Linear and Circular Success Runs

Ken Suman

1 Sampling from a Line

Theorem 1

Suppose the numbers $1, \dots, t$ are arranged in a line in increasing order. Let $(X_1, \dots, X_n)$ be a random sample of size n taken without replacement from these t distinct numbers such that $1 \le X_1 < X_2 < \cdots < X_{n-1} < X_n \le t$.

Define the variables C_j by

$$\begin{aligned} C_1 &= X_1 \\ C_2 &= X_2 - X_1 \\ &\vdots \\ C_n &= X_n - X_{n-1} \\ C_{n+1} &= (t+1) - X_n. \end{aligned}$$

Then

$$\mathrm{E}\big(\Psi(C_1, \dots, C_{n+1})\big) = \frac{1}{\binom{t}{n}(t+1)!} \times \frac{d^{t+1}}{d\theta^{t+1}} \left(\left(\frac{\theta}{1-\theta}\right)^{n+1} \mathrm{E}\big(\Psi(Z_1, \dots, Z_{n+1})\big) \right) \Bigg|_{\theta=0}$$

where Z_1, …,Z_{n+1} are independent and identically distributed 1-shifted geometric$(1-\theta)$ random variables (see Definition 6 in the appendix).

Lemma 2

Let $\mathbb{S}^{n+1}$ be the product space $\{1,2, \dots\} \times \cdots \times \{1,2, \dots\}$ and let $\mathbb{S}^{n+1}_{t+1}$ be the set of all vectors $(s_1, \dots, s_{n+1})$ in $\mathbb{S}^{n+1}$ such that $s_1 + \cdots + s_{n+1} = t + 1$ and where $t + 1 \ge n + 1$.

Let $\mathcal{A} \subset \mathbb{S}^{n+1}$ and define $\mathcal{A}_t = \mathcal{A} \cap \mathbb{S}_{t+1}^{n+1}$. Then for all $t+1 \geq n+1$ we have

$$P\big((C_1, \dots, C_{n+1}) \in \mathcal{A}_t\big) = \frac{1}{\binom{t}{n}(t+1)!} \times \frac{d^{t+1}}{d\theta^{t+1}}\left(\left(\frac{\theta}{1-\theta}\right)^{n+1} P\big((Z_1, \dots, Z_{n+1}) \in \mathcal{A}\big)\right)\Bigg|_{\theta=0}$$

and

$$N\big(\text{selections of } n \text{ numbers from } 1,2,\dots,t \,\big|\, (C_1, \dots, C_{n+1}) \in \mathcal{A}_t\big) = \frac{1}{(t+1)!}\frac{d^{t+1}}{d\theta^{t+1}}\left(\left(\frac{\theta}{1-\theta}\right)^{n+1} P\big((Z_1, \dots, Z_{n+1}) \in \mathcal{A}\big)\right)\Bigg|_{\theta=0}.$$

We will first give some examples illustrating Theorem 1 and will then follow with the proof.

Example 3 (A selection of success run problems.)

Suppose we arrange m S's and n F's (successes and failures) in a line by randomly selecting numbers from $1, \dots, m+n$ to represent the positions in our line of the n F's.

Define the variables

L : length of the longest success run
R : number of success runs of length r
W : number of success runs of length at least r.

Then

$$P(L \leq k) = \sum_{j=0}^{n+1} (-1)^j \frac{\binom{n+m-(k+1)j}{m-(k+1)j}\binom{n+1}{j}}{\binom{n+m}{m}} \tag{1}$$

$$\mathrm{E}\big(L_{(v)}\big) = v \sum_{r=0}^{m} \sum_{j=1}^{n+1} (-1)^{j-1} \frac{\binom{n+m-(r+1)j}{m-(r+1)j}\binom{n+1}{j}}{\binom{n+m}{m}} r_{(v-1)} \tag{2}$$

$$P(R=k)=\frac{1}{\binom{m+n}{m}}\sum_{j=k}^{n+1}(-1)^{j-k}\binom{j}{k}\binom{n+1}{j}\binom{m+n-(r+1)j}{m-rj} \tag{3}$$

$$\mathrm{E}\left(R_{(v)}\right)=\frac{v!\binom{m+n-(r+1)v}{m-rv}\binom{n+1}{v}}{\binom{m+n}{n}} \tag{4}$$

$$P(W=k)=\frac{1}{\binom{m+n}{m}}\sum_{j=k}^{n+1}(-1)^{j-k}\binom{j}{k}\binom{n+1}{j}\binom{m+n-rj}{m-rj} \tag{5}$$

and

$$\mathrm{E}\left(W_{(v)}\right)=\frac{v!\binom{m+n-rv}{m-rv}\binom{n+1}{v}}{\binom{m+n}{n}}. \tag{6}$$

The probabilities $P(L\leq k)$, $P(R=k)$, and $P(W=k)$ are useful in themselves as well as in other calculations. Taking $m=\omega$ and $n=t-\omega$ in this example it follows that

$$P\begin{pmatrix}\text{longest success run of length less than or equal to } k\\ \text{in a series of } t \textit{ iid} \text{ Bernoulli trials with success probability } p\end{pmatrix} \tag{7}$$

$$=\sum_{\omega=0}^{t}P_{\omega}(L\leq k)\binom{t}{\omega}p^{\omega}(1-p)^{t-\omega}$$

and

$$P\begin{pmatrix}\text{longest run of 0's is of length less than or equal to } k\\ \text{in a sample of size } t \text{ from an urn containing } s_1 \text{0's}\\ \text{and } s_2 \text{ 1's using a Markov-Pólya sampling model}\end{pmatrix} \tag{8}$$

$$=\sum_{\omega=0}^{t}P_{\omega}(L\leq k)\frac{\binom{\omega+s_1-1}{s_1-1}\binom{(t-\omega)+s_2-1}{s_2-1}}{\binom{t+s_1+s_2-1}{s_1+s_2-1}}$$

where

$$P_\omega(L \le k) = \sum_{j=0}^{(t-\omega)+1} (-1)^j \frac{\binom{(t-\omega)+\omega-(k+1)j}{\omega-(k+1)j}\binom{t-\omega+1}{j}}{\binom{(t-\omega)+\omega}{\omega}}.$$

The result in (1) is given in David and Barton [?] while results (3) and (5) are given in Bizley [?]. Mood [?] derives formulas for (4) but only for the cases $v = 1$ and $v = 2$. Fu and Koutras [?] have developed an algorithm for computing numerical solutions for problems such as the probability distribution of the longest success run in the more general case of independent but not necessarily identically distributed Bernoulli trials. Results similar to (7) and (8) using $P(R = k)$ or $P(W = k)$ instead of $P(L \le r)$ would follow in the same way.

The purpose of this example is to illustrate how to use Theorem 1 to establish a variety of results from the literature on success runs.

Proof for Example 3

Take $t = n + m$ and let $(X_1, \dots, X_n)$ in Theorem 1 represent the ordered positions of the n F's within $\{1,2, \dots, n+m\}$. For the variables $C_j, j = 1,2 \dots, n+1$ defined as in Theorem 1, $C_j - 1$ counts the number of S's (the length of the success run) between the $(j-1)^{st}$ and j^{th} F.

With these definitions it follows that

$$\begin{aligned} L \le k &\Leftrightarrow \max(C_1 - 1, \dots, C_{n+1} - 1) \le k \\ &\Leftrightarrow C_1 \le k+1, \dots, C_{n+1} \le k+1. \end{aligned}$$

We note that for any indicator function

$$E\big(\mathbb{I}(\text{event } A)\big) = P\big(\text{event } A\big)$$

where we take

$$\mathbb{I}(\text{event } A) = \begin{cases} 1 & \text{if event } A \text{ occurs} \\ 0 & \text{else.} \end{cases}$$

Hence for

$$\Psi(C_1, \dots, C_{n+1}) = \mathbb{I}\big(\max(C_1 - 1, \dots, C_{n+1} - 1) \le k\big)$$

we have

$$\begin{aligned} P(L \leq k) &= P(\max(C_1 - 1, \dots, C_{n+1} - 1) \leq k) \\ &= P(C_1 \leq k+1, \dots, C_{n+1} \leq k+1) \\ &= E(\Psi(C_1, \dots, C_{n+1})). \end{aligned}$$

Now by application of Theorem 1 with $t = m + n$ we have

$$E\big(\Psi(C_1, \dots, C_{n+1})\big)$$

$$= \frac{1}{\binom{m+n}{n}(m+n+1)!} \frac{d^{m+n+1}}{d\theta^{m+n+1}} \left(\left(\frac{\theta}{1-\theta}\right)^{n+1} E\big(\Psi(Z_1, \dots, Z_{n+1})\big) \right) \Bigg|_{\theta=0}$$

$$\begin{aligned} = \frac{1}{\binom{m+n}{n}(m+n+1)!} \frac{d^{m+n+1}}{d\theta^{m+n+1}} \Bigg(\left(\frac{\theta}{1-\theta}\right)^{n+1} \\ \times P(Z_1 \leq k+1) \cdots P(Z_{n+1} \leq k+1) \Bigg) \Bigg|_{\theta=0} \end{aligned}$$

$$= \frac{1}{\binom{m+n}{n}(m+n+1)!} \frac{d^{m+n+1}}{d\theta^{m+n+1}} \left(\left(\frac{\theta}{1-\theta}\right)^{n+1} (1-\theta^{k+1})^{n+1} \right) \Bigg|_{\theta=0}$$

$$\begin{aligned} = \frac{1}{\binom{m+n}{n}(m+n+1)!} \frac{d^{m+n+1}}{d\theta^{m+n+1}} \Bigg(\sum_{j=0}^{n+1} \sum_{i=0}^{\infty} (-1)^j \binom{n+1+i-1}{i} \\ \times \binom{n+1}{j} \theta^{(n+1)+i+j(k+1)} \Bigg) \Bigg|_{\theta=0} \end{aligned}$$

$$\begin{aligned} = \frac{(m+n+1)!}{\binom{m+n}{n}(m+n+1)!} \sum_{j=0}^{n+1} \sum_{i=0}^{\infty} \Bigg((-1)^j \binom{n+1+i-1}{i} \\ \times \binom{n+1}{j} \mathbb{I}\big((n+1) + i + j(k+1) = m + n + 1\big) \Bigg) \end{aligned}$$

$$= \frac{1}{\binom{m+n}{n}} \sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j} \left(\sum_{i=0}^{\infty} \binom{n+i}{i} \mathbb{I}(i = m - j(k+1)) \right)$$

$$= \frac{1}{\binom{m+n}{n}} \sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j} \binom{n+m-j(k+1)}{m-j(k+1)}$$

which proves (1).

Equation (2) follows directly from (1) and Theorem 7 in the appendix (which gives a general formula for factorial moments in terms of the cumulative probability distribution).

Part (c) follows from Theorem ?? and inclusion-exclusion with

$$\Psi(C_1, \dots, C_{n+1}) = \mathbb{I}(\text{exactly } k \text{ of } C_1, \dots, C_{n+1} \text{ equal } r+1).$$

In part (d) it follows from ?? that

$$\mathrm{E}\left(R_{(v)}\right) = v! \binom{n+1}{v} \mathrm{E}\left(\Psi(C_1, \dots, C_{n+1})\right)$$

where $\Psi(C_1, \dots, C_{n+1}) = \mathbb{I}(C_1 = r+1, \dots, C_v = r+1)$ and the final result follows from Theorem ??.

Parts (e) and (f) follow as in (c) and (d) by replacing $P\left(Z_j = k+1\right)$ with $P\left(Z_j \geq k+1\right) = \theta^k$ throughout. Parts (g) and (h) are obvious.

Example 3 Runs of empty urns in a Markov-Pólya Model

Suppose balls are distributed into urns according to the manner and notation of Model 5. Then

$$P(\text{no run of } r+1 \text{ or more empty urns})$$

$$= \frac{1}{\binom{mn+t-1}{t}} \sum_{i=0}^{n} \sum_{j=0}^{n-i+1} \sum_{s=0}^{n-i} (-1)^{j+s} \binom{n-(r+1)j}{i-(r+1)j}$$

$$\times \binom{n-i+1}{j} \binom{n-i}{s} \binom{m(n-i-s)+t-1}{t}.$$

Proof

The result follows from Theorem ?? and part (g) of the previous example with

$$\Psi(C_1, \dots, C_n) = \mathbb{I}\big(\text{there is no } j \text{ such that } C_j = C_{j+1} = \cdots = C_{j+r} = 0\big),$$

and Z_j ~Negative Binomial$(m, 1-p)$, $j = 1, \dots, n$. In part (g) of the previous example we show that

$$P(\text{longest success run of length less than or equal to } r \text{ in a series of } n \textit{ iid} \text{ Bernoulli trials})$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n-i+1} (-1)^j \binom{n-(r+1)j}{i-(r+1)j} \binom{n-i+1}{j} \phi^i (1-\phi)^{n-i}$$

where $\phi = P(\text{success})$. In this problem $\phi(p) = P(Z_i = 0) = (1-p)^m$. Hence,

$$\mathrm{E}\big(\Psi(Z_1, \dots, Z_n)\big) = \sum_{i=0}^{n} \sum_{j=0}^{n-i+1} (-1)^j \binom{n-(r+1)j}{i-(r+1)j} \binom{n-i+1}{j} \big(\phi(p)\big)^i \big(1-\phi(p)\big)^{n-i}$$

and the result follows on simplification.

Proof of Theorem 1

Because $(X_1, \dots, X_n)$ is a random sample of size n taken without replacement from $1, 2, \dots, t$, we have that

$$P(X_1 = x_1, \dots, X_n = x_n) = \frac{1}{\binom{t}{n}}$$

for all $1 \le x_1 < x_2 < \cdots < x_n \le t$.

But for all $(c_1, \dots, c_n, c_{n+1})$ such that $c_j \in \{1, 2, \dots\}$ and $c_1 + \cdots + c_n + c_{n+1} = t + 1$, it follows that

$$\begin{matrix} C_1 = c_1 \\ C_2 = c_2 \\ \vdots \\ C_n = c_n \\ C_{n+1} = c_{n+1} \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} X_1 = c_1 \\ X_2 = c_1 + c_2 \\ \vdots \\ X_n = c_1 + \cdots + c_n \end{matrix}$$

with $1 \leq X_1 < X_2 < \cdots < X_{n-1} < X_n \leq t$. Hence, for all such $(c_1, \dots, c_n, c_{n+1})$

$$P(C_1 = c_1, \dots, C_{n+1} = c_{n+1})$$

$$= \begin{cases} \dfrac{1}{\binom{t}{n}} & (c_1, \dots, c_{n+1}) \in \mathbb{S}_{t+1}^{n+1} \\ 0 & \text{else.} \end{cases}$$

For $Z_1, \dots, Z_{n+1}$ independent and identically distributed 1-shifted geometric$(1-\theta)$ random variables, the joint probability distribution becomes

$$P(Z_1 = z_1, \dots, Z_{n+1} = z_{n+1}) = (1-\theta)^{n+1}\theta^{(z_1+\cdots+z_{n+1})-(n+1)}$$

and by Theorem A?? in the appendix $Z_1 + \cdots + Z_{n+1} \sim (n+1)$-shifted negative binomial $(n+1, 1-\theta)$. Thus,

$$P(Z_1 + \cdots + Z_{n+1} = t+1) = (1-\theta)^{n+1}\theta^{(t+1)-(n+1)}\binom{t}{n}.$$

Now let $(z_1, \dots, z_{n+1})$ be a vector in $\mathbb{S}_{t+1}^{n+1}$. Then

$$P\big(Z_1 = z_1, \dots, Z_{n+1} = z_{n+1} \big| Z_1 + \cdots + Z_{n+1} = t+1\big)$$

$$= \frac{P(Z_1 = z_1, \dots, Z_{n+1} = z_{n+1})}{P(Z_1 + \cdots + Z_{n+1} = t+1)}$$

$$= \frac{(1-\theta)^{n+1}\theta^{(z_1+\cdots+z_{n+1})-(n+1)}}{(1-\theta)^{n+1}\theta^{(t+1)-(n+1)}\binom{t}{n}}$$

$$= \frac{(1-\theta)^{n+1}\theta^{(t+1)-(n+1)}}{(1-\theta)^{n+1}\theta^{(t+1)-(n+1)}\binom{t}{n}}$$

$$= \frac{1}{\binom{t}{n}} = P(C_1 = z_1, \dots, C_{n+1} = z_{n+1}).$$

Therefore,

$$E\big(\Psi(Z_1,\dots,Z_{n+1})\big) = E\left(E\left(\Psi(Z_1,\dots,Z_{n+1})\middle|\sum_{i=1}^{n+1} Z_i\right)\right)$$

$$= \sum_{r=n+1}^{\infty} E\left(\Psi(Z_1,\dots,Z_{n+1})\middle|\sum_{i=1}^{n+1} Z_i = r\right) P\left(\sum_{i=1}^{n+1} Z_i = r\right)$$

$$= \sum_{r=n+1}^{\infty}\left(\sum_{\mathbb{S}_r^{n+1}} \Psi(z_1,\dots,z_{n+1}) P\left(Z_1 = z_1,\dots,Z_{n+1} = z_{n+1}\middle|\sum_{i=1}^{n+1} Z_i = r\right)\right) P\left(\sum_{i=1}^{n+1} Z_i = r\right)$$

$$= \sum_{r=n+1}^{\infty}\left(\sum_{\mathbb{S}_r^{n+1}} \Psi(z_1,\dots,z_{n+1}) P(C_1 = z_1,\dots,C_{n+1} = z_{n+1})\right) P\left(\sum_{i=1}^{n+1} Z_i = r\right)$$

$$= \sum_{r=n+1}^{\infty} E\big(\Psi(C_1,\dots,C_{n+1})\big) P\left(\sum_{i=1}^{n+1} Z_i = r\right)$$

$$= \sum_{r=n+1}^{\infty} E\big(\Psi(C_1,\dots,C_{n+1})\big)(1-\theta)^{n+1}\theta^{r-(n+1)}\binom{r-1}{n}.$$

Thus

$$\left(\frac{\theta}{1-\theta}\right)^{n+1} E\big(\Psi(Z_1,\dots,Z_{n+1})\big) = \sum_{r=n+1}^{\infty} E\big(\Psi(C_1,\dots,C_{n+1})\big)\binom{r-1}{n}\theta^r.$$

It follows that

$$\frac{d^{t+1}}{d\theta^{t+1}}\left(\left(\frac{\theta}{1-\theta}\right)^{n+1} E\big(\Psi(Z_1,\dots,Z_{n+1})\big)\right)\Bigg|_{\theta=0}$$
$$= \frac{d^{t+1}}{d\theta^{t+1}}\left(\sum_{r=n+1}^{\infty} E\big(\Psi(C_1,\dots,C_{n+1})\big)\binom{r-1}{n}\theta^r\right)\Bigg|_{\theta=0}$$
$$= \sum_{r=n+1}^{\infty} E\big(\Psi(C_1,\dots,C_{n+1})\big)\binom{r-1}{n}\left(\frac{d^{t+1}}{d\theta^{t+1}}\theta^r\bigg|_{\theta=0}\right)$$

$$= \sum_{r=n+1}^{\infty} E\big(\Psi(C_1, \dots, C_{n+1})\big) \binom{r-1}{n} \Big((t+1)!\ \mathbb{I}_{\{t+1\}}(r)\Big)$$
$$= E\big(\Psi(C_1, \dots, C_{n+1})\big) \binom{(t+1)-1}{n} (t+1)!.$$

Thus,

$$E\big(\Psi(C_1, \dots, C_{n+1})\big) = \frac{1}{\binom{t}{n}(t+1)!}$$
$$\times \frac{d^{t+1}}{d\theta^{t+1}} \left(\left(\frac{\theta}{1-\theta}\right)^{n+1} E\big(\Psi(Z_1, \dots, Z_{n+1})\big) \right) \Bigg|_{\theta=0}.$$

1.1 Sampling from a circle

Theorem: Sampling from an Oriented Circle Model

Theorem 4

Suppose the numbers $1,2, \dots, t$ are arranged clockwise and in increasing order around a circle. Let $(X_1, \dots, X_n)$ be a random sample of size n taken without replacement from these t distinct numbers such that $1 \le X_1 < X_2 < \cdots < X_{n-1} < X_n \le t$.

