Spherical Trigonometry

Notes

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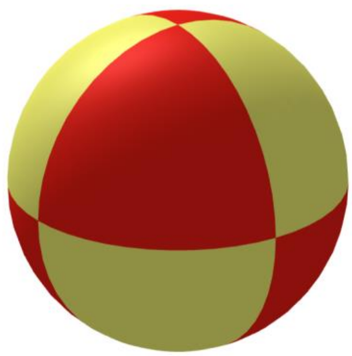


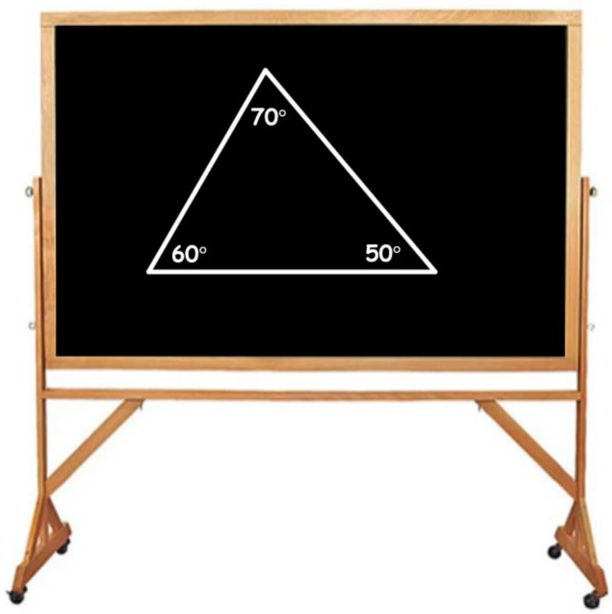
Summer Mathematics Institute

June 23-28, 2024

Minnesota State High School Mathematics League

Lecture 1. Monday Morning, 1st Hour, June 23

To contrast planar trigonometry with spherical trigonometry think about the differences between a triangle drawn on a (planar) chalkboard versus a triangle drawn on a spherical chalkboard.



(You might notice one important difference between the two if you add the degrees of the angles in both cases.)

Introduction

Spherical trigonometry is the study of triangles drawn on the surface of a sphere. The principal applications of spherical trigonometry are in navigation, aviation, and geography on the terrestrial sphere (earth) and in astronomy of the celestial sphere (the apparent surface of the heavens containing the stars).

Here are a dozen spherical trigonometry problems that you will be able to answer after taking this course:

(1) At what longitude will the most direct flight (*i.e.,* shortest path) from Miami to the Aleutians cross latitude $40°0'$ N?

(2) In what latitude is the shortest day of the year just one hour long?

(3) Colorado extends from $37°$ to $40°$ N latitude and from $102°$ to $107°$ W longitude. What is the area of Colorado in square miles?

(4) What is the latitude of three points on the Earth equally distant from each other and from the North Pole?

(5) Where are you if you are 625 miles away from Boise, 690 miles away from Minneapolis and 615 miles away from Tucson by the way the crow flies, *i.e.* how does GPS work?

(6) According to the Islamic faith a follower must offer their prayers facing Mecca by means of the most direct route. If a follower is in Boston, what direction should they face to make their prayers?

(7) If the face angle at the vertex of a regular four-sided pyramid on a square base is $50°$, what is the angle between any two adjacent lateral faces?

(8) How high above the earth would a satellite have to be to see both New York and Los Angeles?

(9) How long does it take the sun to set on October 24th in Duluth, Minnesota?

(10) What is your latitude if on April 13th the shortest shadow of a pole occurs at $12{:}10$ p.m. and is exactly equal to its height?

(11) What is the velocity of Iowa City, Iowa in space due to the rotation of the earth on its axis?

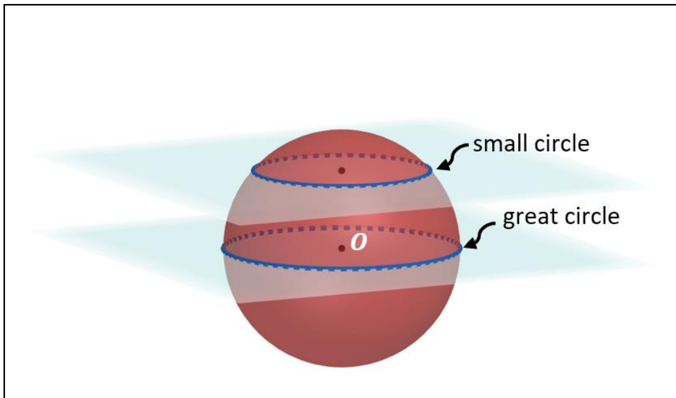
(12) A destroyer is notified that an enemy ship is in a position due south of the destroyer and is steaming at 18 knots on a course bearing $N\ 52°\ E$. On what course should the destroyer set out in order to intercept the enemy ship in 20 hours, if the destroyer makes 27 knots?

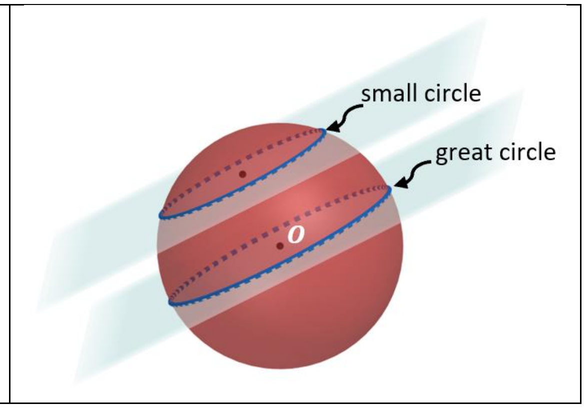
Our first lesson will introduce the terms and theorems of spherical trigonometry. What you will begin to notice is that every theorem in planar trigonometry has a companion result for the sphere.

Great Circles

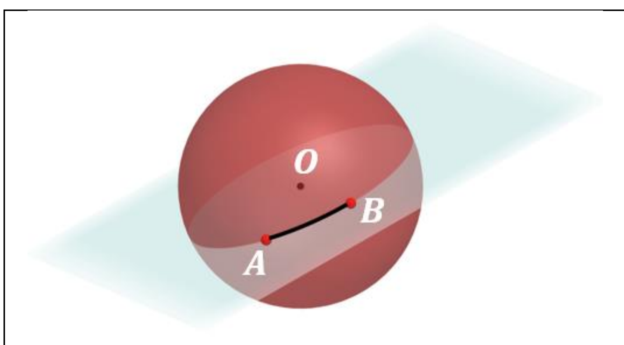
Consider a plane that cuts a sphere in more than one point. Then the intersection of that plane and the sphere is a *circle* on the surface of the sphere.

Any circle on the surface of a sphere whose interior contains the center O of the sphere is called a *great circle* or *orthodrome*. Every circle on a sphere which is not a great circle is called a *small circle***.**

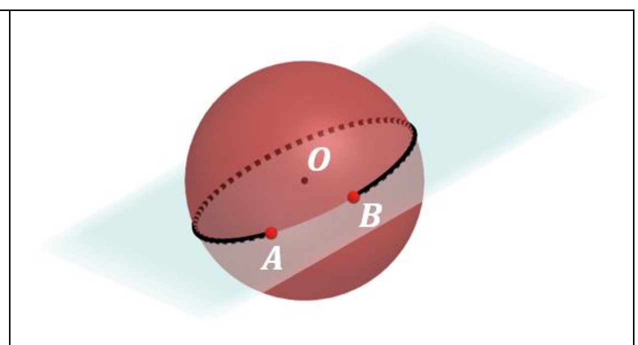




Two points A and B on a circle produce <u>two arcs</u> depending on the direction you take in moving from point A to point B.

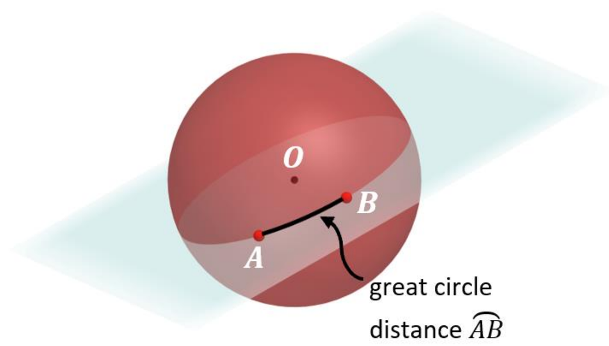


(i)



(ii)

We define arc $\widehat{AB}$ as the *great circle distance* between points A and B if it is the <u>shorter</u> of the two arcs (Case (i)) created by points A and B on a great circle. We will refer to the shorter of the two arcs as the **great circle arc** or more simply the **great arc**.



