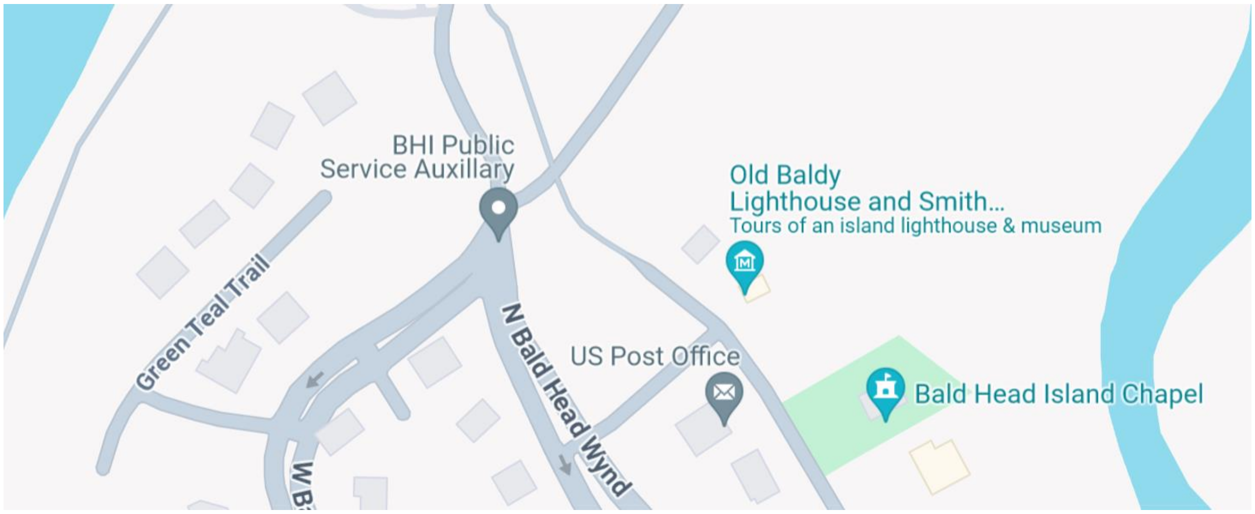
$$\text{latitude} = \phi \approx 0.59128519 \text{ radians} = 33.87815° = 33°53'.$$

But what about longitude? Fortunately, longitude does not require any extra mathematics. Longitude is fixed by the story line because we know the ship is on the Atlantic coastline. So, we need only follow a map along latitude $\phi = 33.87815° = 33°53'$ until we hit the Atlantic coastline.

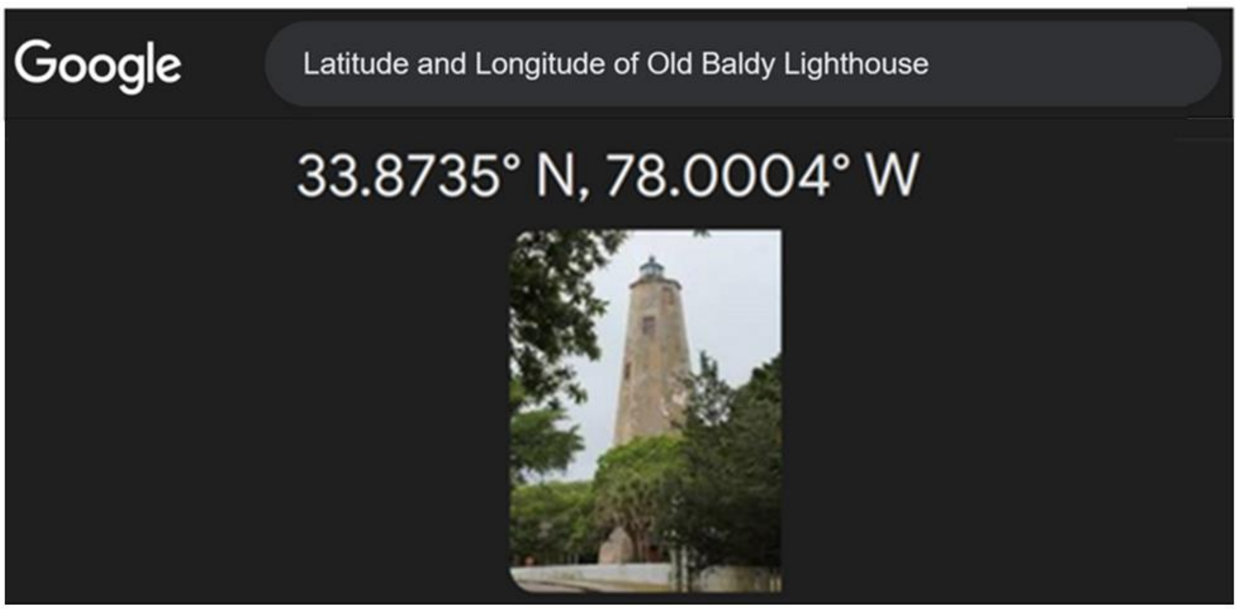
A first cut (using Google Earth) shows that latitude $33°53'$ hits the Atlantic coastline somewhere between Myrtle Beach, SC and Wilmington, NC.