Define the variables C_j by

$$\begin{aligned} C_1 &= X_2 - X_1 \\ C_2 &= X_3 - X_2 \\ &\vdots \\ C_{n-1} &= X_n - X_{n-1} \\ C_n &= t - X_n + X_1. \end{aligned}$$

Then

$$E\big(\Psi(C_1, \dots, C_n)\big) = \frac{1}{\binom{t}{n}(t+1)!} \frac{d^{t+1}}{d\theta^{t+1}} \left(\left(\frac{\theta}{1-\theta}\right)^{n+1} E\big(\Psi(Z_1, \dots, Z_n)\big) \right) \Bigg|_{\theta=0}$$

where Z_1, …,Z_{n-1} are independent and identically distributed 1-shifted geometric$(1-\theta)$

random variables and where the random variable Z_n is independent of $Z_1, \dots, Z_{n-1}$ and follows a 1-shifted negative binomial$(2,1-\theta)$ distribution.

Proof

A critical distinction with how the C_j's are defined in this theorem as opposed to Theorem 12 is that now the C's do not completely determine the X's.

However, it remains true that $(X_1, \dots, X_n)$ is a random sample of size n taken without replacement from $1,2, \dots, t$, so we have that

$$P(X_1 = x_1, \dots, X_n = x_n) = \frac{1}{\binom{t}{n}}$$

for all $1 \leq x_1 < x_2 < \cdots < x_n \leq t$. Now let $(c_1, c_2, \dots, c_n)$ be a vector of positive integers such that $c_1 + c_2 + \cdots + c_n = t$. Then

$$P(C_1 = c_1, \dots, C_{n-1} = c_{n-1}, C_n = c_n)$$

$$= P\big(C_1 = c_1, \dots, C_{n-1} = c_{n-1}, t - (C_1 + \cdots + C_{n-1}) = t - (c_1 + \cdots + c_{n-1})\big)$$

$$= P(C_1 = c_1, \dots, C_{n-1} = c_{n-1})$$

$$= \sum_{\text{all } x_1} P(C_1 = c_1, \dots, C_{n-1} = c_{n-1} \text{ and } X_1 = x_1)$$

$$= \sum_{\text{all } x_1} P(X_1 = x_1, X_2 = x_1 + c_1, X_3 = x_1 + c_1 + c_2, \dots, X_n = x_1 + c_1 + \cdots + c_{n-1}).$$

We can determine the range of possible x_1 values by looking for those values of x_1 where $1 \leq x_1 < x_2 < \cdots < x_{n-1} < x_n \leq t$. The restrictions on x_1 are that $x_1 \geq 1$ and $x_n = x_1 + c_1 + \cdots + c_{n-1} \leq t$. That is, $1 \leq x_1 \leq t - (c_1 + \cdots + c_{n-1}) = c_n$. Thus,

$$P(C_1 = c_1, \dots, C_{n-1} = c_{n-1}, C_n = c_n)$$

$$= \sum_{x_1=1}^{c_n} P(X_1 = x_1, X_2 = x_1 + c_1, \dots, X_n = x_1 + c_1 + \cdots + c_{n-1})$$

$$= \sum_{x_1=1}^{c_n} \frac{1}{\binom{t}{n}} = \frac{c_n}{\binom{t}{n}}.$$

Now we will verify that for any vector $(c_1, c_2, \dots, c_n)$ of positive integers such that $c_1 + c_2 +$

$\cdots + c_n = t$ and for random variables $Z_1, \dots, Z_n$ as defined above that

$$P(C_1 = c_1, \dots, C_n = c_n) = P\big(Z_1 = c_1, \dots, Z_n = c_n \big| \, Z_1 + \cdots + Z_n = t\big).$$

To show this we will first state two standard results from statistical distribution theory. But first recall the definition of the shifted negative binomial distribution with parameters m and p. For $Z \sim c$ - Shifted Negative Binomial Distribution(m, p) we have

$$P(Z = z) = \binom{z - c + m - 1}{m - 1} p^m (1 - p)^{z - c} \qquad z \in \{c, c + 1, \dots\}.$$

(I) c - shifted negative binomial$(1, p) \equiv c$ - shifted geometric(p)

(II) If $Y_1, \dots, Y_n$ are independent and if $Y_j \sim c_j$ - shifted negative binomial$\big(m_j, p\big)$, then $S = Y_1 + \cdots + Y_n \sim c^*$ - shifted negative binomial(m^*, p), where $c^* = c_1 + \cdots + c_n$ and $m^* = m_1 + \cdots + m_n$.

We note that the m - shifted negative binomial(m, p) is simply referred to as the negative binomial distribution in many textbooks. However, the negative binomial and the shifted negative binomial are both used in this paper, sometimes in the same problem. Thus, to avoid confusion, it is necessary for us to delineate between these related models.

Using these two results together we can state that $Z_1 + \cdots + Z_n \sim c^*$ - shifted Negative Binomial$(m^*, 1 - \theta)$ with $c^* = 1 + \cdots + 1 + 1 = n$ and $m^* = 1 + \cdots + 1 + 2 = n + 1$. Therefore,

$$P\big(Z_1 = c_1, \dots, Z_n = c_n \big| \, Z_1 + \cdots + Z_n = t\big)$$

$$= \frac{P(Z_1 = c_1, \dots, Z_n = c_n \text{ and } Z_1 + \cdots + Z_n = t)}{P(Z_1 + \cdots + Z_n = t)}$$

$$= \frac{P(Z_1 = c_1, \dots, Z_n = c_n)}{P(Z_1 + \cdots + Z_n = t)}$$

$$= \frac{\left(\prod_{j=1}^{n-1} (1-\theta)\theta^{c_j-1}\right)\binom{c_n - 1 + 2 - 1}{2-1}(1-\theta)^2\theta^{c_n-1}}{\binom{t-n+(n+1)-1}{(n+1)-1}(1-\theta)^{(n+1)}\theta^{t-n}}$$

$$= \frac{(1-\theta)^{(n+1)}\theta^{c_1+\cdots+c_n-n}\binom{c_n}{1}}{\binom{t}{n}(1-\theta)^{(n+1)}\theta^{t-n}}$$

$$= \frac{(1-\theta)^{(n+1)}\theta^{t-n}\, c_n}{\binom{t}{n}(1-\theta)^{(n+1)}\theta^{t-n}}$$

$$= \frac{c_n}{\binom{t}{n}} = P(C_1 = c_1, \dots, C_n = c_n).$$

Let $\mathbb{S}^n$ be the product space $\{0,1,\dots\} \times \cdots \times \{0,1,\dots\}$ and let $\mathbb{S}_t^n$ be the set of all vectors $(c_1, \dots\ c_n)$ in $\mathbb{S}^n$ such that $c_1 + \dots + c_n = t$. Then

$$E\big(\Psi(Z_1, \dots, Z_n)\big) = E\left(E\left(\Psi(Z_1, \dots, Z_n)\middle|\sum_{i=1}^n Z_i\right)\right)$$
$$= \sum_{r=0}^{\infty} E\left(\Psi(Z_1, \dots, Z_n)\middle|\sum_{i=1}^n Z_i = r\right)P\left(\sum_{i=1}^n Z_i = r\right)$$
$$= \sum_{r=0}^{\infty}\left(\sum\cdots\sum_{(c_1,\dots,c_n)\in\mathbb{S}_r^n}\Psi(c_1, \dots, c_n)P\left(Z_1 = c_1, \dots, Z_n = c_n\middle|\sum_{i=1}^n Z_i = r\right)\right)P\left(\sum_{i=1}^n Z_i = r\right)$$
$$= \sum_{r=0}^{\infty}\left(\sum\cdots\sum_{(c_1,\dots,c_n)\in\mathbb{S}_r^n}\Psi(c_1, \dots, c_n)P(C_1 = c_1, \dots, C_n = c_n)\right)P\left(\sum_{i=1}^n Z_i = r\right)$$
$$= \sum_{r=0}^{\infty} E\big(\Psi(C_1, \dots, C_n)\big)P\left(\sum_{i=1}^n Z_i = r\right)$$
$$= \sum_{r=0}^{\infty} E\big(\Psi(C_1, \dots, C_n)\big)\binom{r}{n}(1-\theta)^{n+1}\theta^{r-n}$$

and thus

$$\left(\frac{\theta}{1-\theta}\right)^{n+1} E\big(\Psi(Z_1,\dots,Z_n)\big) = \sum_{r=0}^{\infty} E\big(\Psi(C_1,\dots,C_n)\big)\binom{r}{n}\theta^{r+1}.$$

It follows that

$$\frac{d^{t+1}}{d\theta^{t+1}}\left(\left(\frac{\theta}{1-\theta}\right)^{n+1} E\big(\Psi(Z_1,\dots,Z_n)\big)\right)\Bigg|_{\theta=0}$$

$$= \frac{d^{t+1}}{d\theta^{t+1}}\left(\sum_{r=0}^{\infty} E\big(\Psi(C_1,\dots,C_n)\big)\binom{r}{n}\theta^{r+1}\right)\Bigg|_{\theta=0}$$

$$= \sum_{r=0}^{\infty} E\big(\Psi(C_1,\dots,C_n)\big)\binom{r}{n}\left(\frac{d^{t+1}}{d\theta^{t+1}}\theta^{r+1}\Bigg|_{\theta=0}\right)$$

$$= \sum_{r=0}^{\infty} E\big(\Psi(C_1,\dots,C_n)\big)\binom{r}{n}\Big((t+1)!\,\mathbb{I}_{\{t+1\}}(r+1)\Big)$$

$$= E\big(\Psi(C_1,\dots,C_n)\big)\binom{t}{n}(t+1)!.$$

Thus,

$$E\big(\Psi(C_1,\dots,C_n)\big) = \frac{1}{\binom{t}{n}(t+1)!}\frac{d^{t+1}}{d\theta^{t+1}}\left(\left(\frac{\theta}{1-\theta}\right)^{n+1} E\big(\Psi(Z_1,\dots,Z_n)\big)\right)\Bigg|_{\theta=0}.$$

Theorem: Sampling from a non-oriented circle model

Theorem 5

Suppose we randomly pick n out of t different colored but unnumbered balls arranged in a circle. Then our sample space consists of $\binom{t}{n}$ equally likely outcomes.

One way to pick n balls at random is to pick all n at once. Alternatively, we could pick 1 ball at random and then pick $n-1$ balls at once from the remaining $t-1$ balls. Both methods assign equal probability to all possible samples of size n.

Consider the latter sampling scheme and suppose that we attach the label t to the first ball picked and attach the labels $1, \dots, t-1$ to the remaining balls going clockwise around the circle. After attaching the labels we randomly pick $n-1$ balls from the balls labeled 1 to $t-1$. Clearly attaching the labels does not change the fact that all possible samples are equally likely to occur.

Let $1 \le X_1 < \cdots < X_{n-1} \le t-1$ be the labels on the $n-1$ balls drawn at this second phase of our sample. Define the variables C_j by

$$\begin{aligned} C_1 &= X_1 \\ C_2 &= X_2 - X_1 \\ &\vdots \\ C_{n-1} &= X_{n-1} - X_{n-2} \\ C_n &= t - X_{n-1}. \end{aligned}$$

Then we have the following two results.

Theorem, Part I

$$E\big(\Psi(C_1, \dots, C_n)\big) = \frac{1}{\binom{t-1}{n-1} t!} \frac{d^t}{d\theta^t}\left(\left(\frac{\theta}{1-\theta}\right)^n E\big(\Psi(Z_1, \dots, Z_n)\big)\right)\bigg|_{\theta=0}$$

where $Z_1, \dots, Z_n$ are independent and identically distributed 1-shifted geometric$(1-\theta)$ random variables.

Theorem, Part II

For some arbitrary set $\mathcal{B}$

$$N\left(\text{selections of } n \text{ of } t \text{ distinct objects arranged in a circle} \,\middle|\, (C_1, \dots, C_n) \in \mathcal{B}\right)$$

$$= \frac{\binom{t}{n}}{t!} \frac{d^t}{d\theta^t}\left(\left(\frac{\theta}{1-\theta}\right)^n P\big((Z_1, \dots, Z_n) \in \mathcal{B}\big)\right)\bigg|_{\theta=0}$$

where the random variables $Z_1, \dots, Z_n$ are defined as in Part I.

Theorem 14 is as an appropriate shortcut on the more general Theorem 13 for situations where the problem does not depend on knowing which particular gap "closes the circle". For example, if the problem asks for the number of selections such that exactly r of the n gaps have property $\mathcal{A}$ then the gap which closes the circle is not distinguished and Theorem 14 would be appropriate. We note that in principle Theorem 13 could also be used to solve this problem but it would be necessary to separate out the cases when the r gaps with property $\mathcal{A}$ included the gap which "closes the circle" and the cases when they do not. Theorem 14 proves to be a considerable shortcut for this and other similar problems.

To appreciate intuitively why the gap which "closes the circle" is distinct, consider that this gap is known to contain the numbers "1" and "t". (We cannot make a similar kind of claim about any other gap.) Now if the gaps were revealed but with the numbers covered up and we were asked to guess which gap contains the numbers "1" and "t", (i.e. closed the circled), we would intuitively pick the longest interval. Formally, we can use Theorem 13 to show for all c that

$$P(C_n \geq c) \geq P(C_j \geq c) \qquad j \in \{1, \dots, n-1\}.$$

In the language of probability, the gap which closes the circle is stochastically larger than any of the other gaps.

The close relationship between the line model and the non-oriented circle model is obvious from comparing Theorems 12 and 14. In particular, if we take $(C_1, \dots, C_{n+1})$ to be the vector of spacings formed when selecting randomly n of t objects arranged in a line and take $(\overline{C}_1, \dots, \overline{C}_{n+1})$ to be the vector of spacings formed when randomly selecting $n+1$ of $t+1$ objects arranged in a non-oriented circle then for general Ψ,

$$E\big(\Psi(C_1, \dots, C_{n+1})\big) = E\left(\Psi\big(\overline{C}_1, \dots, \overline{C}_{n+1}\big)\right).$$

Furthermore, it follows for general $\mathcal{A}$ that

$$N\big(\text{selections of } n+1 \text{ of } t+1 \text{ distinct objects arranged in a circle} \,|\, (\overline{C}_1, \dots, \overline{C}_{n+1}) \in \mathcal{A}\big)$$

$$= \left(\frac{t+1}{n+1}\right) N\left(\text{selections of } n \text{ of } t \text{ distinct objects arranged in a line } \Big| (C_1, \dots, C_{n+1}) \in \mathcal{A}\right)$$

Koutras and Papastavridis [ ] establish this connection between selecting objects arranged in a line and selecting objects arranged in a circle for the special case of sets $\mathcal{A}$ of the form

$$\mathcal{A} = \{\text{exactly } s \text{ of } (a_1, \dots, a_{n+1}) \in \mathcal{B}\}$$

for arbitrary sets $\mathcal{B}$. They point out that Kaplansky [ ] established this connection for sets $\mathcal{A}$ of the form

$$\mathcal{A} = \left\{\text{none of } (a_1, \dots, a_{n+1}) \in \{3,4, \dots\}\right\} \text{ and } n = t - k$$

in his solution of the "Probléme des Ménages".

Whether selecting objects from a line or circle it is important to distinguish whether the question of interest concerns restrictions on the gaps of adjacent selected objects or restrictions on the gaps of any pair of selected objects. The theorems and examples in this section deal with restrictions on the gaps of adjacent selected objects. Some pointers to the literature when restricting the gaps of any pair of selected objects would include Prodinger [ ], Hwang, Korner, and Wei [ ], Hwang [ ], Prodinger [ ], Moser [ ], Konvalina and Liu [ ], [ ].

