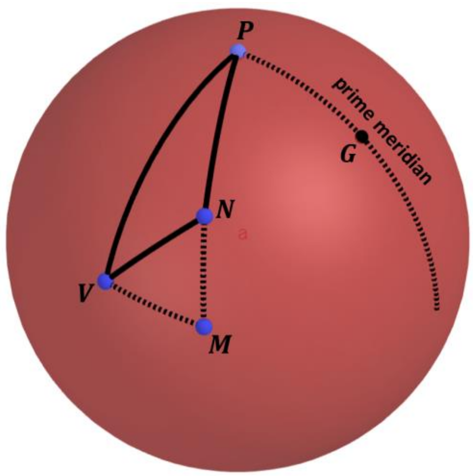
Monday Afternoon, Second Hour

Jack Sparrow and Davy Jones have agreed to parlay at a location in the Atlantic Ocean that is 1920 miles from each of their current locations. The Black Pearl is currently harbored in Nantucket (latitude 41.38333333 N, longitude 70 W), an island off the coast of Massachusetts and the Flying Dutchman is anchored in Vera Cruz, Mexico (latitude 19.18333333 N, longitude 96.06666667 W).

Here is the layout of their situation where V is Vera Cruz, N is Nantucket, M is the parlay location, P is the North Pole and G is Greenwich, England which is located on the Prime Meridian.



What are the latitude and longitude of their parlay location M?

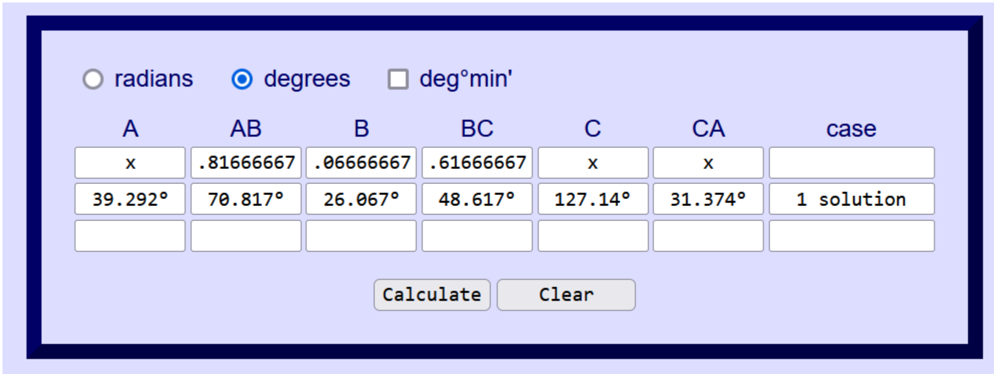
<u>Solution</u>

$$\widehat{VP} = 90° - 19.18333333° = 70.81666667°$$

$$\widehat{NP} = 90° - 41.38333333° = 48.61666667°$$

$$\angle VPN = 96.06666667° - 70° = 26.06666667°$$

So, we have sufficient information (two sides, one angle) to solve the spherical triangle ΔVPN.



https://emf.neocities.org/tr/spherical

From these calculations we see that

$$\angle PVN = 39.292° \text{ and } \widehat{VN} = 31.374.$$

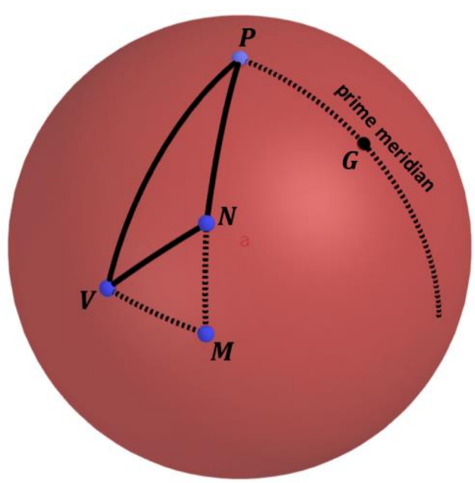
Converting the physical distance of 1920 miles to its angular distance in degrees we find

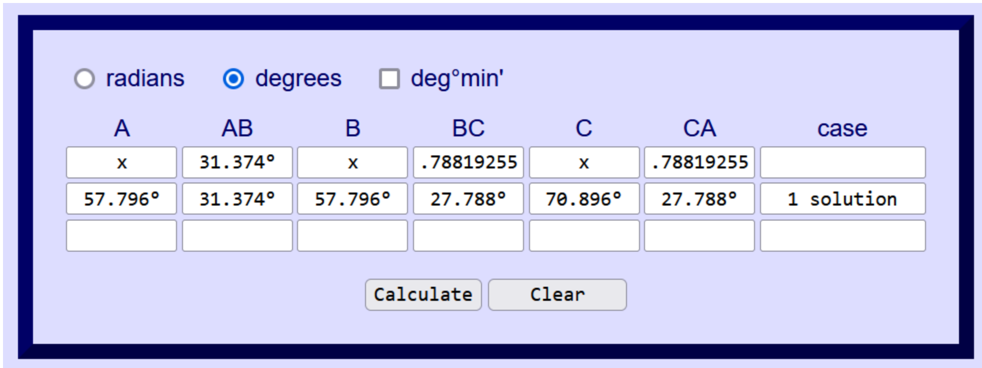
$$\frac{1920}{r} \cdot \frac{180}{\pi} = \frac{1920}{3958.8} \cdot \frac{180}{\pi} = 27.78819255°$$

where r is the radius of Earth. So

$$\widehat{VM} = \widehat{NM} = 27.78819255°.$$

Now we have sufficient information (three sides) to solve the spherical triangle ΔVMN.





radians degrees deg°min'

A	AB	B	BC	C	CA	case
x	31.374°	x	.78819255	x	.78819255	
57.796°	31.374°	57.796°	27.788°	70.896°	27.788°	1 solution

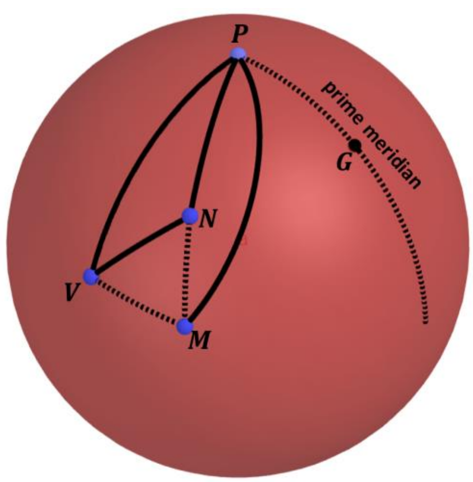
Calculate Clear

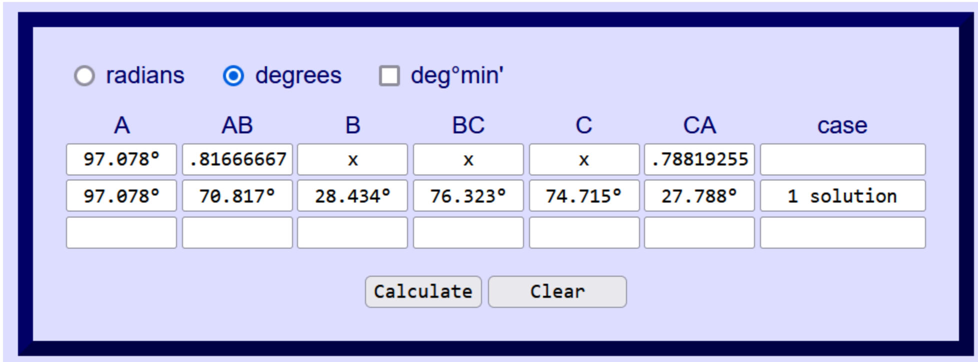
From these calculations we see that $\angle NVM = 57.796°$. Hence

$$\angle PVM = \angle PVN + \angle NVM = 39.292° + 57.796° = 97.078°.$$

Now we have sufficient information (two sides, one angle) to solve the spherical triangle ΔPVM.