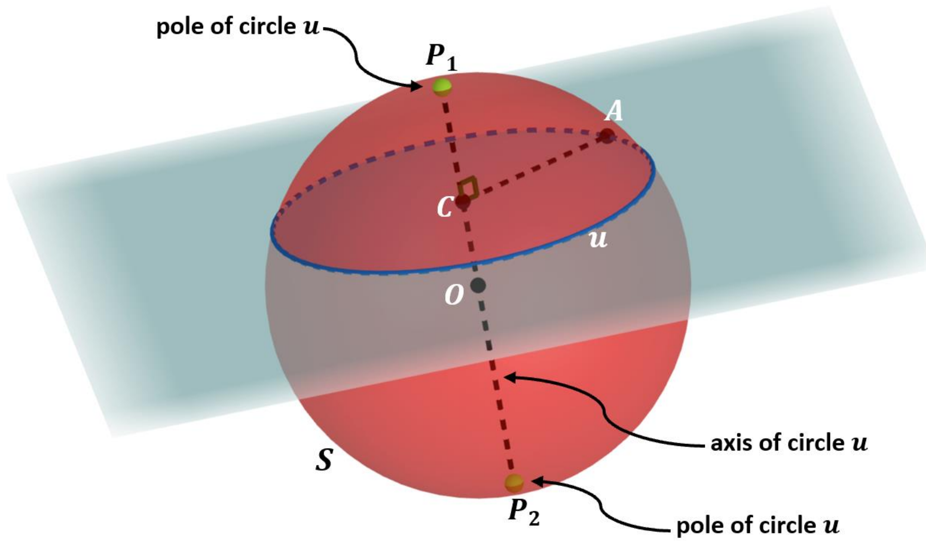
> In the theorems and results that follow in these notes, we will always assume (unless specified otherwise) that $\widehat{AB}$ refers to the great circle arc, the **shorter** of the two arcs created by the points A and B on a sphere.

Axis and Poles of a Circle

Consider any circle (great or small) on sphere S. The ***axis of that circle*** is the diameter of S (*i.e.* goes through the center point O of that sphere) which is perpendicular to any line in the plane containing that circle.

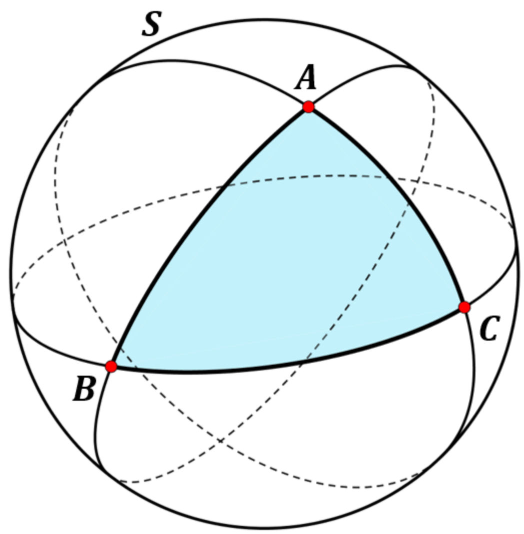
The extremities of the axis (points where the axis intersects the circle) are called the ***poles*** of that circle.

In the example below, diameter $\overline{P_1P_2}$ is the axis of circle u because it is perpendicular to the radii $\overline{CA}$ of circle u (which is a line in that plane).



Spherical Triangles

A spherical triangle is a closed three-sided figure drawn on a sphere where each side is the great circle distance between two points on that sphere.

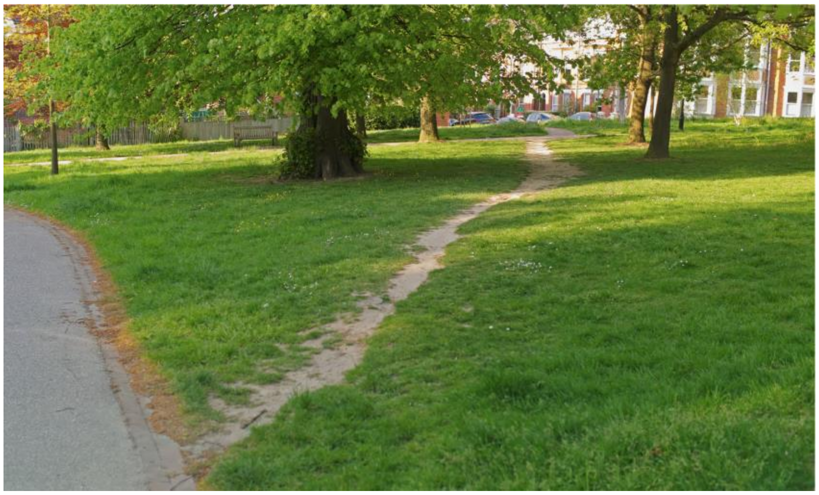


Spherical Triangle $\Delta_s ABC$ with sides $\widehat{AB}$, $\widehat{AC}$ and $\widehat{BC}$.

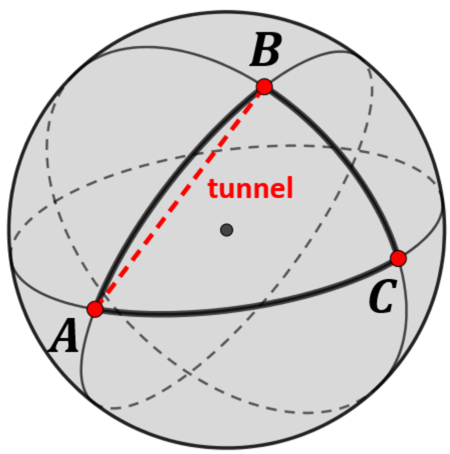
(Remember by our definition that $\widehat{AB}$ always refers to the *shorter* of the two arcs of a great circle connecting points A and B.)

Geodesics on a Sphere

We are generally tempted by the geodesic (shortest possible) path when going from any point A to any point B.

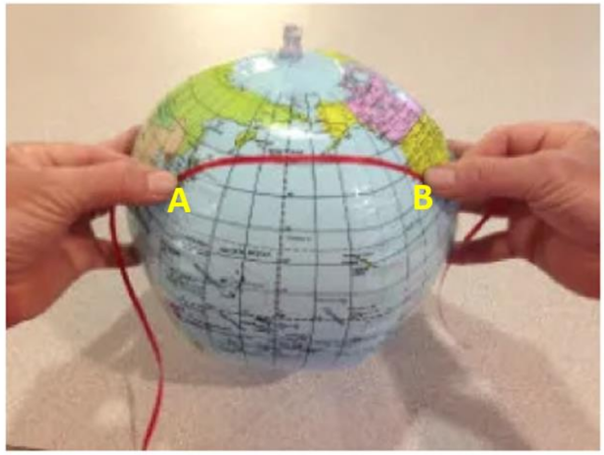


But we would have to dig a tunnel to connect points A and B on a sphere with the shortest possible path.



That is why the assumed surface must be specified when constructing a geodesic path. So, on a sphere, the geodesic is the shortest possible path ***on the surface of a sphere*** between points A and B.

Intuitively, we can find the geodesic route from point A to point B on a globe by stretching a string between the two points and pulling it tight – the intuition being that the shortest route will be the route using the least string.



https://aerosavvy.com/great-circle-routes/

This figure suggests the following theorem. (The proof is in the appendix to this lecture.)

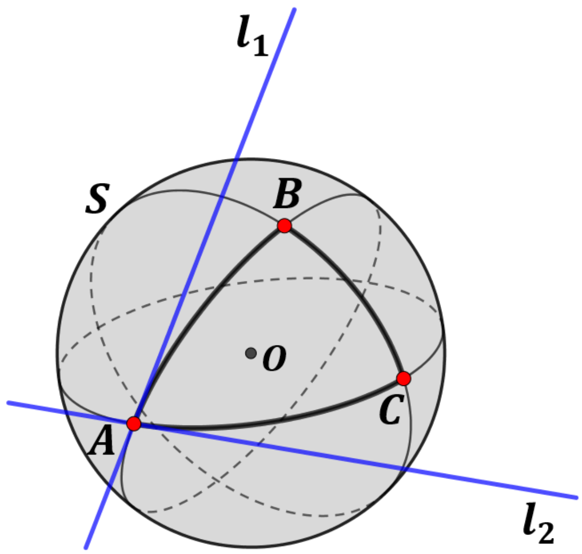
> <u>Theorem 1.</u> The great arc $\widehat{AB}$ connecting points A and B on a sphere is the shortest possible path on a sphere connecting point A to point B.

It follows that each side of $\Delta_s ABC$ is the shortest possible arc between the points A, B and C.

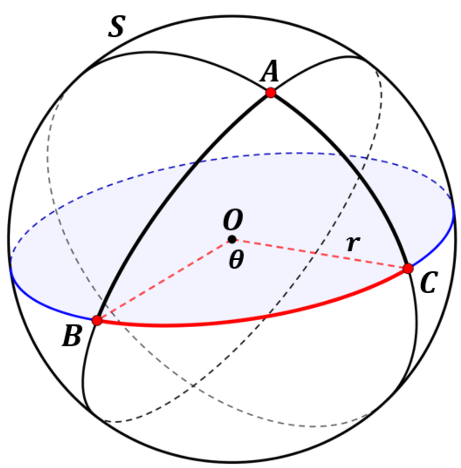
Spherical Angles

Let $\Delta_s ABC$ be a spherical triangle on sphere S with center point O.

The measure of the spherical angle $\sphericalangle BAC$ at vertex A between the great arcs c and $\widehat{CA}$ is defined as the measure of the planar angle between the lines l_1 and l_2 where l_1 is that line tangent to S at A and in the plane containing the great arc $\widehat{BA}$ and l_2 is that line tangent to S at A and in the plane containing the great arc $\widehat{CA}$.