Proof, Part I

Let $h_1, h_2, \dots, h_t$ be the names of the hues of the t different colored balls in this circle. Let H be the name of the hue of the ball initially drawn (and numbered t). Then

$$\begin{gathered} P(X_1 = x_1, \dots, X_{n-1} = x_{n-1}) \\ = \sum_{j=1}^{t} P\left(X_1 = x_1, \dots, X_{n-1} = x_{n-1} \middle| H = h_j\right) P\left(H = h_j\right) \\ = \sum_{j=1}^{t} \frac{1}{\binom{t-1}{n-1}} \cdot \frac{1}{t} = \frac{1}{\binom{t-1}{n-1}}. \end{gathered}$$

But for all $(c_1, \dots, c_n)$ such that $c_j \in \{1,2, \dots\}$ and $c_1 + \dots + c_n = t$, it follows that

$$C_1 = c_1, \dots, C_n = c_n \Leftrightarrow X_1 = c_1, X_2 = c_1 + c_2, \dots, X_{n-1} = c_1 + \dots + c_{n-1}$$

with $1 \leq X_1 < X_2 < \dots < X_{n-1} \leq t - 1$. Hence, for all such $(c_1, \dots, c_n)$

$$P(C_1 = c_1, \dots, C_n = c_n)$$

$$= P(X_1 = c_1, X_2 = c_1 + c_2, \dots, X_{n-1} = c_1 + \dots + c_{n-1})$$

$$= \begin{cases} \dfrac{1}{\binom{t-1}{n-1}} & \begin{array}{l} c_1 + \cdots + c_n = t \\ c_j \in \{1 \ \ldots, \infty\}, j = 1, \ldots, n \\ n \le t \le \infty \end{array} \\ 0 & \text{else.} \end{cases}$$

Now we will verify that for any vector $(c_1, c_2, \ldots, c_n)$ of positive integers such that $c_1 + c_2 + \cdots + c_n = t$ and for random variables $Z_1, \ldots, Z_n$ as defined above that

$$P(C_1 = c_1, \ldots, C_n = c_n) = P\big(Z_1 = c_1, \ldots, Z_n = c_n \big| Z_1 + \cdots + Z_n = t\big).$$

To show this we will first state two standard results from statistical distribution theory. But first recall the definition of the shifted negative binomial distribution with parameters m and p. For $Z \sim c$ - Shifted Negative Binomial Distribution(m, p) we have

$$P(Z = z) = \binom{z - c + m - 1}{m - 1} p^m (1-p)^{z-c} \qquad z \in \{c, c+1, \ldots\}.$$

(I) c - shifted negative binomial$(1, p) \equiv c$ - shifted geometric(p)

(II) If $Y_1, \ldots, Y_n$ are independent and if $Y_j \sim c_j$ - shifted negative binomial$\big(m_j, p\big)$, then $S = Y_1 + \cdots + Y_n \sim c^*$ - shifted negative binomial(m^*, p), where $c^* = c_1 + \cdots + c_n$ and $m^* = m_1 + \cdots + m_n$.

We note that the m - shifted negative binomial(m, p) is simply referred to as the negative binomial distribution in many textbooks. However, the negative binomial and the shifted negative binomial are both used in this paper, sometimes in the same problem. Thus, to avoid confusion, it is necessary for us to delineate between these related models.

Using these two results together we can state that $Z_1 + \cdots + Z_n \sim c^*$ - shifted Negative Binomial$(m^*, 1-\theta)$ with $c^* = 1 + \cdots + 1 = n$ and $m^* = 1 + \cdots + 1 = n$. Therefore

$$P\big(Z_1 = c_1, \ldots, Z_n = c_n \big| Z_1 + \cdots + Z_n = t\big)$$

$$= \frac{P(Z_1 = c_1, \ldots, Z_n = c_n \text{ and } Z_1 + \cdots + Z_n = t)}{P(Z_1 + \cdots + Z_n = t)}$$

$$= \frac{P(Z_1 = c_1, \dots, Z_n = c_n)}{P(Z_1 + \cdots + Z_n = t)}$$

$$= \frac{\displaystyle\prod_{j=1}^{n} (1-\theta)\theta^{c_j - 1}}{\binom{t-n+n-1}{n-1}(1-\theta)^n \theta^{t-n}}$$

$$= \frac{(1-\theta)^n \theta^{(c_1 + \cdots + c_n) - n}}{\binom{t-1}{n-1}(1-\theta)^n \theta^{t-n}}$$

$$= \frac{(1-\theta)^n \theta^{t-n}}{\binom{t-1}{n-1}(1-\theta)^n \theta^{t-n}}$$

$$= \frac{1}{\binom{t-1}{n-1}} = P(C_1 = c_1, \dots, C_n = c_n).$$

Let $\mathbb{S}^n$ be the product space $\{1,2,\dots\} \times \cdots \times \{1,2,\dots\}$ and let $\mathbb{S}_t^n$ be the set of all vectors $(c_1, \dots\ c_n)$ in $\mathbb{S}^n$ such that $c_1 + \dots + c_n = t$. Then

$$E\big(\Psi(Z_1, \dots, Z_n)\big) = E\left(E\left(\Psi(Z_1, \dots, Z_n)\middle| \sum_{i=1}^{n} Z_i\right)\right)$$

$$= \sum_{r=0}^{\infty} E\left(\Psi(Z_1, \dots, Z_n)\middle| \sum_{i=1}^{n} Z_i = r\right) P\left(\sum_{i=1}^{n} Z_i = r\right)$$

$$= \sum_{r=0}^{\infty} \left(\sum \cdots \sum_{(c_1,\dots,c_n)\in\mathbb{S}_r^n} \Psi(c_1, \dots, c_n) P\left(Z_1 = c_1, \dots, Z_n = c_n \middle| \sum_{i=1}^{n} Z_i = r\right)\right) P\left(\sum_{i=1}^{n} Z_i = r\right)$$

$$= \sum_{r=0}^{\infty} \left(\sum \cdots \sum_{(c_1,\dots,c_n)\in\mathbb{S}_r^n} \Psi(c_1, \dots, c_n) P(C_1 = c_1, \dots, C_n = c_n)\right) P\left(\sum_{i=1}^{n} Z_i = r\right)$$

$$= \sum_{r=0}^{\infty} E\big(\Psi(C_1, \dots, C_n)\big) P\left(\sum_{i=1}^{n} Z_i = r\right)$$

$$= \sum_{r=0}^{\infty} E\big(\Psi(C_1, \dots, C_n)\big) \binom{r-1}{n-1} (1-\theta)^n \theta^{r-n}$$

and thus

$$\left(\frac{\theta}{1-\theta}\right)^n E\big(\Psi(Z_1, \dots, Z_n)\big) = \sum_{r=0}^{\infty} E\big(\Psi(C_1, \dots, C_n)\big) \binom{r-1}{n-1} \theta^r.$$

It follows that

$$\frac{d^t}{d\theta^t}\left(\left(\frac{\theta}{1-\theta}\right)^n E\big(\Psi(Z_1, \dots, Z_n)\big)\right)\Bigg|_{\theta=0}$$

$$= \frac{d^t}{d\theta^t}\left(\sum_{r=0}^{\infty} E\big(\Psi(C_1, \dots, C_n)\big) \binom{r-1}{n-1} \theta^r\right)\Bigg|_{\theta=0}$$

$$= \sum_{r=0}^{\infty} E\big(\Psi(C_1, \dots, C_n)\big) \binom{r-1}{n-1} \left(\frac{d^t}{d\theta^t} \theta^r \Bigg|_{\theta=0}\right)$$

$$= \sum_{r=0}^{\infty} E\big(\Psi(C_1, \dots, C_n)\big) \binom{r-1}{n-1} \left(t!\ \mathbb{I}_{\{t\}}(r)\right)$$

$$= E\big(\Psi(C_1, \dots, C_n)\big) \binom{t-1}{n-1} t!.$$

Thus,

$$E\big(\Psi(C_1, \dots, C_n)\big) = \frac{1}{\binom{t-1}{n-1} t!} \frac{d^t}{d\theta^t}\left(\left(\frac{\theta}{1-\theta}\right)^n E\big(\Psi(Z_1, \dots, Z_n)\big)\right)\Bigg|_{\theta=0}.$$

Proof, Part II

Let $\Psi(C_1, \dots, C_n) = \mathbb{I}\big((C_1, \dots, C_n) \in \mathcal{B}\big)$ for some arbitrary set $\mathcal{B}$. Then by the result in Part I,

$$P\big((C_1, \dots, C_n) \in \mathcal{B}\big) = \frac{1}{\binom{t-1}{n-1} t!} \frac{d^t}{d\theta^t}\left(\left(\frac{\theta}{1-\theta}\right)^n P\big((Z_1, \dots, Z_n) \in \mathcal{B}\big)\right)\Bigg|_{\theta=0}.$$

But we know that our sample space consists of $\binom{t}{n}$ equally likely outcomes. So

$$P\big((C_1, \dots, C_n) \in \mathcal{B}\big)$$

$$= \frac{N(\text{selections of } n \text{ of } t \text{ distinct objects arranged in a circle} | (C_1, \dots, C_n) \in \mathcal{B})}{\binom{t}{n}}.$$

Thus,

$$N(\text{selections of } n \text{ of } t \text{ distinct objects arranged in a circle} | (C_1, \dots, C_n) \in \mathcal{B})$$

$$= \frac{\binom{t}{n}}{\binom{t-1}{n-1} t!} \frac{d^t}{d\theta^t} \left(\left(\frac{\theta}{1-\theta} \right)^n P\big((Z_1, \dots, Z_n) \in \mathcal{B}\big) \right) \Bigg|_{\theta=0}$$

$$= \frac{\left(\frac{t}{n}\right)}{t!} \frac{d^t}{d\theta^t} \left(\left(\frac{\theta}{1-\theta} \right)^n P\big((Z_1, \dots, Z_n) \in \mathcal{B}\big) \right) \Bigg|_{\theta=0}.$$

Example 2 Longest circular success run.

Suppose n F's (failures) and $t - n$ S's (successes) are randomly arranged in a circle. Let L be the length of the longest circular success run. Then

$$P(L \le r) = \frac{\left(\frac{t}{n}\right)}{\binom{t}{n}} \sum_{j=0}^{n} (-1)^j \binom{t-(r+1)j-1}{t-(r+1)j-n} \binom{n}{j}.$$

Now consider each ball in this circle to be a component in a circularly connected system where components act independently and have constant probability p of being a successfully working component at the moment the system is turned on. Furthermore assume that the system has built in redundancy such that the system will work as long as there is no run of $r+1$ consecutive failed components. This is referred to as a circular consecutive -$(r+1)$-out of t : F system in the reliability literature [?, ?, ?, ?, ?, ?]. It follows from the above result (after switching the role of successes and failures) that the reliability of this system, that is the probability that the system will work at the moment it is turned on, is

$$\sum_{n=1}^{t}\left(\binom{t}{n}P(L \le r)\right)p^n(1-p)^{t-n}.$$

By switching the order of summation and applying Gould's formula 3.118 [?], this reliability simplifies to

$$\sum_{j=0}^{\left\lfloor\frac{t}{r+2}\right\rfloor}(-1)^j\left(\frac{t}{t-(r+1)j}\right)\binom{t-(r+1)j}{t-(r+2)j}p^j(1-p)^{(r+1)j}-(1-p)^t.$$

Proof

To help establish the connections between the three models (sampling from a line, sampling from an oriented circle, and sampling from a non-oriented circle) we point out how all three models can be used to solve this problem.

If we were starting from scratch then the sampling from a non-oriented circle model would be the easiest approach. We note that $L = \max(C_1 - 1, \dots, C_n - 1)$ and by Theorem ??,

$$P(\max(C_1 - 1, \dots, C_n - 1) \le r) = P(C_1 \le r+1, \dots, C_n \le r+1)$$

$$= \frac{\binom{t}{n}}{\binom{t}{n}t!}\frac{d^t}{d\theta^t}\left(\left(\frac{\theta}{1-\theta}\right)^n P(Z_1 \le r+1, \dots, Z_n \le r+1)\right)\Bigg|_{\theta=0}$$

and the result follows on simplification. Alternatively, we could use the sampling from an oriented circle model. The problem is a special case of Example ??.1 with $a = b = 1$ and $a' = b' = r + 1$. Moser and Abramson [?] take this approach and their work demonstrates that this approach is considerable more involved. Finally, we note that we have already solved this problem for the sampling from a line model in Example ??.2(a). Noting this the final result follows immediately by the connection between the non-oriented circle model and line model as noted in the discussion after Theorem ?? (replacing n with $n - 1$ and t with $t - 1$ throughout and taking $m = t - n$).

Example 3. Sampling from a circle with a fixed number of groupings of minimum length.

The number of ways to select n objects from t distinct objects arranged in a circle such that there are exactly v groups of k or more contiguous objects (an unbroken string of selected objects) in the selection equals

$$\frac{t}{t-n}\binom{t-n}{v}\sum_{i=0}^{t-n-v}(-1)^i\binom{t-k(i+v)-1}{t-n-1}\binom{t-n-v}{i}.$$

Hwang and Yao [?] derive this result for the case $v=0$. The special case $k=1$ simplifies to

$$\frac{t}{t-n}\binom{t-n}{v}\binom{n-1}{v-1}.$$

Proof

The key step is in recognizing that

$N(\text{samples of size } n \text{ such that there are exactly } v \text{ groups of } k \text{ or more contiguous objects})$

$$= N\big(\text{samples of size } n \text{ such that exactly } v \text{ of the gaps between the } t-n \text{ objects } \underline{\text{not}} \text{ selected contain } k \text{ or more objects}\big)$$

$$= N\big(\text{samples of size } t-n \text{ such that exactly } v \text{ of the gaps between the } t-n \text{ objects selected contain } k \text{ or more objects}\big).$$

Restated in this form we see that the problem asks for $\binom{t}{t-n}\mathrm{E}\big(\Psi(C_1,\dots,C_{t-n})\big)$ where

$$\Psi(C_1,\dots,C_{t-n}) = \mathbb{I}\big\{\text{exactly } v \text{ of } C_1,\dots,C_{t-n} \text{ are } > k\big\}$$

with the C_j defined as in Theorem ??. The result follows on applying Theorem ?? (replacing n with $t-n$), recognizing that $P(Z_1>k)=\theta^k$ and using inclusion-exclusion.

1.2 Overlapping and non-overlapping success runs

Theorem: Overlapping and Non-Overlapping Success Runs (Linear and Circular)

We considered problems of counting success runs of length exactly k and success runs of length at least k in Example 2 of Section 4.1. Non-overlapping success runs and overlapping success runs are alternative definitions of runs which are often considered in the statistical and probability literature.

The distinction between these different types of runs can be made clear with the sequence, $S\ S\ S\ S\ F\ F\ S\ S$. We start by considering this as a linear sequence and then consider what changes if this is a circular sequence. That is if we suppose the final S of the linear sequence wraps around and is adjacent to the initial S of the linear sequence.

For this *linear* sequence,

$S\ S\ S\ S\ F\ F\ S\ S$ (linear)	$k = 1$	$k = 2$	$k = 3$	$k = 4$
# success runs of length exactly k	0	1	0	1
# success runs of length at least k	2	2	1	1
# non-overlapping success runs of length k	6	3	1	1
# overlapping success runs of length k	6	4	2	1

To further illustrate the differences we highlight the case $k = 2$.

1 success run of length exactly 2
(1) $S\ S\ S\ S\ F\ F\ \boldsymbol{S\ S}$

2 success runs of length at least 2
(1) $\boldsymbol{S\ S\ S\ S}\ F\ F\ S\ S$
(2) $S\ S\ S\ S\ F\ F\ \boldsymbol{S\ S}$

3 non-overlapping success runs of length 2
(1) $\boldsymbol{S\ S}\ S\ S\ F\ F\ S\ S$
(2) $S\ S\ \boldsymbol{S\ S}\ F\ F\ S\ S$
(3) $S\ S\ S\ S\ F\ F\ \boldsymbol{S\ S}$

4 overlapping success runs of length 2
(1) $\boldsymbol{S\ S}\ S\ S\ F\ F\ S\ S$
(2) $S\ \boldsymbol{S\ S}\ S\ F\ F\ S\ S$
(3) $S\ S\ \boldsymbol{S\ S}\ F\ F\ S\ S$
(4) $S\ S\ S\ S\ F\ F\ \boldsymbol{S\ S}$.

Now we will consider $S\ S\ S\ S\ F\ F\ S\ S$ as a *circular* sequence.

$S\ S\ S\ S\ F\ F\ S\ S$ (circular)	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
# success runs of length exactly k	0	0	0	0	0	1
# success runs of length at least k	1	1	1	1	1	1
# non-overlapping success runs of	6	3	2	1	1	1

length k						
# overlapping success runs of length k	6	5	4	3	2	1

Linear Success Runs

For linear success runs (overlapping and non-overlapping) of fixed length t and with a fixed number of failures n, we suppose the numbers $1, \dots, t$ are arranged in a line in increasing order. Let $(X_1, \dots, X_n)$ be a random sample of size n taken without replacement from these t distinct numbers such that $1 \leq X_1 < X_2 < \cdots < X_{n-1} < X_n \leq t$. The chosen numbers $(X_1, \dots, X_n)$ represent the location of our n failures. The other $t - n$ numbers represent the location of our $t - n$ successes.

Define the variables C_j by

$$\begin{aligned} C_1 &= X_1 \\ C_2 &= X_2 - X_1 \\ &\vdots \\ C_n &= X_n - X_{n-1} \\ C_{n+1} &= (t+1) - X_n. \end{aligned}$$

Part I: Linear Non-Overlapping Success Runs

For any positive integer k define the variables D_j by

$$D_j = \left\lfloor \frac{C_j - 1}{k} \right\rfloor \quad j = 1, \dots, n+1.$$

We note that D_j equals the number of non-overlapping success runs to the left of ball X_1 in the case $j = 1$, the number of non-overlapping success runs between balls X_j and X_{j-1} for $j = 2, \dots, n$ and the number of non-overlapping success runs to the right of ball X_n for the case $j = n + 1$.