> Caution: The great arc $\widehat{PM}$ as drawn below is horribly off. But drawn correctly the arc $\widehat{PM}$ would basically lie on top of $\widehat{PN}$ and $\widehat{NM}$ and you couldn't see what's what. So, I've opted to distort its location for visual purposes.





From these calculations we see that $\widehat{PM} = 76.323°$ and $\angle MPV = 28.434°$. It follows that

$$\angle NPM = \angle MPV - \angle VPN = 28.434° - 26.06666667° = 2.36733333°.$$

Therefore,

Latitude of M	$\lambda_M = 90° - \widehat{PM} = 90° - 76.323° = 13.677°\ \mathrm{N}$
Longitude of M	$\varphi_M = \varphi_N - \angle NPM = 70° - 2.36733333° = 67.63266667°.$

■

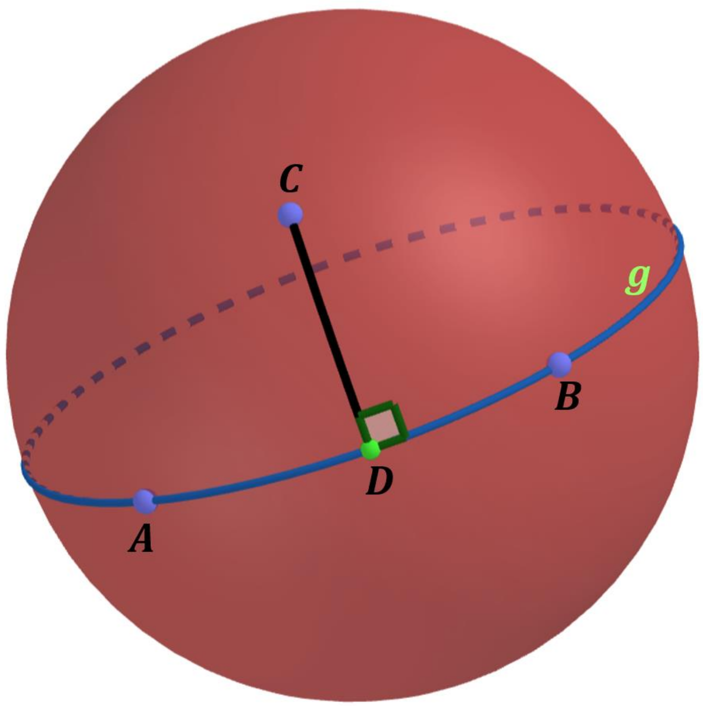
Homework

Meanwhile, Elizabeth Swann learned from her spies about the Sparrow and Jones parlay and feared it bode ill for the safety of her true love Will Turner. Eliabeth and her brigands were in Aruba ($12.5211°$ N, $69.9683°$ W) and much closer to the rendezvous point than either Sparrow or Jones. So, Elizabeth planned to set a trap and capture both Sparrow and Jones.

If Elizabeth's ship, The Empress, could sail at 10 knots how long would it take her to reach the parlay location? Note: a knot equals one nautical mile per hour and one nautical mile equals one minute of latitude.

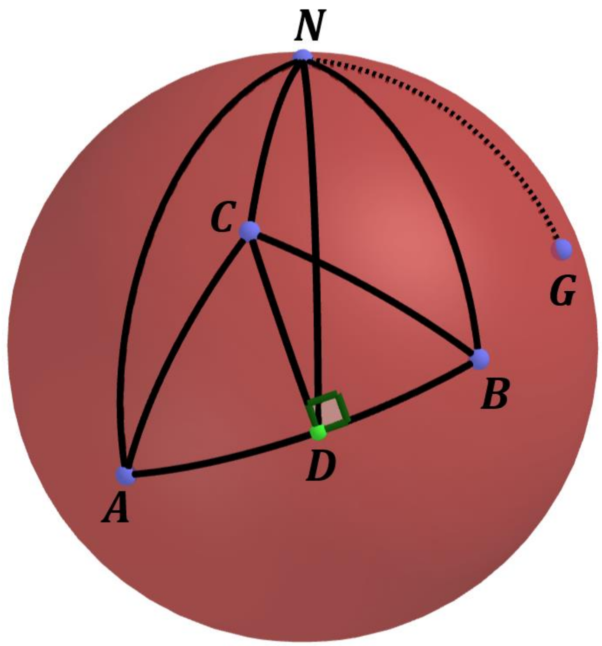
Find Coordinates of Point on Great Circle Nearest Point off Great Circle

Find the latitude and longitude of the point D on the great circle g going through the points A and B that is as close as possible to a third point C which is not on the great circle g. Also find the spherical distance between the points C and D. Assume that the latitude and longitude of A, B and C are known. Note that if D is as close as possible to C, then the great arc $\overset{\frown}{CD}$ will be perpendicular to the great circle g.



Solution

Let N be the north pole and let G be Greenwich, England which is a city on the Prime Meridian.



Let $(\lambda_A, \varphi_A), (\lambda_B, \varphi_B), (\lambda_C, \varphi_C). (\lambda_D, \varphi_D)$ be the (latitude, longitude) of the points A, B, C and D, respectively.

Then,

$\widehat{AN} = 90° - \lambda_A$	$\sphericalangle GNB = \varphi_B$
$\widehat{BN} = 90° - \lambda_B$	$\sphericalangle GNC = \varphi_C$
$\widehat{CN} = 90 - \lambda_C$	$\sphericalangle GNA = \varphi_A$

From this information we can determine that

$\sphericalangle CNB = \varphi_C - \varphi_B$	$\sphericalangle CNA = \varphi_A - \varphi_C$	$\sphericalangle BNA = \varphi_A - \varphi_B.$

Step 1. Use https://emf.neocities.org/tr/spherical to solve ΔANB and ΔANC. [We can solve spherical triangle ΔANB because we know $\widehat{AN}$, $\widehat{BN}$ and $\sphericalangle BNA$ and we can solve spherical triangle ΔANC because we know $\widehat{AN}$, $\widehat{CN}$ and $\sphericalangle CNA$.]

Solving these two triangles determines

$\overset{\frown}{CA}$	$\overset{\frown}{CN}$	$\sphericalangle NAC$	$\sphericalangle NAB$

And with this information we can find $\sphericalangle CAB = \sphericalangle NAB - \sphericalangle NAC$.

Step 2. Use https://emf.neocities.org/tr/spherical to solve ΔBAD. [We can solve ΔCAD because we know $\overset{\frown}{CA}$, $\sphericalangle CAB$ and $\sphericalangle CDA = 90°$.]