Connecting Sides of a Spherical Triangle to Central Angles of a Sphere



The central angle θ in this pair $(\theta, \widehat{AB})$ is called the **angular length** of $\widehat{AB}$. So we can say

$$\text{physical length} = \ r \cdot (\text{angular length}) = r\theta$$

provided θ is measured in radians.

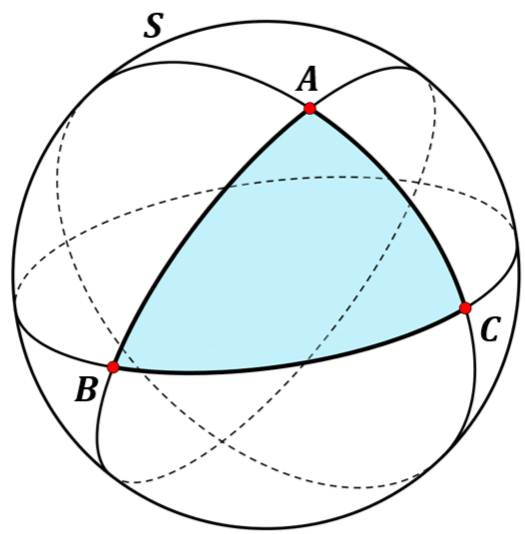
Polar Triangle

Every spherical triangle $\Delta_s ABC$ can be paired with another spherical triangle $\Delta_s A'B'C'$ called the **polar** of $\Delta_s ABC$

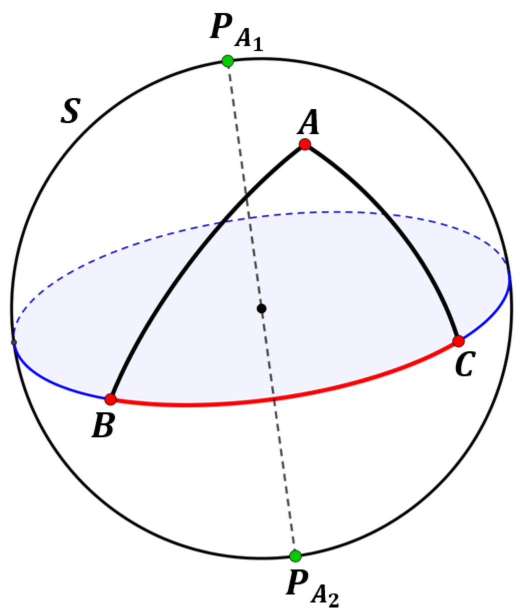
In this section we will demonstrate how to construct $\Delta_s A'B'C'$ and will state some of its most important properties. We will include the proofs of these properties in the appendix to this lecture.

In the lecture that follows, we will see the primary role of the polar triangle $\Delta_s A'B'C'$ in proving two important theorems about its dual spherical triangle $\Delta_s ABC$, namely the spherical law of cosines for angles and the spherical triangle inequality theorem.

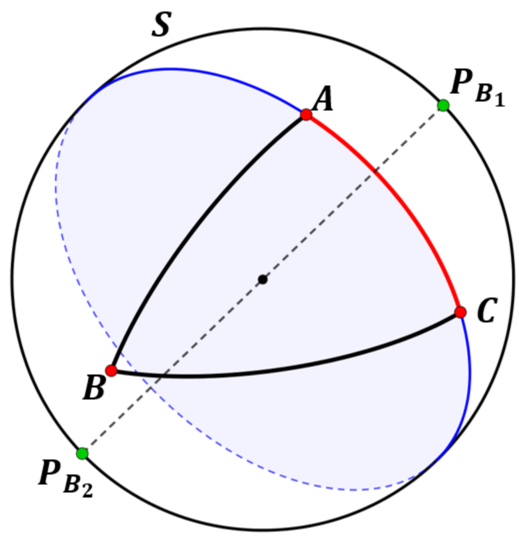
The Construction of the Polar Triangle of $\boldsymbol{\Delta_s ABC}$.



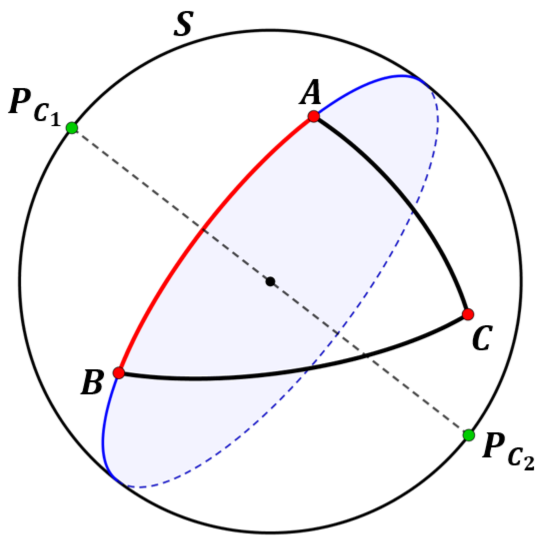
Each *side* of spherical triangle $\Delta_s ABC$ has two poles. In particular, the great circle containing side $\widehat{BC}$ of $\Delta_s ABC$ has poles P_{A_1} and P_{A_2}.



The great circle containing the side $\widehat{AC}$ of $\Delta_s ABC$ has poles P_{B_1} and P_{B_2}.

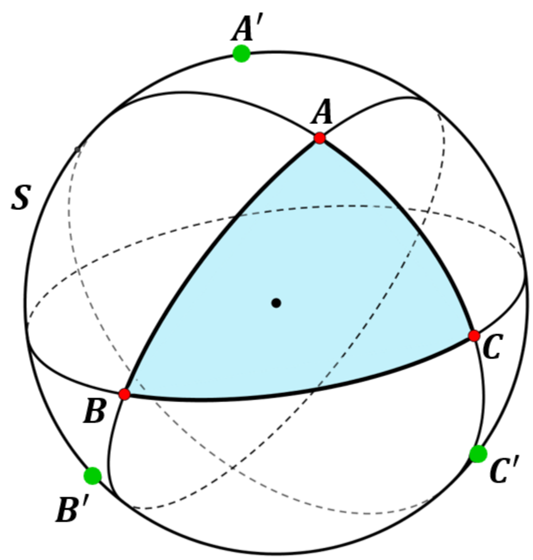


The great circle containing the side $\widehat{AB}$ of $\Delta_s ABC$ has poles P_{C_1} and P_{C_2}.

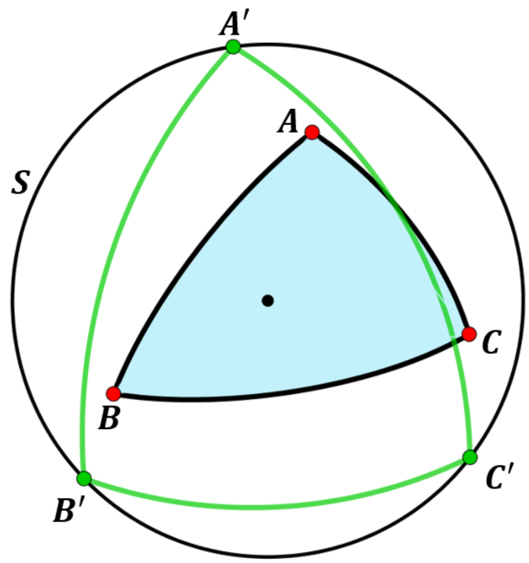


Now suppose that from poles P_{A_1} and P_{A_2} we ***only keep the pole which is closer to point A***. And from poles P_{B_1} and P_{B_2} we only keep the pole which is closer to point B. And from poles P_{C_1} and P_{C_2} we only keep the pole which is closer to point C.

Notice that in our example, this means we would keep poles P_{A_1}, P_{B_2}, and P_{C_2}. The standard labels used in spherical trigonometry textbooks and papers for the three closer poles are A', B', and C'.



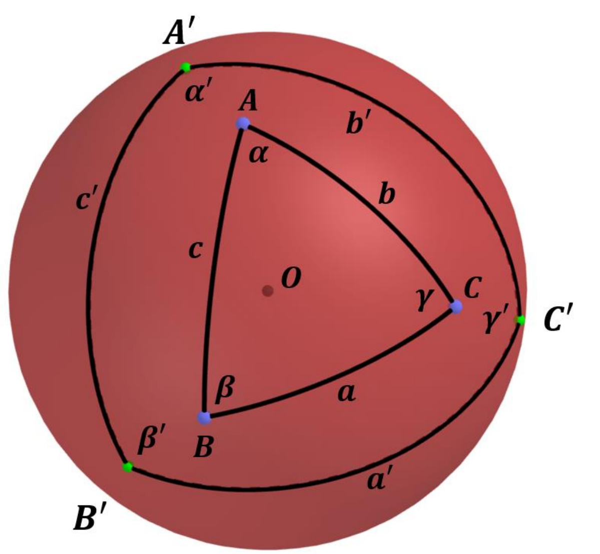
The spherical triangle $\Delta_s A'B'C'$ connecting these three points is called the ***polar*** of spherical triangle $\Delta_s ABC$.



Properties of $\Delta_s A'B'C'$, the polar triangle of $\Delta_s ABC$.