In this case,

$$E\big(\Psi(D_1, \dots, D_{n+1})\big) = \frac{1}{\binom{t}{n}(t+1)!}$$

$$\times \frac{d^{t+1}}{d\theta^{t+1}}\left(\left(\frac{\theta}{1-\theta}\right)^{n+1} E\big(\Psi(Z_1, \dots, Z_{n+1})\big)\right)\Bigg|_{\theta=0}$$

where Z_1, …,Z_{n+1} are independent and identically distributed geometric$(1-\theta^k)$ random variables.

Part II: Linear Overlapping Success Runs

For any positive integer k define the variables G_j by

$$G_j = \begin{cases} C_j - k & C_j \geq k+1 \\ 0 & C_j \leq k \end{cases} \qquad j = 1, \dots, n+1.$$

That is, $G_j = \big(C_j - k\big)\mathbb{I}\big(C_j \geq k+1\big) = max\{C_j - k, 0\}$.

We note that G_j equals the number of overlapping success runs to the left of ball X_1 in the case $j = 1$, the number of overlapping success runs between balls X_j and X_{j-1} for $j = 2, \dots, n$ and the number of overlapping success runs to the right of ball X_n for the case $j = n+1$.

Let Υ_j^{n+1} be the set of all $(n+1)$ dimensional $\{0,1\}$ vectors with exactly j 0's. In this case,

$$E\big(\Psi(G_1, \dots, G_{n+1})\big) = \frac{1}{\binom{t}{n}(t+1)!}$$
$$\times \sum_{j=0}^{n+1} \frac{d^{t+1}}{d\theta^{t+1}}(\left(\frac{\theta}{1-\theta}\right)^{n+1}(1-\theta^k)^j\theta^{k(n+1-j)}$$
$$\times \sum_{v\in\Upsilon_j^{n+1}} E\big(\Psi(v_1 Z_1, \dots, v_{n+1}Z_{n+1})\big))\Bigg|_{\theta=0}$$

where the inner sum is over all vectors $v = (v_1, \dots, v_{n+1}) \in \Upsilon_j^{n+1}$ and Z_1, …,Z_{n+1} are independent and identically distributed 1-shifted geometric$(1-\theta)$ random variables.

If $\Psi(a_1, \dots, a_{n+1})$ is symmetric in its arguments, then this result simplifies to

$$E\big(\Psi(G_1, \dots, G_{n+1})\big)$$

$$= \frac{1}{\binom{t}{n}(t+1)!} \sum_{j=0}^{n+1} \binom{n+1}{j} \frac{d^{t+1}}{d\theta^{t+1}} \left(\left(\frac{\theta}{1-\theta}\right)^{n+1} (1-\theta^k)^j \theta^{k(n+1-j)} \right.$$
$$\left. \times \mathrm{E}\left(\Psi\left(0, \dots, 0, Z_{j+1}, \dots, Z_{n+1}\right)\right) \right)\Bigg|_{\theta=0}$$

where Z_{j+1}, …,Z_{n+1} are independent and identically distributed 1 - shifted geometric$(1-\theta)$ random variables.

Circular Success Runs

For circular success runs (overlapping and non-overlapping) of fixed length t and with a fixed number of failures n, we follow the scheme set up in Theorem 14 and start with t different colored but unnumbered balls arranged in a circle. We pick 1 ball at random and attach the label t to this ball. Then we attach the labels $1, \dots, t-1$ to the remaining balls going clockwise around the circle. After attaching the labels we randomly pick $n-1$ balls from those labeled 1 to $t-1$. Let $1 \leq X_1 < \cdots < X_{n-1} \leq t-1$ be the labels on the $n-1$ balls drawn at this second phase of our sample.

The chosen numbers $(X_1, \dots, X_{n-1})$ along with ball we initially chose and labeled t represent the location of our n failures. The other $t-n$ numbers represent the location of our $t-n$ successes.

Define the variables C_j^* by

$$\begin{aligned} C_1^* &= X_1 \\ C_2^* &= X_2 - X_1 \\ &\vdots \\ C_{n-1}^* &= X_{n-1} - X_{n-2} \\ C_n^* &= t - X_{n-1}. \end{aligned}$$

Part III: Circular Non-Overlapping Success Runs

For any positive integer k define the variables D_j^* by

$$D_j^* = \left\lfloor \frac{C_j^* - 1}{k} \right\rfloor \quad j = 1, \dots, n.$$

We note that D_j^* equals the number of non-overlapping success runs between balls t and X_1 in the case $j = 1$, the number of non-overlapping success runs between balls X_j and X_{j-1} for $j = 2, \dots, n-1$ and the number of non-overlapping success runs between balls X_n and t for $j = n$.

In this case,

$$E\big(\Psi(D_1^*, \dots, D_n^*)\big) = \frac{1}{\binom{t-1}{n-1} t!}$$
$$\times \frac{d^t}{d\theta^t}\left(\left(\frac{\theta}{1-\theta}\right)^n E\big(\Psi(Z_1, \dots, Z_n)\big)\right)\Bigg|_{\theta=0}$$

where Z_1, ...,Z_n are independent and identically distributed geometric$(1-\theta^k)$ random variables.

It is important to note that in Parts III and IV our sample space consists of $\binom{t}{n}$, not $\binom{t-1}{n-1}$, equally likely outcomes.

Part IV: Circular Overlapping Success Runs

For any positive integer k define the variables G_j^* by

$$G_j^* = \begin{cases} C_j^* - k & C_j^* \geq k+1 \\ 0 & C_j^* \leq k \end{cases} \qquad j = 1, \dots, n.$$

That is, $G_j^* = \big(C_j^* - k\big)\mathbb{I}\big(C_j^* \geq k+1\big) = max\{C_j^* - k, 0\}$.

We note that G_j^* equals the number of overlapping success runs between balls t and X_1 in the case $j = 1$, the number of overlapping success runs between balls X_j and X_{j-1} for $j = 2, \dots, n-1$ and the number of overlapping success runs between balls X_{n-1} and t for $j = n$.

In this case,

$$E\big(\Psi(G_1^*, \dots, G_n^*)\big) = \frac{1}{\binom{t-1}{n-1}(t+1)!}$$

$$\times \sum_{j=0}^{n} \frac{d^t}{d\theta^t}\left(\left(\frac{\theta}{1-\theta}\right)^n (1-\theta^k)^j \theta^{k(n-j)} \right.$$

$$\left. \times \sum_{v \in \Upsilon_j^n} E\big(\Psi(v_1 Z_1, \ldots, v_n Z_n)\big) \right)\Bigg|_{\theta=0}$$

where the inner sum is over all vectors $v = (v_1, \ldots, v_n) \in \Upsilon_j^n$, the set of all n dimensional $\{0,1\}$ vectors with exactly j 0's and where Z_1, ...,Z_n are independent and identically distributed 1-shifted geometric$(1-\theta)$ random variables.

If $\Psi(a_1, \ldots, a_n)$ is symmetric in its arguments, then this result simplifies to

$$E\big(\Psi(G_1^*, \ldots, G_n^*)\big)$$

$$= \frac{1}{\binom{t-1}{n-1} t!} \sum_{j=0}^{n} \binom{n}{j} \frac{d^t}{d\theta^t}\left(\left(\frac{\theta}{1-\theta}\right)^n (1-\theta^k)^j \theta^{k(n-j)} \right.$$

$$\left. \times \mathrm{E}\Big(\Psi\big(0, \ldots, 0, Z_{j+1}, \ldots, Z_n\big)\Big) \right)\Bigg|_{\theta=0}$$

where Z_{j+1}, ...,Z_n are independent and identically distributed 1 - shifted geometric$(1-\theta)$ random variables.

Proof, Part I

For the "Sampling from a Line" model in Theorem 12 we showed that for an arbitrary function ζ

$$E\big(\zeta(C_1, \ldots, C_{n+1})\big) = \frac{1}{\binom{t}{n}(t+1)!}$$

$$\times \frac{d^{t+1}}{d\theta^{t+1}}\left(\left(\frac{\theta}{1-\theta}\right)^{n+1} E\big(\zeta(W_1, \ldots, W_{n+1})\big) \right)\Bigg|_{\theta=0}$$

where $W_1, \ldots, W_{n+1}$ are independent and identically distributed 1-shifted geometric$(1-\theta)$ random variables.

For some function Ψ and positive integer k define

$$\zeta(C_1, \ldots, C_{n+1})$$

$$= \Psi\left(\left\lfloor\frac{C_1 - 1}{k}\right\rfloor, \dots, \left\lfloor\frac{C_{n+1} - 1}{k}\right\rfloor\right)$$

$$= \Psi(D_1, \dots, D_{n+1})$$

and define the random variables $Z_j = \left\lfloor\frac{W_j - 1}{k}\right\rfloor$ for $j = 1,2, \dots, n + 1$. In this case Theorem 12 tells us that

$$E\big(\Psi(D_1, \dots, D_{n+1})\big) = \frac{1}{\binom{t}{n}(t+1)!}$$

$$\times \frac{d^{t+1}}{d\theta^{t+1}}\left(\left(\frac{\theta}{1-\theta}\right)^{n+1} E\big(\Psi(Z_1, \dots, Z_{n+1})\big)\right)\Bigg|_{\theta=0}.$$

Now it just remains to determine the distribution of the $Z_j = \left\lfloor\frac{W_j - 1}{k}\right\rfloor$ when the W_j follow a 1 - shifted geometric $(1-\theta)$ distribution.

$$P(Z_j = z) = P\left(\left\lfloor\frac{W_j - 1}{k}\right\rfloor = z\right) \qquad z = 0,1, \dots$$

$$= P\big(zk \le W_j - 1 \le zk + k - 1\big) = P\big(zk + 1 \le W_j \le zk + k\big)$$

$$= \sum_{w=zk+1}^{zk+k} P\big(W_j = w\big) = \sum_{w=zk+1}^{zk+k} (1-\theta)\theta^{w-1}$$

$$= (1-\theta)\sum_{w=zk+1}^{zk+k} \theta^{w-1} = (1-\theta)\left(\frac{\theta^{zk} - \theta^{zk+k}}{1-\theta}\right)$$

$$= (\theta^k)^z(1-\theta^k).$$

But we recognize this as the geometric distribution with parameter $1 - \theta^k$. That is, $Z_j \sim$ geometric$(1-\theta^k)$ for $j = 1, \dots, n + 1$.

Proof, Part II

For the "Sampling from a Line" model in Theorem 12 we showed that for an arbitrary function ζ

$$E\big(\zeta(C_1, \dots, C_{n+1})\big) = \frac{1}{\binom{t}{n}(t+1)!}$$

$$\times \frac{d^{t+1}}{d\theta^{t+1}} \left(\left(\frac{\theta}{1-\theta}\right)^{n+1} E\big(\zeta(W_1, \dots, W_{n+1})\big) \right)\Bigg|_{\theta=0}$$

where $W_1, \dots, W_{n+1}$ are independent and identically distributed 1-shifted geometric$(1-\theta)$ random variables.

For some function Ψ and positive integer k define

$$\zeta(C_1, \dots, C_{n+1})$$

$$= \Psi\big((C_1 - k)\mathbb{I}(C_1 \geq k+1), \dots, (C_{n+1} - k)\mathbb{I}(C_{n+1} \geq k+1)\big)$$

$$= \Psi(G_1, \dots, G_{n+1})$$

and define the random variables $Q_j = Q(W_j) = (W_j - k)\mathbb{I}(W_j \geq k+1)$ for $j = 1, 2, \dots, n+1$. In this case Theorem 12 tells us that

$$E\big(\Psi(G_1, \dots, G_{n+1})\big) = \frac{1}{\binom{t}{n}(t+1)!}$$

$$\times \frac{d^{t+1}}{d\theta^{t+1}} \left(\left(\frac{\theta}{1-\theta}\right)^{n+1} E\Big(\Psi\big(Q(W_1), \dots, Q(W_{n+1})\big)\Big) \right)\Bigg|_{\theta=0}$$

where $W_1, \dots, W_{n+1}$ are independent and identically distributed 1-shifted geometric$(1-\theta)$ random variables.

Now we will focus on simplifying $E\Big(\Psi\big(Q(W_1), \dots, Q(W_{n+1})\big)\Big)$.

$$E\Big(\Psi\big(Q(W_1), \dots, Q(W_{n+1})\big)\Big)$$

$$= \sum_{\substack{\text{all} \\ (w_1, \dots, w_{n+1})}} \cdots \sum \Psi\big(Q(w_1), \dots, Q(w_{n+1})\big) P(W_1 = w_1, \dots, W_{n+1} = w_{n+1})$$

$$= \sum_{j=0}^{n+1} \sum_{\substack{\text{all} \\ (w_1,\ldots,w_{n+1})\ni \\ \text{exactly } j \text{ of} \\ (w_1,\ldots,w_{n+1}) \le k}} \cdots \sum \Psi\big(Q(w_1), \ldots, Q(w_{n+1})\big) P(W_1 = w_1, \ldots, W_{n+1} = w_{n+1}).$$

At this point it is "notationally convenient" to consider what happens in the special case when $w_1 \le k, \ldots, w_j \le k$ and $w_{j+1} \ge k+1, \ldots, w_{n+1} \ge k+1$ and to look back to the general case based on what we see.

$$\sum_{\substack{\text{all} \\ (w_1 \le k,\ldots,w_j \le k) \\ \text{and all} \\ (w_{j+1} \ge k+1,\ldots,w_{n+1} \ge k+1)}} \cdots \sum \Psi\big(Q(w_1), \ldots, Q(w_{n+1})\big) P(W_1 = w_1, \ldots, W_{n+1} = w_{n+1})$$

$$= \sum_{\substack{\text{all} \\ (w_1 \le k,\ldots,w_j \le k) \\ \text{and all} \\ (w_{j+1} \ge k+1,\ldots,w_{n+1} \ge k+1)}} \cdots \sum \Psi\big(0, \ldots, 0, w_{j+1} - k, \ldots, w_{n+1} - k\big) P(W_1 = w_1, \ldots, W_{n+1} = w_{n+1})$$

$$= \sum_{\substack{\text{all} \\ (w_1 \le k,\ldots,w_j \le k) \\ \text{and all} \\ (w_{j+1} \ge k+1,\ldots,w_{n+1} \ge k+1)}} \cdots \sum \Psi\big(0, \ldots, 0, w_{j+1} - k, \ldots, w_{n+1} - k\big) P\big(W_1 = w_1, \ldots, W_j = w_j\big)$$

$$\times P\big(W_{j+1} = w_{j+1}, \ldots, W_{n+1} = w_{n+1}\big)$$

$$= \sum_{\substack{\text{all} \\ (w_1 \le k,\ldots,w_j \le k)}} \cdots \sum \sum_{\substack{\text{all} \\ (w_{j+1} \ge k+1,\ldots,w_{n+1} \ge k+1)}} \cdots \sum \Psi\big(0, \ldots, 0, w_{j+1} - k, \ldots, w_{n+1} - k\big)$$

$$\times P\big(W_1 = w_1, \ldots, W_j = w_j\big)$$

$$\times P\big(W_{j+1} = w_{j+1}, \ldots, W_{n+1} = w_{n+1}\big)$$

$$= \left(\sum_{\substack{\text{all} \\ (w_1 \le k, \dots, w_j \le k)}} \cdots \sum P\left(W_1 = w_1, \dots, W_j = w_j\right) \right)$$

$$\times \left(\sum_{\substack{\text{all} \\ (w_{j+1} \ge k+1, \dots, w_{n+1} \ge k+1)}} \cdots \sum \Psi\left(0, \dots, 0, w_{j+1} - k, \dots, w_{n+1} - k\right) \right)$$

$$\times P\left(W_{j+1} = w_{j+1}, \dots, W_{n+1} = w_{n+1}\right)$$

$$= \left(\sum_{w_1 = 1}^{k} P(W_1 = w_1) \right) \cdots \left(\sum_{w_j = 1}^{k} P\left(W_j = w_j\right) \right)$$

$$\times \left(\sum_{\substack{\text{all} \\ (w_{j+1} \ge k+1, \dots, w_{n+1} \ge k+1)}} \cdots \sum \Psi\left(0, \dots, 0, w_{j+1} - k, \dots, w_{n+1} - k\right) \right)$$

$$\times P\left(W_{j+1} = w_{j+1}, \dots, W_{n+1} = w_{n+1}\right)$$

$$= \left(\sum_{w_1 = 1}^{k} (1 - \theta)\theta^{w_1} \right) \cdots \left(\sum_{w_j = 1}^{k} (1 - \theta)\theta^{w_j} \right)$$

$$\times \left(\sum_{\substack{\text{all} \\ (w_{j+1} \ge k+1, \dots, w_{n+1} \ge k+1)}} \cdots \sum \Psi\left(0, \dots, 0, w_{j+1} - k, \dots, w_{n+1} - k\right) \right)$$

$$\times P\left(W_{j+1} = w_{j+1}, \dots, W_{n+1} = w_{n+1}\right)$$

$$= (1 - \theta^k) \cdots (1 - \theta^k)$$

$$\times \left(\sum_{\substack{\text{all} \\ (w_{j+1} \ge k+1, \dots, w_{n+1} \ge k+1)}} \cdots \sum \Psi\left(0, \dots, 0, w_{j+1} - k, \dots, w_{n+1} - k\right) \right)$$

$$\times P\big(W_{j+1} = w_{j+1}, \dots, W_{n+1} = w_{n+1}\big)$$

$$= (1-\theta^k)^j$$
$$\times \left(\sum_{w_{j+1}=k+1}^{\infty} \cdots \sum_{w_{n+1}=k+1}^{\infty} \Psi\big(0, \dots, 0, w_{j+1}-k, \dots, w_{n+1}-k\big) \times (1-\theta)^{(n+1-j)} \theta^{(w_{j+1}+\cdots+w_{n+1})-(n+1-j)} \right).$$

Now perform a change of variable, letting $y_r = w_r - k$ for $r = j+1, \dots, n+1$. Then

$$\sum_{w_{j+1}=k+1}^{\infty} \cdots \sum_{w_{n+1}=k+1}^{\infty} \Psi\big(0, \dots, 0, w_{j+1}-k, \dots, w_{n+1}-k\big) \times (1-\theta)^{(n+1-j)} \theta^{(w_{j+1}+\cdots+w_{n+1})-(n+1-j)}$$

$$= \theta^{(n+1-j)k} \cdot \left(\sum_{y_{j+1}=1}^{\infty} \cdots \sum_{y_{n+1}=1}^{\infty} \Psi\big(0, \dots, 0, y_{j+1}, \dots, y_{n+1}\big) \times (1-\theta)^{(n+1-j)} \theta^{(y_{j+1}+\cdots+y_{n+1})-(n+1-j)} \right).$$

But we recognize this latter factor is just $\mathrm{E}\left(\Psi\big(0, \dots, 0, Z_{j+1}, \dots, Z_{n+1}\big)\right)$ where $Z_{j+1}, \dots, Z_{n+1}$ are independent and identically 1 - shifted geometric$(1-\theta)$ distributed random variables.