Solving this triangle determines

$\overset{\frown}{CD}$	$\overset{\frown}{AD}$

Step 3. Use https://emf.neocities.org/tr/spherical to solve ΔNAD. [We can solve ΔNAD because we know $\overset{\frown}{NA}$, $\overset{\frown}{AD}$ and $\sphericalangle NAD = \sphericalangle NAB$.]

Solving this triangle determines

$\overset{\frown}{ND}$	$\sphericalangle AND$

Now we have all the information we need to finish.

$$\text{Distance from } C \text{ to } D = \overset{\frown}{CD}$$

$$\lambda_D = \text{Latitude of } D = 90° - \overset{\frown}{ND}$$

$$\varphi_D = \text{Longitude of } D = \sphericalangle GND = \sphericalangle GNA - \sphericalangle AND.$$

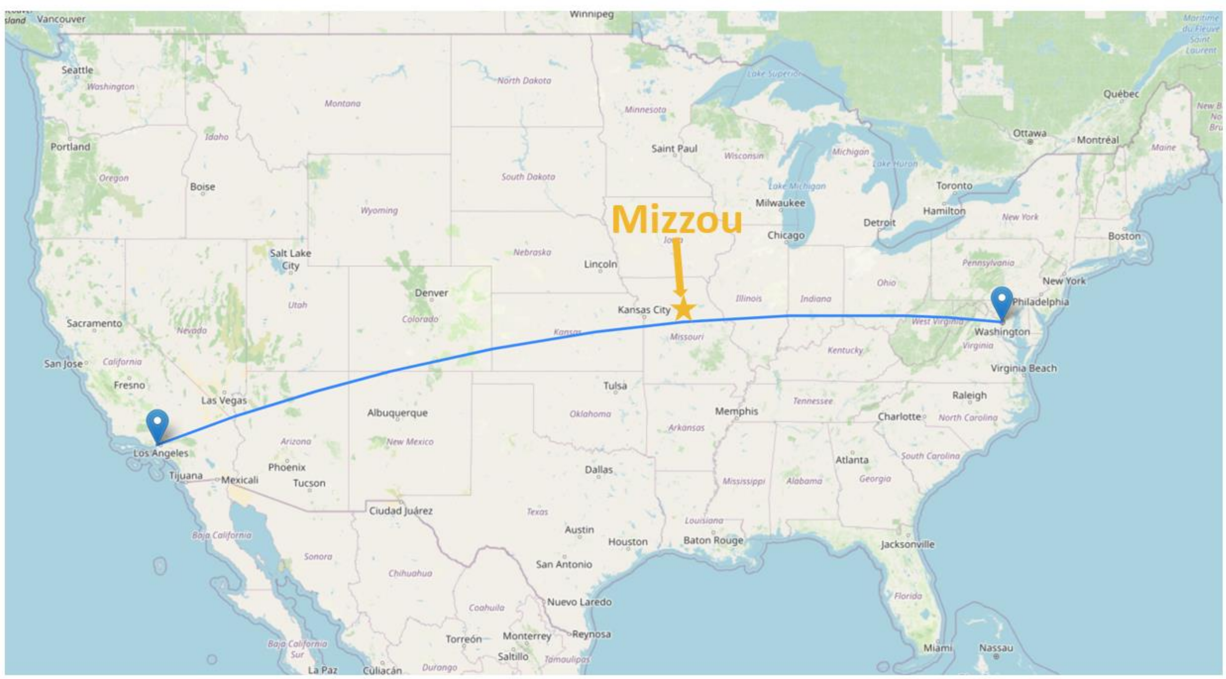
■

Exercise

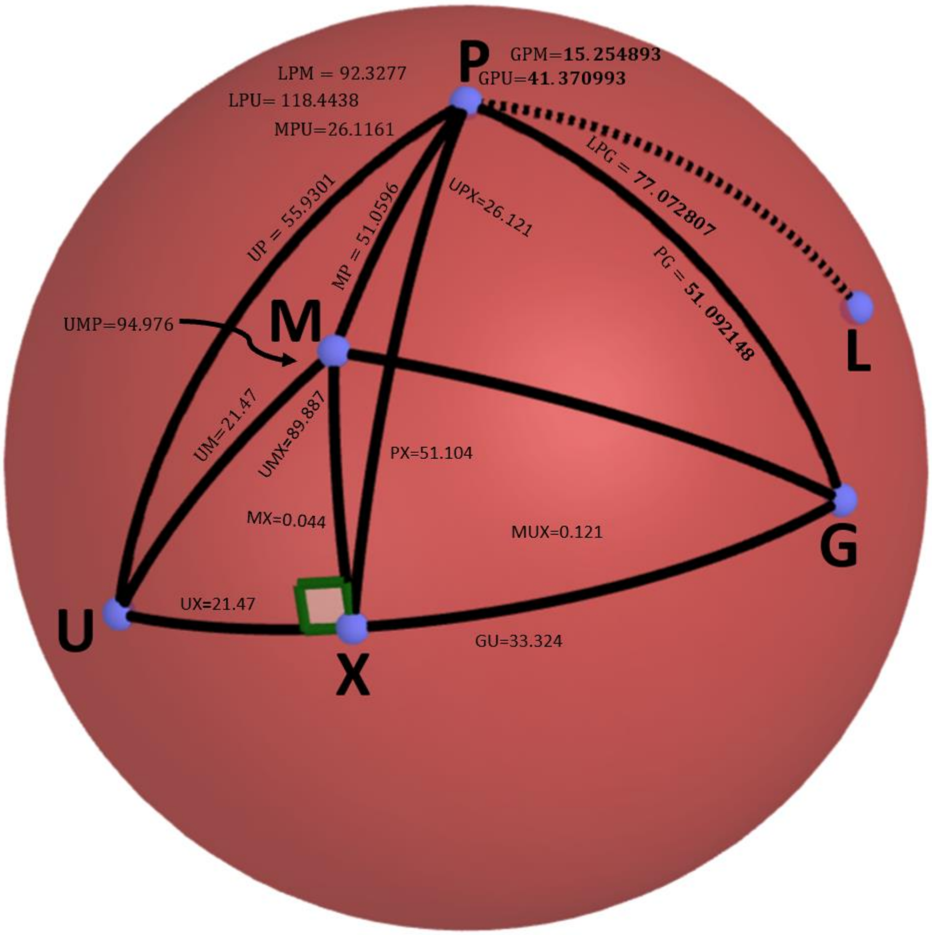
Suppose you are a student at UCLA (latitude 34.0699° N, longitude 118.4438° W) and need to drive to Georgetown University (38.907852° N**,** 77.072807° W) to start a summer math REU (Research Experiences for Undergraduates) program. Your plan is to take the shortest possible route. You have a friend at the University of Missouri at Columbia who will also attend this REU and you have offered to pick them up along the way. Your Mizzou friend is an avid hiker and has decided to walk from their campus location (38.9404° N, 92.3277° W) to the closest point on your route so you will not have to lose any time picking them up.

(1) How long (one-way, in miles) will your road trip be?
(2) How long will your friend's hike be?
(3) What are the coordinates (latitude, longitude) where you will pick up your friend?