Zooming in further we find that latitude $33°53'$ hits the Atlantic Ocean at Bald Head Island, NC.

We can pinpoint the lighthouse on Bald Head Island using Google Maps.



Finally, we can ask Google for the coordinates of Old Baldy Lighthouse.



Reminder: It is possible to solve the equation

$$A\sin(x) + B\cos(x) = C$$

analytically for known A, B and C. [You might recall that getting an explicit solution for this equation is part of the MSHSML Test 3 Event C curriculum.]

The strategy comes down to seeing that there exists a unique θ in $0 \leq \theta < 2\pi$ such that

$$A\sin(x) + B\cos(x) = \sqrt{A^2 + B^2} \cdot \sin(x + \theta)$$

where θ is determined from the equation

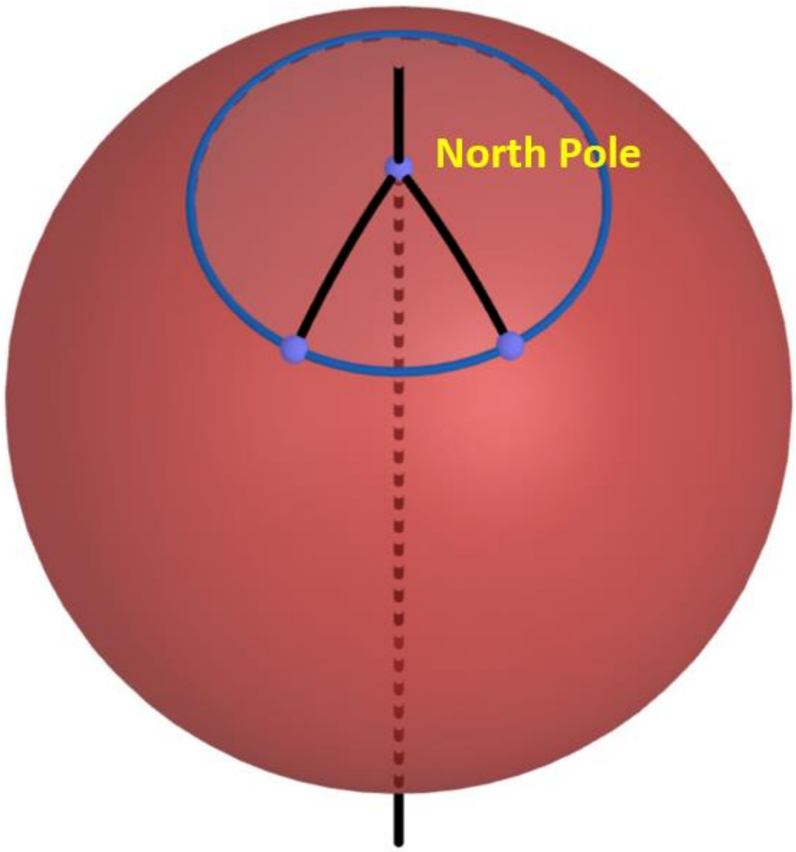
$$\tan(\theta) = \frac{B}{A}$$

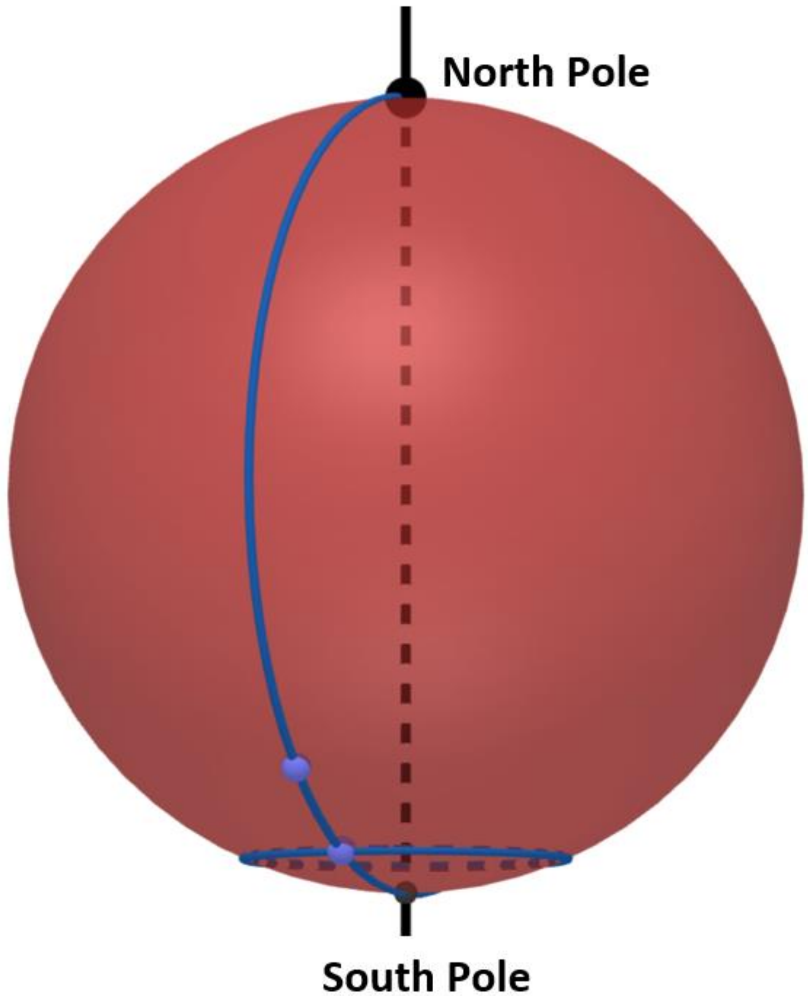
with the quadrant of θ determined by the pair of constraints

$$\cos(\theta) = \frac{A}{\sqrt{A^2 + B^2}} \text{ and } \sin(\theta) = \frac{B}{\sqrt{A^2 + B^2}}.$$

Problem 10

Suppose I start at some point A and hike one mile due south. Then I decided to hike one mile due west and finally hiked another mile due north. If I find myself back at point A, where did I start?


North Pole