That is, in the special case where $w_1 \le k, \dots, w_j \le k$ and $w_{j+1} \ge k+1, \dots, w_{n+1} \ge k+1$,

$$\sum_{\substack{\text{all} \\ (w_1 \le k, \dots, w_j \le k) \\ \text{and all} \\ (w_{j+1} \ge k+1, \dots, w_{n+1} \ge k+1)}} \cdots \sum \Psi\big(Q(w_1), \dots, Q(w_{n+1})\big) P(W_1 = w_1, \dots, W_{n+1} = w_{n+1})$$

$$= (1-\theta^k)^j \, \theta^{(n+1-j)k} \, \mathrm{E}\left(\Psi\big(0, \dots, 0, Z_{j+1}, \dots, Z_{n+1}\big)\right)$$

where $Z_{j+1}, \dots, Z_{n+1}$ are independent and identically distributed 1 - shifted geometric$(1-\theta)$ random variables.

Recall that we defined Υ_j^{n+1} be the set of all $(n+1)$ dimensional $\{0,1\}$ vectors with exactly j 0's. Let $(v_1, \dots, v_{n+1})$ be a vector in Υ_j^{n+1}. Using this notation, we can express the solution to the special case we just considered in the form

$$(1-\theta^k)^j\,\theta^{(n+1-j)k}\,\mathrm{E}\left(\Psi\big(0,\dots,0,Z_{j+1},\dots,Z_{n+1}\big)\right)$$

$$=(1-\theta^k)^j\,\theta^{(n+1-j)k}\,\mathrm{E}\left(\Psi\big(v_1Z_1,\dots,v_jZ_j,v_{j+1}Z_{j+1},\dots,v_{n+1}Z_{n+1}\big)\right)$$

with $v_1 = \cdots = v_j = 0$ and $v_{j+1} = \cdots = v_{n+1} = 1$ where $Z_1, \dots, Z_{n+1}$ are independent and identically distributed 1 - shifted geometric$(1-\theta)$ random variables. In fact, all cases can be covered with this notation and we can say that

$$E\left(\Psi\big(Q(W_1),\dots,Q(W_{n+1})\big)\right)$$

$$=\sum_{j=0}^{n+1}\ \underset{\substack{\text{all}\\ (w_1,\dots,w_{n+1})\ni\\ \text{exactly } j \text{ of}\\ (w_1,\dots,w_{n+1})\le k}}{\sum\cdots\sum}\ \Psi\big(Q(w_1),\dots,Q(w_{n+1})\big)P(W_1=w_1,\dots,W_{n+1}=w_{n+1})$$

$$=\sum_{j=0}^{n+1}\sum_{v\in\Upsilon_j^{n+1}}(1-\theta^k)^j\,\theta^{(n+1-j)k}\,\mathrm{E}\big(\Psi(v_1Z_1,\dots,v_{n+1}Z_{n+1})\big)$$

where the inner sum is over all vectors $v = (v_1, \dots, v_{n+1}) \in \Upsilon_j^{n+1}$ and $Z_1, \dots, Z_{n+1}$ are independent and identically distributed 1-shifted geometric$(1-\theta)$ random variables.

So can conclude that

$$E\big(\Psi(G_1,\dots,G_{n+1})\big)=\frac{1}{\binom{t}{n}(t+1)!}$$
$$\times\frac{d^{t+1}}{d\theta^{t+1}}\left(\left(\frac{\theta}{1-\theta}\right)^{n+1}E\left(\Psi\big(Q(W_1),\dots,Q(W_{n+1})\big)\right)\right)\Bigg|_{\theta=0}$$

$$= \frac{1}{\binom{t}{n}(t+1)!} \frac{d^{t+1}}{d\theta^{t+1}} \left(\left(\frac{\theta}{1-\theta} \right)^{n+1} \right.$$

$$\left. \times \sum_{j=0}^{n+1} \sum_{v \in \Upsilon_j^{n+1}} (1-\theta^k)^j \theta^{k(n+1-j)} E\big(\Psi(v_1 Z_1, \dots, v_{n+1} Z_{n+1})\big) \right) \Bigg|_{\theta=0}$$

$$= \frac{1}{\binom{t}{n}(t+1)!} \sum_{j=0}^{n+1} \frac{d^{t+1}}{d\theta^{t+1}} \left(\left(\frac{\theta}{1-\theta} \right)^{n+1} (1-\theta^k)^j \theta^{k(n+1-j)} \right.$$

$$\left. \times \sum_{v \in \Upsilon_j^{n+1}} E\big(\Psi(v_1 Z_1, \dots, v_{n+1} Z_{n+1})\big) \right) \Bigg|_{\theta=0}$$

where the inner sum is over all vectors $v = (v_1, \dots, v_{n+1}) \in \Upsilon_j^{n+1}$ and Z_1, …,Z_{n+1} are independent and identically distributed 1-shifted geometric$(1-\theta)$ random variables.

If $\Psi(a_1, \dots, a_{n+1})$ is symmetric in its arguments, then this result simplifies to

$$E\big(\Psi(G_1, \dots, G_{n+1})\big)$$

$$= \frac{1}{\binom{t}{n}(t+1)!} \sum_{j=0}^{n+1} \frac{d^{t+1}}{d\theta^{t+1}} \left(\left(\frac{\theta}{1-\theta} \right)^{n+1} (1-\theta^k)^j \theta^{k(n+1-j)} \right.$$

$$\left. \times \binom{n+1}{j} \mathrm{E}\Big(\Psi\big(0, \dots, 0, Z_{j+1}, \dots, Z_{n+1}\big)\Big) \right) \Bigg|_{\theta=0}$$

$$= \frac{1}{\binom{t}{n}(t+1)!} \sum_{j=0}^{n+1} \binom{n+1}{j} \frac{d^{t+1}}{d\theta^{t+1}} \left(\left(\frac{\theta}{1-\theta} \right)^{n+1} (1-\theta^k)^j \theta^{k(n+1-j)} \right.$$

$$\left. \times \mathrm{E}\Big(\Psi\big(0, \dots, 0, Z_{j+1}, \dots, Z_{n+1}\big)\Big) \right) \Bigg|_{\theta=0}$$

where Z_{j+1}, …,Z_{n+1} are independent and identically distributed 1 - shifted geometric$(1-\theta)$

random variables.

Proof, Part III

By Theorem 14, Part I, we know that for an arbitrary function ζ

$$E\big(\zeta(C_1^*, \dots, C_n^*)\big) = \frac{1}{\binom{t-1}{n-1} t!} \frac{d^t}{d\theta^t}\left(\left(\frac{\theta}{1-\theta}\right)^n E\big(\zeta(W_1, \dots, W_n)\big)\right)\Bigg|_{\theta=0}$$

where $W_1, \dots, W_n$ are independent and identically distributed 1-shifted geometric$(1-\theta)$ random variables. For some function Ψ and positive integer k define

$$\begin{aligned}&\zeta(C_1^*, \dots, C_n^*)\\ &= \Psi\left(\left\lfloor\frac{C_1^*-1}{k}\right\rfloor, \dots, \left\lfloor\frac{C_n^*-1}{k}\right\rfloor\right)\\ &= \Psi(D_1^*, \dots, D_n^*).\end{aligned}$$

In this case we can apply Theorem 15, Part I (after replacing $n+1$ with n and $t+1$ with t) to show that

$$\begin{aligned}E\big(\Psi(D_1^*, \dots, D_n^*)\big) &= \frac{1}{\binom{t-1}{n-1} t!}\\ &\times \frac{d^t}{d\theta^t}\left(\left(\frac{\theta}{1-\theta}\right)^n E\big(\Psi(Z_1, \dots, Z_n)\big)\right)\Bigg|_{\theta=0}\end{aligned}$$

where Z_1, …,Z_n are independent and identically distributed geometric$(1-\theta^k)$ random variables.

Proof, Part IV

By Theorem 14, Part I, we know that for an arbitrary function ζ

$$E\big(\zeta(C_1^*, \dots, C_n^*)\big) = \frac{1}{\binom{t-1}{n-1} t!} \frac{d^t}{d\theta^t}\left(\left(\frac{\theta}{1-\theta}\right)^n E\big(\zeta(W_1, \dots, W_n)\big)\right)\Bigg|_{\theta=0}$$

where $W_1, \dots, W_n$ are independent and identically distributed 1-shifted geometric$(1-\theta)$ random variables. For some function Ψ and positive integer k define

$$\begin{gathered}\zeta(C_1^*, \dots, C_n^*) \\ = \Psi\big((C_1^* - k)\mathbb{I}(C_1^* \geq k+1), \dots, (C_n^* - k)\mathbb{I}(C_n^* \geq k+1)\big) \\ = \Psi(G_1^*, \dots, G_n^*).\end{gathered}$$

In this case we can apply Theorem 15, Part II (after replacing $n+1$ with n and $t+1$ with t) to show that

$$\begin{gathered}E\big(\Psi(G_1^*, \dots, G_n^*)\big) = \frac{1}{\binom{t-1}{n-1} t!} \\ \times \sum_{j=0}^{n} \frac{d^t}{d\theta^t}\left(\left(\frac{\theta}{1-\theta}\right)^n (1-\theta^k)^j \theta^{k(n-j)}\right. \\ \left.\left. \times \sum_{\upsilon \in \Upsilon_j^n} E\big(\Psi(\upsilon_1 Z_1, \dots, \upsilon_n Z_n)\big)\right)\right|_{\theta=0}\end{gathered}$$

where the inner sum is over all vectors $\upsilon = (\upsilon_1, \dots, \upsilon_n) \in \Upsilon_j^n$, the set of all n dimensional $\{0,1\}$ vectors with exactly j 0's and $Z_1, \dots, Z_n$ are independent and identically distributed 1-shifted geometric$(1-\theta)$ random variables.

If $\Psi(a_1, \dots, a_n)$ is symmetric in its arguments, then this result simplifies to

$$\begin{gathered}E\big(\Psi(G_1^*, \dots, G_n^*)\big) = \frac{1}{\binom{t-1}{n-1} t!} \sum_{j=0}^{n} \binom{n}{j} \frac{d^t}{d\theta^t}\left(\left(\frac{\theta}{1-\theta}\right)^n\right. \\ \left.\left. \times (1-\theta^k)^j \theta^{k(n-j)} E\left(\Psi\big(0, \dots, 0, Z_j, \dots, Z_n\big)\right)\right)\right|_{\theta=0}\end{gathered}$$

where Z_j,…,Z_n are independent and identically distributed 1 - shifted geometric$(1-\theta)$ random variables.

Example 1 .... A selection of non-overlapping and overlapping success run problems.

(a) Let $N_{t,k,r,n}$ equal the number of arrangements of n failures and $t-n$ successes with r non-overlapping successes runs of length k.

Then

$$N_{t,k,r,n} = \sum_{i=0}^{\left\lfloor \frac{t-n}{k} \right\rfloor - r} (-1)^i \binom{n+1}{i} \binom{n+r}{r} \binom{t-k(r+i)}{t-k(r+i)-n}$$

for $kr + n \leq t < k(r+n+1)$ and 0 else.

(b) Let $N_{t,k}$ equal the number of non-overlapping success runs of length k in a sequence of t independent Bernoulli trials with constant success probability p.

Then

$$P(N_{t,k} = r) = \sum_{n=\left\lfloor \frac{t}{k} \right\rfloor - r}^{t-kr} N_{t,k,r,n}\, p^{t-n}(1-p)^n$$

and

$$\mathrm{E}\left(\left(N_{t,k}\right)_{(w)}\right) = \sum_{n=0}^{t-kw} \sum_{i=0}^{\left\lfloor \frac{t-n}{k} \right\rfloor - w} \left(\binom{t-k(w+i)}{t-n-k(w+i)} \right.$$

$$\left. \times \binom{w+i-1}{i} (n+1)^{[w]}\, p^{(t-n)}(1-p)^n \right).$$

The probability distribution of $N_{t,k}$ was derived by Hirano [?] and Philippou and Makri [?] and was simplified by Godbole [?]. The form given here agrees with Godbole's simplified solution. Antzoulakos and Chadjiconstantinidis [?] derive the falling factorial moment result in a different but equivalent form than given here.

Let $M_{t,k,r,n}$ equal the number of arrangements of n failures and $t-n$ successes with r overlapping successes runs of length k.

Then for $r \geq 1$ and $\left\lfloor \frac{t-r}{k} \right\rfloor \leq n \leq t-r$

$$M_{t,k,r,n} = \binom{r-1}{n} \mathbb{I}_{\{t-k(n+1)=r-1\}} +$$

$$\sum_{j=n+1-\left\lfloor \frac{t-r}{k} \right\rfloor}^{n+1} \sum_{i=0}^{j-\left(n+1-\left\lfloor \frac{t-r}{k} \right\rfloor\right)} \left((-1)^i \binom{j}{i} \binom{n+1}{j} \right.$$

$$\times \binom{r-1}{n-j} \binom{t-r-k(n+1-j+i)}{t-r-k(n+1-j+i)-j+1} \Bigg) \Bigg)$$

and for $r = 0$ and $n \geq \left\lfloor \frac{t}{k} \right\rfloor$

$$M_{t,k,0,n} = \sum_{j=0}^{\left\lfloor \frac{t}{k} \right\rfloor} (-1)^j \binom{n+1}{j} \binom{t-kj}{t-kj-n}.$$

We note that $M_{t,k,r,n} = 0$ for $r \geq 0$ and $n < \left\lfloor \frac{t-r}{k} \right\rfloor$

(d) Let $M_{t,k}$ equal the number of overlapping success runs of length k in a sequence of t independent Bernoulli trials with constant success probability p. Then

$$P(M_{t,k} = r) = \sum_{n=\left\lfloor \frac{t-r}{k} \right\rfloor}^{t-r} M_{t,k,r,n}\, p^{t-n}(1-p)^n$$

and

$$\mathrm{E}\left(\left(M_{t,k}\right)^{[w]} \right) = \sum_{n=0}^{t} \sum_{j=0}^{n+1} \sum_{i=0}^{j} \Bigg((-1)^i \binom{j}{i} \binom{n+1}{j}$$

$$\times \binom{w+t-k(n+1-j+i)}{t-n-k(n+1-j+i)} (n-j+1)^{[w]} p^{t-n}(1-p)^n \Bigg)$$

The probability distribution of $M_{t,k}$ was derived by Ling [?] and was simplified by Godbole [?]. The solution given here is similar in form to Godbole's solution but is slightly more compact. $M_{t,k,r,n}$ can be stated more compactly than given here by using the standard extension of binomial coefficients to allow for a negative argument. Antzoulakos and Chadjiconstantinidis [?] state a formula for the falling factorial moment. The result given here is for the rising factorial moment.

(e) Let $N_{t,k}$ equal the number of non-overlapping success runs of length k in a sample of size t drawn from an urn with a white balls and b black balls according to the Markov-P'olya sampling scheme. Then

$$P\left(N_{t,k}=r\right)=\sum_{n=\left\lfloor\frac{t}{k}\right\rfloor-r}^{t-kr} N_{t,k,r,n}\frac{\binom{n+a-1}{a-1}\binom{(t-n)+b-1}{b-1}}{\binom{t+a+b-1}{a+b-1}}.$$

Problems of this type are considered by Sen, Agarwal and Chakraborty [?].

Proof

For part (a) if we define C_j and D_j as in Theorem ??, Line ?? and consider $(X_1, \ldots, X_n)$ as the position of n failures then D_j will equal the number of non-overlapping success runs of length k between the $(j-1)^{st}$ and j^{th} failures. (It is understood that D_1 will equal the number of non-overlapping success runs of length k before the first failure and D_{n+1} the number after the last failure.) It then follows that $D_1+\cdots+D_{n+1}$ equals the total number of non-overlapping success runs of length k in the random arrangement of n failures and $t-n$ successes.

If we take

$$\Psi(C_1, \ldots, C_{n+1})=\mathbb{I}(C_1+\cdots+C_{n+1}=r)$$

in Theorem ??, Line ??, then

$$N_{t,k,r,n}=\frac{1}{(t+1)!}\frac{d^{t+1}}{d\theta^{t+1}}\left(\left(\frac{\theta}{1-\theta}\right)^{n+1} P(Z_1+\cdots+Z_{n+1}=r)\right)\Bigg|_{\theta=0}$$

where $Z_1, \ldots, Z_{n+1}$ are independent and identically distributed (iid) geometric$(1-\theta^k)$ random variables. However the sum of iid geometric random variables follows a negative binomial distribution$(n+1,1-\theta^k)$ and the final result follows on simplification. That $N_{t,k,r,n}$ equals 0 for $t-k(r+n+1)\geq 0$ follows from identity 3.150 of Gould [?].