(i) If $\Delta_s A'B'C'$ is the polar triangle of $\Delta_s ABC$, then $\Delta_s ABC$ is the polar triangle of the polar triangle $\Delta_s A'B'C'$. Similar to how the complement of the complement of a set A is again the set A.

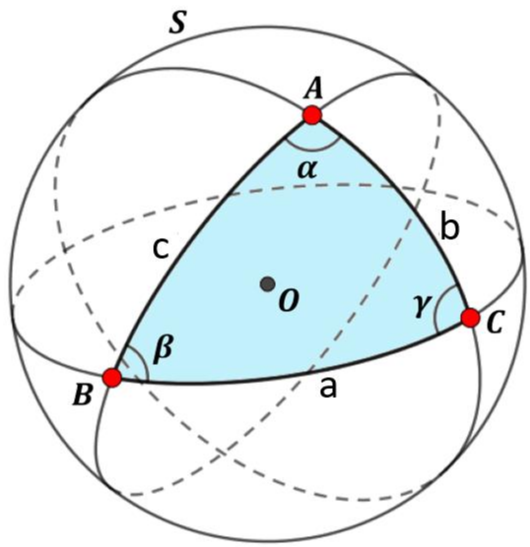
(ii) Let $\Delta_s A'B'C'$ be the polar triangle of $\Delta_s ABC$ on a sphere and let the angles and sides of $\Delta_s ABC$ and $\Delta_s A'B'C'$ be labeled as indicated as in figure below.



Then the sides and angles of these two triangles are related as given in the following table.

$a' = r(\pi - \alpha)$	$b' = r(\pi - \beta)$	$c' = r(\pi - \gamma)$
$\alpha' = \pi - \frac{a}{r}$	$\beta' = \pi - \frac{b}{r}$	$\gamma' = \pi - \frac{c}{r}$

Theorem – Spherical Law of Cosines for Sides



Let r equal the radius of sphere S. Then

$$\cos\left(\frac{\mathrm{a}}{r}\right) = \cos\left(\frac{\mathrm{b}}{r}\right)\cos\left(\frac{c}{r}\right) + \sin\left(\frac{\mathrm{b}}{r}\right)\sin\left(\frac{\mathrm{c}}{r}\right)\cos(\alpha)$$

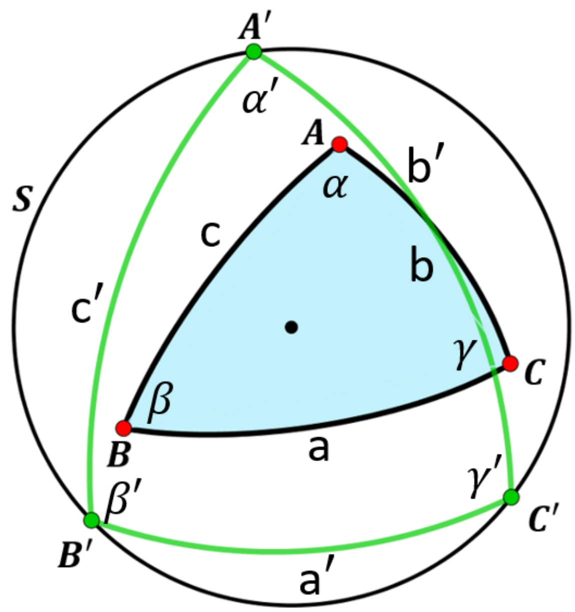
$$\cos\left(\frac{\mathrm{b}}{r}\right) = \cos\left(\frac{\mathrm{a}}{r}\right)\cos\left(\frac{\mathrm{c}}{r}\right) + \sin\left(\frac{\mathrm{a}}{r}\right)\sin\left(\frac{\mathrm{c}}{r}\right)\cos(\beta)$$

and

$$\cos\left(\frac{\mathrm{c}}{r}\right) = \cos\left(\frac{\mathrm{a}}{r}\right)\cos\left(\frac{\mathrm{b}}{r}\right) + \sin\left(\frac{\mathrm{a}}{r}\right)\sin\left(\frac{\mathrm{b}}{r}\right)\cos(\gamma)$$

assuming we are working in radian mode.

Theorem – Spherical Law of Cosines <u>for Angles</u>



Recall the Spherical Law of Cosines for Sides applied to spherical triangle ΔABC

$$\cos\left(\frac{\text{a}}{r}\right) = \cos\left(\frac{\text{b}}{r}\right)\cos\left(\frac{c}{r}\right) + \sin\left(\frac{\text{b}}{r}\right)\sin\left(\frac{\text{c}}{r}\right)\cos(\alpha)$$

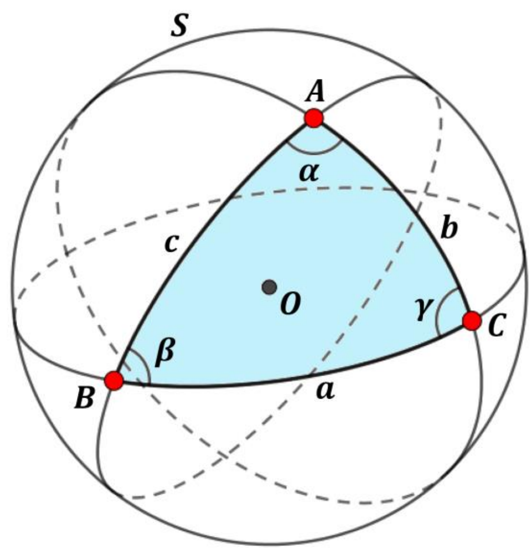
and the Polar Spherical Triangle to "Regular" Spherical Triangle translation table.

$$\text{a}' = r(\pi - \alpha) \qquad \text{b}' = r(\pi - \beta) \qquad \text{c}' = r(\pi - \gamma)$$

$$\alpha' = \pi - \frac{\text{a}}{r} \qquad \beta' = \pi - \frac{\text{b}}{r} \qquad \gamma' = \pi - \frac{\text{c}}{r}$$

Now apply the Spherical Law of Cosines for Sides to the polar spherical triangle $\Delta A'B'C'$ and simplify. What do you get?

Theorem – Spherical Law of Sines

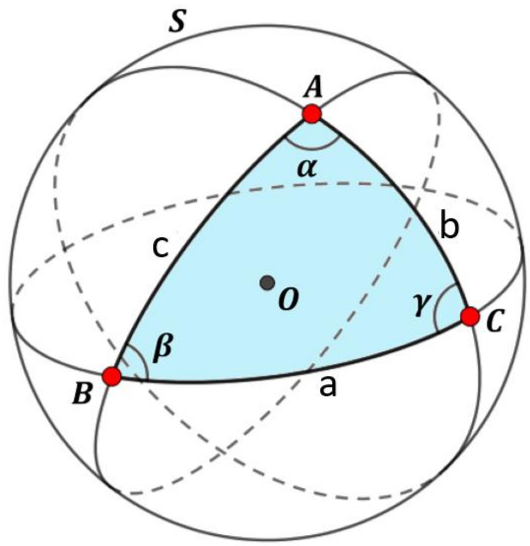


Let r equal the radius of sphere S. Then

$$\frac{\sin(\alpha)}{\sin\left(\frac{a}{r}\right)} = \frac{\sin(\beta)}{\sin\left(\frac{b}{r}\right)} = \frac{\sin(\gamma)}{\sin\left(\frac{c}{r}\right)}$$

where a, b and c are measured in the same units as the radius of the sphere.

"Sum of the Three Sides" and "Sum of the Three Angles"



For any spherical triangle $\Delta_S(ABC)$ on a sphere with radius r,

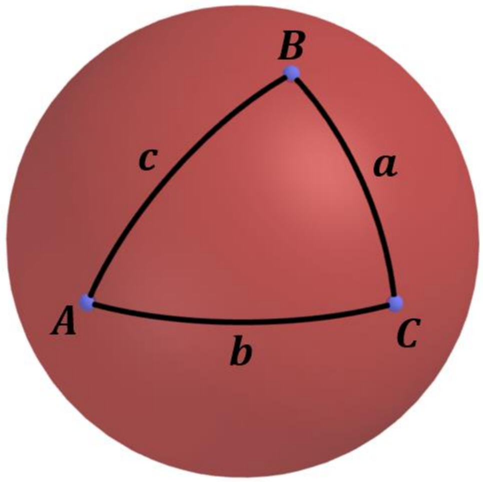
$$a + b + c < 2\pi r \qquad (i)$$

and

$$\pi < \alpha + \beta + \gamma \leq 3\pi. \qquad (ii)$$

Sum of the Three Sides of a Spherical Triangle is Less Than $2\pi r$.

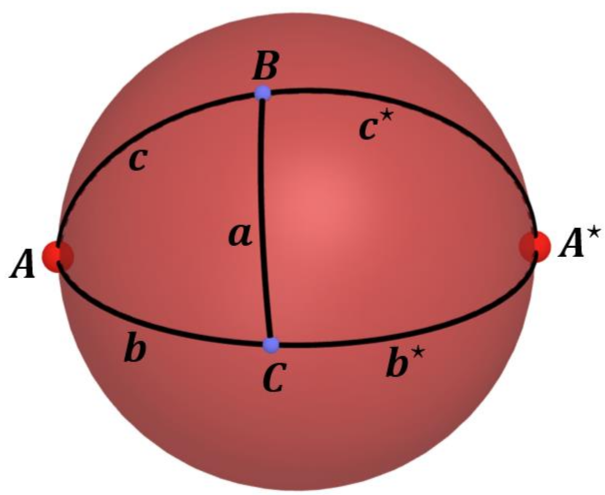
For any spherical triangle $\Delta_S(ABC)$ on a sphere with radius r, $a + b + c < 2\pi r$.