(Assume that all roads and hiking trails are conveniently designed as great circle routes.)



https://emf.neocities.org/tr/spherical



Hike Distance $\widehat{MX} = 0.044°$

Road Trip Distance $\widehat{GU} = 33.324$

$\text{Long} = \text{GPX} + \text{LPG} = (\text{GPU} - \text{UPX}) + \text{LPG}$

$= (41.370993 - 26.121) + 77.07807 = 92.328063°$

$\text{Lat} = 90 - \text{PX} = 90 - 51.104 = 38.896°$

(38.896N,92.328063W) coordinates of point X

Hike Distance $\widehat{MX} = 0.044°$ angular distance

Physical Distance

$$0.044° \cdot \frac{\pi}{180} \cdot r = 0.044° \cdot \frac{\pi}{180} \cdot 3958.8 = 3.040140155 \text{ miles}$$

Road Trip Distance $\widehat{GU} = 33.324$ angular distance

Physical Distance

$$33.324 * \frac{\pi}{180} * 3958.8 = 2302.491603 \text{ miles}$$

Equation of Great Circle with Vertex Coordinates (Latitude,Longitude) $(\boldsymbol{\phi_V}, \boldsymbol{\lambda_V})$

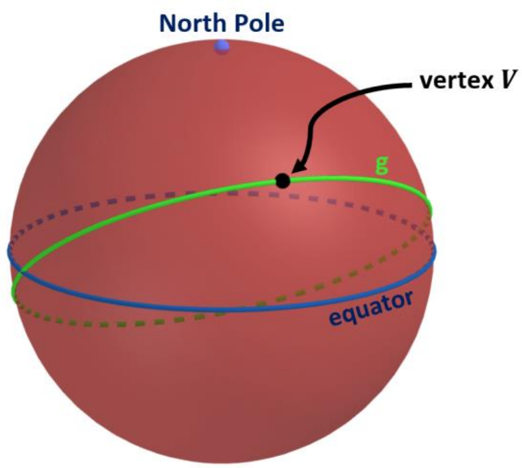
A point (ϕ, λ) on sphere S will be on the great circle g with vertex (latitude, longitude) given by (ϕ, λ) if and only if

$$\tan(\phi) = \tan(\phi_V)\cos(\lambda_V - \lambda)$$

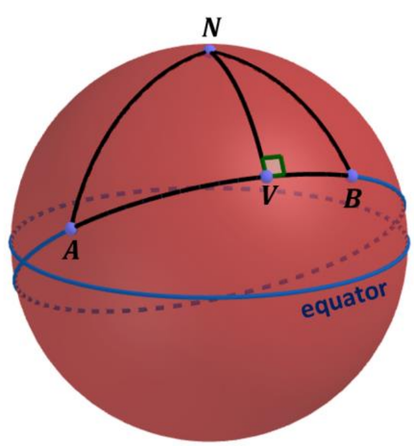
Step 1. Find the vertex

Finding $V(\phi, \lambda)$

The ***vertex*** of the great circle g is that point V which is ***closest to the North Pole*** N (or equivalently the farthest away from the equator).



Another way to phrase this is that V is the "northern most" point on the great circle g.

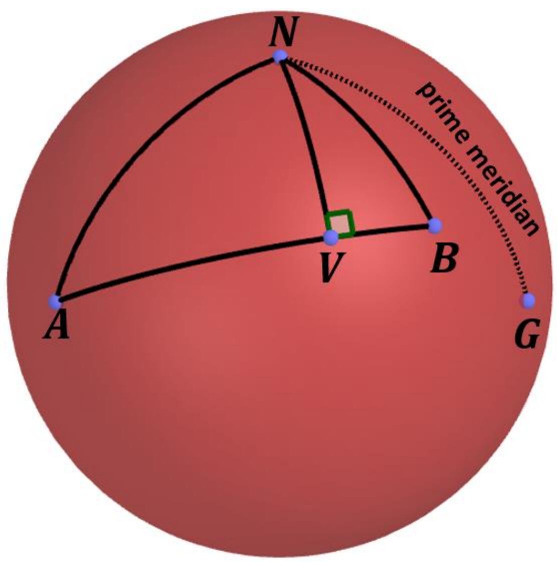


From basic geometric principles we know that if V is the point on g that is as close as possible to N (North Pole), then ***the great arc $\widehat{NV}$ is perpendicular to the great circle g***.

Algorithm[*] for finding (λ_V, φ_V), the latitude and longitude of the vertex V.

([*]Note: For simplicity of notation, this demonstration is for the special case when V is between A and B and where the prime meridian is east of both A and B. But this algorithm can be modified to handle all cases.)

Let (λ_A, φ_A) be the latitude and longitude of point A and let (λ_B, φ_B) be latitude and longitude of point B.



From our previous examples we can now just state that $\widehat{AN} = 90° - \lambda_A$, $\widehat{BN} = 90° - \lambda_B$, $\angle GNB = \varphi_B$ and $\angle GNA = \varphi_A$. It follows that $\angle BNA = \varphi_A - \varphi_B$.

Because we know three pieces of information about spherical triangle ΔANB we can use spherical trigonometry (mainly the spherical law of cosines for sides) to find all the missing pieces.

So, let's assume we've done that (by hand, if need be or using the computer program at https://emf.neocities.org/tr/spherical)

So, we can assume that we know $\theta = \angle NAB$.

This morning we looked at the spherical version of the Pythagorean theorem. Now we get to use it! Remember that following the "primary" version we listed 9 variations.

Variations (R2) and (R10) applied to the right spherical triangle ΔANV above give us the results we want.

$\sin(\widehat{NV}) = \sin(\widehat{NA}) \sin(\angle NAV)$	Variation (R2)
$\cos(\widehat{NA}) = \cot(\angle NAV) \cot(\angle ANV)$	Variation (R10)

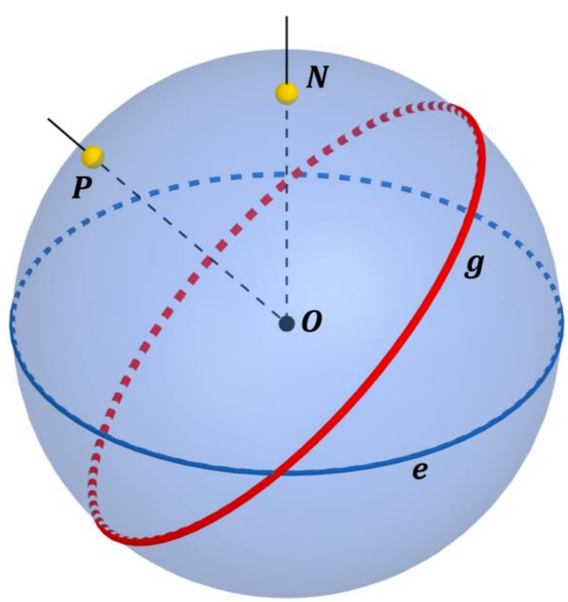
We know $\widehat{NA} = 90° - \lambda_A$ and we just discussed how to find $\angle NAV$ by solving ΔBNA. Hence, from these two variations we can find $\widehat{NV}$ and $\angle ANV$. We see that

$$\text{Latitude of } V = \lambda_V = 90° - \widehat{NV}$$

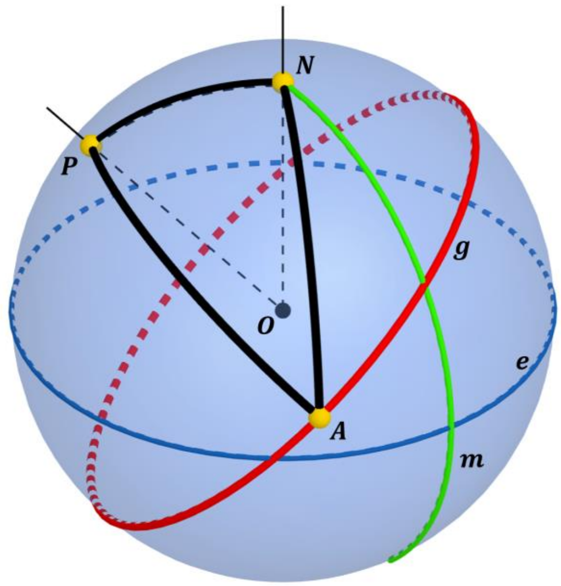
$$\text{Longitude of } V = \varphi_V = \varphi_A - \angle ANV.$$

Step 2. Equation for the great circle with a given pole.