For the first result in part (b), clearly

$$P\left(N_{t,k}=r\right)=\sum_{n=0}^{t} N_{t,k,r,n}\, p^{t-n}(1-p)^n$$

and the more restrictive bounds on n follow from the given bounds on t.

The falling factorial moment in part (b) could be computed from definition using the distribution calculated in part (a). However, applying Theorem ??, Line ?? results in a simpler form.

Let F equal the number of failures in t independent Bernoulli trials with constant success probability p. Then

$$\mathrm{E}\left(\left(N_{t,k}\right)_{(w)}\right) = \mathrm{E}\left(\mathrm{E}\left(\left(N_{t,k}\right)_{(w)}\middle| F = n\right)\right)$$

$$= \sum_{n=0}^{t} \mathrm{E}\left(\left(N_{t,k}\right)_{(w)}\middle| F = n\right)\binom{t}{n} p^{(t-n)}(1-p)^{n}.$$

By Theorem ??, Line ??,

$$\mathrm{E}\left(\left(N_{t,k}\right)_{(w)}\middle| F = n\right) = \mathrm{E}\left((D_1 + \cdots + D_{n+1})_{(w)}\right)$$

$$= \frac{1}{\binom{t}{n}(t+1)!}\frac{d^{t+1}}{d\theta^{t+1}}\left(\left(\frac{\theta}{1-\theta}\right)^{n+1}\mathrm{E}\big((Z_1 + \cdots + Z_{n+1})_{(w)}\big)\right)\Bigg|_{\theta=0}$$

$$= \frac{1}{\binom{t}{n}(t+1)!}\frac{d^{t+1}}{d\theta^{t+1}}\left(\left(\frac{\theta}{1-\theta}\right)^{n+1}\left(\frac{\theta^k}{1-\theta^k}\right)^{w}(n+1)^{[w]}\right)\Bigg|_{\theta=0}.$$

This last equality follows using A??, the formula for the falling factorial moment of a negative binomial distribution. The final form follows on simplification.

For part (c) if we define C_j and G_j as in Theorem ??, Line ?? and consider $(X_1, \dots, X_n)$ as the position of n failures then G_j will equal the number of overlapping success runs of length k between the $(j-1)^{st}$ and j^{th} failures. (It is understood that G_1 will equal the number of overlapping success runs of length k before the first failure and G_{n+1} the number after the last failure.)

Accordingly, $G_1 + \cdots + G_{n+1}$ equals the total number of overlapping success runs of length k in the random arrangement of n failures and $t-n$ successes.

If we take

$$\Psi(C_1, \dots, C_{n+1}) = \mathbb{I}(C_1 + \cdots + C_{n+1} = r)$$

in Theorem ??, Line ??, then

$$M_{t,k,r,n} = \frac{1}{(t+1)!}\sum_{j=0}^{n+1} \frac{d^{t+1}}{d\theta^{t+1}}\left(\left(\frac{\theta}{1-\theta}\right)^{n+1} (1-\theta^k)^j \theta^{k(n+1-j)}\right.$$

$$\left.\left.\times\left(\sum_{v\in\Upsilon_j^{n+1}} P(v_1 Z_1 + \cdots + v_{n+1}Z_{n+1} = r)\right)\right)\right|_{\theta=0}$$

where Z_1, ...,Z_{n+1} are independent and identically distributed 1-shifted geometric$(1-\theta)$ random variables.

We note that for $r > 0$ and $j < n+1$

$$\sum_{v\in\Upsilon_j^{n+1}} P(v_1 Z_1 + \cdots + v_{n+1}Z_{n+1} = r)$$

$$= \binom{n+1}{j} P\big(Z_1 + \cdots + Z_{n+1-j} = r\big)$$

$$= \binom{n+1}{j}\left(\binom{r-1}{n-j}(1-\theta)^{n+1-j}\,\theta^{r-(n+1-j)}\right).$$

For $r > 0$ and $j = n+1$ we have

$$\sum_{v\in\Upsilon_{n+1}^{n+1}} P(v_1 Z_1 + \cdots + v_{n+1}Z_{n+1} = r)$$

$$= P(0\cdot Z_1 + \cdots + 0\cdot Z_{n+1} = r) = 0$$

and for $r = 0$

$$\sum_{v\in\Upsilon_j^{n+1}} P(v_1 Z_1 + \cdots + v_{n+1}Z_{n+1} = 0) = \mathbb{I}(j = n+1).$$

The final result for part (c) then follows on simplification. The first result in part (d) follows as in (b). We can apply Theorem ??, Line ?? to compute the rising factorial moment in part (d).

Let F equal the number of failures in t independent Bernoulli trials with constant success probability p. Then

$$\mathrm{E}\left(\left(M_{t,k}\right)_{(w)}\right)=\sum_{n=0}^{t}\mathrm{E}\left(\left(M_{t,k}\right)_{(w)}\middle|F=n\right)\binom{t}{n}p^{(t-n)}(1-p)^{n}.$$

By Theorem ??, Line ??,

$$\mathrm{E}\left(\left(M_{t,k}\right)^{[w]}\middle|F=n\right)=\mathrm{E}\left((D_1+\cdots+D_{n+1})^{[w]}\right)$$

$$=\frac{1}{\binom{t}{n}(t+1)!}\sum_{j=0}^{n+1}\frac{d^{t+1}}{d\theta^{t+1}}\left(\left(\frac{\theta}{1-\theta}\right)^{n+1}(1-\theta^{k})^{j}\right.$$

$$\left.\left.\times\,\theta^{k(n+1-j)}\left(\sum_{v\in\Upsilon_j^{n+1}}\mathrm{E}\big((v_1Z_1+\cdots+v_{n+1}Z_{n+1})^{[w]}\big)\right)\right)\right|_{\theta=0}.$$

Let $S=v_1Z_1+\ldots+v_{n+1}Z_{n+1}$. It follows from A. that for all $v\in\Upsilon_j^{n+1}$, $S\sim(n+1-j)$-shifted negative binomial$(n+1-j,1-\theta)$. Therefore by A., for all $v\in\Upsilon_j^{n+1}$,

$$\mathrm{E}\left((v_1Z_1+\cdots+v_{n+1}Z_{n+1})^{[w]}\right)=\left(\frac{1}{1-\theta}\right)^{w}(n-j+1)^{[w]}$$

and the final form follows on simplification.

1.3 Coverage of the Line and Circle

Theorem 19, Part I, Random Coverage of the Line

Consider a line of t urns numbered $1,\ldots,t$. Let $(X_1,\ldots,X_n)$ be a random sample (of urns) of size n taken without replacement from the t numbered urns such that $1\le X_1<X_2<\cdots<X_{n-1}<X_n\le t$.

Now suppose that we put a ball in urn X_1 and in each of the $k-1$ urns to the left of urn X_1, a ball in urn X_2 and in each of the $k-1$ urns to the left of urn X_2, …, and a ball in urn X_n and in each of the $k-1$ urns to the left of urn X_n and also in each of the $k-1$ urns to the right of urn X_n.

If there are not enough urns at any point in this distribution scheme (*e.g.* if there are fewer than $k-1$ urns to the left of urn X_1), the extra balls are just set aside.

We will refer to this distribution scheme as the discrete random coverage of the line model.

Clearly there is the potential that some urns will receive more than one ball, but that is not important to us here. Our only interest in this section is whether or not an urn is empty, that is whether an urn has been "covered" or not.

As an example, consider a row of $t = 15$ urns numbered from 1 to 15 in increasing order. Suppose we select $n = 4$ of these urns at random and happen to select Urn 3, Urn 4, Urn 10 and Urn 14. Then $X_1 = 3$, $X_2 = 4$, $X_3 = 10$ and $X_4 = 14$.

Suppose we are interested in the case $k = 4$. Then according to the directions, we put a ball in Urn 3 and the $k-1 = 4-1 = 3$ urns to the left of Urn 3. Because there are only two urns to the left of Urn 3 we just set the one extra ball aside. So, at this point Urns 1, 2 and 3 are "covered" (not empty). Continuing, we put a ball in Urn 4 and the $k-1 = 3$ urns to the left of Urn 4. So now Urns 1, 2, 3 and 4 are "covered". Note that Urns 1, 2 and 3 now each have two balls in them while Urn 4 only has one ball. But we don't need to keep track of this. Again, we are only interested in whether an urn is empty or not empty.

After we put a ball in Urn 10 and the $k-1 = 3$ to the left of Urn 10, Urns 1, 2, 3, 4, 7, 8, 9 and 10 are covered.

At the final step we put a ball in Urn 14 and the $k-1 = 3$ urns to the left and to the right of Urn 14. So, we put a ball in Urns 11, 12, 13 and 15. Because there is only one urn to the right of Urn 14 we just set the two extra balls aside. Remember that it is only at this final step that we also distribute balls to the right of a selected urn.

So, in this example, Urns 1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13, 14 and 15 are covered. Urns 5 and 6 are the only two urns which are not covered (i.e. are left empty).

The continuous analogue of the line coverage problem dates back to Whitworth [ ] and was also developed in a series of papers by Domb [ ].

For $j = 1, \dots, n+1$ define the variables C_j by

$$
\begin{aligned}
C_1 &= X_1 \\
C_2 &= X_2 - X_1 \\
&\vdots \\
C_n &= X_n - X_{n-1} \\
C_{n+1} &= (t+1) - X_n.
\end{aligned}
$$

and the variables G_j by

$$
G_j = \begin{cases} C_j - k & C_j \geq k+1 \\ 0 & C_j \leq k. \end{cases}
$$

Continuing with our example, where $t = 15, n = 4$ and $k = 4$ and where $X_1 = 3$, $X_2 = 4$, $X_3 = 10$ and $X_4 = 14$, we get $C_1 = X_1 = 3$, $C_2 = X_2 - X_1 = 1, C_3 = X_3 - X_2 = 6, C_4 = X_4 - X_3 = 4$ and $C_5 = (t+1) - X_n = (15+1) - X_4 = 2$.

Hence,

$$
\begin{aligned}
G_1 &= \max\{C_1 - k, 0\} = \max\{-1,0\} = 0 \\
G_2 &= \max\{C_2 - k, 0\} = \max\{-3,0\} = 0 \\
G_3 &= \max\{C_3 - k, 0\} = \max\{2,0\} = 2 \\
G_4 &= \max\{C_4 - k, 0\} = \max\{0,0\} = 0 \\
G_5 &= max\{C_4 - k, 0\} = max\{-2,0\} = 0.
\end{aligned}
$$

We note that G_1 counts the number of empty (uncovered) urns to the left of Urn $X_1 = 3$, G_2 counts the number of empty urns between Urns $X_1 = 3$ and $X_2 = 4$, G_3 counts the number of empty urns between Urns $X_2 = 4$ and $X_3 = 10$, G_4 counts the number of empty urns between Urns $X_3 = 10$ and $X_4 = 14$ and G_5 counts the number of empty urns to the right of Urn $X_n = X_4 = 14$.

It is not hard to see that the random coverage of the line model is identical to the model for linear overlapping success runs if we let a chosen urn represent a "failure" and an unchosen urn a "success". Then each uncovered (empty) urn corresponds to where an linear overlapping success run of length k starts.

It follows that the formula for $E\big(\Psi(G_1, \dots, G_{n+1})\big)$ given in Theorem 15, Part II for linear overlapping success runs holds for the random coverage of the line distribution. That is, for the random coverage of the line model,

$$E\big(\Psi(G_1, \dots, G_{n+1})\big) = \frac{1}{\binom{t}{n}(t+1)!}$$
$$\times \sum_{j=0}^{n+1} \frac{d^{t+1}}{d\theta^{t+1}} \left(\left(\frac{\theta}{1-\theta}\right)^{n+1} (1-\theta^k)^j \theta^{k(n+1-j)} \right.$$
$$\left. \times \sum_{\upsilon \in \Upsilon_j^{n+1}} E\big(\Psi(\upsilon_1 Z_1, \dots, \upsilon_{n+1} Z_{n+1})\big) \right) \Bigg|_{\theta=0}$$

where the inner sum is over all vectors $\upsilon = (\upsilon_1, \dots, \upsilon_{n+1}) \in \Upsilon_j^{n+1}$ and Z_1, …,Z_{n+1} are independent and identically distributed 1-shifted geometric$(1-\theta)$ random variables.

Furthermore, if $\Psi(a_1, \dots, a_{n+1})$ is symmetric in its arguments, then this result simplifies to

$$E\big(\Psi(G_1, \dots, G_{n+1})\big)$$
$$= \frac{1}{0.8\binom{t}{n}(t+1)!} \sum_{j=0}^{n+1} \left(\binom{n+1}{j} \frac{d^{t+1}}{d\theta^{t+1}} \left(\frac{\theta}{1-\theta}\right)^{n+1} (1-\theta^k)^j \theta^{k(n+1-j)} \right.$$
$$\left. \times \mathrm{E}\Big(\Psi\big(0, \dots, 0, Z_{j+1}, \dots, Z_{n+1}\big)\Big) \right) \Bigg|_{\theta=0}$$

where Z_{j+1}, …,Z_{n+1} are independent and identically distributed 1 - shifted geometric$(1-\theta)$ random variables.

Theorem 19, Part II, Random Coverage of the Circle

Consider t urns arranged in a circle. Suppose we pick one of these urns at random and label that urn with the number t and then attach the labels $1, \dots, t-1$ to the remaining urns as we go around the circle clockwise from urn t. Let $(X_1, \dots, X_{n-1})$ be a random sample (of urns) of size $n-1$ taken without replacement from the urns numbered $1, \dots, t-1$ such that $1 \le X_1 < X_2 < \dots < X_{n-2} < X_{n-1} \le t-1$.

Now suppose that we put a ball in urn X_1 and in each of the $k-1$ urns going counterclockwise from urn X_1, a ball in urn X_2 and in each of the $k-1$ urns going counterclockwise from urn X_2, …, and a ball in urn t and in each of the $k-1$ going counterclockwise from urn t. We will refer to this distribution scheme as the discrete random coverage of the circle model.

Clearly there is the potential in this distribution scheme that some urns will receive more than one ball. However, in the theorem and examples which follow our interest will only be in whether urns or empty or not, that is whether an urn has been “covered” or not.

The continuous analogue of the circle coverage problem dates back to Stevens [ ]. Solomon [ ] devotes a chapter to the continuous circle coverage problem. Holst [ ] and Ivchenko [ ] consider the discrete version.

For $j = 1, \dots, n$ define the variables C_j^* by

$$\begin{aligned} C_1^* &= X_1 \\ C_2^* &= X_2 - X_1 \\ &\vdots \\ C_{n-1}^* &= X_{n-1} - X_{n-2} \\ C_n^* &= t - X_{n-1}. \end{aligned}$$

and the variables G_j^* by

$$G_j^* = \begin{cases} C_j^* - k & C_j^* \geq k+1 \\ 0 & C_j^* \leq k. \end{cases}$$

We note that G_j^* equals the number of empty urns between the selected urns t and X_1 in the case $j = 1$, the number of empty urns between the selected urns X_j and X_{j-1} for $j = 2, \dots, n - 1$ and the number of empty urns between the selected urns X_{n-1} and t for $j = n$.

It is not hard to see that the random coverage of the circle model is identical to the model for circular overlapping success runs if we let a chosen urn represent a “failure” and an unchosen urn a “success”. Then each uncovered (empty) urn corresponds to where an circular overlapping success run of length k starts (going clockwise).

It follows that the formula for $E\big(\Psi(G_1^*, \dots, G_n^*)\big)$ given in Theorem 15, Part IV for circular overlapping success runs holds for the random coverage of the circle distribution. That is, for the random coverage of the circle model,

$$E\big(\Psi(G_1^*, \dots, G_n^*)\big) = \frac{1}{\binom{t-1}{n-1} t!} \sum_{j=0}^{n} \frac{d^t}{d\theta^t} \Bigg(\left(\frac{\theta}{1-\theta}\right)^n \\ \times (1-\theta^k)^j \theta^{k(n-j)} \sum_{v \in \Upsilon_j^n} E\big(\Psi(v_1 Z_1, \dots, v_n Z_n)\big) \Bigg)\Bigg|_{\theta=0}$$

where the inner sum is over all vectors $v = (v_1, \dots, v_n) \in \Upsilon_j^n$, the set of all n dimensional $\{0,1\}$ vectors with exactly j 0's and where Z_1, …,Z_n are independent and identically distributed 1-shifted geometric$(1-\theta)$ random variables.

Furthermore, if $\Psi(a_1, \dots, a_n)$ is symmetric in its arguments, then this result simplifies to

$$E\big(\Psi(G_1^*, \dots, G_n^*)\big) = \frac{1}{\binom{t-1}{n-1} t!} \sum_{j=0}^{n} \binom{n}{j} \frac{d^t}{d\theta^t}\Bigg(\left(\frac{\theta}{1-\theta}\right)^n \\ \times (1-\theta^k)^j \theta^{k(n-j)} E\left(\Psi\big(0, \dots, 0, Z_j, \dots, Z_n\big)\right)\Bigg)\Bigg|_{\theta=0}$$

where Z_j,…,Z_n are independent and identically distributed 1 - shifted geometric$(1-\theta)$ random variables.

Example 1 Number of empty urns

Let R equal the number of empty urns in the discrete random coverage of the circle distribution as described above. (That is, we randomly select n of t urns arranged in a circle and place a ball in each selected urn and in the $k-1$ urns following clockwise from each selected urn.) In this case:

If $r \geq 1$, then

$$P(R=r) = \frac{1}{\binom{t}{n}} \sum_{j=0}^{n-1} \sum_{i=0}^{j} (-1)^i \left(\frac{t}{n}\right) \binom{n}{j} \binom{j}{i} \binom{r-1}{n-j-1} \binom{t-(n-j+i)k-r-1}{t-(n-j+i)k-r-j}$$

If $r = 0$, then

$$P(R=r) = \frac{1}{\binom{t}{n}} \sum_{i=0}^{n} (-1)^i \left(\frac{t}{n}\right) \binom{n}{i} \binom{t-ik-1}{t-ik-n}$$

and

$$\mathrm{E}\big(R^{[u]}\big) = \frac{1}{\binom{t}{n}} \sum_{j=0}^{n} \sum_{i=0}^{j} (-1)^i \left(\frac{t}{n}\right) \binom{n}{j} \binom{j}{i} \binom{n-j+u-1}{u} \binom{u+t-k(n-j+i)-1}{t-k(n-j+i)-n}.$$

The probability distribution for R is result is given in Holst [?].