Proof

Extend $\widehat{AB}$ and $\widehat{AC}$ out from B and C until they intersect for the second time at $A^\star$.

Let $\widehat{BA^\star} = c^\star$ and $\widehat{CA^\star} = b^\star$.



We know that A and $A^\star$ are ***antipodal*** points which means that $\widehat{ABA^\star}$ and $\widehat{ACA^\star}$ are meridians (*i.e.* semicircles on a sphere).

Therefore,

$$\overset{\frown}{ABA^\star} = \frac{1}{2}(2\pi r) = \pi r \quad \text{and} \quad \overset{\frown}{ACA^\star} = \frac{1}{2}(2\pi r) = \pi r.$$

Now consider the spherical triangle $\Delta_S(A^\star BC)$. From the spherical triangle inequality theorem we know that

$$a < b^\star + c^\star.$$

Thus,

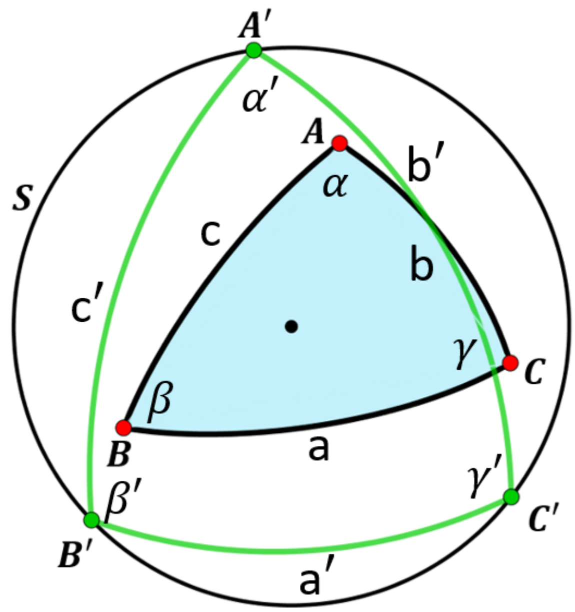
$$\begin{aligned} a + b + c &< (b^\star + c^\star) + b + c \\ &= (b^\star + b) + (c^\star + c) \\ &= \overset{\frown}{ACA^\star} + \overset{\frown}{ABA^\star} \\ &= \pi r + \pi r \\ &= 2\pi r. \end{aligned}$$

■

Having proven (i), we can prove (ii) by using the same "angle-side" swap that we used to establish the Spherical Law of Cosines for Angles.

Proof

Let $\Delta_s A'B'C'$ be the polar triangle of $\Delta_s ABC$ with the angles and sides of each labeled as follows.



Recall the relationships between the sides and angles of these two spherical triangles.

$\mathrm{a}' = r(\pi - \alpha)$	$\mathrm{b}' = r(\pi - \beta)$	$\mathrm{c}' = r(\pi - \gamma)$
$\alpha' = \pi - \frac{\mathrm{a}}{r}$	$\beta' = \pi - \frac{\mathrm{b}}{r}$	$\gamma' = \pi - \frac{\mathrm{c}}{r}$

Hence,

$$\mathrm{a}' + \mathrm{b}' + \mathrm{c}' = 3\pi r - r(\alpha + \beta + \gamma)$$

or

$$\alpha + \beta + \gamma = 3\pi - \frac{1}{r}(\mathrm{a}' + \mathrm{b}' + \mathrm{c}').$$

Hence,

$$\alpha + \beta + \gamma < 3\pi.$$

Now we determined in the last theorem that for any spherical triangle on a sphere with radius r the sum of the sides is less than $2\pi r$.

So

$$\frac{1}{r}(\mathrm{a}' + \mathrm{b}' + \mathrm{c}') < 2\pi.$$

Hence

$$\alpha + \beta + \gamma = 3\pi - \frac{1}{r}(\mathrm{a}' + \mathrm{b}' + \mathrm{c}') > 3\pi - 2\pi = \pi.$$

■

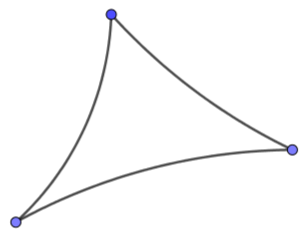
By the way, looking one more time at our table

$\mathrm{a}' = r(\pi - \alpha)$	$\mathrm{b}' = r(\pi - \beta)$	$\mathrm{c}' = r(\pi - \gamma)$
$\alpha' = \pi - \frac{\mathrm{a}}{r}$	$\beta' = \pi - \frac{\mathrm{b}}{r}$	$\gamma' = \pi - \frac{\mathrm{c}}{r}$

is it possible for an ***individual*** angle in a spherical triangle to equal or exceed $180°$? Why or why not?

In a <u>Spherical </u>Triangle the sum of the angles is strictly larger than $180°$.

Does this has you wondering whether there is another type of triangle where the sum of the angles is *strictly less than* $180°$?



There *are* and they are called ***hyperbolic triangles***. We will not be discussing them in these notes but they are important too. The theory of special relativity depends on hyperbolic triangles.

Can a spherical triangle have two right angles?
Can a spherical triangle have three right angles?

The *only* restrictions on the size of angles in a spherical triangle are

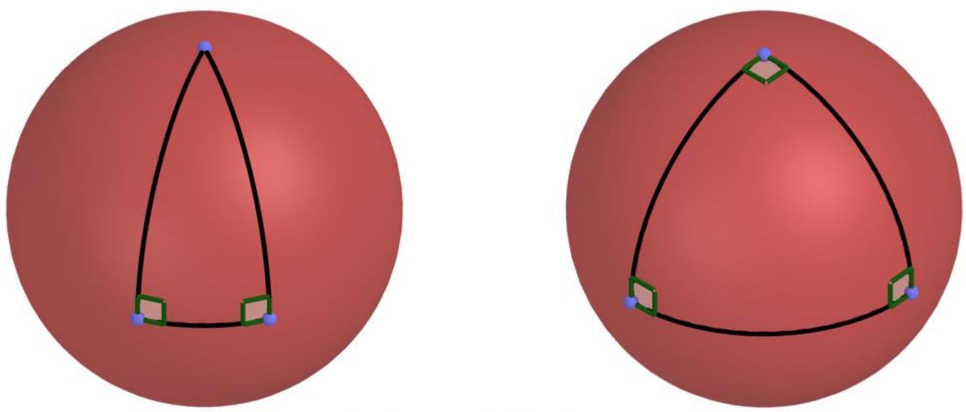
$$0 < \alpha < 180°, 0 < \beta < 180°, 0 < \gamma < 180°$$

and

$$180° < \alpha + \beta + \gamma < 540°.$$

Can $\alpha = 90°$ and $\beta = 90°$ and $\gamma \neq 90°$ without violating the above rules?

Can $\alpha = 90°$ and $\beta = 90°$ and $\gamma = 90°$ without violating the above rules?

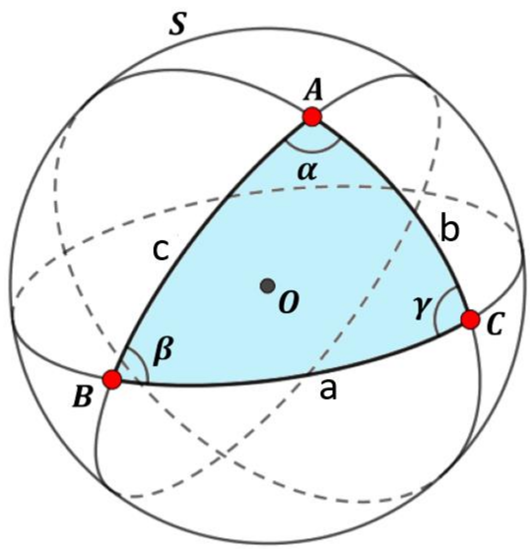


Seeing is believing.

But there's more. Multiple angles of $90°$ will force restrictions on the sides.

Theorem

Suppose $\alpha = 90°$ and $\beta = 90°$ and suppose γ might be a right angle but does not have to be.