Consider a sphere S with radius r, center O, geographic North Pole N, and equator e. Let P be a point on S with coordinates (latitude, longitude) $= (\phi_P, \lambda_P)$. Let g be the great circle of radius r of all points on S whose distance from P is $\pi r/2$.



Let $A = (\phi, \lambda)$ be some point on the great circle g and draw the spherical triangle ΔPNA. For simplicity the latitudes ϕ_P and ϕ are both north of the equator and the longitudes λ_P and λ are both west of the prime meridian m such that $0 \leq \lambda \leq \lambda_P$ as illustrated in the figure below.



Then $NP = r(\pi/2 - \phi_P)$, $NA = r(\pi/2 - \phi)$ and $PA = \pi r/2$. In this configuration, $\angle ANP = \lambda_P - \lambda$.

Applying the Spherical Law of Cosines to the spherical triangle ΔPNA we find

$$\cos\left(\frac{PA}{r}\right) = \cos\left(\frac{NP}{r}\right)\cos\left(\frac{NA}{r}\right) + \sin\left(\frac{NP}{r}\right)\sin\left(\frac{NA}{r}\right)\cos(\angle ANP).$$

Substituting the known values leads us to

$$\cos\left(\frac{\pi}{2}\right) = \cos(\pi/2 - \phi_P)\cos(\pi/2 - \phi) + \sin(\pi/2 - \phi_P)\sin(\pi/2 - \phi)\cos(\lambda_P - \lambda).$$

On simplification we have

$$0 = 1 + \tan(\pi/2 - \phi_P)\tan(\pi/2 - \phi)\cos(\lambda_P - \lambda)$$

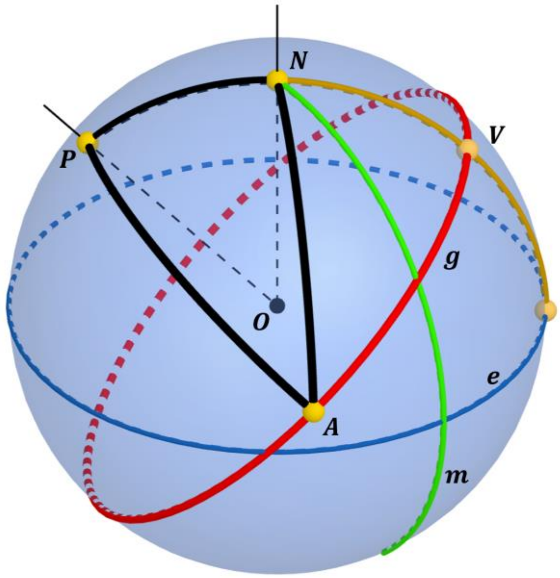
$$-\cot(\pi/2 - \phi_P)\cot(\pi/2 - \phi) = \cos(\lambda_P - \lambda)$$

$$-\tan(\phi_P)\tan(\phi) = \cos(\lambda_P - \lambda).$$

This final result is the equation of the great circle g with pole (ϕ_P, λ_P) on S. More specifically, a point (ϕ, λ) on S will be on the great circle g if and only if

$$-\tan(\phi_P)\tan(\phi) = \cos(\lambda_P - \lambda).$$

Other configurations of the points P and A relative to the equator and the prime meridian will lead to different but analogous equations.



Step 3. Express the Coordinates of the Pole in Terms of the Coordinates of the Vertex

Let V be the vertex of great circle g (which has a pole P). Because V is the vertex of g, $\widehat{NV}$ is necessarily perpendicular to g at V The great circle g plays the role of the "equator" for pole P. Hence, $\widehat{PV}$ is perpendicular to g at V. But there is only one great arc that is perpendicular to g at V. Thus, it follows that N (the earth's North Pole) must be on the great arc $\widehat{PV}$.

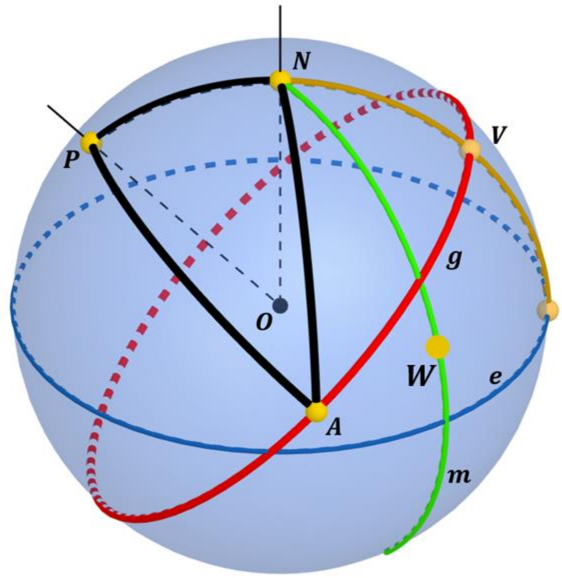
Because g is the "equator" for pole P, $\widehat{PV} = 90°$.

Now we also are given the information that the latitude of V is ϕ_V and the latitude of P is ϕ_P. So

$$\phi_V + \phi_P = \widehat{VN} + \widehat{NP} = \widehat{PV} = 90°$$

or

$$\phi_P = 90° - \phi_V.$$



We can also see that $\angle PNV = \angle WNP + \angle WNV$ where W is some arbitrary point on the prime meridian m (the green arc).

But $\angle WNP = \lambda_P$, the longitude of P and $\angle WNV = -\lambda_V$ because λ_V is negative because V is east of the prime meridian m.