Proof

The first result follows on direct application of Theorem ?? with

$$\Psi(G_1, \dots, G_n) = \mathbb{I}(G_1 + \cdots + G_n = r).$$

For $r \geq 1$ we note that

$$\sum_{\upsilon \in \Upsilon_j^n} \mathrm{E}\big(\Psi(\upsilon_1 Z_1, \dots, \upsilon_n Z_n)\big) = \begin{cases} \binom{n}{j} P\big(Z_1 + \cdots + Z_{n-j} = r\big) & 0 \leq j \leq n-1 \\ 0 & j = n. \end{cases}$$

It follows from A. and A. that $Z_1 + \cdots + Z_{n-j} \sim (n-j)$ shifted negative binomial$(n-j, 1-\theta)$. Hence,

$$P\big(Z_1 + \cdots + Z_{n-j} = r\big) = \binom{z-1}{n-j-1}(1-\theta)^{n-j}\theta^{z-(n-j)}.$$

For $r = 0$ we see that

$$\sum_{\upsilon \in \Upsilon_j^n} \mathrm{E}\big(\Psi(\upsilon_1 Z_1, \dots, \upsilon_n Z_n)\big) = \begin{cases} 0 & 0 \leq j \leq n-1 \\ 1 & j = n. \end{cases}$$

The final form for the first result follows on simplification. The second result follows on direct application of Theorem ?? with

$$\Psi(G_1, \dots, G_n) = (G_1 + \cdots + G_n)^{[u]}.$$

In this case we have that

$$\sum_{\upsilon \in \Upsilon_j^n} \mathrm{E}\big(\Psi(\upsilon_1 Z_1, \dots, \upsilon_n Z_n)\big) = \begin{cases} \binom{n}{j} \mathrm{E}\left(\left(Z_1 + \cdots + Z_{n-j}\right)^{[u]}\right) & 0 \leq j \leq n-1 \\ 0 & j = n. \end{cases}$$

As already noted $Z_1 + \cdots + Z_{n-j} \sim (n-j)$ shifted negative binomial$(n-j, 1-\theta)$.

Therefore by A.?? we have

$$\mathrm{E}\left(\left(Z_1 + \cdots + Z_{n-j}\right)^{[u]}\right) = (n-j)^{[u]}\left(\frac{1}{1-\theta}\right)^u.$$

The final form for the second result then follows on simplification.

Example 2 Number of runs of empty urns

Let W equal the number of runs of empty urns in the discrete random coverage of the circle distribution as described above. In this case

$$P(W=w)=\frac{1}{\binom{t}{n}}\sum_{i=0}^{n-w}(-1)^i\left(\frac{t}{n}\right)\binom{n-w}{i}\binom{n}{w}\binom{t-(i+w)k-1}{t-(i+w)k-n}.$$

This result is given in Holst [?].

Proof

The result follows on direct application of Theorem ?? with

$$\Psi(G_1,\ldots,G_n)=\mathbb{I}(\text{exactly } w \text{ of events } G_1\geq 1,\ldots,G_n\geq 1 \text{ are true}).$$

In this case we have that

$$\sum_{v\in \Upsilon_j^n}\mathrm{E}\big(\Psi(v_1Z_1,\ldots,v_nZ_n)\big)$$

$$=\binom{n}{j}P\big(\text{exactly } w \text{ of events } Z_1\geq 1,\ldots,Z_{n-j}\geq 1 \text{ are true}\big)$$

$$=\begin{cases}\binom{n}{n-w}\cdot 1 & j=n-w\\ \binom{n}{j}\cdot 0 & j\neq n-w\end{cases}$$

and the final result follows on simplification.

Example 3 Shortest run of empty urns

Let S equal the shortest run of empty urns in the discrete random coverage of the circle distribution as described above. If there are no empty urns then we take $S=0$. Then for $s\geq 1$

$$P(S=s)=\frac{1}{\binom{t}{n}}\sum_{j=0}^{n}\sum_{i=0}^{j}(-1)^i\left(\frac{t}{n}\right)\binom{n}{j}\binom{j}{i}\binom{t-(n-j)(k+s-1)-ik-1}{t-(n-j)(k+s-1)-ik-n}$$

$$-\frac{1}{\binom{t}{n}}\sum_{j=0}^{n}\sum_{i=0}^{j}(-1)^i\binom{t}{n}\binom{n}{j}\binom{j}{i}\binom{t-(n-j)(k+s)-ik-1}{t-(n-j)(k+s)-ik-n}.$$

In the case $s = 0$ we have

$$P(S=0)=\frac{1}{\binom{t}{n}}\sum_{i=0}^{n}(-1)^i\binom{t}{n}\binom{n}{i}\binom{t-ik-1}{t-ik-n}.$$

Proof

The result follows on direct application of Theorem ?? with

$$\Psi(G_1,\dots,G_n)=\begin{cases}\mathbb{I}(\text{smallest of all positive } G_i\text{'s} = s) & \text{not all } G_i\text{'s equal } 0\\ \\ 1 & \text{all } G_i\text{'s equal } 0.\end{cases}$$

For $s \geq 1$ we note that

$$\sum_{\upsilon\in\Upsilon_j^n}\mathrm{E}\big(\Psi(\upsilon_1 Z_1,\dots,\upsilon_n Z_n)\big)=\begin{cases}\binom{n}{j}P\big(\min(Z_1,\dots,Z_{n-j})=s\big) & 0\leq j\leq n-1\\ \\ \binom{n}{n}\cdot 1 & j=n.\end{cases}$$

It is easily verified that $\min(Z_1,\dots,Z_{n-j}) \sim$ 1-shifted geometric$\left(1-\theta^{n-j}\right)$. Therefore

$$P\big(\min(Z_1,\dots,Z_{n-j})=s\big)=\left(1-\theta^{n-j}\right)\left(\theta^{n-j}\right)^{s-1}.$$

For $s = 0$ we see that

$$\sum_{\upsilon\in\Upsilon_j^n}\mathrm{E}\big(\Psi(\upsilon_1 Z_1,\dots,\upsilon_n Z_n)\big)=\begin{cases}\binom{n}{j}\cdot 0 & 0\leq j\leq n-1\\ \\ \binom{n}{n}\cdot 1 & j=n.\end{cases}$$

The final result follows on simplification.

Example 1. Number of empty urns.

Let R equal the number of empty urns in the discrete random coverage of the line distribution. In the case $r \geq 1$ and $\left\lfloor \frac{t-r}{k} \right\rfloor \leq n \leq t-r$ we have

$$P(R=r) = \frac{\binom{r-1}{n}}{\binom{t}{n}} \mathbb{I}_{\{t-k(n+1)=r-1\}}$$

$$+ \frac{1}{\binom{t}{n}} \sum_{j=n+1-\left\lfloor \frac{t-r}{k} \right\rfloor}^{n+1} \sum_{i=0}^{j-\left(n+1-\left\lfloor \frac{t-r}{k} \right\rfloor\right)} \left((-1)^i \binom{j}{i} \binom{n+1}{j} \right.$$

$$\left. \times \binom{r-1}{n-j} \binom{t-r-k(n+1-j+i)}{t-r-k(n+1-j+i)-j+1} \right)$$

and for $r=0$ and $n \geq \left\lfloor \frac{t}{k} \right\rfloor$

$$P(R=r) = \frac{1}{\binom{t}{n}} \sum_{j=0}^{\left\lfloor \frac{t}{k} \right\rfloor} (-1)^j \binom{n+1}{j} \binom{t-kj}{t-kj-n}.$$

We note that $P(R=r)=0$ for $r \geq 0$ and $n < \left\lfloor \frac{t-r}{k} \right\rfloor$.

Proof

Follows immediately from Example 1c of Section 4.3.

Problem 4.

The probability of no run of $r+1$ or more sparse urns when distributing balls according to a grouped Bose-Einstein distribution equals

$$\frac{1}{\binom{M+t-1}{t}} \sum_{i=0}^{n} \sum_{j=0}^{n-i+1} \sum_{\phi=0}^{i} \sum_{s=0}^{n-i} \sum_{\varphi=0}^{s} (-1)^{i-\phi+t+\varphi+j} \left(\binom{n-(r+1)j}{i-(r+1)j} \right.$$
$$\left. \times \binom{n-i+1}{j} \binom{i}{\phi} \binom{n-i}{s} \binom{s}{\varphi} \right.$$

$$\times \binom{m(i+s-n)+(i-\phi+s-\varphi)}{t}(m+1)^{\phi+\varphi} m^{i-\phi+s-\varphi} \Bigg)$$

provided $t \leq m(i+s-n)+(i-\phi+s-\varphi)$

(A ***grouped Bose-Einstein distribution*** results when n urns contain m cells each, the number of balls that can go into any cell is unrestricted and all possible distributions of balls into the mn cells are equally likely. A ***sparse urn*** is defined as an urn with at most 1 ball in total.)

Proof

P(longest success run of length less than or equal to r in a series of n *iid* Bernoulli trials)

$$= \sum_{i=0}^{n} \sum_{j=0}^{n-i+1} (-1)^j \binom{n-(r+1)j}{i-(r+1)j} \binom{n-i+1}{j} p^i (1-p)^{n-i}$$

success ≡ event $Y_j = \{0 \text{ or } 1\}$ where $Y_1, \ldots, Y_n$ are assumed to be independent negative binomial random variables such that $Y_i \sim$Negative Binomial(m, θ).

i.e.

$$P(Y_i = y) = \binom{y+m-1}{m-1} \theta^m (1-\theta)^y \qquad y \in \{0,1,\ldots\} \text{ and } 0 \leq \theta \leq 1$$

so

$$p = P(Y_i \leq 1) = \binom{0+m-1}{m-1} \theta^m (1-\theta)^0 + \binom{1+m-1}{m-1} \theta^m (1-\theta)^1$$

$$= (m+1)\theta^m - m\theta^{m+1} = g(\theta)$$

$$M = mn$$

$$P\left((X_{1,t}, \ldots, X_{n,t}) \in \mathcal{A}_t\right) = \frac{(-1)^t}{\binom{M+t-1}{t} t!} \frac{d^t}{d\theta^t} \left(\left(\frac{1}{\theta}\right)^M P((Y_1, \ldots, Y_n) \in \mathcal{A}) \right) \Bigg|_{\theta=1}$$

$$= \frac{(-1)^t}{\binom{M+t-1}{t} t!} \frac{d^t}{d\theta^t}\left(\left(\frac{1}{\theta}\right)^M \sum_{i=0}^{n} \sum_{j=0}^{n-i+1} \left((-1)^j \binom{n-(r+1)j}{i-(r+1)j}\binom{n-i+1}{j} \times \left(g(\theta)\right)^i \left(1-g(\theta)\right)^{n-i}\right)\right)\Bigg|_{\theta=1}$$

$$= \frac{(-1)^t}{\binom{M+t-1}{t} t!} \sum_{i=0}^{n} \sum_{j=0}^{n-i+1} \left((-1)^j \binom{n-(r+1)j}{i-(r+1)j}\binom{n-i+1}{j} \times \left(\frac{d^t}{d\theta^t}\left(\left(\frac{1}{\theta}\right)^M \left(g(\theta)\right)^i \left(1-g(\theta)\right)^{n-i}\right)\Bigg|_{\theta=1}\right)\right)$$

$M = mn$

$g(\theta) = (m+1)\theta^m - m\theta^{m+1}$

$$\frac{d^t}{d\theta^t}\left(\left(\frac{1}{\theta}\right)^M \left(g(\theta)\right)^i \left(1-g(\theta)\right)^{n-i}\right)\Bigg|_{\theta=1}$$

$$= \frac{d^t}{d\theta^t}\left(\left(\frac{1}{\theta}\right)^{mn} \left((m+1)\theta^m - m\theta^{m+1}\right)^i (1-(m+1)\theta^m + m\theta^{m+1})^{n-i}\right)\Bigg|_{\theta=1}$$

$$= \frac{d^t}{d\theta^t}\left(\left(\frac{1}{\theta}\right)^{mn} \sum_{\phi=0}^{i} \sum_{s=0}^{n-i} \sum_{\varphi=0}^{s} \left((-1)^{i-\phi+\varphi} \binom{i}{\phi}\binom{n-i}{s}\binom{s}{\varphi} \times \left((m+1)\theta^m\right)^{\phi+\varphi} \left(m\theta^{m+1}\right)^{i-\phi+s-\varphi}\right)\right)\Bigg|_{\theta=1}$$

$$= \frac{d^t}{d\theta^t}\left(\sum_{\phi=0}^{i} \sum_{s=0}^{n-i} \sum_{\varphi=0}^{s} \left((-1)^{i-\phi+\varphi} \binom{i}{\phi} \binom{n-i}{s} \binom{s}{\varphi} \right.\right.$$

$$\left.\left.\left. \times\ (m+1)^{\phi+\varphi}\, m^{i-\phi+s-\varphi}\, \theta^{m(i+s-n)+(i-\phi+s-\varphi)} \right)\right)\right|_{\theta=1}$$

$$= \frac{d^t}{d\theta^t}\left(\sum_{\phi=0}^{i} \sum_{s=0}^{n-i} \sum_{\varphi=0}^{s} \sum_{u=0}^{m(i+s-n)+(i-\phi+s-\varphi)} \left((-1)^{i-\phi+\varphi+u} \binom{i}{\phi} \binom{n-i}{s} \binom{s}{\varphi} \right.\right.$$

$$\left.\left.\left. \times \binom{m(i+s-n)+(i-\phi+s-\varphi)}{u} (m+1)^{\phi+\varphi}\, m^{i-\phi+s-\varphi} (1-\theta)^u \right)\right)\right|_{\theta=1}$$

$$= \sum_{\phi=0}^{i} \sum_{s=0}^{n-i} \sum_{\varphi=0}^{s} \sum_{u=0}^{m(i+s-n)+(i-\phi+s-\varphi)} \left(\left((-1)^{i-\phi+\varphi+u} \binom{i}{\phi} \binom{n-i}{s} \binom{s}{\varphi} \right.\right.$$

$$\left. \left. \times \binom{m(i+s-n)+(i-\phi+s-\varphi)}{u} (m+1)^{\phi+\varphi}\, m^{i-\phi+s-\varphi} \right) \left(\left(\frac{d^t}{d\theta^t} (1-\theta)^u \right)\bigg|_{\theta=1} \right) \right)$$

$$= \sum_{\phi=0}^{i} \sum_{s=0}^{n-i} \sum_{\varphi=0}^{s} \sum_{u=0}^{m(i+s-n)+(i-\phi+s-\varphi)} \left(\left((-1)^{i-\phi+\varphi+u} \binom{i}{\phi} \binom{n-i}{s} \binom{s}{\varphi} \right.\right.$$

$$\left. \left. \times \binom{m(i+s-n)+(i-\phi+s-\varphi)}{u} (m+1)^{\phi+\varphi}\, m^{i-\phi+s-\varphi} \right) \left(t!\,(-1)^t \mathrm{I}_{\{u\}}(t) \right) \right)$$

$$= \sum_{\phi=0}^{i} \sum_{s=0}^{n-i} \sum_{\varphi=0}^{s} \left((-1)^{i-\phi+\varphi+t} \binom{i}{\phi} \binom{n-i}{s} \binom{s}{\varphi} \binom{m(i+s-n)+(i-\phi+s-\varphi)}{t} \right.$$

$$\left. \times\ t!\,(m+1)^{\phi+\varphi}\, m^{i-\phi+s-\varphi} I\big(t \le m(i+s-n)+(i-\phi+s-\varphi)\big) \right)$$

Therefore,

$$P\left(\left(X_{1,t},\dots,X_{n,t}\right)\in\mathcal{A}_t\right)$$

$$=\frac{(-1)^t}{\binom{M+t-1}{t}t!}\sum_{i=0}^{n}\sum_{j=0}^{n-i+1}\left((-1)^j\binom{n-(r+1)j}{i-(r+1)j}\binom{n-i+1}{j}\right.$$

$$\left.\times\left(\frac{d^t}{d\theta^t}\left(\left(\frac{1}{\theta}\right)^M\left(g(\theta)\right)^i\left(1-g(\theta)\right)^{n-i}\right)\Bigg|_{\theta=1}\right)\right)$$

$$=\frac{(-1)^t}{\binom{M+t-1}{t}t!}\sum_{i=0}^{n}\sum_{j=0}^{n-i+1}\left((-1)^j\binom{n-(r+1)j}{i-(r+1)j}\binom{n-i+1}{j}\right.$$

$$\times\left(\sum_{\phi=0}^{i}\sum_{s=0}^{n-i}\sum_{\varphi=0}^{s}\left((-1)^{i-\phi+\varphi+t}\binom{i}{\phi}\binom{n-i}{s}\binom{s}{\varphi}\binom{m(i+s-n)+(i-\phi+s-\varphi)}{t}\right.\right.$$

$$\left.\left.\left.\times\, t!\,(m+1)^{\phi+\varphi}\,m^{i-\phi+s-\varphi}I\left(t\le m(i+s-n)+(i-\phi+s-\varphi)\right)\right)\right)\right)$$

$$=\frac{1}{\binom{M+t-1}{t}}\sum_{i=0}^{n}\sum_{j=0}^{n-i+1}\sum_{\phi=0}^{i}\sum_{s=0}^{n-i}\sum_{\varphi=0}^{s}(-1)^{i-\phi+t+\varphi+j}\left(\binom{n-(r+1)j}{i-(r+1)j}\right.$$

$$\times\binom{n-i+1}{j}\binom{i}{\phi}\binom{n-i}{s}\binom{s}{\varphi}$$

$$\times \binom{m(i+s-n)+(i-\phi+s-\varphi)}{t}(m+1)^{\phi+\varphi} m^{i-\phi+s-\varphi}\Bigg).$$

References

Companion results can be found in

(i) "Sparse and Crowded Cells and Dirichlet Distributions", Sobel and Uppuluri, The Annals of Statistics, 1974, Vol. 2, No. 5, pages 977 - 987.