By the Spherical Law of Cosines for Angles we know that

$$\cos(\alpha) = -\cos(\beta)\cos(\gamma) + \sin(\beta)\sin(\gamma)\cos\left(\frac{a}{r}\right)$$

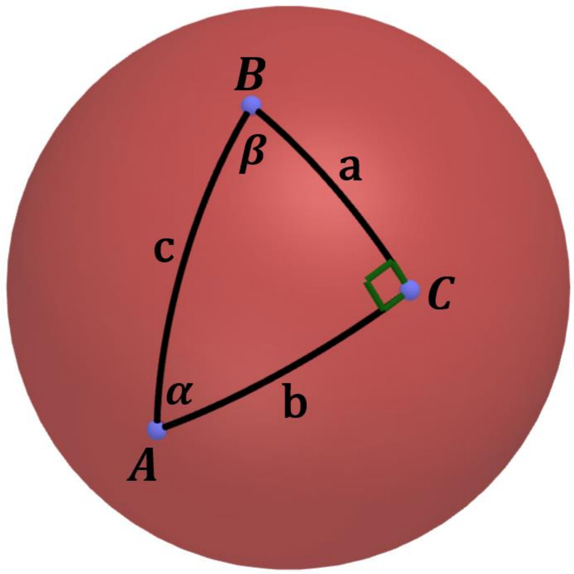
$$\cos(\beta) = -\cos(\alpha)\cos(\gamma) + \sin(\alpha)\sin(\gamma)\cos\left(\frac{b}{r}\right)$$

$$\cos(\gamma) = -\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)\cos\left(\frac{c}{r}\right).$$

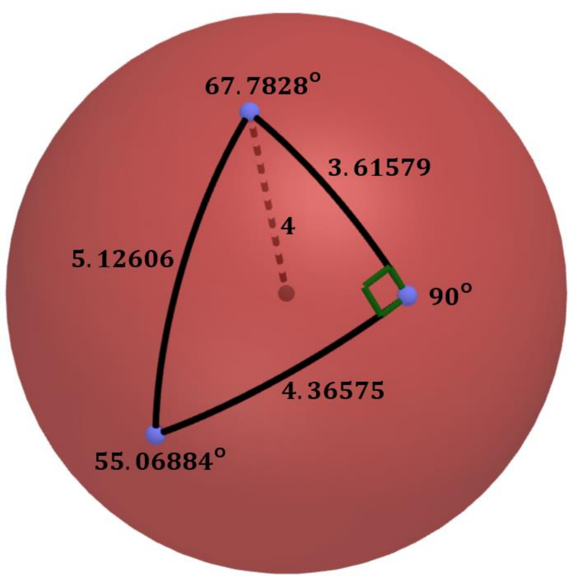
Plug in $\alpha = 90°$ and $\beta = 90°$. What can you determine about the sides a, b and c?

Spherical Pythagorean Theorem

It's ***not*** $c^2 = a^2 + b^2$. Can you figure out what it should be? Look back at the theorems and stated results listed so far. Which one(s) look relevant?

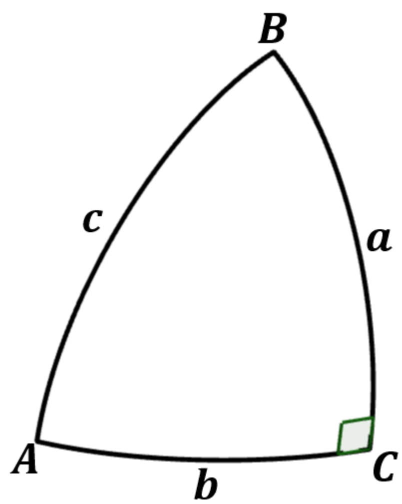


Use the following data to test any of your conjectures.



Consequences of the Spherical Pythagorean Theorem

Suppose $\Delta_S(ABC)$ is a right spherical triangle on a sphere with radius r and with a right angle at C as labeled in the figure below.



Then

(R1) $\cos\left(\frac{c}{r}\right) = \cos\left(\frac{a}{r}\right)\cos\left(\frac{b}{r}\right)$

(R2) $\sin\left(\frac{a}{r}\right) = \sin(A)\sin\left(\frac{c}{r}\right)$

(R3) $\sin\left(\frac{b}{r}\right) = \sin(B)\sin\left(\frac{c}{r}\right)$

(R4) $\tan\left(\frac{a}{r}\right) = \tan(A)\sin\left(\frac{b}{r}\right)$

(R5) $\tan\left(\frac{b}{r}\right) = \tan(B)\sin\left(\frac{a}{r}\right)$

(R6) $\tan\left(\frac{b}{r}\right) = \cos(A)\tan\left(\frac{c}{r}\right)$

(R7) $\tan\left(\frac{a}{r}\right) = \cos(B)\tan\left(\frac{c}{r}\right)$

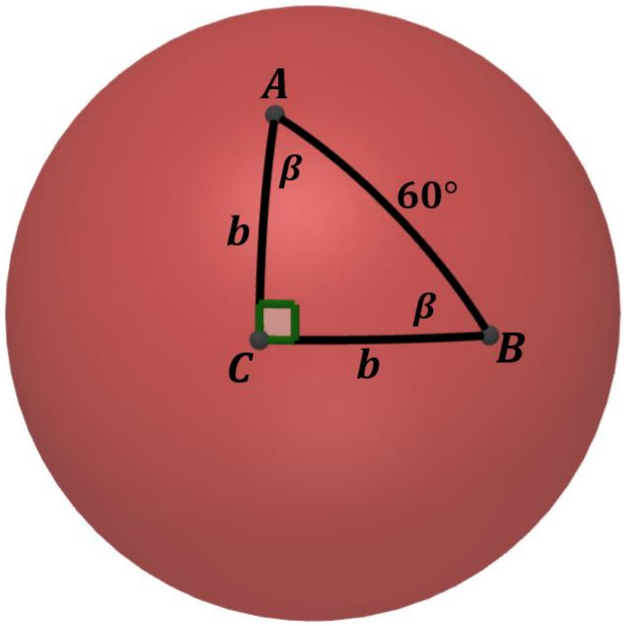
(R8) $\cos(A) = \sin(B)\cos\left(\frac{a}{r}\right)$

(R9) $\cos(B) = \sin(A)\cos\left(\frac{b}{r}\right)$

(R10) $\cos\left(\frac{c}{r}\right) = \cot(A)\cot(B)$

Exercise

Find b and β in the right isosceles spherical triangle ΔABC in the diagram below if the sphere has radius $r = 1$.



<u>Solution</u>

From (R1) we know $\cos\left(\frac{\pi/3}{1}\right) = \cos\left(\frac{b}{1}\right)\cos\left(\frac{b}{1}\right)$. So

$$\cos^2(b) = \cos\left(\frac{\pi}{3}\right) = \cos(60°) = \frac{1}{2}$$

$$\cos(b) = \frac{1}{\sqrt{2}} \Longrightarrow b = 45°.$$

From (R10) we know $\cos\left(\frac{\pi/3}{1}\right) = \cot(\beta)\cot(\beta)$. So

$$\cot(\beta) = \frac{1}{\sqrt{2}}.$$

or

$$\tan(\beta) = \sqrt{2}.$$

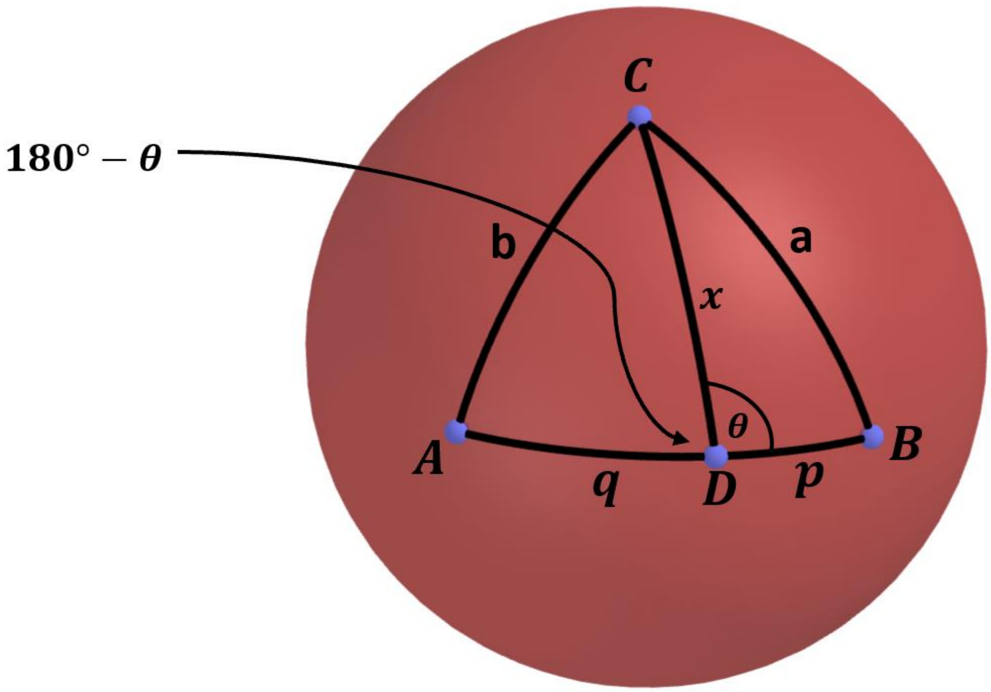
Therefore,

$$\beta = \tan^{-1}(\sqrt{2}) \approx 54.73561032°.$$

■

Spherical Stewart's Theorem

Consider the spherical triangle ABC as labeled in the figure below. Let $\widehat{CD}$ be a great arc drawn from C to $\widehat{AB}$. Let r be the radius of the sphere.