Furthermore, we see that $\angle PNV = 180°$. Therefore,

$$180° = \lambda_P + (-\lambda_V)$$

or

$$\lambda_P = 180° + \lambda_V.$$

Step 4. Write Down the Equation of Great Circle g in Terms of the Vertex of g

Making these substitutions in the equation for the great circle with pole (ϕ_P, λ_P), we have

$$-\tan(\phi_P)\tan(\phi) = \cos(\lambda_P - \lambda)$$

$$-\tan\left(\frac{\pi}{2} - \phi_V\right)\tan(\phi) = \cos(\pi + \lambda_V - \lambda)$$

$$-\cot(\phi_V)\tan(\phi) = -\cos(\lambda_V - \lambda)$$

$$\cot(\phi_V)\tan(\phi) = \cos(\lambda_V - \lambda)$$

$$\tan(\phi) = \tan(\phi_V)\cos(\lambda_V - \lambda).$$

Hence,

$$\tan(\phi) = \tan(\phi_V)\cos(\lambda_V - \lambda).$$

A point (ϕ, λ) on sphere S will be on the great circle g if and only if

$$\tan(\phi) = \tan(\phi_V)\cos(\lambda_V - \lambda)$$

where (ϕ_V, λ_V) is the vertex of great circle g.

$$\tan(\phi) = \tan(\phi_V)\cos(\lambda_V - \lambda)$$

is the equation for the great circle with vertex (ϕ_V, λ_V). More specifically, a point (ϕ, λ) on S will be on the great circle g if and only if

$$\tan(\phi) = \tan(\phi_V)\cos(\lambda_V - \lambda).$$

Example

A plane takes a great circle course from Los Angeles via LAX $(33°56'33''N\ 118°24'2''W)$ to Paris via CDG $(49°00'35''N\ 2°32'5''E)$. At what longitude will the flight enter Canada?

Solution

Let's start by putting each of these coordinates in decimal form.

$$118°24'29'' = 118 + \frac{24}{60} + \frac{29}{60^2} = 118.4080555555556$$

$$2°32'52'' = 2 + \frac{32}{60} + \frac{54}{60^2} = 2.548333333333333$$

$$33°56'33'' = 33 + \frac{56}{60} + \frac{52}{60^2} = 33.94777777777777$$

$$49°00'35'' = 49 + \frac{0}{60} + \frac{35}{60^2} = 49.00972222222222.$$

$$(L_V, \lambda_V) = (61.87665142152449°, 49.49590643076745°)$$

The equation of the great circle with vertex $(latitude, longitude) = (L_V, \lambda_V)$ is

$$\tan(L) = \tan(L_V)\cos(\lambda_V - \lambda)$$

$$\tan(L) = \tan(61.87665142152449°)\cos(49.49590643076745° - \lambda)$$

Let $L = 49°$ (49th parallel)

$$\frac{\tan(49°)}{\tan(61.87665142152449°)} = \cos(49.49590643076745° - \lambda)$$

$$0.6148420950280024 = \cos(49.49590643076745° - \lambda)$$

$$\tan(X) = \tan(X_m)\cos(Y_m - Y)$$

$$\mathrm{acos}(0.6148420950280024) = 52.05955247055633°$$

$$49.49590643076745° - \lambda = 52.05955247055633°$$

$$\lambda = 49.49590643076745 - 52.05955247055633$$
$$= -2.563646039788885$$

$$\lambda = 49.49590643076745 - (-52.05955247055633)$$
$$= 101.5554589013238°$$

$$(Latitude, Longitude) = (49\ N, 101.5554589013238\ W)$$

$$101.5554589013238° = 101°33'20''$$

Output Location:
Monument 663, 109th Street Northwest, Sherwood, Renville County, North Dakota, 58782, USA

Output Latitude, Longitude:
48.999389°, -101.558523°

109th Street Northwest, Sherwood, North Dakota

Monument 663, Sherwood, ND

$$L_V = 61.87665142152449°$$

Homework

(1) At what latitude will the great-circle track from Miami (Lat. $25°46'$N., Long. $80°\ 12'$W.) to the Aleutians (Lat. $52°40'$N., Long. $175°\ 0'$W.) cross longitude $120°0'$ W?

(2) At what longitude will the great-circle track from Miami to the Aleutians cross latitude $40°0'$ N?

(*Suggestion*: You will get two solutions to this ambiguous case. One can be rejected because it is not a point between Miami and the Aleutians on the great-circle track.)

Ans: $96°15'$ W, $311°53'$.

(3) At what longitude will the great-circle track from Capetown (Lat. $33°55'$S., Long. $18°\ 22'$W.) to Delhi (Lat. $28°29'$N., Long. $77°\ 15'$E.) cross the equator? What will be the course at that point?

Pursuit Problem

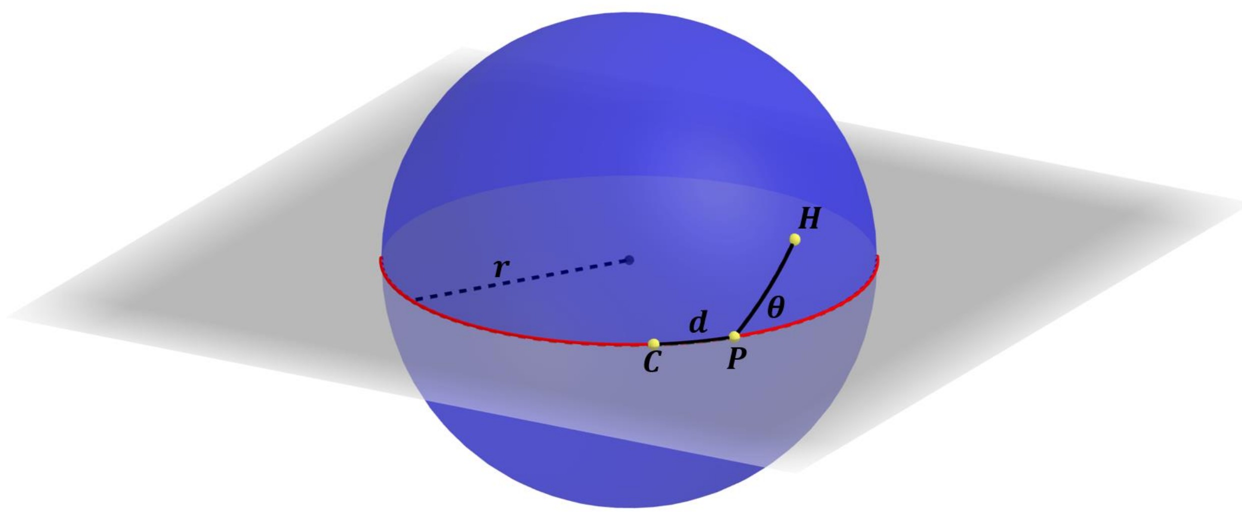
A pirate ship P is initially separated from a chase ship C by d miles measured along a great circle of a sphere of radius r. (By rotational symmetry, we can without loss of generality, assume that the pirate ship is initially located due east of the chase ship on the equator of the sphere.)

Assume the pirate ship sets sail with an initial angle of θ degrees off the equator with the plan of reaching an island hideout at H as fast as possible.

At that same moment the chase ship begins its pursuit at full speed with the purpose of intercepting the pirate ship as quickly as possible.