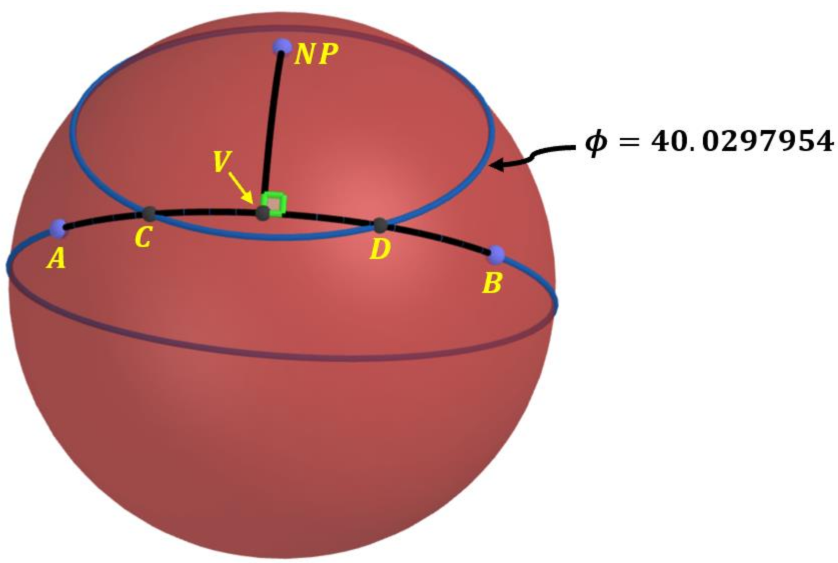

North Pole
South Pole

Homework

You are captaining a hover ship (it can sail over land or water) that is set to sail the great circle route from point B (latitude 23.91040°N, longitude 0.016490327°W) to point A (latitude 26.756156°N, longitude 123.627661°W).

But right after taking off your hover ship developed an engine problem that causes the engines to fail if you ever go north of the 40.0297954 north parallel.

Your new course plan is to sail the great circle path between A and B until this path reaches the 40.0297954 north parallel (point D on the diagram below). Then you will stay on the 40.0297954 north parallel until you get to point where the great circle path is once again below the 40.0297954 north parallel (point C). Then you will be able to safely continue on the great circle path all the way to point A.



(1) Find the latitude and longitude of V, the vertex of the great circle connecting points A and B.

(2) Find the latitude and longitude of points C and D.

Answers:

V (latitude 45.180439° N, longitude 63.694547° W)

C (latitude 40.0297954° N, longitude 97.1086735° W)

D (latitude 40.0297954° N, longitude 30.2804212462° W)