In this case,

$$\sin\left(\frac{p+q}{r}\right)\cos\left(\frac{x}{r}\right) = \sin\left(\frac{p}{r}\right)\cos\left(\frac{b}{r}\right) + \sin\left(\frac{q}{r}\right)\cos\left(\frac{a}{r}\right)$$

or

$$\cos\left(\frac{x}{r}\right) = \frac{\sin\left(\frac{p}{r}\right)\cos\left(\frac{b}{r}\right) + \sin\left(\frac{q}{r}\right)\cos\left(\frac{a}{r}\right)}{\sin\left(\frac{p+q}{r}\right)}.$$

Proof

From the spherical law of cosines for sides applied to $\Delta_S(BCD)$ and $\Delta_S(ACD)$ we have

$$\cos\left(\frac{a}{r}\right) = \cos\left(\frac{x}{r}\right)\cos\left(\frac{q}{r}\right) + \sin\left(\frac{x}{r}\right)\sin\left(\frac{q}{r}\right)\cos(\theta)$$

and

$$\cos\left(\frac{b}{r}\right) = \cos\left(\frac{x}{r}\right)\cos\left(\frac{p}{r}\right) + \sin\left(\frac{x}{r}\right)\sin\left(\frac{p}{r}\right)\cos(180-\theta)$$

$$= \cos\left(\frac{x}{r}\right)\cos\left(\frac{p}{r}\right) - \sin\left(\frac{x}{r}\right)\sin\left(\frac{p}{r}\right)\cos(\theta)\,.$$

Now multiply the first equation by $\sin(p/r)$ and the second equation by $\sin(q/r)$ to get

$$\cos\left(\frac{a}{r}\right)\sin\left(\frac{p}{r}\right) = \cos\left(\frac{x}{r}\right)\cos\left(\frac{q}{r}\right)\sin\left(\frac{p}{r}\right) + \sin\left(\frac{x}{r}\right)\sin\left(\frac{q}{r}\right)\sin\left(\frac{p}{r}\right)\cos(\theta)$$

$$\cos\left(\frac{b}{r}\right)\sin\left(\frac{q}{r}\right) = \cos\left(\frac{x}{r}\right)\cos\left(\frac{p}{r}\right)\sin\left(\frac{q}{r}\right) - \sin\left(\frac{x}{r}\right)\sin\left(\frac{p}{r}\right)\sin\left(\frac{q}{r}\right)\cos(\theta).$$

Adding these two resulting equations and simplifying we find

$$\begin{aligned}
&\cos\left(\frac{a}{r}\right)\sin\left(\frac{p}{r}\right) + \cos\left(\frac{b}{r}\right)\sin\left(\frac{q}{r}\right) \\
&\quad = \left(\cos\left(\frac{x}{r}\right)\cos\left(\frac{q}{r}\right)\sin\left(\frac{p}{r}\right) + \cos\left(\frac{x}{r}\right)\cos\left(\frac{p}{r}\right)\sin\left(\frac{q}{r}\right)\right) \\
&\quad = \cos\left(\frac{x}{r}\right)\left(\cos\left(\frac{q}{r}\right)\sin\left(\frac{p}{r}\right) + \cos\left(\frac{p}{r}\right)\sin\left(\frac{q}{r}\right)\right) \\
&\qquad\quad + \left(\sin\left(\frac{x}{r}\right)\sin\left(\frac{q}{r}\right)\sin\left(\frac{p}{r}\right)\cos(\theta) - \sin\left(\frac{x}{r}\right)\sin\left(\frac{p}{r}\right)\sin\left(\frac{q}{r}\right)\cos(\theta)\right) \\
&\quad = \cos\left(\frac{x}{r}\right)\left(\cos\left(\frac{q}{r}\right)\sin\left(\frac{p}{r}\right) + \cos\left(\frac{p}{r}\right)\sin\left(\frac{q}{r}\right)\right) \\
&\quad = \cos\left(\frac{x}{r}\right)\sin\left(\frac{p+q}{r}\right) \\
&\quad = \cos\left(\frac{x}{r}\right)\sin\left(\frac{c}{r}\right).
\end{aligned}$$

Thus

$$\cos\left(\frac{x}{r}\right) = \frac{\cos\left(\frac{a}{r}\right)\sin\left(\frac{p}{r}\right) + \cos\left(\frac{b}{r}\right)\sin\left(\frac{q}{r}\right)}{\sin\left(\frac{c}{r}\right)}.$$

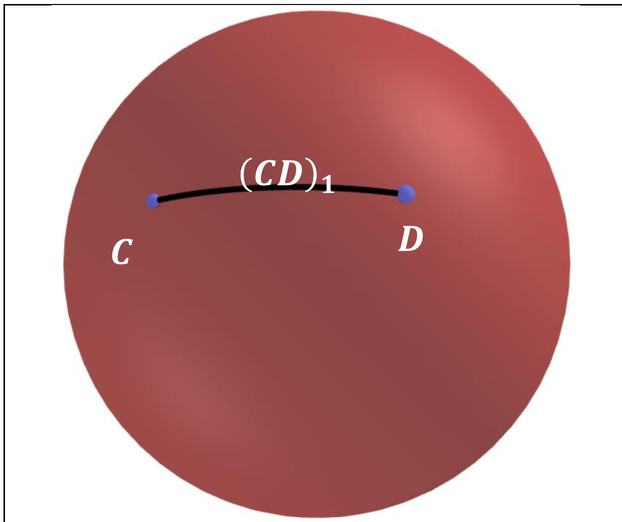
■

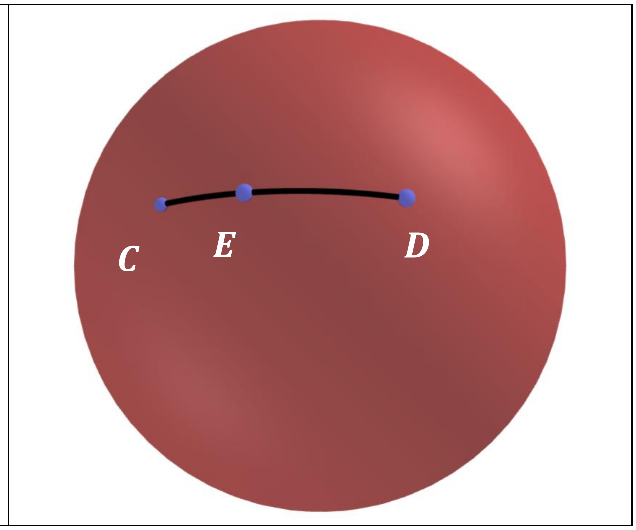
Appendix

<u>Theorem</u> The great arc $\widehat{AB}$ connecting points A and B on a sphere is the shortest possible path on a sphere connecting point A to point B.

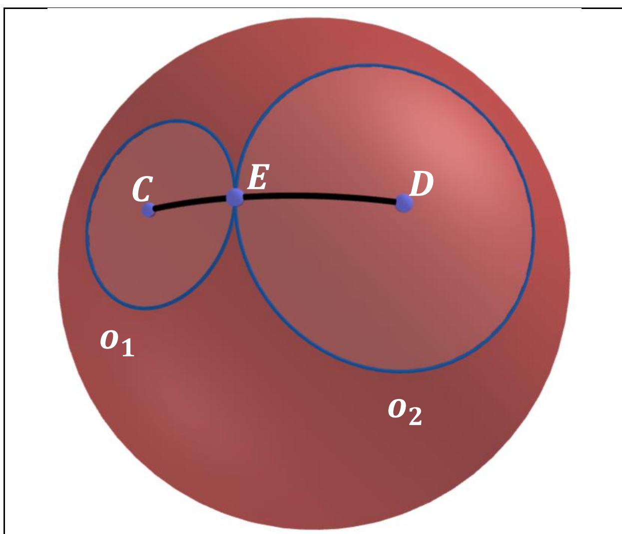
<u>Proof</u>

Let $(CD)_1$ be the (unique) ***great arc*** connecting points C and D and let E be some point on $(CD)_1$.

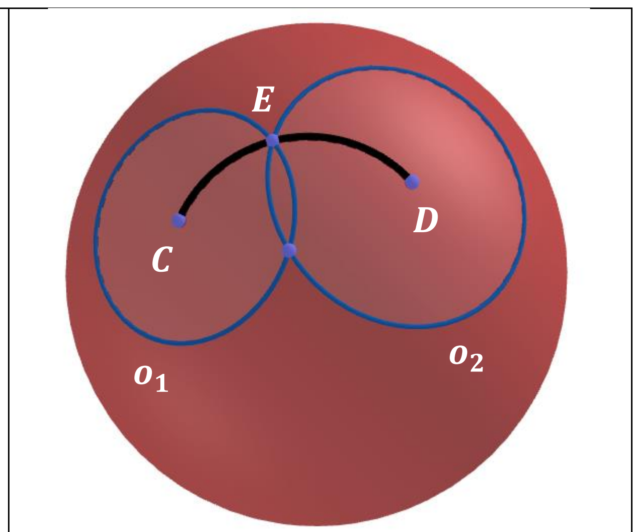




Draw the small circle o_1 centered at C and containing the point E and then draw the small circle o_2 centered at D and containing the point E. There are two potential cases.