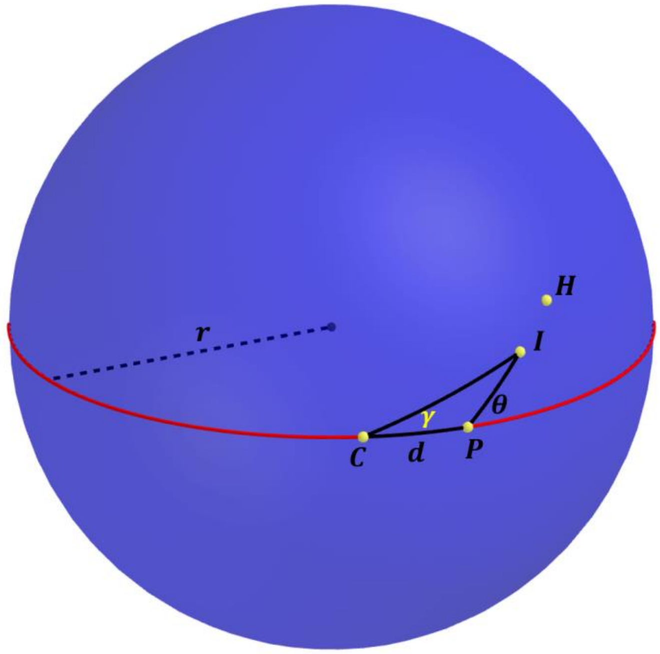
If the chase ship C can sail k times faster than the pirate ship P, what is the optimal pursuit track for the chase ship?

Assume that the distances are great enough that it is necessary to take the curvature of the sphere into account in all calculations. Also assume that both ships can maintain their full speed regardless of external conditions such as winds and obstacles in their path.



Solution

Both ships maintain their maximum speed at all times. Therefore, the goal of minimizing time is equivalent to minimizing distance. Therefore, both ships will travel along a great circle.



So, the problem comes down to finding the initial angle γ for the chase ship so that the two ships will collide at some point I on $\widehat{PH}$.

If we t represent the length of the arc $\widehat{PI}$ then the length of arc $\widehat{CI}$ is kt. From the spherical law of cosines for sides applied to the spherical triangle ΔCPI we have that

$$\cos\left(\frac{kt}{r}\right) = \cos\left(\frac{t}{r}\right)\cos\left(\frac{d}{r}\right) + \sin\left(\frac{t}{r}\right)\sin\left(\frac{d}{r}\right)\cos(180° - \theta)$$

$$= \cos\left(\frac{t}{r}\right)\cos\left(\frac{d}{r}\right) - \sin\left(\frac{t}{r}\right)\sin\left(\frac{d}{r}\right)\cos(\theta).$$

Solving for $\cos(\theta)$ we have

$$\cos(\theta) = \frac{\cos\left(\frac{t}{r}\right)\cos\left(\frac{d}{r}\right) - \cos\left(\frac{kt}{r}\right)}{\sin\left(\frac{t}{r}\right)\sin\left(\frac{d}{r}\right)}.$$

The only unknow in this equation is t and from this equation we can numerically solve for t.

By a second application of the spherical law of cosines for sides we have

$$\cos\left(\frac{t}{r}\right) = \cos\left(\frac{kt}{r}\right)\cos\left(\frac{d}{r}\right) + \sin\left(\frac{kt}{r}\right)\sin\left(\frac{d}{r}\right)\cos(\gamma).$$

Solving for $\cos(\gamma)$ we have

$$\cos(\gamma) = \frac{\cos\left(\frac{t}{r}\right) - \cos\left(\frac{kt}{r}\right)\cos\left(\frac{d}{r}\right)}{\sin\left(\frac{kt}{r}\right)\sin\left(\frac{d}{r}\right)}.$$

Because we solved for t in the previous step, the only unknown in this equation is the angle γ which can be explicated solved by

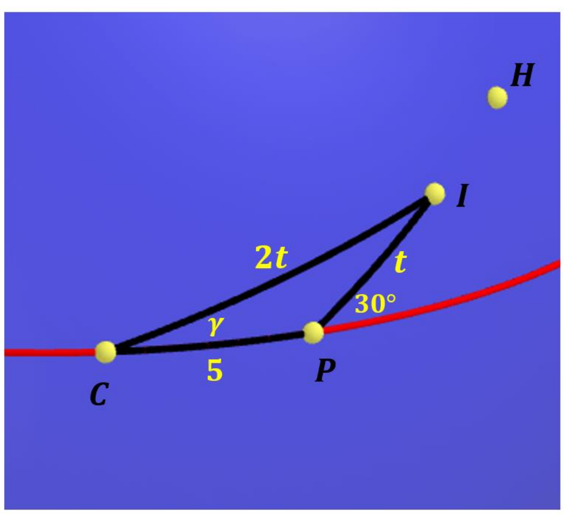
$$\gamma = \cos^{-1}\left(\frac{\cos\left(\frac{t}{r}\right) - \cos\left(\frac{kt}{r}\right)\cos\left(\frac{d}{r}\right)}{\sin\left(\frac{kt}{r}\right)\sin\left(\frac{d}{r}\right)}\right).$$

■

Example 1.

Following the set up as detailed above, assume the sphere for the two ships is our earth with radius $r = 3959$ miles, that the two ships start out 5 miles apart, that the chase ship's maximum speed is twice that of the pirate ship and that the pirate ship sets out with an initial angle of $30°$ off the equator. Find the initial angle γ that the chase ship should take so as to overtake the pirate ship as quickly as possible. How far will the pirate ship go and how far will the chase ship go before colliding?

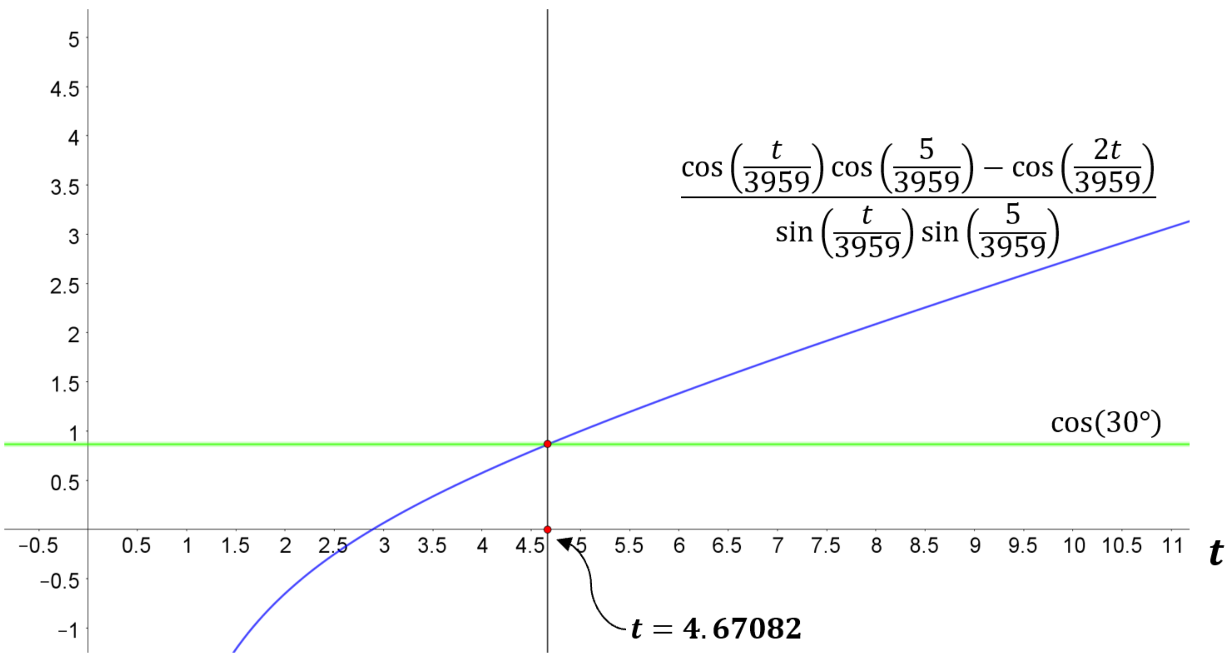
<u>Solution</u>



The first step in the solution is to numerically solve for t in the equation

$$\cos(30°) = \frac{\cos\left(\frac{t}{3959}\right)\cos\left(\frac{5}{3959}\right) - \cos\left(\frac{2t}{3959}\right)}{\sin\left(\frac{t}{3959}\right)\sin\left(\frac{5}{3959}\right)}.$$

You can you use any numerical software package to do this. I used the software Geogebra to graph the left-hand side expression and then determined where this graph equaled $\cos(30°)$. In this way I found $t = 4.67082$ miles. (I have ignored all potential issues of numerical accuracy in this calculation.)