(ii) "Exact Null Distributions of Runs Statistics for Occupancy Models with Applications to Disease Cluster Analysis", Mancuso, Ph.D. Dissertation, SUNY Stony Brook, 1998.

where they consider the same question but when distributing balls according to a multinomial distribution.

Problem 5.

The probability of observing r empty urns among the first n when distributing t identical balls according to a grouped Bose-Einstein distribution where the first n urns each contain m cells and the last urn contains s cells equals

$$\frac{\binom{n}{r}}{\binom{mn+s+t-1}{t}} \sum_{i=0}^{n-r} (-1)^{n-r-i} \binom{n-r}{i} \binom{mi+s+t-1}{t}$$

$$= \frac{n_{(n-r)}}{(mn+s)^{(t)}} |G(t, n-r; -m, -s)|$$

where $G(t, n; m, s)$ are the Gould-Hopper defined by

$$G(t, n; m, s) = \frac{1}{n!} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (jm+s)_{(t)}$$

in Gould, H.W. and Hopper, A.T., “Operational Formulas Connected with Two Generalizations of Hermite Polynomials”, Duke Mathematics Journal, Vol 29, 1962, pages 51 - 63.

References

This result can be found in Charalambides, Ch. A. and Koutras, M., “On the Differences of the Generalized Factorials at an Arbitrary Point and Their Combinatorial Applications”, Discrete Mathematics, Vol. 47, 1983, pages 183-201.

(b) The u^{th} descending factorial moment of the number of empty urns among the first n urns when distributing t balls among $n+1$ urns according to a grouped Bose-Einstein distribution where each of the first n urns contains m cells and the last urn contains s cells equals

$$\frac{n_{(u)}(m(n-u)+s)^{(t)}}{(mn+s)^{(t)}}.$$

provided $0 \leq u \leq n.$

References

In the paper Fu and Koutras, “Distribution Theory of Runs: A Markov Chain Approach”, Journal of the American Statistical Association, Vol. 89, No. 427, September 1994, pages 1050-1058, the authors develop a computer algorithm for finding the probabilities in (a), (c), and (d) for any specified values of the parameters in the more general case where the Bernoulli trials are independent but not necessarily identically distributed. Their approach consists of imbedding the problems into a finite Markov chain and expressing the probabilities in terms of transition probabilities of the Markov chain. The authors indicate that they will make their computer program for calculating these transition probabilities available on request.

In the abstract to their paper the authors state, “For almost a century, even in the simplest case of independent and identically distributed Bernoulli trials, the exact distribution of many run statistics still remain unknown.” However the solution in part (a) can be found in Combinatorial Chance, F.N. David and D.E. Barton, (1962), page 230. Solutions to parts (b) and (c) can be found in “A Note on Restricted Selections”, M.T.L. Bizley, Journal of the Institute of Actuaries, Students Society, Vol. 16, 1969, pages 333-345, equations (4) and (5).

(d) Let $t_{m,n}(k)$ equal the number of different sequences of length n that can be constructed using a k letter alphabet if sequences cannot contain any runs of length m or greater of the first letter of the alphabet. Assume letters can be reused. Show

$$t_{m,n}(k) = \sum_{i=0}^{n} \sum_{j=0}^{n-i+1} (-1)^j \binom{n-mj}{i-mj} \binom{n-i+1}{j} (k-1)^{n-i}$$

J. Gani considers this problem in his paper On Sequences of Events with Repetitons, Communications in Statistics - Stochastic Models, Vol. 14, no. 1&2, 1998, pages 265-271. His solution requires the calculation of eigenvalues of an $m \times m$ matrix and hence is appropriate when m is not too large. Gani points out the $t_{2,n}(2)$ equals the Fibonacci number F_{n+2}. K. Suman considers the problem of constructing sequences where no letter of the alphabet can occur in runs of length m or greater in his paper The Longest Run of Any Letter in a Randomly Generated Word, Runs and Patterns in Probability, A.P. Godbole and S.G. Papastravridis (editors), pages 119-130.

(e) Let $w_{m,n}(k)$ equal the number of different distributions of k identical balls into n distinguishable urns such that there are no runs of empty urns of length m or greater.

$$w_{m,n}(k) = \sum_{i=0}^{n} \sum_{j=0}^{n-i+1} (-1)^j \binom{n-mj}{i-mj} \binom{n-i+1}{j} \binom{k-1}{k+i-n}$$

Problem 6.

Suppose we randomly arrange m X's and n Y's in a line. Find the $P(\text{longest run of } X'\text{s} \leq k)$.

Solution

One method of randomly arranging the m X's and n Y's is to consider placing an urn before, after, and between each Y and to then distribute the m X's into these $n+1$ urns in such a manner that all

$$\binom{(n+1)+m-1}{m} = \binom{n+m}{m}$$

distributions are equally likely.

Let $X_{j,m} =$ the number of X's that are put into the j^{th} urn, $j = 1,2, \dots, n+1$. It follows that $(X_{1,m}, X_{2,m}, \dots, X_{n+1,m})$ is a random vector which is equally likely to be any value in $\mathbb{S}_m^{n+1}$.

Therefore,

$$P(\text{longest run of } X'\text{s} \le k)$$
$$= P\big(X_{1,m} \le k, \dots, X_{n+1,m} \le k\big)$$
$$= P(\big(X_{1,m}, X_{2,m}, \dots, X_{n+1,m}\big) \in \mathcal{A}_m)$$

where

$$\mathcal{A}_m = \{(s_1, s_2, \dots, s_{n+1}) | 0 \le s_1 \le k, \dots, 0 \le s_{n+1} \le k \text{ and } s_1 + \cdots + s_{n+1} = m\}.$$

Define

$$\mathcal{A} = \{(s_1, s_2, \dots, s_{n+1}) | 0 \le s_1 \le k, \dots, 0 \le s_{n+1} \le k\}.$$

Then by the Geometric Randomization Theorem,

$$P\big((C_1, \dots, C_{n+1}) \in \mathcal{A}_t\big) = \frac{1}{\binom{t}{n}(t+1)!}$$
$$\times \frac{d^{t+1}}{d\theta^{t+1}}\left(\left(\frac{\theta}{1-\theta}\right)^{n+1} P\big((Z_1, \dots, Z_{n+1}) \in \mathcal{A}\big)\right)\Bigg|_{\theta=0}$$

$$P(\text{longest run of } X'\text{s} \le k) = P\left(\big(X_{1,m}, X_{2,m}, \dots, X_{n+1,m}\big) \in \mathcal{A}_m\right)$$

$$= \frac{(-1)^m}{\binom{n+m}{m} m!} \sum_{j=0}^{\infty} \binom{n+j}{j} \frac{d^m}{dp^m}\Big((1-p)^j P\big((Y_1, \dots, Y_{n+1}) \in \mathcal{A}\big)\Big)\Big|_{p=1}$$

$$= \frac{(-1)^m}{\binom{n+m}{m} m!} \sum_{j=0}^{\infty} \binom{n+j}{j} \frac{d^m}{dp^m}\Big((1-p)^j (P(Y_1 \le k))^{n+1})\Big)\Big|_{p=1}$$

$$= \frac{(-1)^m}{\binom{n+m}{m} m!} \sum_{j=0}^{\infty} \binom{n+j}{j} \frac{d^m}{dp^m}\big((1-p)^j (1-(1-p)^{k+1})^{n+1}\big)\Big|_{p=1}$$

$$= \frac{(-1)^m}{\binom{n+m}{m} m!} \sum_{j=0}^{\infty} \binom{n+j}{j} \frac{d^m}{dp^m} \left(\sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} (1-p)^{j+(k+1)i} \right) \Bigg|_{p=1}$$

$$= \frac{(-1)^m}{\binom{n+m}{m} m!} \sum_{j=0}^{\infty} \binom{n+j}{j} \left(\sum_{i=0}^{n+1} (-1)^{i+m} m! \binom{n+1}{i} I_{\{(k+1)i+j\}}(m) \right)$$

$$= \frac{1}{\binom{n+m}{m} m!} \sum_{i=0}^{n+1} \sum_{j=0}^{\infty} (-1)^i \binom{n+j}{j} \binom{n+1}{i} I_{\{m-(k+1)i\}}(j)$$

$$= \frac{1}{\binom{n+m}{m} m!} \sum_{i=0}^{n+1} (-1)^i \binom{n+m-(k+1)i}{m-(k+1)i} \binom{n+1}{i}.$$

References

2 Appendix

Definition 6

If W ~negative binomial(m, p), then $Z = W + c \sim c$-shifted negative binomial(m, p). The c-shifted geometric(p) is the special case $m = 1$ of the c-shifted negative binomial.

Theorems

If $Z \sim c$-Shifted Negative Binomial Distribution(m, p) then

$$P(Z = z) = \binom{z-c+m-1}{m-1} p^m (1-p)^{z-c} \qquad z \in \{c, c+1, \dots\}.$$

If $Z_1, \dots, Z_n$ are independent and if $Z_j \sim c_j$-shifted negative binomial(m_j, p), then $S = Z_1 + \cdots + Z_n \sim c^*$-shifted negative binomial(m^*, p), where $c^* = c_1 + \cdots + c_n$ and $m^* = m_1 + \cdots +$

m_n.

Theorem 7

Let X be a discrete random variable defined on $\mathbb{S} \subseteq \{0,1,\ldots\}$. Then

$$\mu_{[r]} = r\sum_{n=0}^{\infty} P(X>n)n_{[r-1]}.$$

In the special case $r = 1$ we have the familiar result

$$\mu = \sum_{n=0}^{\infty} P(X>n).$$

Proof

$$\begin{aligned}
\mu_{[r]} = E(X_{[r]}) &= \sum_{n=0}^{\infty} P(X=n)n_{[r]} \\
&= \sum_{n=0}^{\infty} P(X=n)\left(\frac{d^r}{dt^r}t^n\bigg|_{t=1}\right) \\
&= \left(\frac{d^r}{dt^r}\sum_{n=0}^{\infty} P(X=n)t^n\right)\Bigg|_{t=1} \\
&= \left(\frac{d^r}{dt^r}g(t)\right)\Bigg|_{t=1} \quad \text{with} \quad g(t) = \sum_{n=0}^{\infty} P(X=n)t^n.
\end{aligned}$$

Now define $h(t) = \sum_{n=0}^{\infty} P(X>n)t^n$. We note that

$$\begin{aligned}
&1 + (t-1)h(t) \\
&= 1 + (t-1)\left(\sum_{n=0}^{\infty} P(X>n)t^n\right) \\
&= \big(1 - P(X>0)\big)t^0 + \big(P(X>0) - P(X>1)\big)t^1 + \big(P(X>1) - P(X>2)\big)t^2 + \\
&= \big(P(X=0)\big)t^0 + \big(P(X=1)\big)t^1 + \big(P(X=2)\big)t^2 + \cdots \\
&= g(t).
\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d^r}{dt^r}g(t) &= \frac{d^r}{dt^r}\big(1+(t-1)h(t)\big)\\ &= \frac{d^r}{dt^r}\big((t-1)h(t)\big)\\ &= \sum_{j=0}^{r}\binom{r}{j}\left(\frac{d^j}{dt^j}(t-1)\right)\left(\frac{d^{r-j}}{dt^{r-j}}h(t)\right)\end{aligned}$$

[Leibniz's Product Rule for Differentiation]

$$= \binom{r}{0}\big((t-1)\big)\left(\frac{d^r}{dt^r}h(t)\right)+\binom{r}{1}\left(\frac{d^{r-1}}{dt^{r-1}}h(t)\right)+0.$$

Hence,

$$\begin{aligned}E(X_{[r]}) &= \left(\frac{d^r}{dt^r}g(t)\right)\Bigg|_{t=1}\\ &= r\left(\frac{d^{r-1}}{dt^{r-1}}h(t)\right)\Bigg|_{t=1}\\ &= r\left(\frac{d^{r-1}}{dt^{r-1}}\sum_{n=0}^{\infty}P(X>n)t^n\right)\Bigg|_{t=1}\\ &= r\sum_{n=0}^{\infty}P(X>n)\left(\frac{d^{r-1}}{dt^{r-1}}t^n\Bigg|_{t=1}\right)\\ &= r\sum_{n=0}^{\infty}P(X>n)n_{[r-1]}.\end{aligned}$$

Appendix

A.1

$\boldsymbol{k^{th}}$ descending factorial moment

$$E\big((X)_{(k)}\big) = E\big(X(X-1)\cdots(X-k+1)\big)$$

A.2

$\boldsymbol{k^{th}}$ ascending factorial moment

$$E\left((X)^{[k]}\right) = E\left(X(X+1)\cdots(X+k-1)\right)$$

A.3

$\boldsymbol{k^{th}}$ descending factorial moment of sum of indicator variables

Consider an experiment with probability space $(\Omega, \mathcal{A}, P)$ and suppose $A_1,\ldots,A_n$ are all events within $\mathcal{A}$. Suppose the experiment is performed and let $\omega \in \Omega$ be the outcome of this experiment. Define

$$X = \text{number of events among } A_1,\ldots,A_n \text{ that } \omega \text{ is an element of} = \mathrm{I}_{A_1}(\omega) + \cdots + \mathrm{I}_{A_n}(\omega).$$

Then,

$$\frac{1}{r!}E\left(X_{(r)}\right) = \sum_{(j_1,\ldots,j_r)\in\mathbb{C}_r} P\left(A_{j_1} \cap \cdots \cap A_{j_r}\right)$$

where $\mathbb{C}_r$ is the set of all r subsets of $\{1,2,\ldots,n\}$.

A.4

For a discrete random variable X defined on $\mathbb{S} \subseteq \{0,1,\ldots\}$

$$E\left(X_{(v)}\right) = v\sum_{r=0}^{\infty}\left(1 - P(X \le r)\right) r_{(v-1)}$$

A.5

$\boldsymbol{Z}$ ~Geometric($\boldsymbol{p}$)

$$P(Z = z) = p(1-p)^z \qquad z = 0,1,\ldots \text{ and } 0 \le p \le 1$$

A.6

$$P(Z \ge c) = (1-p)^c$$

A.7

$\boldsymbol{Z}$ ~c-shifted Geometric($\boldsymbol{p}$)

$$P(Z = z) = p(1-p)^{z-c} \qquad z = c, c+1, \ldots \text{ and } 0 \le p \le 1$$

A.8

If $W \sim$ geometric(p), then $Z = W + c \sim c$-shifted geometric(p).

A.9

$\boldsymbol{Z}$ ~Left & right truncated Geometric($\boldsymbol{p}$) on $\boldsymbol{a}, \ldots, \boldsymbol{b}$

$$P(Z = z) = \frac{p(1-p)^z}{(1-p)^a - (1-p)^{b+1}} \qquad z \in \{a, \ldots, b\} \text{ and } 0 \le p \le 1.$$

A.10

If $W \sim \text{Geometric}(p)$,

then $Z = W \mid a \le W \le b \sim$ Left & right truncated geometric(p)

A.11

c-shifted geometric (p) ≡ Left & right truncated geometric(p) on $c, \ldots, \infty$

A.12

$\boldsymbol{Z}$ ~Negative Binomial($\boldsymbol{m}, \boldsymbol{p}$)

$$P(Z = z) = \binom{z+m-1}{m-1} p^m (1-p)^z \qquad z \in \{0, 1, \ldots\} \text{ and } 0 \le p \le 1.$$

A.13

$$E(Z_{(r)}) = \left(\frac{1-p}{p}\right)^r m^{[r]}$$

A.14

Negative Binomial$(1, p) \equiv$ Geometric(p)

A.15

If $Z_1, \dots, Z_n$ are independent random variables such that $Z_j \sim$Negative Binomial$\left(m_j, p\right)$ and if we take $S = Z_1 + \cdots + Z_n$ and $M = m_1 + \cdots + m_n$, then $S \sim$Negative Binomial(M, p).

A.16

$\boldsymbol{Z} \sim \boldsymbol{c}$-Shifted Negative Binomial Distribution$(\boldsymbol{m}, \boldsymbol{p})$

$$P(Z = z) = \binom{z - c + m - 1}{m - 1} p^m (1 - p)^{z - c} \qquad z \in \{c, c + 1, \dots\}.$$

A.17

$$E\left((Z - c + m)^{[r]}\right) = m^{[r]} \left(\frac{1}{p}\right)^r.$$

Note: The m-shifted negative binomial(m, p) is simply referred to as the negative binomial distribution in many textbooks. However the negative binomial and the shifted negative binomial are both used in this article, sometimes in the same problem. Thus to avoid confusion, it is necessary for us to delineate between these related models.

A.18

If $W \sim$negative binomial(m, p), then $Z = W + c \sim c$-shifted negative binomial(m, p).

A.19

c-shifted negative binomial$(1, p) \equiv c$-shifted geometric(p)

A.20

If $Z_1, \dots, Z_n$ are independent and if $Z_j \sim c_j$-shifted negative binomial$\left(m_j, p\right)$, then $S = Z_1 + \cdots + Z_n \sim c^*$-shifted negative binomial(m^*, p), where $c^* = c_1 + \cdots + c_n$ and $m^* = m_1 + \cdots + m_n$.

A.21

$\boldsymbol{Z} \sim$Poisson$(\boldsymbol{\theta})$

$$P(Z = z) = \frac{e^{-\theta} \theta^z}{z!} \qquad z \in \{0,1, \dots\} \text{ and } \theta > 0$$

A.22

$$E\big((Z)_{(r)}\big) = \boldsymbol{\theta}^r$$