Case 1.
Small Circles o_1 and o_2 Have <u>One</u> Point of Intersect

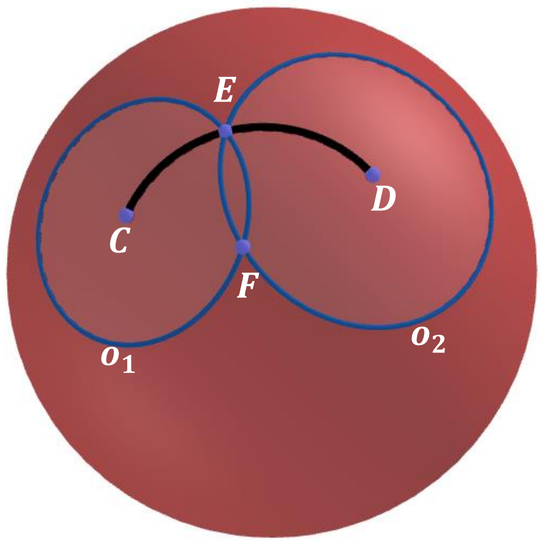


Case 2.
Small Circles o_1 and o_2 Have <u>Two</u> Points of Intersection

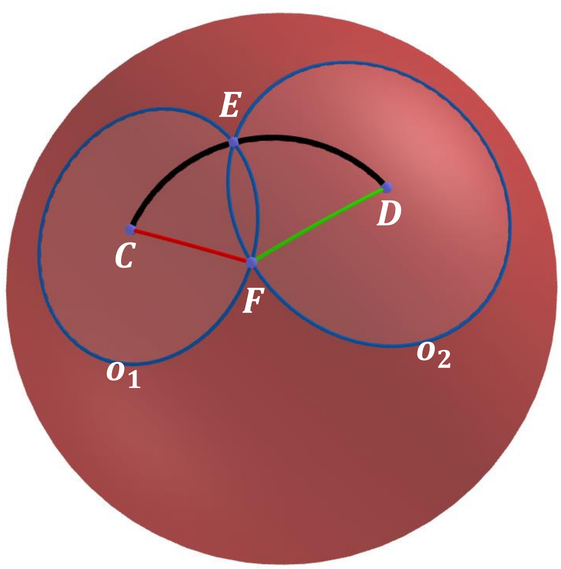
The first step in our proof is to eliminate Case 2. That is to show that the figure drawn below cannot really "in the real world".

We will prove this by contradiction. That is, we will assume Case 2 is possible and then show that this assumption leads us into a contradiction.

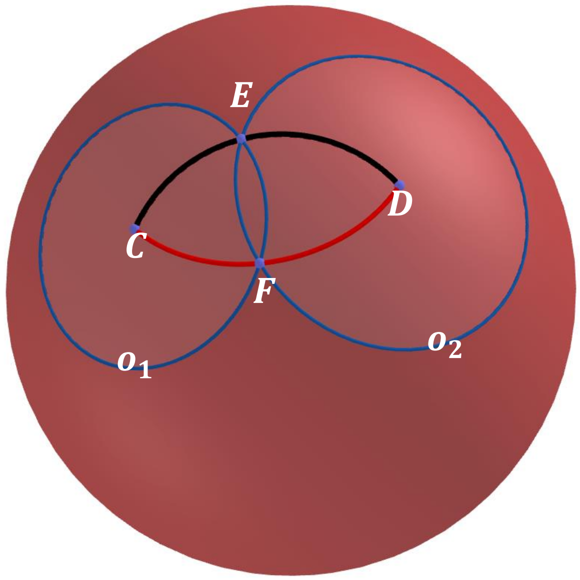
Label the second point of intersection of small circles o_1 and o_2 in Case 2 as point F.



Draw the great arc connecting C and F and the great arc connecting D and F.

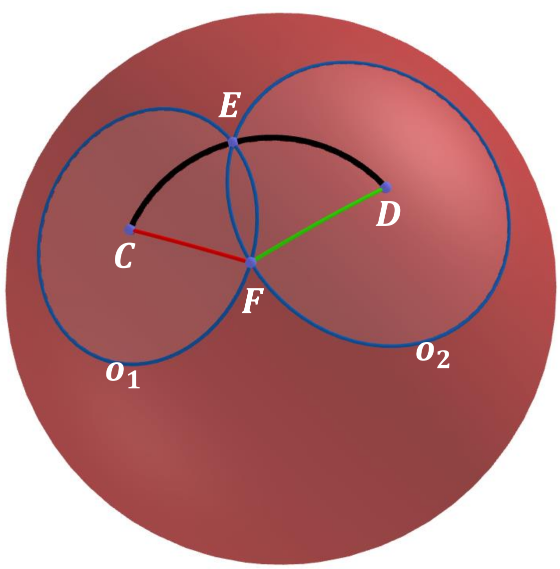


This yields the spherical triangle CDF (the three sides are CD, DF and FC) *unless* CF and DF are arcs on the *same* great circle



(which would mean "triangle" CDF only has two sides). But this cannot be true because if it were then CED and CFD would be distinct great circles connecting the points C and D, which violates the fact that great circles through two points are *unique*.

But could we have the situation where CF and DF are arcs on different great circles and CDF is a three-sided spherical triangle?



Let's consider this. We have already established the triangle inequality for spherical triangles. That is,

$$CF + FC > CD.$$

On the other hand, all arcs of small circles have the same length. That is,

$$CE = CF \text{ and } DE = DF.$$

But this means

$$CF + FC = CE + DE = CD.$$

And this contradicts the previous result that

$$CF + FC > CD.$$

So, we can rule out Case 2. It cannot possibly occur.

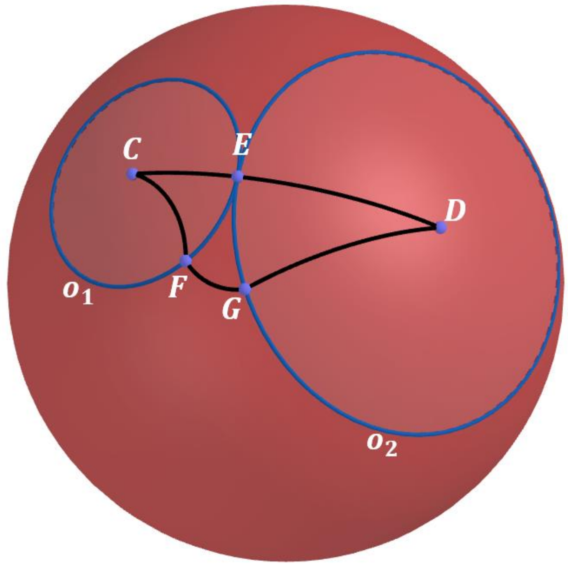
Let $(CD)_1$ be the great circle path connecting points C and D and let $(CD)_2$ be the shortest possible path connecting point C and D.

Let E be a point on the great circle path $(CD)_1$ and draw the small circle O_1 centered at C and containing the point E and the small circle O_2 centered at D and containing the point E.

Claim. The point E must be on path $(CD)_2$.

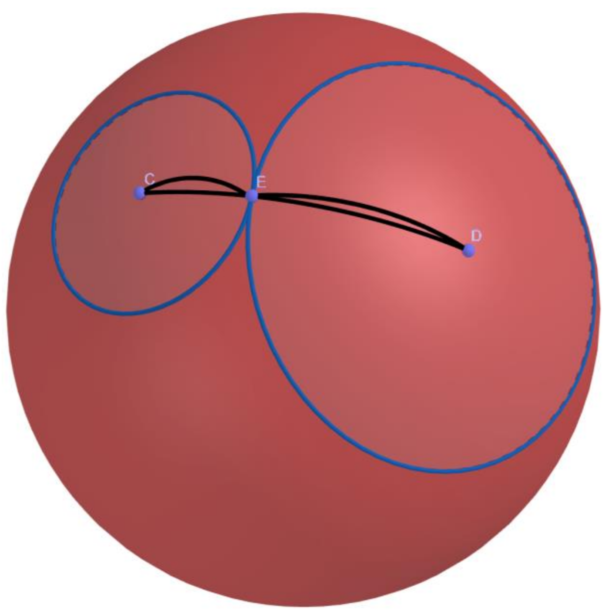
We will prove this claim by contradiction. So, assume E is not a point on $(CD)_2$.

Obviously, every path from C to D must eventually cross over small circle O_1 (at least once). Let F be the point where $(CD)_2$ crosses over O_1 for the *final time*. Similarly, let G be the point where $(CD)_2$ crosses over O_2 for the *final time*.



Because we are assuming that E is not on $(CD)_2$, the segment FG on $(CD)_2$ has positive length.

Now construct the path $(C[FE])_2$ by rotating the path $(CF)_2$ about the point C until F matches up with E. In the same way construct the path $(D[GE])_2$ by rotating the path $(DG)_2$ about the point D until G matches up with E. (illustrated below)



The new path $(C[FE])_2 + (D[GE])_2$ is still a path connecting C to D and is shorter than the path $CFGD$ which did not include the point E.

Thus, the path $CFGD$ which did not include the point E cannot possibly be the shortest path from C to D. That is, the point E on the great circle arc from C to D must be on $(CD)_2$, the theoretical shortest path from C to D.

But E is an arbitrary point on the great circle arc $\widehat{CD}$. So, the above argument shows that *every* point on the great circle arc $\widehat{CD}$ must be on $(CD)_2$.

And if $(CD)_2$ contains any extra points that are not on $\widehat{CD}$, then $(CD)_2$ would necessarily be longer than $\widehat{CD}$, which would contradiction the definition that $(CD)_2$ is the theoretical shortest path from C to D.

Hence, $(CD)_2 = \widehat{CD}$.

■