The second step is to find the initial angle γ of the chase ship from the formula

$$\gamma = \cos^{-1}\left(\frac{\cos\left(\frac{4.67082}{3959}\right) - \cos\left(\frac{2\cdot 4.67082}{3959}\right)\cos\left(\frac{5}{3959}\right)}{\sin\left(\frac{2\cdot 4.67082}{3959}\right)\sin\left(\frac{5}{3959}\right)}\right)$$

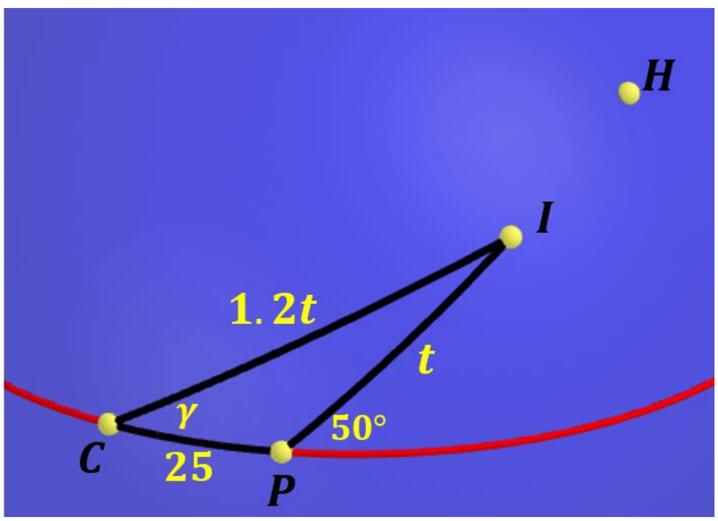
$$= 14.47843°.$$

So, we have determined that in order for the chase ship to intercept the pirate ship as fast as possible, the chase ship should steer an initial angle of $14.47843°$ off the equator, the pirate ship will go $t = 4.67082$ miles and the chase ship will go $2t = 9.34164$ miles before the two ships collide.

Example 2.

Once again, we will follow the set up as detailed above, assume the sphere for the two ships is our earth with radius $r = 3959$ miles. But in this example assume that the two ships start out 25 miles apart, that the chase ship's maximum speed is only 1.2 times that of the pirate ship and that the pirate ship sets out with an initial angle of $50°$ off the equator. Again, the question is to find the initial angle γ that the chase ship should take so as to overtake the pirate ship as quickly as possible. How far will the pirate ship go and how far will the chase ship go before colliding?

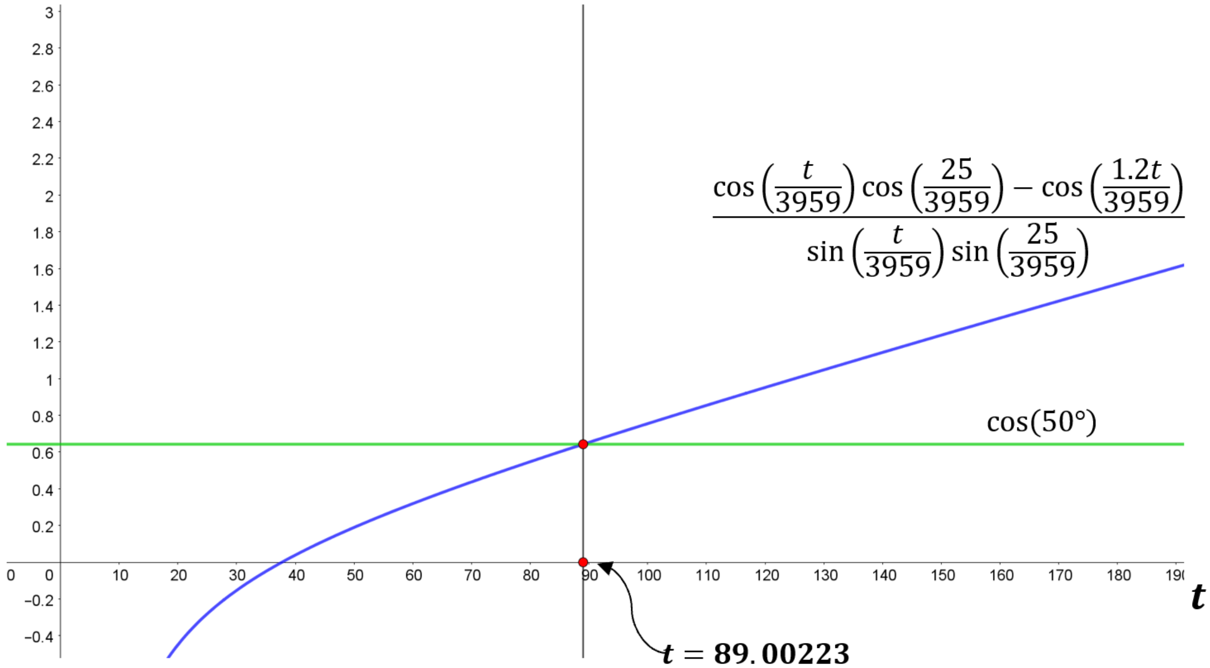
<u>Solution</u>



As in the previous example the first step in the solution is to numerically solve for t in the equation

$$\cos(50°) = \frac{\cos\left(\frac{t}{3959}\right)\cos\left(\frac{25}{3959}\right) - \cos\left(\frac{1.2t}{3959}\right)}{\sin\left(\frac{t}{3959}\right)\sin\left(\frac{25}{3959}\right)}.$$

You can you use any numerical software package to do this. I used the software Geogebra to graph the left-hand side expression and then determined where this graph equaled $\cos(30°)$. In this way I found $t = 4.67082$ miles. (Once again, I have ignored all potential issues of numerical accuracy in this calculation.)



The second step is to find the initial angle γ of the chase ship from the formula

$$\gamma = \cos^{-1}\left(\frac{\cos\left(\frac{89.00223}{3959}\right) - \cos\left(\frac{1.2\cdot 89.00223}{3959}\right)\cos\left(\frac{25}{3959}\right)}{\sin\left(\frac{1.2\cdot 89.00223}{3959}\right)\sin\left(\frac{25}{3959}\right)}\right)$$

$$= 39.67217°.$$

So, we have determined that in order for the chase ship to intercept the pirate ship as fast as possible, the chase ship should steer an initial angle of 39.67217° off the equator, the pirate ship will go $t = 89.00223$ miles and the chase ship will go $1.2t = 106.802676$ miles before the two ships collide.

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